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On differential invariants of geometric structures

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Abstract. We prove that if the fibre dimension m of a bundle of geometric structures exceeds the dimension n of its base, then the number of sufficiently general functionally independent local differential invariants of the bundle increases to infinity as the differential degree of these invariants grows. For $m \leqslant n$ we describe all but two canonical forms to which every sufficiently general geometric structure can be reduced by an appropriate coordinate change on the base. The results obtained may be generalized.

§ 1. Introduction

In §§ 1–5 of this paper, all objects (functions, manifolds and so on) are assumed to be real and infinitely differentiable unless otherwise stated. Bundles are assumed to be locally trivial. The term "smooth" means "infinitely differentiable".

A typical example of a bundle of geometric objects over an n-dimensional manifold X is the tangent bundle $T(X)$. This bundle may be obtained as follows. The coframe bundle $\operatorname{Rep}_1(X)$ over X is a (right) principal bundle for the group GL_n of all real non-singular matrices of order n. We consider the (left) regular representation of GL_n on the real n-dimensional vector space $\mathbb{R}^n$. Then $T(X)$ is obtained from the direct product $\operatorname{Rep}_1(X) \times \mathbb{R}^n$ by identifying points $(e, r) \sim (eg, g^{-1}r)$ for all $g \in G$, $e \in \operatorname{Rep}_1(X)$, $r \in \mathbb{R}^n$. A bundle of geometric objects is a generalization of this construction. First, we replace $\operatorname{Rep}_1(X)$ by the bundle $\operatorname{Rep}_q(X)$ of coframes of order q. This is a principal bundle for the differential group $G^q(n)$ of order q (and dimension $n = \dim X$). Second, let α be an action of the connected component $G^q(n)^0$ of the identity in $G^q(n)$ on a manifold D. Then the bundle P of geometric objects with base X and generic fibre D is obtained from the direct product $\operatorname{Rep}_q(X) \times D$ by identifying points $(e, r) \sim (eg, g^{-1}r)$ for all $g \in G$, $e \in \operatorname{Rep}_q(X)$, $r \in \mathbb{R}^n$. The sections of P are called *geometric objects* (*structures*).

Every choice of a coordinate system F near a point $b \in X$ determines an isomorphism $t_F \colon D_b \to D$, where D_b is the fibre of the bundle $P \to X$ at b and D is the generic fibre. If F, G are coordinate system of the same orientation near b, then the passage from F to G at b determines an element $g \in G^q(n)^0$, and the corresponding transformation $\alpha(g) \colon D \to D$ coincides with $t_F t_G^{-1}$. One usually constructs geometric objects from an action of $G^q(n)$ (not $G^q(n)^0$). Then the transformation rule α of the geometric object is defined for all non-singular coordinate changes. Our definition of the bundle of geometric structures corresponds to the case when the base X is oriented and the transformation rule is defined only for orientation-preserving coordinate changes. We denote by $\mathscr{V}$ the group of all

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orientation-preserving diffeomorphisms of X. Each element of this group induces a diffeomorphism of the bundle P of geometric objects over X.

Important examples of geometric objects are Riemannian metrics, and a typical example of a differential invariant is the scalar curvature. This is a scalar function on a Riemannian manifold, and its value in any local coordinate system is given by the same expression depending on the components g_{ij} of the metric tensor and its first and second derivatives $\frac{\partial g_{ij}}{\partial x^k}$, $\frac{\partial^2 g_{ij}}{\partial x^k \partial x^r}$, where $x^1, x^2, \dots, x^n$ are the local coordinates on the manifold. More generally, a function of the components of a geometric object and their derivatives up to order k is a *differential invariant of degree* k if the calculation of this function at a given point gives the same result for all local coordinate systems. All considerations in this paper are local (in a neighbourhood of a given point), and differential invariants are also defined locally, in a neighbourhood of a given value of the arguments. A natural domain for a differential invariant is the manifold of jets of sections of the corresponding bundle of geometric objects. As a rule, we consider differential invariants only at sufficiently general points of the jet manifold. We define the notion of a sufficiently general point and prove that such points form an open dense invariant subset of the corresponding manifold.

The study of differential invariants was initiated in the classical works [1], [2] and continued in [3], [4]. (An essential part of [4] is explained in § 6 of the textbook [5].) Differential invariants were also studied in [6]–[8] (including invariants for finite-dimensional Lie groups).

In § 2 of this paper we prove (Theorem 1) that if $P \to X$ is a bundle of geometric objects such that the dimension of its generic fibre D exceeds that of its base X, then the number $t(k)$ of functionally independent differential invariants of degree k at any sufficiently general point tends to infinity as k grows. We also give an asymptotic lower bound for $t(k)$. The exact value of $t(k)$ for sufficiently general Riemannian metrics and other concrete geometric objects is given in [9]. At the end of § 3 we prove Theorem 2, which generalizes Theorem 1. One can also generalize Theorem 1 to the case of bundles attached to flat G-structures. The author hopes to return to this question in the future.

In § 5 we consider bundles P of geometric objects such that the dimension m of their generic fibre Y does not exceed $n = \dim X$. Such manifolds Y are said to be *special*, and the corresponding bundles P are called *special bundles*. To keep the notation simple, we study only the case $n > 2$. The case $n = 1$ is trivial, and one can easily make the necessary changes for $n = 2$.

There are two exceptional special manifolds:

a) the 4-dimensional manifold Y_{gr}, which is the Grassmannian of 2-dimensional planes in a 4-dimensional space with the natural action of GL_4;

b) the 3-dimensional manifold Y_{fl} of flags of type $(1, 2)$ in a 3-dimensional space with the natural action of GL_3, whose stabilizers are its Borel subgroups.

Among the other special manifolds, we distinguish 17 types of (non-exceptional) *sample manifolds* $Y_1, \dots, Y_{17}$. They determine 17 types of (non-exceptional) special bundles $P_1, \dots, P_{17}$, which are referred to as *sample bundles*. The manifolds Y_{gr} and Y_{fl} determine (exceptional) sample bundles P_{gr} and P_{fl}, giving two further (exceptional) types of sample bundles. The (partial) action of $\mathscr{V}$ near a regular point of any special bundle (exceptional or not) has the same structure as its action near an appropriate point of some sample bundle.

It is known that a sufficiently general differential form on X can be locally reduced to a canonical form by an appropriate choice of coordinate system (Darboux' theorem). One can also straighten a non-zero vector field on X. A similar fact holds for sufficiently general sections of any non-exceptional special bundle: the section reduces to a canonical form (depending on the type) in an appropriate coordinate system on X. This is proved in Theorem 5. The manifolds Y_{gr}, Y_{fl} and the bundles P_{gr}, P_{fl} will be studied in a future paper, where the results of this paper will be used to prove that there are only finitely many functionally independent differential invariants at every sufficiently general point of any special manifold.

In § 4 and § 7 we list all connected Lie subgroups of codimension at most n in SL_n and $G^q(n)^0$ for $n > 2$. These results are used in § 5.

The results of this paper have applications to a theorem of Tresse [3], which we hope to publish separately.

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§ 2. Preliminaries

2.1. We introduce the following notation. Let $\mathbb{R}$ be the field of real numbers, $\mathbb{R}^n$ the n-dimensional arithmetic space, $\mathbb{R}^{n*}$ its dual space and $\mathbf{0}$ the zero vector in $\mathbb{R}^n$ or $\mathbb{R}^{n*}$. The standard coordinates of an element t in $\mathbb{R}^n$ are denoted by $t^1, \dots, t^i, \dots, t^n$, and the element itself is written as $t = (t^i)$. The standard coordinates of an element t in $\mathbb{R}^{n*}$ are denoted by $t_1, \dots, t_i, \dots, t_n$, and the element itself is written as $t = (t_i)$. Unless otherwise stated, we assume summation over repeated upper and lower indices. We denote by $\mathbb{R}^0$ the group of positive real numbers under multiplication.

Let $\dim X$ be the dimension of a manifold (or a vector space) X, and let $\operatorname{codim}(X, Y)$ be the codimension of a submanifold X in a manifold Y (resp. of a vector subspace X in a vector space Y). All neighbourhoods are assumed to be open. The closure of a set W is denoted by $\overline{W}$.

By a Lie group we mean a real finite-dimensional Lie group. We denote the connected component of the identity in a Lie group G by G^0, and the centre of G is denoted by $Z = Z(G)$. A manifold X acted on by a Lie group G is called a *G-manifold*. We denote the stabilizer of a point $b \in X$ by G_b. The connected component of the identity in G_b is denoted by G_b^0. We write GU for the image of a set U under the action of G. The Lie algebra of a Lie group G (the tangent space at the identity) is denoted by $\operatorname{Lie}(G)$. If H is a Lie subgroup of G, and $\mathcal{H}$, $\mathcal{G}$ are their Lie algebras, then $\operatorname{codim}(G, H) = \operatorname{codim}(\mathcal{G}, \mathcal{H})$. The commutator of elements x and y in a Lie algebra is denoted by $[x, y]$. We write G/H for the space of left cosets of G with respect to a subgroup H. As usual, Lie subgroups H of G may be non-closed in G. Given an element g of a Lie group G, we denote by $\operatorname{Ad}(g)$ the inner automorphism of $\operatorname{Lie}(G)$ induced by the inner automorphism $x \to gxg^{-1}$ of G. Let GL_n (resp. SL_n) be the group of real matrices (resp. the subgroup of unimodular matrices) of order n, and let $GL_n,\ SL_n$ be their Lie algebras. The connected component of the identity in GL_n is denoted by GL_n^0. When we write $A = (a^i_j)$, the superscript is the row number and the subscript is the column number of the matrix A. Given a matrix A, we denote the inverse (resp. transposed) matrix by A^{-1} (resp. A^T). Let E_n be the identity matrix of order n. We write $\operatorname{diag}(a_1, \dots, a_i, \dots, a_n)$ for the diagonal matrix with elements $a_i = a^i_i$ on the diagonal. If $a_i = a$ for all i, then we denote this matrix by $\mathrm{sk}_n(a)$.

Let L be a vector space of dimension n. We denote the dual space by L^*. Given a basis of L, we write $t = (t^i)$ for the vector $t \in L$ with coordinates $(t^1, \dots, t^i, \dots, t^n)$ in this basis. Let ST_n be the simplest (identity) representation of the group GL_n on the space $\mathbb{R}^n$: the action of a matrix $A = (a^i_j)$ on a vector $t = (t^i) \in \mathbb{R}^n$ (regarded as a column) is given by multiplying the column by the matrix. This yields the column vector $(a) = A(t) = At$ with the coordinates

$$a^i = a^i_j t^j$$

(we recall that this notation implies summation over j).

We consider the representation of GL_n on $\mathbb{R}^{n*}$ that is dual to the standard (identity) representation: the action of a matrix $A = (a^i_j)$ on a vector $t = (t_i) \in \mathbb{R}^{n*}$ (regarded as a row) is given by multiplying this row by A^{-1}. This yields the row vector $a = A(t) = tA^{-1}$. Writing $A^{-1} = (p^i_j)$, we have

$$a_j = t_i p^i_j.$$

We denote this representation by ST_n^*.

The direct product of groups (manifolds and so on) G_1 and G_2 is written $G_1 \times G_2$. Elements $g \in G_1 \times G_2$ are identified with pairs (g_1, g_2), where g_i is the projection of g onto the ith factor. We write $g = (g_1, g_2)$. The semidirect product of a normal subgroup G_2 and a subgroup G_1 is denoted by $G_1 \leftthreetimes G_2$. We write δ^i_j for the Kronecker delta and denote the trace and determinant of a matrix A by $\operatorname{Sp} A$ and $\det A$ respectively.

The *jet of order* K (or simply the k*-jet*) *of a function* $f\colon X \to \mathbb{R}$ at a point b of a manifold X is the class of all functions that are tangent of order k to f at b. Clearly, the notion of a jet is independent of the choice of a coordinate system on X and may also be defined for maps of X to another manifold Y. The jet of order k of a function (map) f at b is denoted by $j_k f(b)$ or $f_k(b)$. If we choose a coordinate system on X, then the jet $f_k(b)$ is determined by those terms of the Taylor series of f at b whose total degree does not exceed k. The homogeneous jet of order k is determined by the terms of total degree k in the Taylor series. (This notion depends on the choice of coordinate system on X.)

A map of a set X to a set Y that sends a point $b \in X$ to a point $d \in Y$ is said to be *pointed* and is denoted by $(X, b) \to (Y, d)$. The kth order jets of pointed maps $(\mathbb{R}^n, \mathbf{0}) \to (\mathbb{R}^n, \mathbf{0})$ form a Lie group, which is denoted by $G^q(n)$ and called the *differential group of dimension* n *and order* q. The connected component of the identity in $G^q(n)$ is denoted by $G^q(n)^0$. Clearly, $G^1(n) = \mathrm{GL}_n$. The tangent space to a manifold X at a point b is denoted by $T_b(X)$, and the tangent bundle of X by $T(X)$. Let $\varphi\colon X \to Y$ be a map and let g be a function on Y.

Then the function sending every $b \in X$ to $g\bigl(\varphi(b)\bigr)$ is said to be *induced* by φ and denoted by $\varphi^*(g)$.

Let b be a point of a manifold X. As usual, a *germ of a function* (map and so on) at b is determined by an infinitely differentiable function (map and so on) f defined on a neighbourhood U_f of b. By definition, functions f, g determine the same germ at b if there is a neighbourhood V of b such that $U_f \cap U_g \supset V$ and $f = g$ on V. The ring of germs at b of all functions on X is denoted by $O_b(X)$ and called the *local ring of the point b on X*. Every map $\varphi\colon X \to Y$ induces a map $\varphi^*\colon O_b(Y) \to O_b(X)$.

