Exercises for Day 14: Completeness

1. METRIC COMPLETENESS. Show that if a Riemannian manifold (M, g) is geodesically complete, then it is metrically complete. (Hint: It is enough to show that, for every $p \in M$ and every $\delta > 0$, the closure of the ball $B_\delta(p)$ in M is compact. Now use the Hopf-Rinow theorem to write this ball closure as a continuous image of a compact set.)

2. UNIFORMITY OF GEODESICS. Show that if K is a compact subset of M, then there exists a $\delta > 0$ such that, for every $p \in K$ and every $v \in T_pM$ with $|v| < \delta$, the geodesic with initial velocity v is defined on the interval $(-1, 1)$. In particular, if M is compact, then every geodesic can be extended to infinite length in both directions.

3. FAILURE OF COMPLETENESS. Show that if $\gamma : (a, b) \to M$ is a geodesic that cannot be extended beyond (a, b), then, for any compact set $K \subset M$, there is an $\epsilon > 0$ such that $\gamma(t)$ does not lie in K whenever $t < a + \epsilon$ or $t > b - \epsilon$. Consequently, show that there is a Cauchy sequence in M that does not converge. Conclude that, if M is metrically complete, then it is geodesically complete.