Exercises for Day 10: Riemannian Metrics: Examples

1. Metric Comparisons. Suppose that g and g^* are two Riemannian metrics on M. We say that g and g^* are *comparable* on M, if there exists a constant $\lambda \geq 1$ such that $\lambda^{-2}\, g(v) \leq g^*(v) \leq \lambda^2\, g(v)$ for all $v \in TM$. Show that comparability is an equivalence relation and that if g and g^* are comparable with comparison constant λ, then the metrics d and d^* are comparable, in the sense that $\lambda^{-1}\, d(p,q) \leq d^*(p,q) \leq \lambda\, d(p,q)$.

Suppose now that g is a Riemannian metric on M^n and fix $p \in M$. Choose a coordinate chart (V, ϕ) with $p \in V$ such that $\phi(p) = 0$ and $\phi(V)$ contains $B_2(0) \subset \mathbb{R}^n$. Show that there is a constant $\lambda > 1$ such that

$$\lambda^{-2}\, |\phi'(p)(v)|^2 \leq g(v) \leq \lambda^2\, |\phi'(p)(v)|^2$$

for all $v \in T_pM$ with $|\phi(p)| \leq 1$. Conclude that, if $\gamma : [0,1] \to M$ is a piecewise C^1 curve with $\gamma(0) = p$ such that $\gamma([0,1])$ is not contained in $\phi^{-1}\big(B_1(0)\big) \subset V$, then the length of γ is at least $1/\lambda$. In particular, for every $\delta < \lambda^{-1}$, we have $B_\delta(p) \subset \phi^{-1}\big(B_1(0)\big)$, where $B_\delta(p) = \{q \in M \,\big|\, d(p,q) < \delta\}$. (Hint: You will need to use the 'obvious' fact that, for the standard metric in $\mathbb{R}^n$, $d(x,y) = |x - y|$. For the proof of this, see Exercise 1 from Day 11.) More generally, show that

$$B_{\lambda^{-1}\delta}(0) \subset \phi\big(B_\delta(p)\big) \subset B_{\lambda\delta}(0)$$

for all $\delta < 1/\lambda$. Conclude that the metric topology of (M, d) is the same as the manifold topology. In particular, $B_\delta(p)$ is an open subset of M for all $\delta > 0$ and the function on M defined as the d-distance from p is continuous. Moreover, if M is compact (as a manifold), then (M, d) is complete, i.e., any d-Cauchy sequence in M converges.

2. The Poincaré upper half-plane. Let $H^+ = \{\,(x,y) \in \mathbb{R}^2 \,\big|\, y > 0\,\}$ and let $g = y^{-2}\big(\mathrm{d}x^2 + \mathrm{d}y^2\big)$. Show that the following maps from H^+ into itself are isometries of (H^+, g):

(a) $f(x,y) = (x + a, y)$ for any constant $a \in \mathbb{R}$.

(b) $f(x,y) = (rx, ry)$ for any constant $r \in \mathbb{R}^+$.

(c) $f(x,y) = \big(\dfrac{x}{x^2+y^2}, \dfrac{y}{x^2+y^2}\big)$. (Hint: If you are having trouble with the calculations, show that, for any differentiable curve $\gamma(t) = \big(x(t), y(t)\big)$, the curves γ and $f \circ \gamma$ have the same speed with respect to g.)

(d) Show that each of the above transformations (a–c) carries the set of vertical lines and semicircles meeting the x-axis orthogonally onto itself. Moreover, show that, for any two distinct points p and q in H^+, either they lie on a common vertical line or else they lie on a unique circle with its center on the x-axis.

(e) Show that, for any two distinct points p and q in H^+, there is an isometry $f : H^+ \to H^+$ such that $f(p) = (0,1)$ and $f(q) = (0,r)$ for some $r > 1$. (Hint: Use (d) together with the fact that any composition of g-isometries is a g-isometry.)

3. A left-invariant metric. Recall that, since $\mathrm{GL}(n,\mathbb{R})$ is an open subset of the vector space $M_{n,n}(\mathbb{R})$, we can identify $T\mathrm{GL}(n,\mathbb{R})$ with $\mathrm{GL}(n,\mathbb{R}) \times M_{n,n}(\mathbb{R})$. Define $g : T\mathrm{GL}(n,\mathbb{R})$ by

$$g(a,v) = \mathrm{tr}\big({}^t(a^{-1}v)\, a^{-1}v\big).$$

Verify that g is a Riemannian metric on $\mathrm{GL}(n,\mathbb{R})$ and that it has the *left-invariance property* that the map $L_a : \mathrm{GL}(n,\mathbb{R}) \to \mathrm{GL}(n,\mathbb{R})$ defined by $L_a(b) = ab$ is a g-isometry. (Hint: First, show that $L'_a(b,v) = (ab, av)$.) Show that $R_a : \mathrm{GL}(n,\mathbb{R}) \to \mathrm{GL}(n,\mathbb{R})$ defined by $R_a(b) = ba$ is not necessarily a g-isometry unless a lies in $O(n)$.

4. Homogeneous metrics. A Riemannian manifold (M, g) is said to be *homogeneous* if for any two points p and q in M, there exists a g-isometry $f : M \to M$ such that $f(p) = q$. Show that a homogeneous

Riemannian manifold is complete. (Hint: You'll need to use Exercise 1.) Conclude that the left-invariant metric defined on $\mathrm{GL}(n, \mathbb{R})$ in Exercise 3 is complete and that the Poincaré metric on the upper half-plane defined in Exercise 2 is complete.

5. A METRIC FROM CALCULUS. Recall that, for any positive C^1-function $f : [a, b] \to \mathbb{R}^+$, the area of the surface got by revolving the graph $y = f(x)$ about the x-axis is given by the formula

$$A(f) = 2\pi \int_a^b f(x)\sqrt{1 + f'(x)^2}\,\mathrm{d}x.$$

Notice that $A(f)$ is 2π times the g-length of the curve $y = f(x)$ where g is the Riemannian metric $g = y^2\left(\mathrm{d}x^2 + \mathrm{d}y^2\right)$ on the upper half-plane H^+. Show that this metric is *not* complete on H^+.

6. A METRIC FROM PHYSICS. The *Brachistochrone problem* is the famous problem of finding the shape of a wire joining two points lying in a vertical plane that will allow a bead sliding without friction on the wire and subject only to the acceleration due to gravity to go from the initial point to the final point in least time. A simple analysis (which you are invited to do) shows that, if the wire is represented by the graph $y = -f(x)$ for $f \geq 0$, then the time needed to slide from $(0, 0)$ to $\big(a, -f(a)\big)$ is given by the integral

$$T(f) = \frac{1}{\sqrt{2g}} \int_0^a \frac{\sqrt{1 + f'(x)^2}}{\sqrt{f(x)}}\,\mathrm{d}x$$

Notice that $T(f)$ is $(2g)^{-1/2}$ times the g-length of the curve $y = f(x)$ where g is the Riemannian metric $g = y^{-1}\left(\mathrm{d}x^2 + \mathrm{d}y^2\right)$ on the upper half-plane H^+. Show that this metric is also *not* complete on H^+.