A chart $F = (U, t^i)$ on a manifold X consists of a neighbourhood $U \subset X$ and local coordinates $t^1, \dots, t^i, \dots, t^n$ in U. We denote differentiation with respect to the ith coordinate by ∂_i, and the corresponding tangent vector by $\frac{\partial}{\partial f^i}$. The notation F for a chart always means that we have a coordinate neighbourhood U_F and local coordinates f^i on U_F. In other words, by simply writing F, we mean $F = (U_F, f^i)$. Similarly, $H = (U_H, h^i)$ and so on. We write $\mathrm{or}(F) = \mathrm{or}(H)$ $\bigl($resp. $\mathrm{or}(F) = \mathrm{or}(X)\bigr)$ if the charts F and H (resp. the chart F and the manifold X) have the same orientation. By writing $b \in F$, we mean that $b \in U_F$. We omit the subscript of U if it is clear which chart on X is meant. A covering of X by charts is called an *atlas* on X. A chart F is said to be *centred at b* (and called a b-*chart*) if its local coordinates f^i satisfy $f^i(b) = 0$. We always assume that the base X of any bundle of geometric objects is oriented and include the condition $\mathrm{or}(F) = \mathrm{or}(X)$ in the definition of a b-chart.

Differential forms will often be referred to simply as *forms*, and 1-*forms* are differential forms of degree 1. We shall use the invariance of the sign of a non-degenerate top-degree form on X under orientation-preserving diffeomorphisms of X. The fibre $\pi^{-1}(b)$ of a bundle $\pi\colon P \to X$ over a point b is denoted by P_b.

2.2. We shall use the notion of a geometric structure (see the survey [10]). Every manifold X of dimension n invariantly determines a manifold $\mathrm{Rep}_q(X)$ of coframes of order q. This is a (right) principal bundle over X with an action of the differential group $G^q(n)$ of order q on the fibre. We consider only arcwise-connected manifolds X. Given a (left) action $\alpha\colon G^q(n)^0 \times D \to D$ of $G^q(n)^0$ on D, we consider the bundle $\pi^D\colon P = P_X(D) \to X$ associated to $\mathrm{Rep}_q(X)$. We call P the *bundle of geometric objects of type D* and *order q*, or simply the D-*bundle*, and D is called the *generic fibre* of P. A section $s\colon X \to P$ defined on some neighbourhood in X is called a (*local*) *geometric structure* or a (*local*) *geometric object*.

Remark 1. For any positive integers q, s with $q \geqslant s$, there is a natural projection $g(q,s)\colon G^q(n)^0 \to G^s(n)^0$. We define the *exact order* of the action α to be the minimal s such that α is induced by an action of $G^s(n)^0$ on D.

Remark 2. Unless it affects a proof, we shrink X whenever necessary and do not indicate this explicitly. Hence we shall always assume that all local geometric objects, functions, coordinate systems and so on are defined on the whole of X.

Definition 1. Let α be an action of a group G on a manifold D. The α-*rank* (α-*corank*) of a point $d \in D$ is the dimension (resp. codimension in D) of the α-orbit of G passing through d. We denote the α-rank and α-corank of d by $\mathrm{rk}_\alpha(d)$ and $\mathrm{cor}_\alpha(d)$ respectively. The point d is said to be α-*regular* if the function $\mathrm{rk}_\alpha(d)$ (equivalently, $\mathrm{cor}_\alpha(d)$) is constant in some neighbourhood of d. The function $\mathrm{rk}_\alpha(d)$ $\big($resp. $\mathrm{cor}_\alpha(d)\big)$ is lower (resp. upper) semicontinuous.

Clearly, the orbit of an α-regular point consists entirely of α-regular points. Such an orbit is said to be *α-regular.* If a manifold D consists only of α-regular points, it is said to be *α-regular.*

Definition 2. Let α be an action of a Lie group G on a manifold D and let d be a point of D. A germ of a function f on D with domain $U_f \ni d$ is called an *α-invariant at d* (or an *α-invariant defined at d*) if f is constant on every connected component of the intersection of D with each orbit of G. Clearly, the set of all α-invariants at d is a subring of $O_d(D)$. We denote this subring by $I_d(D;\alpha)$ and call it the (*local*) *ring of α-invariants* (at d). One can also say that $I_d(D;\alpha)$ is the local ring of invariants of the point d with respect to the induced action of the Lie algebra $\mathrm{Lie}(G)$ on X. If β is an action of G on a manifold Y and $\varphi\colon D \to Y$ is a map commuting with the action of G, then φ induces a map $\varphi^*\colon I_y(Y;\beta) \to I_d(D;\alpha)$ (where $y = \varphi(d)$).

Remark 3. When it is clear which action α is considered, we omit the symbol α in our notation and use the terms "rank", "corank", "regular point", "ring $I_d(D)$ of invariants" instead of "α-rank", "α-corank", "α-regular point", "ring $I_d(D;\alpha)$ of α-invariants". This also applies to all the concepts and notation (to be introduced below) depending on an action α of G on D.

Lemma 1. *The regular points of any G-manifold D form a subset W which is open, dense in D and invariant under the action of G.*

Proof. (We recall that G, D and the action of G on D are assumed to be infinitely differentiable.) Let W^1 be the set of points belonging to orbits of maximal dimension. Being upper semicontinuous, the dimension of the orbit can only increase after a small perturbation of the point. Hence the set W^1 is open (and invariant). Now consider the complement $D_1 = D - \overline{W}^1$ of the closure of W^1, and let W^2 be the set of points belonging to orbits of maximal dimension in D_1. As for W_1, one can prove that W^2 is open. Putting $D_2 = D - \overline{W}^2$ and repeating this process, we get an open invariant set $W = W^1 \cup W^2 \cup \dots \cup W^n$ which is dense in D, as required.

Remark 4. If D is an analytic manifold, then W^1 is dense in D. This is because W^1 is given locally by the condition that some (non-square) matrix of analytic functions on D must have maximal rank. These functions determine the action of $\mathrm{Lie}(G)$ on the tangent bundle of D. We do not go into details.

2.3. For every bundle $\pi\colon P \to X$ we have a manifold J^kP of k-jets. (This manifold is formed by the k-jets of local sections $s\colon X \to P$.) We shall denote the manifold of k-jets of the bundle $P = P_X(D) \to X$ by $J^k_X(D)$, $J^k(D)$ or J^k, and we put $J^0 = P$, $J^{-1} = X$. For every pair of integers k, s with $k \geqslant s \geqslant -1$, there is a natural projection $\pi(k,s)^D_X\colon J^k_X(D) \to J^s_X(D)$. These projections satisfy the following condition for every triple k, s, d of integers with $k \geqslant s \geqslant d \geqslant -1$:

$$\pi(s,d)^D_X \pi(k,s)^D_X = \pi(k,d)^D_X.$$

We put $\pi(k,-1)^D_X = \pi^D_k$ and $\pi(k,0)^D_X = \beta^D_k$. Then π^D_k is the projection of J^k onto X, β^D_k is the projection of J^k onto P and $\pi(0,-1)^D_X$ is the projection $\pi\colon P\to X$. We have

$$\pi^D_s\pi(k,s)^D_X = \pi^D_k, \qquad \beta^D_s\pi(k,s)^D_X = \beta^D_k, \qquad \pi\beta^D_k = \pi^D_k.$$

We omit the superscript D and the subscript X in the notation $\pi(k,s)^D_X$, π^D_k, β^D_k whenever it is clear which manifolds X, D are being considered.

The construction of the manifold of k-jets is functorial. In other words, for every $k\geqslant 1$ and every diffeomorphism $\varphi\colon X\to T$ there is a map $\varphi(k)^D\colon J^k_X(D)\to J^k_T(D)$ with the usual functorial properties. Let $\mathscr{W}(T)$ be the group of diffeomorphisms of an arbitrary manifold T, $\mathscr{V}(T)$ the subgroup of all orientation-preserving diffeomorphisms in $\mathscr{W}(T)$ and $\mathscr{V}_b(T)$ the subgroup of all diffeomorphisms in $\mathscr{V}(T)$ that fix the point b. We write $\mathscr{W}$, $\mathscr{V}$, $\mathscr{V}_b$ instead of $\mathscr{W}(X)$, $\mathscr{V}(X)$, $\mathscr{V}_b(X)$ respectively. The group $\mathscr{V}$ acts on the manifold $P = P_X(D)$. For every $\varphi\in\mathscr{V}$ there is a diffeomorphism $\psi = \psi(D,\varphi)\colon P\to P$ generating a commutative diagram

$$\begin{array}{ccc} \psi\colon P & \longrightarrow & P \\ \big\downarrow & & \big\downarrow \\ \varphi\colon X & \longrightarrow & X \end{array}$$

The diffeomorphism φ acts on a section $s\colon X\to P$ by sending it to the section $s_\varphi\colon X\to P$ determined by $\psi s = s_\varphi\varphi$. Thus generates an action $\varphi\to\varphi(k)^D$ of $\mathscr{V}$ on the manifolds $J^k_X(D)$, and this action commutes with $\pi(k,s)$, π_k, β_k. We put $\varphi(-1)^D = \varphi$ and $\varphi(0)^D = \psi(D,\varphi)$.

2.4. The following definitions are analogous to Definitions 1 and 2. The relation between these definitions will be clarified in § 3.2. We consider differential invariants (see Definition 4) only locally, in a neighbourhood of a point $a\in J^k_X(D)$.

Definition 3. A point $a\in J^k = J^k_X(D)$ is said to be $\mathscr{V}$-*regular* if it has a neighbourhood U such that the dimension $d(b)$ of the $\mathscr{V}$-orbit passing through b is the same for all $b\in U$. The number $\operatorname{rk}_{\mathscr{V}}(a) = d(a)-n$ is called the $\mathscr{V}$-*rank* of a regular point a. The codimension of the $\mathscr{V}$-orbit passing through a is called the $\mathscr{V}$-*corank* of the regular point a and is denoted by $\operatorname{cor}_{\mathscr{V}}(a)$. Clearly, $\operatorname{cor}_{\mathscr{V}}(a) = \dim J^k - d(a)$.

Definition 4. Let a be a point of the manifold $J^k = J^k_X(D)$. The germ of a function f on J^k defined on $U_f\ni a$ is called a *differential invariant* (defined) at a (or a $\mathscr{V}$-*invariant*) if f is constant on every connected component of the intersection of U_f and every orbit of $\mathscr{V}$. The number k is called the *degree of the differential invariant.*

Clearly, the set of all $\mathscr{V}$-invariants at a is a subring of $O_a(J^k)$. It is denoted by $I_a(J^k;\mathscr{V})$ and called the (*local*) *ring of differential invariants* (of degree k) or $\mathscr{V}$-*invariants* (at a). Since the maps $\pi(k,s)\colon J^k\to J^s$ commute with the action of $\mathscr{V}$, they induce maps $\pi(k,s)^*\colon I_a(J^k;\mathscr{V})\to I_u(J^s;\mathscr{V})$, where $\pi(k,s)(a) = u$. The *exact degree* of a differential invariant $f\in I_a(J^k;\mathscr{V})$ is defined as the minimal s such that there is $g\in I_u(J^s;\mathscr{V})$ with $\pi(k,s)^*(g) = f$, where $\pi(k,s)(a) = u$. We shall usually omit the symbol $\mathscr{V}$ in notation relating to the action of $\mathscr{V}$ when it is clear which action is considered. This also applies to all the notation and definitions below concerning actions of $\mathscr{V}$.

2.5. Let a be a $\mathscr{V}$-regular point of J^k and d a positive integer. Any set of d differential invariants F_i at a is called an $(a; d; \mathscr{V})$-*set* (or a *set* for short) and is denoted by $F = (F^i)$. We define the *rank* of the set F at a (denoted by $\mathrm{rk}_F(a)$) as the rank of the Jacobian matrix $\bigl(\frac{\partial F^i}{\partial z^j(a)}\bigr)$, where $z^1, \dots, z^s$ is any local coordinate system in a neighbourhood of a. Clearly, $\mathrm{rk}_F(a)$ is independent of the choice of z^i. The set F is said to be *functionally independent at the point* a if $\mathrm{rk}_F(a) = d$. We always have $\mathrm{cor}(a) \geqslant d$ for a functionally independent set. A functionally independent set is said to be *complete at* a if $\mathrm{cor}(a) = d$.

Suppose that α is an action of a Lie group G on a manifold D and a is an α-regular point of D. Given an $(a; d; \alpha)$-*set* (or simply a *set*) F, which consists of d invariants (defined at a) with respect to the action of G on D, we literally transfer to F the definitions of rank, functional independence and completeness at a. Clearly, for every $\mathscr{V}$-regular point of J^k and every α-regular point of D, one can find a complete functionally independent set of invariants defined at this point.

§ 3. Invariants of non-special structures

3.1. As usual, $n! = 1\cdot 2 \dots n$. Let $C_n^k = n!\,/\bigl(k!\,(n-k)!\bigr)$ be the binomial coefficient. Using Pascal's triangle, we get the formula

$$C_n^0 + C_{n+1}^1 + \dots + C_{n+i}^i = C_{n+i+1}^i.$$

Now we can state the first main result.

