Exercises for Day 8: Vector Fields

1. Change of Variables. Suppose that (V,ϕ) and (V,ψ) are coordinate charts on the n-manifold M and that $\phi = (x^i)$ while $\psi = (y^i)$. (The indices i, j, k, etc., run from 1 to n.) Show that if a smooth vector field X on V is expanded in the form

$$X = a^i \frac{\partial}{\partial x^i}$$

for some functions $a^i \in C^\infty(V)$, then

$$X = a^i \frac{\partial y^j}{\partial x^i} \frac{\partial}{\partial y^j}.$$

(Of course, the summation convention is implicit in these formulae.) (Hint: Remember that $a^i = Xx^i$.)

2. The Lie bracket. Recall that if X and Y are smooth vector fields on M, then we defined the *Lie bracket* of X and Y to be the unique vector field $[X,Y]$ such that $[X,Y]f = X(Yf) - Y(Xf)$ for all $f \in C^\infty(M)$. Verify the following properties of the Lie bracket, where X, Y, and Z are smooth vector fields on M and f and g are smooth functions on M

(a) $[X,Y] = -[Y,X]$.
(b) $[X,Y+Z] = [X,Y] + [X,Z]$.
(c) $[X,fY] = (Xf)Y + f[X,Y]$.
(d) $\bigl[X,[Y,Z]\bigr] = \bigl[[X,Y],Z\bigr] + \bigl[Y,[X,Z]\bigr]$.

This last identity, a sort of Leibnitz rule for the Lie bracket as a multiplication, is known as the *Jacobi identity* and is often written in the more symmetric form $\bigl[X,[Y,Z]\bigr] + \bigl[Y,[Z,X]\bigr] + \bigl[Z,[X,Y]\bigr] = 0$. Show that this alternate form is implied by items (a) and (d).

3. The group $\mathrm{GL}(n,\mathbb{R})$. Remember that $\mathrm{GL}(n,\mathbb{R})$ is an open subset of $M_{n,n}(\mathbb{R})$ and hence that, as a smooth manifold, we can identify $T_A\mathrm{GL}(n,\mathbb{R})$ with $M_{n,n}(\mathbb{R})$. For each $a \in M_{n,n}(\mathbb{R})$, define a vector field X^a on $\mathrm{GL}(n,\mathbb{R})$ by $X^a(g) = ga$. Show that X^a has the following properties:

(a) For any $h \in \mathrm{GL}(n,\mathbb{R})$, define $L_h : \mathrm{GL}(n,\mathbb{R}) \to \mathrm{GL}(n,\mathbb{R})$ by $L_h(g) = hg$. Then $L'_h(g)\bigl(X^a(g)\bigr) = X^a(hg) = X^a\bigl(L_h(g)\bigr)$. (This property is known as *left invariance* because it says that X^a is unchanged under the mapping induced by 'left translation'.)
(b) X^a is complete on $\mathrm{GL}(n,\mathbb{R})$. In fact, the flow $\Phi^a : \mathbb{R}\times\mathrm{GL}(n,\mathbb{R}) \to \mathrm{GL}(n,\mathbb{R})$ of X^a is given by $\Phi^a(t,g) = g\,e^{ta} = g\exp(ta)$.
(c) For any $a,b \in M_{n,n}(\mathbb{R})$, we have $[X^a,X^b] = X^{[a,b]}$ (where $[a,b] = ab - ba$).
(d) If $a = -{}^ta$, then X^a is tangent to $O(n)$ in the sense that, for every $g \in O(n) \subset \mathrm{GL}(n,\mathbb{R})$, we have $X^a(g) \in T_gO(n)$. (Note, also, that, if $b = -{}^tb$ as well, then $[a,b] = -{}^t[a,b]$.)

4. Show that, if M is compact, then any smooth vector field X on M is complete. More generally, show that if the support[1] of X is compact, then it is complete. (Hint: First, establish that there is an $\varepsilon > 0$ such that the flow of X is defined for all $|t| < \varepsilon$. Then show that this property alone implies that the flow of X is defined for all time t.

5. Let X and Y be smooth vector fields on M and let $\Phi_X : U_X \to M$ and $\Phi_Y : U_Y \to M$ be their (locally defined) flows. We say that their flows *commute* if $\Phi_X(s,\Phi_Y(t,p)) = \Phi_Y(t,\Phi_X(s,p))$ for all $p \in M$ and for all s and t satisfying $|s|,|t| < \varepsilon$ for some $\varepsilon > 0$. Show that, if their flows commute, then $[X,Y] = 0$. Hint: for any function f defined on M and any point $p \in M$, define the function $F(s,t) = \Phi_X(s,\Phi_Y(t,p)) = \Phi_Y(t,\Phi_X(s,p))$ on the open set in the st-plane defined by $|s|,|t| < \varepsilon$. Now compute $[X,Y]f$ evaluated at p directly from the definitions in terms of F.

[1] The support of a vector field is the closure of the set of points at which it is nonzero.