Theorem 1. *Let $P \to X$ be a bundle of geometric structures of type D and order q, where $\dim D = m$, $\dim X = n$ and $m > n$. Then, for every $\mathscr{V}$-regular point $a \in J_X^k(D)$, there are at least $t(k) = mC_{n+k}^k - n\bigl(C_{n+k+q}^{k+q} - 1\bigr)$ functionally independent differential invariants of degree k at a. Letting $k \to \infty$, we can write*

$$t(k) = (m-n)C_{n+k}^k - \varepsilon(k)C_{n+k}^k,$$

where $\varepsilon(k)$ tends to zero as k grows. Thus $t(k)$ tends to infinity as k grows. The number $\varepsilon(k)$ depends only on m, n, q and k.

Before proving this, we recall some facts from [10], [11] (see §§ 3.2–3.6 below).

3.2. Given an action $\alpha\colon G^q(n)^0 \times D \to D$ of $G^q(n)^0$ on D and a positive integer k, one can canonically define a manifold D^k (the k-extension of the manifold D) and an action α_k (the k-extension of the action α) of the group $G^{q+k}(n)^0$ on D_k in such a way that there are canonical identifications

$$\gamma(k)_X^D : J_X^k(D) \approx P_X(D^k).$$

We recall from § 2.2 that the bundle $P_X(D^k)$ is associated with the principal coframe bundle Rep_{q+k} of order $q+k$ by means of the action α_k. We put $D^0 = D$. Thus the fibre J_b^k over any point $b \in X$ is non-canonically identified with D^k. This construction is functorial in the following sense.

1. For all integers k,s with $k \geqslant s \geqslant 0$, the natural projection of groups $g(k,s)\colon G^k(n)^0 \to G^s(n)^0$ commutes with the actions α_k, α_s of these groups on D^k, D^s under the natural projection $\mu^D(k,s)\colon D^k \to D^s$. We put $\mu^D(k,0) = \mu^D_k$.

2. The natural projection $r(k+q,s+q)\colon \mathrm{Rep}_{k+q}(X) \to \mathrm{Rep}_{s+q}(X)$ of coframe spaces induces a projection $p^D_X(k,s)\colon\ P_X(D^{k+q}) \to P_X(D^{s+q})$ of the associated bundles. The identifications $\gamma(k)^D_X$ and $\gamma(s)^D_X$ transform $p^D_X(k,s)$ to the projection $\pi(k,s)^D_X$. We shall omit the superscript D and subscript X in this notation if it is clear which manifolds X, D are being considered.

3. A diffeomorphism $\varphi\colon X \to T$ induces diffeomorphisms $\varphi(R,k)\colon \mathrm{Rep}_k(X) \to \mathrm{Rep}_k(T)$, $\varphi(k)^D\colon J^k_X(D) \to J^k_T(D)$, $\psi(D^k,\varphi)\colon P_X(D^k) \to P_T(D^k)$ which commute with the identifications $\gamma(k)$ and projections $r(k,s)$, $\pi^D(k,s)$, $\mu^D(k,s)$ for the appropriate values of k,s. For example,

$$\gamma(k)^D_T\varphi(k)^D = \psi(D^k,\varphi)\gamma(k)^D_X.$$

3.3. Let the D-bundle $P = P_X(D)$ be determined by an action of the differential group $G^q(n)^0$ on the generic fibre D. We take any point $a \in J^k = J^k_X(D)$ and put $b = \pi_k(a) \in X$, $G = G^{q+k}(n)^0$. It is known that any choice of a chart $F = (U,f^i)$ on X determines a trivialization

$$\mathrm{tr}_F\colon J^k_U \to D^k \times U.$$

Here J^k_U is regarded as the open subset $\pi_k^{-1}(U)$ of J^k. Now we choose a b-chart F on X such that the functions f^i of F determine a diffeomorphism (denoted by f) of the neighbourhood U onto a neighbourhood in $\mathbb{R}^n$. We denote the open ball of radius 1 in $\mathbb{R}^n$ by E^1 and assume that $f(U) = E^1$. We denote by ν_i the projection of the direct product $D^k \times U$ onto the ith factor and put $\mathrm{tr}_F(a) = t$, $w = \nu_1(t) \in D^k$. It is clear that $b = \nu_2(t)$, $t = (w,b)$ and $f(b) = \mathbf{0}$. Let $O(a)$ (resp. $O_1(a)$) be the orbit of a in J^k under the action of $\mathscr{V}$ (resp. $\mathscr{V}_b$). The orbit $O_1(a)$ lies in the fibre J^k_b of the bundle J^k over b. We put $\pi_k^{-1}(U) \cap O(a) = O_U(a)$. Moreover, put $\mathrm{tr}_F\bigl(O_U(a)\bigr) = O(a)^*$ and $\mathrm{tr}_F\bigl(O_1(a)\bigr) = O_1(a)^*$. For every vector $r \in E^1$, it is easy to construct an orientation-preserving diffeomorphism $E^1 \to E^1$ which equals the identity map near the boundary of E^1 and equals the shift by r near the point $\mathbf{0}$. Combining this diffeomorphism with the map f^{-1}, we get a diffeomorphism of U that can be extended to X. Hence we easily obtain the following assertions.

1) We have $O(a)^* = O_1(a)^* \times U$, whence $O_U(a)$ is a trivial bundle over U.

2) The orbit $O(a)$ is a locally trivial bundle over X.

3) Under the trivialization tr_F, the action of $\mathscr{V}_b$ on the fibre J^k_b is transformed to an action α_k of $G = G^{q+k}(n)^0$ on the k-extension D^k of the generic fibre D.

4) The $\mathscr{V}$-regularity of $a \in J^k$ is equivalent to the α_k-regularity of w. If a is $\mathscr{V}$-regular, then we have $\mathrm{rk}(a) = \mathrm{rk}(w)$ and $\mathrm{cor}(a) = \mathrm{cor}(w)$. Thus, although the point w depends on the choice of trivialization, the α_k-regularity of w is independent of this choice. (In what follows we often use the term "regularity" for $\mathscr{V}$-regularity and α_k-regularity.)

5) If R' is the set of α_k-regular points in D^k, then the set R_U of $\mathscr{V}$-regular points in J^k_U is given by $R_U = \mathrm{tr}_F^{-1}(R' \times U)$. Since R' is open and dense in D^k, the set R_U is open and dense in J^k_U. Hence the set of all $\mathscr{V}$-regular points in J^k is open and everywhere dense.

Now suppose that a is $\mathscr{V}$-regular. Then we easily get the following assertions.

$1'$) The map tr_F induces an isomorphism i^* of the ring $I_a(J^k;\mathscr{V})$ of $\mathscr{V}$-invariants onto a subring of $O_t(D^k\times U)$. We denote this subring by I_t. Each element of I_t is determined by the germ at t of a function $S(y,x)$, where $y\in D^k$, $x\in X$ and S is constant on the orbits of $\mathscr{V}_0$. However, we easily see that $S=S(y)$ depends only on $y\in D^k$ and, therefore, S is determined by its restriction s to the fibre D^k. The function s is determined in an appropriate neighbourhood T of the point $w\in D^k$ and is constant on the connected components of the intersection of T with the orbits of G on D^k. Hence it is clear that the map $S\to s$ determines an isomorphism $i^*\colon I_a(J^k;\mathscr{V})\to I_t\approx I_w(D^k;\alpha_k)$ of the ring of differential invariants at a onto the ring of α_k-invariants at w for the action of G on D^k.

$2'$) Suppose that $F=(F^i)$ is an $(a;d;\mathscr{V})$-set (see § 2.5), $i^*(F^i)=S^i$ and s^i is the restriction of S^i to D^k. Then the germs of the s^i are α_k-invariants at w. If s is the $(w;d;\alpha_k)$-set formed by them, then $\mathrm{rk}_F(a)=\mathrm{rk}_s(w)$. Hence the set F is functionally independent (complete) at t if and only if the set s is functionally independent (complete) at w.

$3'$) If we choose another b-chart H on X instead of F, then $\mathrm{tr}_H(w)=(w_H,b)$ determines another point w_H in D^k. We have $w_H=gw$, where the element $g\in G$ is determined by the $(k+q)$-jet of the transition map from F to H at b. The set of elements w_H corresponding to all b-charts H is the orbit $Gw=O_1(a)^*$ of the G-action α_k on D^k. This orbit depends only on a (and is independent of the choice of F). For any choice of F, the trivialization tr_F sends $O_1(a)$ to $O_1(a)^*$. Thus there is a natural one-to-one correspondence between M and M_b, where M is the set of orbits of the G-action α_k on D^k and M_b is the set of $\mathscr{V}_b$-orbits on the fibre J^k_b of the bundle J^k over any point $b\in X$. In what follows we denote the correspondence $M\to M_b$ by λ_b. Let MJ be the set of all $\mathscr{V}$-orbits on J^k. If X is connected, then, by sending each orbit in MJ to its restriction to the fibre J^k_b, we get a bijection of MJ onto M_b over any point $b\in X$.

Thus we see that to describe the differential invariants at the regular points of $J^k_X(D)$, it suffices to describe the α_k-invariants at the regular points of D^k. This is the problem we wish to consider.

3.4. Let F be an arbitrary chart on X containing the point b. We preserve the notation introduced at the beginning of § 3.3. The manifold D^k decomposes canonically into the direct product

$$D^k=D\times D_1\times\cdots\times D_k. \tag{1}$$

We write the corresponding decomposition of any element $d\in D^k$ as $d=(d_0,d_1,\dots,d_k)$. Equation (1) corresponds to the splitting of the jet $f\colon\mathbb{R}^n\to D$ into a sum of homogeneous summands. This splitting is compatible with the projections $\mu(k,s)$, that is, $\mu(k,s)(d)=(d_0,d_1,\dots,d_s)$. The spaces D_s are vector spaces of dimensions mC^s_{n+s-1}.

We consider the projection $\mu_k\colon D^k\to D$ and put $\mu_k(w_F)=w^0_F$. Choose any chart $Q=(N,y^c)$, $1\leqslant c\leqslant m$, on D with $N\ni w^0_F$. Then the inverse image $\mu_k^{-1}(N)=N^k$ of the neighbourhood N is given by the direct product

$$N^k=N\times D_1\times\cdots\times D_k. \tag{2}$$

This decomposition induces a coordinate system (y) on the neighbourhood N^k:

$$(y) = (y^c, y^c_{j_1}, \dots, y^c_{j_1,\dots,j_k}), \qquad 1 \leqslant c \leqslant m, \quad 1 \leqslant j_s \leqslant n, \quad 1 \leqslant s \leqslant k.$$

For $s > 0$ the coordinates $y^c_{j_1,\dots,j_s}$ are global coordinates on D_s.

We put $N^k(F) = \mathrm{tr}_F^{-1}(N^k \times U) \subset J^k$. Combining the coordinates f^i of the chart F on X and the coordinates (y) on N^k and using the trivialization tr_F, we construct the coordinate system $\pi_k^*(f^i)$, (y_F) on $N^k(F)$, where

$$(y_F) = (y^c_F, y^c_{F\,j_1}, \dots, y^c_{F\,j_1,\dots,j_k}), \qquad 1 \leqslant c \leqslant m, \quad 1 \leqslant j_s \leqslant n, \quad 1 \leqslant s \leqslant k.$$

Here we write $y^c_{F\,j_1,\dots,j_s} = y^c_{j_1,\dots,j_s}\,\mathrm{tr}_F$. The subscript F indicates the dependence on the chart F. We omit this subscript whenever F is fixed or this dependence is inessential. To simplify the notation, we write f^i instead of $\pi_k^*(f^i)$. We assign weight 0 to the coordinates (y^c_F) and symbolically write $y_{F\,0} = (y^c_F)$. We assign weight s to each coordinate $y^c_{F\,j_1,\dots,j_s}$ and denote the set of all coordinates of weight s symbolically by $y_{F\,s} = \bigl(y^c_{F\,j_1,\dots,j_s}\bigr)$. We also write $(y_F) = (y_{F\,0}, \dots, y_{F\,k})$. One similarly defines the weights and symbolic notation for the coordinates (y) on N^k. Let $Q^k(F)$ be the chart on J^k consisting of the neighbourhood $N^k(F)$ and the coordinate system (f^i), (y_F) and let Q^k be the chart on Y^k consisting of the neighbourhood N^k and the coordinate system (y).

3.5. If $k = 0$, we write simply N, Q, $Q(F)$ and $N(F) = (N, F)$ instead of N^0, Q^0, $Q^0(F)$ and $N^0(F)$. The coordinates of a point $a \in J^k$ in the chart $Q^k(F)$ are denoted by $(a^i_{-1}, a^c, a^c_{j_1}, \dots, a^c_{j_1,\dots,j_k})$. We shall write $a = (a^i_{-1}, a^c, a^c_{j_1}, \dots, a^c_{j_1,\dots,j_k})$, where we put $f^i\bigl(\pi_k(a)\bigr) = a^i_{-1}$. We also symbolically write $a = (a_{-1}, a_0, \dots, a_k)$. The corresponding notation $d = (d^c, d^c_{j_1}, \dots, d^c_{j_1,\dots,j_k}) = (d_0, d_1, \dots, d_k)$ is used for points $d \in N^k$.

3.6. We recall that the differential group $G = G^{q+k}(n)^0$ consists of $(k+q)$-jets at $\mathbf{0}$ of orientation-preserving pointed diffeomorphisms $(\mathbb{R}^n, \mathbf{0}) \to (\mathbb{R}^n, \mathbf{0})$. The product $\varphi_{k+q}(\mathbf{0})\psi_{k+q}(\mathbf{0})$ of the jets of φ and ψ is defined as the jet $\theta_{k+q}(\mathbf{0})$ of the map $\theta = \varphi(\psi)$. As a topological space, G is a direct product of manifolds,

$$G^{k+q}(n)^0 = G_1 \times \dots \times G_{k+q}, \tag{3}$$

corresponding to the splitting of jets into homogeneous summands. Coordinates on G_1 are given by the linear part of the jet, whence G_1 is identified with GL_n^0. We note that the G_s are not subgroups of G for $s > 1$, but the $G_s \times \dots \times G_{k+q}$ are normal subgroups of G for any s. We put $N = N(k+q) = G_2 \times \dots \times G_{k+q}$. (Later we shall see that the group N is nilpotent.) The group G may be written as a semidirect product

$$G = G^{q+k}(n)^0 = \mathrm{GL}_n^0 \leftthreetimes N. \tag{4}$$

This decomposition is called the *algebraic Levi decomposition*. The genuine Levi decomposition into a semisimple factor S and a soluble radical R is obtained as follows. Let

$$\mathrm{GL}_n^0 = \mathrm{SL}_n \times Z^0 \tag{5}$$

be the standard decomposition of GL_n^0 into the direct product of SL_n and the connected component Z^0 of the centre. (Here Z^0 is the set of scalar matrices with positive entries.) Then the soluble radical $R = R(k+q)$ of G is the semidirect product of the subgroup Z^0 of $\mathrm{GL}_n^0 = G_1$ and the normal subgroup N:

$$R = Z^0 \leftthreetimes N. \tag{6}$$

The semidirect factor S in the Levi decomposition of G is the subgroup SL_n of G_1. Thus the Levi decomposition of G is given by

$$G = G^{q+k}(n)^0 = \mathrm{SL}_n \leftthreetimes R. \tag{7}$$

Coordinates on G_s are given by the Taylor coefficients $g^i_{j_1,\dots,j_s}$ of degree s. This enables us to identify G_s with a vector space of dimension $\lambda(i) = nC^s_{n+s-1}$. We symbolically denote the resulting coordinate system on G by $(g_1,\dots,g_{k+q})$. The corresponding coordinate system on the Lie algebra $\mathcal{G}(k+q) = \mathcal{G}$ of the group $G^{q+k}(n)^0$ is denoted by $\bigl(b^i_{j_1},\dots,b^i_{j_1,\dots,j_{q+k}}\bigr)$, or symbolically by $(b_1,\dots,b_{k+q})$. The decomposition (3) induces the decomposition

$$\mathcal{G} = \mathcal{G}_1 \times \dots \times \mathcal{G}_{k+q}, \tag{8}$$

where $\mathcal{G}_i$ is the tangent space to G_i at the identity. We note that the $\mathcal{G}_i$ are not subalgebras for $i > 1$, but the $\mathcal{G}_i \times \dots \times \mathcal{G}_{k+q}$ are normal subalgebras of $\mathcal{G}$. We use the notation $\mathcal{N} = \mathcal{N}(k+q) = \mathcal{G}_2 \times \dots \times \mathcal{G}_{k+q}$. Put $\mathcal{Z} = \mathrm{Lie}(Z^0)$, $\mathcal{R} = \mathcal{R}(k+q) = \mathrm{Lie}(R)$. Clearly, $\mathcal{G}_1 = GL_n$, and we have decompositions of Lie algebras corresponding to the decompositions of groups

$$GL_n = SL_n \times \mathcal{Z}, \tag{9}$$
$$\mathcal{R} = \mathcal{Z} \leftthreetimes \mathcal{N}. \tag{10}$$

We also have the Levi decomposition

$$\mathcal{G} = SL_n \leftthreetimes \mathcal{R} \tag{11}$$

corresponding to the decomposition (7), and the algebraic Levi decomposition corresponding to (4):

$$\mathcal{G} = \mathcal{G}(k+q) = GL_n \leftthreetimes \mathcal{N}. \tag{12}$$

The structure of the group $G^s(n)^0$ and its Lie algebra $\mathcal{G}(s)$ will be described in more detail later. Now we calculate the dimensions of the group $G^k(n)$ and the space D^k.

Lemma 2. *If* $\dim D = m$ *and* $\dim X = n$, *then* $\dim G^k(n) = n(C^k_{n+k} - 1)$ *and* $\dim D^k = mC^k_{n+k}$.

Since the dimensions of G_s and D_s are known (see above), the proof follows directly from (1), (3) and Pascal's triangle (see § 3.1).

Lemma 3. *For any fixed* q *and* n, *the quotient of* C^k_{n+k} *and* C^{k+q}_{n+k+q} *tends to* 1 *as* $k \to \infty$.

Proof. We have

$$\frac{C_{n+k}^{k}}{C_{n+k+q}^{k+q}} = \frac{(n+k)!\,(k+q)!\,n!}{n!\,k!\,(n+k+q)!} = \frac{(k+1)\dots(k+n)}{(q+k+1)\dots(q+k+n)}\,,$$

and each of the n quotients $(k+i)/(q+k+i)$ tends to 1 as $k\to\infty$.

Proof of Theorem 1. Remark 1′) of § 3.3 reduces the consideration of $t(k)$ to the analogous question for the action α_k. Near a regular point $w\in D^k$, the number $t(k)$ of functionally independent α_k-invariants is estimated as follows. All orbits in the neighbourhood of w have the same dimension d (and $d\leqslant \dim G^{k+q}(n)$). The invariants are determined by germs of functions that are constant on the orbits. If we take a submanifold S that passes through w and is transversal to the orbits near w, then the invariants are determined by their restrictions to S. Hence we get

$$t(k)\geqslant \dim D^k-\dim G^{k+q}(n).$$

We put $C_{n+k+q}^{k+q}/C_{n+k}^{k}=p(k)$. Then

$$\begin{aligned} t(k)&\geqslant mC_{n+k}^{k}-n\bigl(C_{n+k+q}^{k+q}-1\bigr)\geqslant mC_{n+k}^{k}-nC_{n+k}^{k}p(k)\\ &=C_{n+k}^{k}\bigl(m-np(k)\bigr)=(m-n)C_{n+k}^{k}+nC_{n+k}^{k}\bigl(1-p(k)\bigr). \end{aligned}$$

The theorem is proved.

3.7. Theorem 1 may be generalized to the case when we consider scalar invariants of several (at least two) geometric objects. More precisely, suppose that the differential group $G=G^q(n)^0$ acts on several manifolds $D_1,\dots,D_r$. We fix a tuple $K=\bigl(k(1),\dots,k(r)\bigr)$ of positive integers and consider the fibred product $J_K=J^{k(1)}(D_1)\times\dots\times J^{k(r)}(D_r)$ over X. The group $\mathscr{V}$ of orientation-preserving diffeomorphisms of X acts on all factors of this fibred product. Hence we have a diagonal action of $\mathscr{V}$ on J_K. The differential invariants and $\mathscr{V}$-regular points are defined in this case in the same way as in Definitions 3 and 4. The number $k=\max\bigl(k(j),1\leqslant j\leqslant r\bigr)$ is called the *degree of the differential invariant.* We put $m_j=\dim D_j$, $s=\dim Z$, $m=\sum_{j=1}^{r} m_j$.

Theorem 2. *For every $\mathscr{V}$-regular point $a\in J_K(D)$ there are at least*

$$t(K)=\sum_{j=1}^{r} m_j C_{n+k(j)}^{k(j)}-n\bigl(C_{n+k+q}^{k+q}-1\bigr)$$

functionally independent differential invariants of degree k defined at a.

In particular, take the set $K=(k,k,\dots,k)$. It follows that

$$t(K)=(m-n)C_{n+k}^{k}-\varepsilon(K)C_{n+k}^{k},$$

where $\varepsilon(K)$ tends to zero as k grows. Hence $t(K)$ tends to infinity as k grows. For this set K, the number $\varepsilon(K)$ depends only on m, n, q and k.

Proof. Consider the manifold $D_K = D^{k(1)} \times \cdots \times D^{k(r)}$, which is the direct product of $k(j)$-extensions of D_j. Choose a coordinate system on X (shrinking X if necessary). This yields a trivialization $J_K \approx D_K \times X$. Since $\dim D^{k(j)} = m_j C^{k(j)}_{n+k(j)}$, we complete the proof by a dimension count as in the proof of Theorem 1.

Definition 5. Let $G = G^k(n)^0$. A G-manifold D is said to be *non-special* (resp. *special*) if $\dim D > n$ (resp. $\dim D \leqslant n$). A bundle $P = P_X(D)$ of geometric objects is said to be *non-special* if the dimension $m = \dim D$ of the generic fibre exceeds the dimension $n = \dim X$ of the base. If $m \leqslant n$, then P is said to be *special.*

§ 4. Subgroups of small codimension in $G^q(n)$

4.1. In this section we list all connected Lie subgroups of GL^0_n whose codimension does not exceed n. As mentioned in § 1, we consider only the case $n > 2$.

Let σ be the outer automorphism $A \to (A^T)^{-1}$ of the group GL^0_n. We assume that $\mathbb{R}^n$ is endowed with the standard basis e_i and GL^0_n acts on $\mathbb{R}^n$ by the representation ST_n. We distinguish the following connected subgroups of SL_n.

1) The subgroup T_1 consists of all matrices that multiply the vector e_1 by a positive number (depending on the matrix).

2) The subgroup $T_2 = \sigma(T_1)$ consists of all matrices that multiply the form x_1 by a positive number (depending on the matrix). The codimensions of T_1 and T_2 in SL_n are equal to $n-1$.

3) The subgroup T_3 consists of all matrices that fix the vector e_1.

4) The subgroup $T_4 = \sigma(T_3)$ consists of all matrices that fix the form x_1. The codimensions of T_3 and T_4 in SL_n are equal to n.

Remark 5. Each of the T_i contains the subgroup SL_{n-1} that fixes the vector e_1 and the complementary space spanned by $e_2, \dots, e_n$. It is well known that no two of the subgroups T_i are conjugate for $n > 2$. If $n = 2$, then T_1 is conjugate to T_2, and T_3 is conjugate to T_4. Clearly, the subgroups T_i are connected and closed, T_3 is a normal subgroup of T_1, and T_4 is a normal subgroup of T_2.

We shall use the following fact, which will be proved in § 6.

Assertion 1. *Every proper connected Lie subgroup $H \subset \mathrm{SL}_n$ of codimension at most n in SL_n is conjugate by an appropriate inner automorphism either to one of the subgroups T_i, $1 \leqslant i \leqslant 4$, or (for $n = 4$) to the subgroup P_{24} consisting of all matrices that fix the plane spanned by e_1, e_2, or (for $n = 3$) to the Borel subgroup B_3 (of upper or lower triangular matrices) of* SL_3.

In what follows we assume that $n > 2$. We put $\mathrm{Lie}(T_i) = \mathcal{T}_i$.

Lemma 4. *If $n > 3$, then T_3 is the unique Lie subgroup in T_1 of codimension* 1. *If $n = 3$, then T_1 contains precisely one (up to conjugacy) subgroup of codimension* 1 *other than T_3. This is the (non-normal) subgroup B_3 of upper triangular matrices.*

Proof. We give a proof in terms of Lie algebras. The matrices in $\mathcal{T}_1$ have block form $\left(\begin{smallmatrix} \lambda & a \\ 0 & B \end{smallmatrix}\right)$, where λ is a number, a is a row of length n, B is a square matrix of order $n-1$ and $\operatorname{tr} B + \lambda = 0$. We consider the Levi decomposition $\mathcal{T}_1 = SL_{n-1} \leftthreetimes \mathcal{R}$, where the factor SL_{n-1} is determined by the conditions $a = 0$, $\lambda = 0$, and the ideal $\mathcal{R}$ is generated by the ideal $\mathcal{R}'$ (defined by the equations $\lambda = 0$, $B = 0$) and the one-dimensional subalgebra spanned by the matrix $\operatorname{diag}(1-n, 1, \dots, 1)$. Let $\mathcal{H}$ be a subalgebra of codimension 1 in $\mathcal{T}_1$ and $\mathcal{H} = \mathcal{S} \leftthreetimes \mathcal{B}$ its Levi decomposition. By Malcev's theorem ([12], Ch. 6, Theorem 3), the subalgebra $\mathcal{S}$ is conjugate by an element of T_1 to a subalgebra of SL_{n-1}. Therefore we can assume that $\mathcal{S} \subset SL_{n-1}$. Consider the projection θ of $\mathcal{T}_1$ to the subalgebra SL_{n-1} induced by the Levi decomposition. We put $\theta(\mathcal{H}) = \mathcal{H}'$ and $\theta(\mathcal{B}) = \mathcal{B}'$. Clearly, $\mathcal{H}' = \mathcal{S} \leftthreetimes \mathcal{B}'$ is the Levi decomposition of $\mathcal{H}'$.

Case 1. Suppose that $\mathcal{H}' = SL_{n-1}$. Then $\mathcal{S} = SL_{n-1} \subset \mathcal{H}$. The subalgebra $\mathcal{H} \cap \mathcal{R}'$ is of codimension at most 1 and, therefore, contains a non-zero vector h. Since the action of SL_{n-1} by conjugation on $\mathcal{R}'$ is isomorphic to ST_{n-1} (that is, to the ordinary action of SL_{n-1} on $\mathbb{R}_{n-1}$), it is clear that conjugation of any non-zero vector h by elements of the subgroup SL_{n-1} yield the whole subalgebra $\mathcal{R}'$. Hence $\mathcal{H}$ contains the subalgebra $SL_{n-1} \leftthreetimes \mathcal{R}' = \mathcal{T}_3$.

Case 2. Suppose that $\mathcal{H}'$ has codimension 1 in SL_{n-1}. This is possible only for $n = 3$. Then $\mathcal{H}'$ is conjugate by an element of SL_{n-1} to the subalgebra of upper triangular matrices, and $\mathcal{H} = \theta^{-1}(\mathcal{H}')$ is conjugate to the Borel subalgebra $\mathcal{B}_3$ of upper triangular matrices. The lemma is proved.

4.2. Consider the decompositions (5) and (9): $\mathrm{GL}_n^0 = \mathrm{SL}_n \times Z^0$ and $GL_n = SL_n + \mathcal{Z}$. Let ν_1 and ν_2 be the projections of GL_n^0 onto SL_n and Z^0 respectively. The same notation is used for the projections of GL_n onto SL_n and $\mathcal{Z}$. Let H be a proper connected Lie subgroup of GL_n^0 of codimension at most n and let $\mathcal{H}$ be its Lie algebra. We consider $\mathcal{H}_1 = \nu_1(\mathcal{H}) \subset SL_n$. Three cases may arise.

1°. The kernel of the projection $\nu_1\colon \mathcal{H} \to \mathcal{H}_1$ contains $\mathcal{Z}$. Then the codimension of the subalgebra $\mathcal{H}_1$ in SL_n equals the codimension of $\mathcal{H}$ in GL_n.

2°. The projection $\nu_1\colon \mathcal{H} \to \mathcal{H}_1$ is an isomorphism, and $\mathcal{H}$ is contained in SL_n.

3°. The projection $\nu_1\colon \mathcal{H} \to \mathcal{H}_1$ is an isomorphism, and $\mathcal{H}$ is not contained in SL_n. Then we put $\mathcal{H}_2 = \mathcal{H} \cap SL_n$. Clearly, $\mathcal{H}_2$ is a normal subalgebra of codimension 1 in $\mathcal{H}_1$ and $\mathcal{H}$. It corresponds to the subgroup $H_2 = H \cap \mathrm{SL}_n$.

We have $\operatorname{codim}(\mathcal{H}, GL_n) = \operatorname{codim}(\mathcal{H}_1, SL_n) + 1$ in cases 2° and 3°.

In case 1° the subgroup H is conjugate to one of the following subgroups: $T_i \times Z^0$ for an appropriate i, $1 \leqslant i \leqslant 4$, or $P_{24} \times Z^0$ (for $n = 4$), or $B_3 \times Z^0$ (for $n = 3$). Hence the subgroup H is closed.

In case 2° we have either $H = \mathrm{SL}_n$, or H is conjugate to T_1 or T_2. Hence the subgroup H is closed.

In case 3° it is clear that $\mathcal{H}_1$ cannot coincide with SL_n (for otherwise the subalgebra $\mathcal{H}_1$, which is isomorphic to SL_n, has a non-trivial homomorphism onto the one-dimensional subalgebra $\mathcal{Z}$, which is impossible). Then $\operatorname{codim}(\mathcal{H}, GL_n) = n$. (The case $\operatorname{codim}(\mathcal{H}, GL_n) \leqslant n-1$ is impossible since it follows that $\operatorname{codim}(\mathcal{H}_1, SL_n) \leqslant n-2$ contrary to § 4.1.) Using § 4.1 again, we can assume (after conjugating by an appropriate inner automorphism of the group SL_n) that

(i) either $H_1 = T_1$,

(ii) or $H_1 = T_2$.

One can easily deduce from this that H is closed (we do not give a proof).

4.3. Consider case (i). Since H_2 is normal in H_1, we see from Lemma 4 that $H_2 = T_3$, $\mathcal{H}_1 = \mathcal{T}_1$, $\mathcal{H}_2 = \mathcal{T}_3$. Since $\operatorname{codim}(H, \mathrm{GL}_n) = \operatorname{codim}(T_3, \mathrm{SL}_n) = n$, the embedding $\mathrm{SL}_n \to \mathrm{GL}_n^0$ induces a diffeomorphism $i\colon\ \mathbb{R}^n - \mathbf{0} = \mathrm{SL}_n/T_3 \approx \mathrm{GL}_n^0/H$ of homogeneous spaces, and i is compatible with the action of SL_n. (Here $\mathbb{R}^n$ is endowed with the standard action ST_n.) Let us calculate the action on $\mathbb{R}^n$ obtained in this way from the action of Z^0 on GL_n^0/H. To do this, we consider the one-parameter subgroup t in T_1 consisting of the elements $t(s) = \operatorname{diag}(s^{-1}, s, 1, \dots, 1)$, where $s \in \mathbb{R}^0$. Since $\nu_1\colon H \to H_1$ is an isomorphism, there is a one-parameter subgroup h of H such that ν_1 is an isomorphism of h onto t. Since t does not lie in T_3, we see that h does not lie in SL_n, whence ν_2 isomorphically maps h onto Z^0. Thus the map $\nu_2\nu_1^{-1}\colon t \to Z^0$ is an isomorphism. Clearly, this map is given by $\operatorname{diag}(s^{-1}, s, 1, \dots, 1) \to \mathrm{sk}_n(s^a)$ for an appropriate real number $a = a(h) \neq 0$.

We consider an element $h(s)$ in h such that $\nu_1\bigl[h(s)\bigr] = t(s)$ and put $\nu_2\bigl[h(s)\bigr] = z(s) = \mathrm{sk}_n(s^a)$. Then $h(s) = t(s)z(s)$. Every vector $x \in \mathbb{R}^n - \mathbf{0}$ corresponds to a coset gT_3, where the matrix $g \in \mathrm{SL}_n$ is determined by the condition $ge_1 = x$. The correspondence i sends the coset gT_3 to the coset gH, and the action of $z(s)$ sends gH to

$$z(s)gH = gz(s)H = gt^{-1}(s)h(s)H = gt^{-1}(s)H.$$

Since $gt^{-1}(s) \in \mathrm{SL}_n$, we see that $z(s)gH$ is the image (under the diffeomorphism i) of the coset $gt^{-1}(s)T_3$ corresponding to the vector $sx \in \mathbb{R}^n - \mathbf{0}$. Hence the matrix $\mathrm{sk}_n(s)$ acts on the vector x as multiplication by s^d, where $d = 1/a$. We note that the resulting map $h \to d$ is a one-to-one correspondence between the one-parameter subgroups h of T_1Z^0 with $\nu_1 h = t$ and the real numbers $d \neq 0$. It is also clear that $H = hT_3$. Hence we may assign the subgroup h_d and the coset $H_{1d} = h_dT_3$ to every real $d \neq 0$. We also put $H_{10} = T_3 \times Z^0$. This subgroup was already encountered in case 1° in § 4.2.

We denote the homogeneous space GL_n^0/H_{1d} by $\mathbb{R}^n_d$. The spaces $\mathbb{R}^n_d$ with different d are pairwise inequivalent as GL_n^0-spaces. (By definition, an equivalence $r\colon D_1 \to D_2$ of GL_n^0-spaces is a diffeomorphism commuting with the action of GL_n^0.) Indeed, consider an equivalence $r\colon \mathbb{R}^n_d \to \mathbb{R}^n_b$ as a map of SL_n-spaces and identify $\mathbb{R}^n_d$ with $\mathbb{R}^n - \mathbf{0}$ by the above diffeomorphism i. If we define $r(\mathbf{0}) = \mathbf{0}$, then r maps $\mathbb{R}^n = (\mathrm{SL}_n/T_3) \cup \mathbf{0}$ to itself. Every G-map of a homogeneous space G/H to itself sends a coset gH to a coset gnH, where n is a fixed element of the normalizer of H. In our case, the vector $x \in \mathbb{R}^n - \mathbf{0} = \mathrm{SL}_n/T_3$ is represented by the matrix g, and the vector $r(x)$ is represented by the matrix gn. Since the coordinates of x are equal to the entries of the first column of g and the normalizer of T_3 (being equal to T_1) consists of matrices whose first column is zero except for the first entry, we see that $r(x)$ is just the product of x and this non-zero entry. Hence r is multiplication by a scalar in $\mathbb{R}^n$. Clearly, this map does not commute with the action of Z^0 if $b \neq d$. Hence, in case (i), the subgroup H is conjugate to one of the subgroups H_{1d} (for $d \neq 0$).

Remark 6. The action of GL_n^0 on $\mathbb{R}^n_d$ is described in § 5 below (see Type XII). The subgroup H_{1d} is the stabilizer of the column vector $e_1 = (1, 0, \dots, 0)^T$ under this action.

4.4. Case (ii) reduces to case (i) by applying the outer automorphism σ. This also yields that H is conjugate to one of the subgroups H_{2d} (for $d \neq 0$), where $H_{2d} = \sigma(H_{1d})$. If $d = 0$, then we get one of the subgroups of case 1° in § 4.2. The corresponding action of GL_n^0 is described in § 5 below. The subgroup H_{2d} is the stabilizer of the row vector $e_1 = (1, 0, \dots, 0)$ under this action.

Thus we have the following theorem.

Theorem 3. *If $n > 2$, then every connected Lie subgroup of GL_n of codimension at most n is closed and conjugate by an inner automorphism to one and only one Lie subgroup in the following list*:

1) *the group GL_n^0 of codimension* 0;
2) *the group SL_n of codimension* 1;
3) *the group $H_1 = T_1 \times Z^0$ of codimension $n-1$*;
4) *the group $H_2 = T_2 \times Z^0$ of codimension $n-1$*;
5) *the group T_1 of codimension n*;
6) *the group T_2 of codimension n*;
7) *the group H_{1d} of codimension n, where d is any real number*;
8) *the group H_{2d} of codimension n, where d is any real number*;
9) *the group $P_{24} \times Z^0$ of codimension 4, $n = 4$*;
10) *the group $B_3 \times Z^0$ of codimension 3, $n = 3$.*

(*We recall that Z^0 is the connected component of the identity in the centre of GL_n, the subgroups T_i are introduced in* § 4.1, *and the subgroups H_{1d} and H_{2d} are defined in* § 4.3 *and* § 4.4 *respectively. All codimensions are with respect to GL_n.*)

4.5. Here we describe all connected subgroups of codimension at most n in the groups $G^q(n)$, where $q \geqslant 2$ and $n > 2$.

Theorem 4. 1) *If $n > 2$, then the group $G^2(n)$ contains two non-conjugate connected closed subgroups of codimension n. Every connected subgroup of codimension at most n either is conjugate to one of these subgroups, or is the full pre-image of a similar subgroup of GL_n* (*that is, a connected closed subgroup of GL_n of codimension at most n*) *under the natural projection $g(2,1)\colon G^2(n) \to G^1(n) = \mathrm{GL}_n$.*

2) *If $q \geqslant 3$ and $n > 2$, then every connected subgroup of codimension at most n in $G^q(n)$ is the full pre-image of a similar subgroup of $G^2(n)$ with respect to the projection $g(q,2)\colon G^q(n) \to G^2(n)$.*

The connected subgroups of codimension at most n in the group $G^1(n) = \mathrm{GL}_n$ have already been described in Theorem 3.

Proof. We consider the following representation ρ_i of the group GL_n on the space $T_i^{1\prime}$ of real tensors with one upper and i lower indices: every tensor $t = \bigl(t^u_{j_1,\dots,j_i}\bigr)$ is mapped to the tensor $\rho_i(h)t = f = \bigl(f^u_{j_1,\dots,j_i}\bigr)$ defined by

$$f^u_{j_1,\dots,j_i} = h^u_s t^s_{a_1,\dots,a_i} g^{a_1}_{j_1} g^{a_2}_{j_2} \dots g^{a_i}_{j_i}. \tag{13}$$

Here $g = (g^a_j)$, $h = (h^u_s)$ and $gh = E$. Let G_i' be the subspace of $T_i^{1\prime}$ formed by tensors that are symmetric in the lower indices. We denote the restriction of ρ_i to G_i' by ρ_i'.

Lemma 5. *The representation ρ_i' is the sum of two irreducible invariant subspaces*:

$$G_i' = V_i^{1\prime} + V_i^{2\prime}. \tag{14}$$

Here the space $V_i^{1\prime}$ consists of all tensors $\left(t^u_{j_1,\dots,j_i}\right)$ such that the convolution of the upper index with the last lower index is the zero tensor (*then the same holds for any lower index since the tensor is symmetric in the lower indices*):

$$V_i^{1\prime} = \left\{\left(t^u_{j_1,\dots,j_i}\right) \mid \left(t^u_{j_1,\dots,j_{i-1},u}\right) = 0\right\}.$$

The subspace $V_i^{2\prime}$ is the unique invariant complement of $V_i^{1\prime}$. The existence of $V_i^{2\prime}$ follows since GL_n is reductive, and the uniqueness follows since $V_i^{1\prime}$ and $V_i^{2\prime}$ are irreducible and inequivalent. The restriction of ρ_i' to $V_i^{2\prime}$ is isomorphic to the representation of GL_n on the space T_{i-1} of symmetric tensors $(t_{j_1,\dots,j_{i-1}})$ with $i-1$ lower indices. There is a projection map $\lambda_i'\colon G_i' \to T_{i-1}$ commuting with the action of GL_n. This map is given by convolution: *if $t = \left(t^u_{j_1,\dots,j_i}\right)$ and $g = \lambda_i'(t) = (g_{j_1,\dots,j_{i-1}})$, then $g_{j_1,\dots,j_{i-1}} = t^u_{j_1,\dots,j_{i-1},u}$. The kernel of λ_i' equals $V_i^{1\prime}$, and the restriction of λ_i' to $V_i^{2\prime}$ is an isomorphism of $V_i^{2\prime}$ onto T_{i-1}. The dimension of $V_i^{2\prime}$ is equal to C^{i-1}_{n+i-2}. The dimension of $V_i^{1\prime}$ is equal to $nC^i_{n+i-1} - C^{i-1}_{n+i-2}$.*

Lemma 5 remains valid if we replace the group GL_n by SL_n.

We omit the proof of Lemma 5 since it is standard: it is mentioned in [12], Table 5, Russian p. 312.

4.6. The following Levi decompositions of the group $G = G^q(n)^0$ and its algebra $\mathcal{G}$ were constructed in § 3.6 (we put $k = 0$ in formulae (4)–(12)):

$$\begin{aligned} &G = \mathrm{GL}_n \leftthreetimes N, \qquad G = \mathrm{SL}_n \leftthreetimes R, \qquad R = R(q) = Z^0 \leftthreetimes N, \\ &\mathcal{G} = GL_n \leftthreetimes \mathcal{N}, \qquad \mathcal{G} = SL_n \leftthreetimes \mathcal{R}, \qquad \mathcal{R} = \mathcal{R}(q) = \mathcal{Z} \leftthreetimes \mathcal{N}, \\ &N = N(q) = G_2 \times \dots \times G_q, \qquad \mathcal{N} = \mathcal{N}(q) = \mathcal{G}_2 \times \dots \times \mathcal{G}_q. \end{aligned} \tag{15}$$

Using the definition of multiplication in G, we see that every element $h \in \mathrm{GL}_n^0$ is represented by a linear map $h(x) = h^s_j x^j$ with constant coefficients h^s_j, and conjugation by h sends each element $t = (t^u_{j_1,\dots,j_i}) \in G_i$ to the element $f = hth^{-1}$ whose coordinates $f^u_{j_1,\dots,j_i}$ are given by (13). It follows that the manifold G_i and its tangent space $\mathcal{G}_i$ at the identity are invariant under conjugation by GL_n^0, and the inner automorphism $\mathrm{Ad}(h)$ maps every element $b = (b^u_{j_1,\dots,j_i}) \in \mathcal{G}_i$ to the element $d = d^u_{j_1,\dots,j_i}$ given by the analogous formula

$$d^u_{j_1,\dots,j_i} = h^u_s b^s_{a_1,\dots,a_i} g^{a_1}_{j_1} g^{a_2}_{j_2} \dots g^{a_i}_{j_i}.$$

Hence, replacing each element $b \in \mathcal{G}_i$ by the set of its coordinates $b^u_{j_1,\dots,j_i}$ and considering the tensor $b' \in G_i'$ with these coordinates, we get an identification $\chi\colon \mathcal{G}_i \approx G_i'$ that transforms the action ρ_i of GL_n^0 by inner automorphisms on $\mathcal{G}_i$ to the action ρ_i' on G_i'. We also get a decomposition

$$\mathcal{G}_i = \mathcal{V}_i^1 \times \mathcal{V}_i^2 \tag{16}$$

induced by the decomposition (14).

Put $\lambda_i = \lambda'_i \chi$. It is known that every element $b \in \mathcal{G}_i$ may be represented by the germ (at the origin $\mathbf{0}$) of some vector field $\mathcal{B}$ on $\mathbb{R}^n$ with $\mathcal{B}(\mathbf{0}) = \mathbf{0}$. Namely, the element $b = (b^u_j, \dots, b^u_{j_1,\dots,j_i})$ is tangent to the one-parameter family of germs of transformations given by

$$y^u(\varepsilon) = x^u + \varepsilon\bigl(b^u_j x^j + b^u_{j_1,j_2} x^{j_1} x^{j_2} + \dots + b^u_{j_1,\dots,j_q} x^{j_1} \dots x^{j_q}\bigr).$$

Here ε is the parameter of the family, and $dy/d\varepsilon|_{\varepsilon=0}$ is the germ (at $\mathbf{0}$) of the vector field $\mathcal{B}$ corresponding to the element b. We see that homogeneous elements b_s correspond to germs $\mathcal{B}_s$ that are given by homogeneous polynomials of degree s:

$$\mathcal{B}^u_s = b^u_{j_1,\dots,j_s} x^{j_1} \dots x^{j_s}.$$

One can also verify that the Lie bracket of $\mathcal{G}$ corresponds to the commutator of vector fields. It follows that $[\mathcal{G}_k, \mathcal{G}_i] \subseteq \mathcal{G}_{i+k-1}$ for $i \geqslant 2$ and $k \geqslant 2$. Hence the $\mathcal{G}_i$ form a grading of the Lie algebra $\mathcal{N} = \mathcal{G}_2 \times \dots \times \mathcal{G}_q$. In particular, it follows that the algebra $\mathcal{N}$ is nilpotent.

The following lemma will be used only when $i = 2$.

Lemma 6. *We have $[\mathcal{G}_k, \mathcal{G}_i] = \mathcal{G}_{i+k-1}$ for $i \geqslant 2$ and $k \geqslant 2$.*

Proof. Consider decomposition (16) of the space $\mathcal{G}_{i+k-1}$. For simplicity, we put $\mathcal{V}^1_{i+k-1} = \mathcal{V}^1$, $\mathcal{V}^2_{i+k-1} = \mathcal{V}^2$ and $\mathcal{W} = [\mathcal{G}_k, \mathcal{G}_i]$. Since $\mathcal{W}$ is invariant with respect to $\mathrm{Ad}(\mathrm{GL}_n)$, it suffices to prove that the projections $\mathcal{W}^1$, $\mathcal{W}^2$ of $\mathcal{W}$ to $\mathcal{V}^1$, $\mathcal{V}^2$ are non-zero.

Let us calculate the commutator of the vector fields $X_1 = (x^1)^i \frac{\partial}{\partial x^1}$ and $Y_1 = x^1 (x^2)^{k-1} \frac{\partial}{\partial x^1}$. By definition, the commutator of the vector fields $X = \sum_j X^j \frac{\partial}{\partial x^j}$ and $Y = \sum_j Y^j \frac{\partial}{\partial x^j}$ equals

$$Z = [X, Y] = \sum_{j,p} \left(\frac{\partial X^p}{\partial x^j} Y^j - \frac{\partial Y^p}{\partial x^j} X^j \right) \frac{\partial}{\partial x^p}\,.$$

Hence $Z_1 = [X_1, Y_1] = (i-1)(x^1)^i (x^2)^{k-1} \frac{\partial}{\partial x^1}$. The vector field Z_1 corresponds to a tensor $b = b^u_{j_1,\dots,j_{i+k-1}} \in \mathcal{G}_{i+k-1}$ which is symmetric in the lower indices and is given by the following conditions:

$$b^1_{j_1,\dots,j_{i+k-1}} = \frac{(i-1) i!\,(k-1)!}{(i+k-1)!}$$

if precisely i elements of the set $j_1, \dots, j_{i+k-1}$ are equal to 1 and the remaining $k-1$ elements are equal to 2, and $b^u_{j_1,\dots,j_{i+k-1}} = 0$ otherwise. Clearly, the image $\lambda_{i+k-1}(b)$ under the convolution map $\lambda_{i+k-1} \colon \mathcal{G}_{i+k-1} \to T_{i+k-2}$ is non-zero. Hence $\mathcal{W}^2$ is non-zero.

Now we consider $X_2 = (x^2)^i \frac{\partial}{\partial x^1}$ and $Y_2 = (x^2)^k \frac{\partial}{\partial x^2}$. Then $Z_2 = [X_2, Y_2] = i(x^2)^{i+k-1} \frac{\partial}{\partial x^1}$. The vector field Z_2 corresponds to the tensor $d = d^u_{j_1,\dots,j_{i+k-1}} \in \mathcal{G}_{i+k-1}$ such that $d^1_{2,\dots,2} = i$ and all other components of d are equal to zero. Clearly, $\lambda_{i+k-1}(d) = 0$. Hence d belongs to $\mathcal{W}^1$. The lemma is proved.

We now return to the proof of Theorem 4. To prove assertion 1) of the theorem, we put $G=G^2(n)^0$ and describe the multiplication in G in terms of the coordinates (g^u_j,g^u_{jk}) introduced in § 3.6. An element g, which corresponds to the 2-jet of a pointed orientation-preserving diffeomorphism $\psi\colon(\mathbb{R}^n,\mathbf{0})\to(\mathbb{R}^n,\mathbf{0})$, is determined by the Taylor coefficients of the map $\psi=(\psi^u)$ up to order 2:

$$y^u=\psi^u(x)=g^u_jx^j+g^u_{jk}x^jx^k+o(x^2).$$

Here we assume that ψ maps the space $\mathbb{R}^n_x$ of standard variables (x^u) to the space $\mathbb{R}^n_y$ of standard variables (y^u).

Let $\varphi\colon(\mathbb{R}^n_y,\mathbf{0})\to(\mathbb{R}^n_z,\mathbf{0})$ be a map from $\mathbb{R}^n_y$ to the space of the standard variables (z^u):

$$z^u=\varphi^u(y)=h^u_jy^j+h^u_{jk}y^jy^k+o(y^2),$$

where $h\in G$ is the element corresponding to φ. By definition, the product $t=hg$ is the 2-jet of the map $\theta=\varphi(\psi)$ at the point $\mathbf{0}$. We have

$$\begin{aligned}z^u(x)&=h^u_s\bigl(g^s_jx^j+g^s_{jk}x^jx^k\bigr)+h^u_{sk}\bigl(g^s_jx^j\bigr)\bigl(g^k_rx^r\bigr)+o(x^2)\\&=\bigl(h^u_sg^s_j\bigr)x^j+\bigl(h^u_sg^s_{jk}+h^u_{sr}g^s_jg^r_k\bigr)x^jx^k+o(x^2).\end{aligned}$$

Thus $t=(t^u_j,t^u_{jk})$, where

$$t^u_j=h^u_sg^s_j,\qquad t^u_{jk}=h^u_sg^s_{jk}+h^u_{sr}g^s_jg^r_k. \tag{17}$$

We shall sometimes denote the elements $h=(h^u_j,h^u_{jk})\in G$ by $h=\bigl(H,h^u_{jk}\bigr)$, where H is the non-singular matrix (h^u_j).

It is easy to see that the subgroups GL^0_n and R in the algebraic Levi decomposition of G are identified with the sets of elements of the form $(H,0)$ and (E,h^u_{jk}) respectively. We shall write (h^u_{jk}) instead of (E,h^u_{jk}) and use the notation H or (h^u_j) for $(H,0)$, where $H=(h^u_j)$. The normal subgroup R and the corresponding subalgebra $\mathcal{R}$ are Abelian. Therefore the exponential map exp is an isomorphism of the vector space $\mathcal{R}$ and the group R. Clearly, if $b=(b^u_{jk})\in\mathcal{R}$, then $\exp(b)=h$ is the element $h=(h^u_{jk})$ with $h^u_{jk}=b^u_{jk}$. The action of GL^0_n by conjugation on R corresponds to the action Ad on $\mathcal{R}$. Therefore, putting $\exp(\mathcal{V}^1_2)=V^1$ and $\exp(\mathcal{V}^2_2)=V^2$ in the decomposition (16), we decompose R into the sum of its subgroups V^1 and V^2, which are normal in G. It follows from Lemma 5 that $\dim V^1=n^2(n+1)/2-n$, $\dim V^2=n$.

We denote the natural projection of G onto $\mathrm{Af}_n=G/V^1$ by $\phi\colon G\to\mathrm{Af}_n$. The group Af_n has dimension n^2+n and is an extension of the n-dimensional vector space $T_1=R/V^1$ by the group GL^0_n. Every element $p\in\mathrm{Af}_n$ is uniquely represented as

$$p=Aa, \tag{18}$$

where $A\in\mathrm{GL}^0_n$, $a\in T_1$. Using formula (17) for $H=(h^u_j)\in\mathrm{GL}^0_n$ and $t=(t^u_{jk})\in R$, we get

$$H^{-1}tH=\bigl(f^u_pt^p_{sr}h^s_ih^r_k\bigr), \tag{19}$$

where $H^{-1}=(f^u_p)$. Given $t=(t^u_{jk})\in R$, we take the functions $a_j=t^u_{ju}$ as the coordinates of the element $\phi(t)$ in T_1. Then (19) yields that the action of $H=(h^u_j)\in\mathrm{GL}^0_n$ on $a=(a_j)\in T_1$ is given in these coordinates by

$$H^{-1}aH=c=(c_j),\qquad c_j=a_uh^u_j.$$

4.7. To introduce coordinates in Af_n, we consider the decomposition (18) of any element $p \in \mathrm{Af}_n$ and take the set (A, a) as the coordinates of p. Here $A = (h^u_j)$ is a non-singular matrix, and the element $a \in T_1$ is regarded as a row vector of length n using the coordinates introduced on T_1. We identify elements of the group Af_n with such pairs (A, a). The image $\phi(h)$ of an element $h = (h^u_j, h^u_{jk}) \in G$ is calculated as follows.

1) By (17), the algebraic Levi decomposition of h is given by $h = At$, where $A = (h^u_j) \in \mathrm{GL}^0_n$, the matrix $D = (d^u_j)$ is inverse to A, $t = (t^u_{jk}) \in R$ and $t^u_{jk} = d^u_s h^s_{jk}$.

2) The element $\phi(h)$ corresponds to (A, a), where

$$a = (a_j) = t^u_{ju} = d^u_s h^s_{ju}. \tag{20}$$

The group operation in Af_n is given by

$$(A_1, a_1)(A_2, a_2) = (A_1A_2, a_1A_2 + a_2).$$

Here and in what follows, a_1A_2 is the product of a row and a matrix. The group Af_n is the affine group of an n-dimensional vector space.

We consider the space $\mathbb{R}^{n*}$ with the following right action of Af_n: if $t = (t_i)$ is a row in $\mathbb{R}^{n*}$ and $p = (A, a) \in \mathrm{Af}_n$, then

$$tp = tA + a. \tag{21}$$

Replacing p by $p^{-1} = (A^{-1}, -aA^{-1})$, we get the corresponding left action η' of Af_n on $\mathbb{R}^{n*}$:

$$pt = tp^{-1} = tA^{-1} - aA^{-1}. \tag{22}$$

Under the projection ϕ, this action induces an action η of the group G on $\mathbb{R}^{n*}$.

4.8. The stabilizer H'_1 of any vector in $\mathbb{R}^{n*}$ under the action η' has codimension n in Af_n. Hence the full pre-image H_1 of H'_1 in G also has codimension n. For any hyperplane L of dimension $n-1$ in $\mathbb{R}^{n*}$, the elements of Af_n that leave L fixed form a subgroup H'_2 of codimension n. The full pre-image H_2 of H'_2 in G also has codimension n in G.

Thus, for $q = 2$, we have found two closed connected subgroups of codimension n in $G = G^2(n)^0$, H_1 and H_2.

Now we take an arbitrary $q \geqslant 2$. Let H be a connected subgroup of $G = G^q(n)^0$ that does not contain G_q (see § 4.6), let $\mathcal{H} = \mathrm{Lie}(H)$ be its Lie subalgebra and let (15) be the Levi decomposition of $\mathcal{G}$. Then

1) the intersection $\mathcal{W} = \mathcal{H} \cap \mathcal{G}_q$ is a vector subspace of codimension at most n in $\mathcal{G}_q$,

2) the subalgebra $\mathcal{S}$ in the Levi decomposition $\mathcal{H} = \mathcal{S} \leftthreetimes \mathcal{B}$ of $\mathcal{H}$ is conjugate by an element of G to a subalgebra of SL_n (Malcev's theorem: see [12], Ch. 6, Theorem 3). Therefore we can assume that $\mathcal{S} \subset SL_n$.

Consider the projection θ of the algebra $\mathcal{G}$ onto the subalgebra GL_n along $\mathcal{N}$ according to the algebraic Levi decomposition (15). Put $\theta(\mathcal{H}) = \mathcal{H}'$ and $\theta(\mathcal{B}) = \mathcal{B}'$. Clearly, $\mathcal{H}' = \mathcal{S} \leftthreetimes \mathcal{B}'$ is the Levi decomposition for $\mathcal{H}'$, and the codimension of $\mathcal{H}'$ in GL_n does not exceed n.

By Theorem 3 we have the following cases for $\mathcal{H}'$.

1. Suppose that $\mathcal{H}' = SL_n$ or $\mathcal{H}' = GL_n$. Then $S = \mathrm{SL}_n$ and inner automorphisms of SL_n leave $\mathcal{W}$ fixed. By (16), $\mathcal{G}_q = \mathcal{V}_q^1 + \mathcal{V}_q^2$. If $\mathcal{W} \neq 0$, then Lemma 5 yields that $\mathcal{W}$ must coincide with either $\mathcal{V}_q^1$ or $\mathcal{V}_q^2$. By the same lemma, we have $\operatorname{codim}(\mathcal{V}_q^1, \mathcal{G}_q) > n$ and $\operatorname{codim}(\mathcal{V}_q^2, \mathcal{G}_q) \geqslant n$, with equality only for $q = 2$. Hence this case is possible only when $q = 2$, $S = \mathrm{SL}_n$ and $\mathcal{W} = \mathcal{V}_q^2$. In this case, H is conjugate to the subgroup H_1.

2. Suppose that $\mathcal{H}' \neq SL_n$ and $\mathcal{H}' \neq GL_n$. Using the list in Theorem 3, we see that $\operatorname{codim}(\mathcal{H}', GL_n)$ must be equal to n or $n-1$. Since we are assuming that $\mathcal{G}_q \neq \mathcal{W}$, it follows that $\operatorname{codim}(\mathcal{H}', GL_n) = n - 1$ and $\operatorname{codim}(\mathcal{W}, \mathcal{G}_q) = \operatorname{codim}(H_1, \mathcal{N}) = 1$, where $H_1 = H_n\mathcal{N}$. Therefore the list in Theorem 3 shows that we have either $H' = T_1 \times Z^0$ or $H' = T_2 \times Z^0$ (after an appropriate conjugation of H by an element of GL_n^0). In both cases, S contains the subgroup SL_{n-1} that leaves the coordinate x^1 (in the representation ST_n) fixed and transforms the set of coordinates $x^2, \dots, x^n$ to itself.

Let $\mathbb{R}^n = \mathbb{R}^1 + \mathbb{R}^{n-1}$ be the corresponding decomposition of $\mathbb{R}^n$ into subspaces invariant under SL_{n-1}.

The subgroup $T = H \cap \mathrm{SL}_n$ is contained in $T_i = H' \cap \mathrm{SL}_n$, $i = 1, 2$. Since $\operatorname{codim}(T, \mathrm{SL}_n) \leqslant n$, we have $\operatorname{codim}(T, T_i) \leqslant 1$. If $n > 3$, then the proof of Lemma 4 shows that T contains either T_3 (for $i = 1$) or T_4 (for $i = 2$). If $n = 3$, then there are additional possibilities: after an appropriate conjugation by an element of $\mathrm{SL}_2 \subseteq S \subseteq H$, the subgroup T may contain the group of upper (for $i = 1$) or lower (for $i = 2$) triangular matrices. For any n, the subgroup T contains either s_1 (if $i = 1$) or s_2 (if $i = 2$), where

$$s_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & E_{n-2} \end{pmatrix}, \qquad s_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & E_{n-2} \end{pmatrix}.$$

This will be used later.

1. We start with the case $q = 2$. In this case $\mathcal{N} = \mathcal{G}_2$. The representation ρ_2 of GL_n on $\mathcal{G}_2$ equals $\mathrm{Sym}^2[\mathrm{ST}_n^*] \otimes \mathrm{ST}_n$, where Sym^i stands for the ith symmetric power (of a representation or vector space) and where

$$\mathrm{Sym}^2[\mathrm{ST}_n^*] = \mathrm{Sym}^2\bigl(\mathbb{R}^{1*}\bigr) + \bigl(\mathbb{R}^1 \otimes \mathbb{R}^{n-1}\bigr)^* + \mathrm{Sym}^2\bigl(\mathbb{R}^{n-1*}\bigr).$$

Hence we can write

$$\begin{aligned} \mathcal{G}_2 = {} & \mathrm{Sym}^2\bigl(\mathbb{R}^{1*}\bigr) \otimes \mathbb{R}^1 + \mathrm{Sym}^2\bigl(\mathbb{R}^{1*}\bigr) \otimes \mathbb{R}^{n-1} + \bigl(\mathbb{R}^1 \otimes \mathbb{R}^{n-1}\bigr)^* \otimes \mathbb{R}^1 \\ & + \bigl(\mathbb{R}^1 \otimes \mathbb{R}^{n-1}\bigr)^* \otimes \mathbb{R}^{n-1} + \mathrm{Sym}^2\bigl(\mathbb{R}^{n-1*}\bigr) \otimes \mathbb{R}^1 + \mathrm{Sym}^2\bigl(\mathbb{R}^{n-1*}\bigr) \otimes \mathbb{R}^{n-1}. \end{aligned} \tag{23}$$

We denote the ith term of this decomposition by $\mathcal{V}_i$. All the $\mathcal{V}_i$ are invariant under the subgroup SL_{n-1} of S. The space $\mathcal{V}_1$ is one-dimensional. The spaces $\mathcal{V}_2$ and $\mathcal{V}_3$ are irreducible (with respect to SL_{n-1}) and $(n-1)$-dimensional. By Lemma 5, the space $\mathcal{V}_6$ is the direct sum of two irreducible invariant subspaces of dimension $n - 1$ and $(n-1)^2 n/2 - n + 1$ respectively. It is well known (and easily proved) that $\mathcal{V}_5$ is irreducible and has dimension $n(n-1)/2$ while $\mathcal{V}_4$ is the sum of two irreducible subspaces, $\mathcal{V}_4^1$ and $\mathcal{V}_4^2$. The space $\mathcal{V}_4$ consists of tensors $t = (t^u_{jk}) \in \mathcal{G}_2$ whose only non-zero components are t^u_{1k} with $2 \leqslant u \leqslant n,\ 2 \leqslant k \leqslant n$. The subspace $\mathcal{V}_4^1$ has dimension $(n-1)^2 - 1$ and is defined by the extra equation

$$\sum_{u=2}^{n} t^u_{1u} = 0. \tag{24}$$

The subspace $\mathcal{V}_4^2$ is one-dimensional. It is spanned by any tensor t with $t^2_{12} = t^3_{13} = \cdots = t^n_{1n}$ and all other components vanishing. Since $\operatorname{codim}(\mathcal{W}, \mathcal{G}_2) = 1$ and $\mathcal{W}$ is invariant, we have only two possibilities (we recall that $n > 2$).

1) $\mathcal{W} = \mathcal{V}_2 + \mathcal{V}_3 + \mathcal{V}_4 + \mathcal{V}_5 + \mathcal{V}_6$. It is easy to verify that this case is realized for $H = H_2$.

2) $\mathcal{W} = \mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_3 + \mathcal{V}_4^2 + \mathcal{V}_5 + \mathcal{V}_6$. Then $\mathcal{W}$ is given by one equation (24) in $\mathcal{G}_2$.

Let us prove that case 2) is impossible.

We claim that equation (24) is not invariant under the action of s_1 and s_2. Indeed, given $t = (t^u_{jk}) \in \mathcal{G}_2$, we put

$$\rho_2(s_1)[t] = f, \qquad \rho_2(s_2)[t] = g.$$

We have $f^2_{12} = t^2_{12} - t^2_{11},\ f^i_{1i} = t^i_{1i}$ for $i \geqslant 3$ and $g^2_{12} = t^1_{12} - t^1_{22} + t^2_{12} - t^2_{22},\ g^i_{1i} = t^i_{1i} - t^i_{2i}$ for $i \geqslant 3$. The condition (24) is not preserved by these transformations. This proves assertion 1) of Theorem 4.

2. Now we consider the case when $q > 2$. We recall that $\mathcal{N} = \mathcal{N}(q) = \mathcal{G}_2 \times \cdots \times \mathcal{G}_q$. If the projection of $\mathcal{W}$ onto $\mathcal{G}_2$ along $\mathcal{G}_3 \times \cdots \times \mathcal{G}_q$ does not coincide with $\mathcal{G}_2$, then $\mathcal{W}$ contains $\mathcal{G}_3 \times \cdots \times \mathcal{G}_q$ because $\operatorname{codim}(\mathcal{W}, \mathcal{N}) = 1$. Hence this projection coincides with $\mathcal{G}_2$. (We recall that $\mathcal{W}$ does not contain $\mathcal{G}_q$.) Consider

$$\mathcal{W} \cap (\mathcal{G}_{q-1} \times \mathcal{G}_q) = \mathcal{W}_1.$$

Clearly, the codimension of $\mathcal{W}_1$ in $(\mathcal{G}_{q-1} \times \mathcal{G}_q)$ equals 1. The projection of $\mathcal{W}_1$ onto $\mathcal{G}_{q-1}$ along $\mathcal{G}_q$ coincides with $\mathcal{G}_{q-1}$ (for otherwise the kernel of this projection contains $\mathcal{G}_q$ contrary to the assumption that $\mathcal{G}_q \neq \mathcal{W}$). We have $[\mathcal{G}_2, \mathcal{G}_{q-1}] = \mathcal{G}_q$ by Lemma 6. Then $[\mathcal{W}_1, \mathcal{W}] = G_q$ because the value of $[w, u]$ for $w \in \mathcal{W}_1$ and $u \in \mathcal{W}$ is determined by the values of $w \bmod \mathcal{G}_q$ and $u \bmod \mathcal{G}_3 \times \cdots \times \mathcal{G}_q$ respectively. Since $\mathcal{W}$ is a subalgebra, we get $\mathcal{G}_q \subset \mathcal{W}$ contrary to the assumption that $\mathcal{G}_q \neq \mathcal{W}$. The theorem is proved.

§ 5. Local classification of special structures

5.1. We put $G = G^q(n)^0$ and recall that a G-manifold D is said to be *special* if $\dim D \leqslant n$.

Let D be any G-manifold, w a regular point of D and L_w the tangent subspace to the G-orbit through w. It is known that the family of all L_w is an involutive distribution in an appropriate neighbourhood W of the point w. This means that the dimension of L_w is constant in W and the commutator of any two vector fields with values in the family of L_w takes its values in this family. Applying Frobenius' theorem and shrinking W if necessary, we see that W is diffeomorphic to the direct product $S{\times}A$ of the submanifolds S, A passing through w and, moreover, the image of every point $z \in S$ under the action of an appropriate neighbourhood of the identity in G equals $z \times A$. We consider the subset $M = \bigcup_{z\in S}(z, G_z^0)$ of the direct product $S \times G$, where G_z^0 is the connected component of the identity in the stabilizer of z.

Definition 6. We say that G_z^0 *depends smoothly on* z if the Lie algebra $\mathcal{G}_z = \mathrm{Lie}(G_z)$ depends smoothly on z. This in turn means that every regular point of D possesses a neighbourhood W such that the subalgebra $\mathrm{Lie}(G_z)$ has a basis $X_1(z), \dots, X_t(z)$ depending smoothly on $z \in W$.

Lemma 7. *The group G_z^0 depends smoothly on $z \in S$. The subset M is a smooth submanifold of $S \times G$.*

Proof. This follows from the smooth dependence of the Lie subalgebra $\mathcal{G}_z = \mathrm{Lie}(G_z)$ on z, which is proved as follows.

Let $e_1, \dots, e_f$ be tangent vectors to G at the identity that form a basis of the Lie algebra $\mathcal{G} = \mathrm{Lie}(G)$. Their images under the action of G on D determine f vector fields on W: $X_1(z), \dots, X_f(z)$, where $z \in W$. We may assume that the rank of the point $z \in W$ is constant and equals r. Then there are $f - r$ independent relations between the $X_j(z)$. For $z \in S$, every relation

$$a_1(z)X_1(z) + \dots + a_f(z)X_f(z) = 0$$

determines a vector $a_1(z)e_1 + \dots + a_f(z)e_f$ belonging to $\mathcal{G}_z$. Since the $X_i(z)$ depend smoothly on z, we can choose $f - r$ independent relations depending smoothly on z (shrinking W if necessary). This completes the proof.

5.2. Consider the bundle θ: $Y = G \times S/M \to S$ obtained from the direct product $G \times S \to S$ by taking the quotient of each fibre $G \times z$ over $z \in S$ by G_z^0. Then Y has a natural structure of a smooth manifold such that θ becomes a smooth map with epimorphic differentials and the fibrewise action of G on Y is smooth. This can be justified as follows. There is a smooth map ν: $G \times S \to D$ sending a point (g, z) to $gz \in D$. For every $z \in S \subset D$, the restriction of ν to the fibre $G \times z$ maps G onto the homogeneous space G/G_z by the formula $g \to gz$. It is known that this map has a smooth local section $G/G_z \to G$ in a neighbourhood of z. Therefore G contains a smooth submanifold L (depending on z) which passes through the identity e and is mapped diffeomorphically onto its image in G/G_z. Then the differential at $e \times z$ of the restriction of ν to $L \times S$ is an isomorphism of the tangent spaces $T_{e\times z}(L \times S)$ and $T_z(D)$. Here $T_{e\times z}(L \times S)$ is the tangent space to $L \times S$ at the point $e \times z \in L \times S$ and $T_z(D)$ is the tangent space to D at z. Hence we can shrink L to L_1 and S to S_1 (preserving the conditions $e \in L_1$, $z \in S_1$) in such a way that ν is a diffeomorphic (that is, bijective) map of $S_1 \times L_1$ onto its image in D.

Let μ: $G \times S \to Y$ be the map sending each point (g, z) to the point of Y determined by the coset gG_z^0 of the fibre $G \times z$ over z. Left multiplication by elements of G determines actions of G on $G \times S$ and Y that are compatible with μ. Consider the map ε from Y to D that sends each coset gG_z^0 of the fibre $G \times z$ over z to the point $gz \in D$. The map ε is well defined, commutes with the action of G and satisfies $\nu = \varepsilon\mu$. Hence μ maps $L_1 \times S_1$ bijectively onto $\mu(L_1 \times S_1) = W'$, and ε maps W' bijectively onto $\nu(S_1 \times L_1) = W$. By definition, the smooth structure on W' is transferred from $L_1 \times S_1$. The union of the shifts gW' of W' over all $z \in S$ and $g \in G$ coincides with Y. The smooth structure of gW' is transferred from W'. This determines a smooth structure on Y such that $\theta\colon Y \to S$ is a smooth projection, the action of G on Y is smooth and $\varepsilon(Y)$ is an open subset of D which is invariant under the action of G and contains the orbit passing through w.

We define a section $\gamma\colon S \to Y$ of the projection $\theta\colon Y \to S$ by sending $z \in S$ to $\gamma(z) = \mu(e \times z) \in Y$. The construction of Y shows that, for every point $z \in \varepsilon(Y) \subset D$, the point $\gamma(z) \in Y$ has a neighbourhood $U \subset Y$ such that ε is a diffeomorphism of U onto its image. Since ε commutes with the action of G on Y and D, we get a local isomorphism of the action of G on Y in a neighbourhood of $\gamma(z)$ and the action of G on D in a neighbourhood of z.

The resulting covering $\varepsilon\colon Y \to \varepsilon(Y) \subset D$ is called an *unfolding* (or an *S-unfolding*) of D. Clearly, the covering ε induces coverings $\varepsilon^k\colon Y^k \to D^k$. Any points $y \in Y^k$ and $d \in D^k$ with $\varepsilon^k(y) = d$ have neighbourhoods that are diffeomorphically mapped to each other by ε^k.

Lemma 8. *Let $\varepsilon_1\colon Y_1 \to D_1 \subset D$ and $\varepsilon_2\colon Y_2 \to D_2 \subset D$ be unfoldings of a regular G-manifold D. Take points $c_1 \in Y_1$ and $c_2 \in Y_2$ such that $\varepsilon_1(c_1) = \varepsilon_2(c_2) = c$. Then one can find an open G-invariant neighbourhood $D_3 \subset D_1 \cap D_2$ of the point c such that the coverings $Y_1' = \varepsilon_1^{-1}(D_3)$ and $Y_2' = \varepsilon_2^{-1}(D_3)$ are related by a G-isomorphism χ with $\chi(c_1) = c_2$.*

Proof. Consider open neighbourhoods U_1, U_2 and U of the points c_1, c_2 and c respectively such that the restrictions $\bar\varepsilon_1$ and $\bar\varepsilon_2$ of ε_1 and ε_2 to U_1 and U_2 are diffeomorphisms onto U. The map $\bar\chi = \bar\varepsilon_2^{-1}\bar\varepsilon_1$ is a diffeomorphism of U_1 onto U_2. Consider any point u_1 in U_1 and its image $u_2 = \bar\chi(u_1)$. Then $\varepsilon_1(u_1) = \varepsilon_2(u_2) = u$. Clearly, $G_{u_i} = G_u^0$ by the construction of Y_i. This means that $\bar\chi$ may be extended to a G-invariant map $\chi\colon Y_1' \to Y_2'$, where $Y_1' = GU_1$ and $Y_2' = GU_2$. It is clear that χ is an isomorphism and $Y_i' = \varepsilon_i(D_3)$, where $D_3 = GU$. This proves the lemma.

Remark 7. Using the space Y of an unfolding $\varepsilon\colon Y \to D$, we may construct a bundle $\pi^Y\colon P = P_X(Y) \to X$ of geometric objects with generic fibre Y. The covering ε determines a covering $\varepsilon_X\colon P_X(Y) \to P_X(D)$ and a map $\varepsilon_X^k\colon J^k(Y) \to J^k(D)$ of the corresponding jet spaces. The bundle $P_X(Y)$ is called an *unfolding* of the bundle $P_X(D)$. Clearly, for every regular point $d \in P_X(D)$ there is an unfolding $\varepsilon\colon Y \to D$ such that d lies in the image of $\varepsilon_X\colon P_X(Y) \to P_X(D)$. Moreover, any points $y \in P_X(Y)$ and $d \in P_X(D)$ with $\varepsilon_X(y) = d$ have neighbourhoods U and V such that ε_X is a diffeomorphism of U onto V commuting with the action of the group $\mathscr{V}$ of orientation-preserving diffeomorphisms. For every pair of points $y \in J^k(Y)$ and $d \in J^k(D)$ with $\varepsilon_X^k(y) = d$ one can similarly find neighbourhoods U and V such that ε_X^k is a diffeomorphism commuting with the action of $\mathscr{V}$.

Remark 8. The usual example of a non-closed one-parameter subgroup of the torus shows that D may contain orbits whose intersection with S is dense. In the S-unfolding of D, there is precisely one orbit through each point of S.

Lemma 9. *Suppose that the assumptions of § 5.1 hold and all subgroups G_z^0 of the family M are conjugate to a fixed subgroup K of G. Then one can locally choose a conjugating element that depends smoothly on $z \in S$.*

Proof. The subgroup K is obviously closed. The set of all subgroups in G conjugate to K forms the homogeneous space $T = G/N(K)$, where $N(K)$ is the normalizer of K. The family G_z^0 determines a smooth path in T. It is well known that the bundle $G \to G/H$ is locally trivial for every Lie group G and every closed subgroup H. This proves the lemma.

Remark 9. Under the assumptions of § 5.1 and Lemma 9, let $g(z)\colon\ S \to G$ be a smooth function that conjugates G_z^0 to K in a neighbourhood of w, that is, $g(z)G_z^0 g^{-1}(z) = K$. Replacing S by the manifold S_0 consisting of all points $\bigl(z, g^{-1}(w)g(z)z\bigr)$, where $z \in S$, we can assume that the family G_z^0 is constant in a neighbourhood of the point $w \in S$. More precisely, the following lemma holds.

Lemma 10. *Consider a map $\theta\colon Y \to S$ admitting a section $\gamma\colon S \to Y$ and an action of the Lie group G on Y such that each fibre of the projection θ is precisely one orbit of the action of G. Suppose that the stabilizer G_y of any point $y \in Y$ is conjugate to the same connected subgroup K of G. Let $w \in \gamma(S)$ be an arbitrary point. Then there is a neighbourhood $S_0 \subset S$ of $\theta(w)$ such that the manifold $Y_1 = \theta^{-1}(S_0)$ admits an isomorphism $\nu\colon Y_1 \to S_0 \times Y_0$ onto the direct product of S_0 and $Y_0 = G/G_w$ commuting with the action of G. Here Y_0 is the homogeneous space of G by the stabilizer G_w of w.*

Remark 10. Since S can be taken arbitrarily small, we assume in what follows that ν is an isomorphism between Y and $S \times Y_0$.

Proof of Lemma 10. By Lemma 9 there is a neighbourhood S_0 of the point $\theta(w)$ such that we have a smooth function $g(z)\colon S_1 \to G$ conjugating G_z to K on the set $S_1 = \gamma(S_0)$:

$$g(z)G_z g^{-1}(z) = K. \tag{25}$$

Let S_2 be the image of S_0 under the map $u \in S_0 \to g^{-1}(w)g\bigl(\gamma(u)\bigr)\gamma(u)$. Then the stabilizers G_p are equal to G_w for all points p of the manifold S_2. Indeed, we have $p = g^{-1}(w)g\bigl(\gamma(u)\bigr)\gamma(u)$ for some point $u \in S_0$. Applying the formula $G_{ax} = aG_x a^{-1}$ to $x = \gamma(u)$ and $a = g^{-1}(w)g\bigl(\gamma(u)\bigr)$, we get

$$G_p = g^{-1}(w)g\bigl(\gamma(u)\bigr)G_{\gamma(u)}g^{-1}\bigl(\gamma(u)\bigr)g(w).$$

Since $K = g\bigl(\gamma(u)\bigr)G_{\gamma(u)}g^{-1}\bigl(\gamma(u)\bigr)$ by formula (25), we have $G_p = g^{-1}(w)Kg(w) = G_w$ by the same formula. We now define a map $S_0 \times Y_0 \to Y_1$ by $(p, gG_w) \to gp$. This map is well defined. We easily see that its inverse is the desired map ν.

5.3. From now until the end of this section, we fix an oriented manifold X, a point $b \in X$ and an orientation of X. Using this orientation, we can define the sign of a non-degenerate form of top degree on X. Up to the end of the section, we consider the bundles $\pi^Y : P = P_X(Y) \to X$ of geometric structures whose generic fibre Y is an S-unfolding of a special G-manifold D near a regular point.

Let $\theta\colon Y \to S$ and $\gamma\colon S \to Y$ be the corresponding projection and section (see § 5.2). We shall define the property of a section $s\colon X \to P$ of being "sufficiently general" in a neighbourhood of the point $b \in X$, introduce the notion of a canonical form for such sections and show how to reduce a sufficiently general section to canonical form.

We use the following notation. Let $F = (U_F, f^i)$ and $H = (U_H, h^i)$ be charts on X. We write $J(H/F)$ for the Jacobian matrix $(\partial h^i/\partial f^j)$ of the transition map from F to H. Choose a chart $Q = (T, y^c)$ on Y. A geometric object (section) $s\colon X \to P$ is given in the chart (Q, F) by a map $s_F\colon X \to Y$, that is, by $m = \dim Y$ real functions $y^c = s^c_F(f^i)$ of $f^1, \dots, f^n$. The functions s^c_F are called the *components* of s in the chart (Q, F). We say that a section $s\colon X \to P$, which is defined in a neighbourhood of b, is representable in a chart V of Y if $s_F(b) \in V$. The dependence on Q is not indicated in this notation because, by definition, the reduction of s to canonical form is done by choosing an appropriate chart F on X (depending on s) with $\operatorname{or}(F) = \operatorname{or}(X)$ while the chart Q on Y remains fixed independently of s. A chart on X is said to be *canonical* (for the section s) if s has a canonical form in this chart. Let $s^c_F(b)$ be the value of the function s^c_F at the point $b \in X$, that is, $s^c_F(f^i)$ for $f^i = f^i(b)$. The vector field e and the differential form ω are written in the chart F on X as $e_F = e^i_F(\partial/\partial f^i)$ and $\omega_F = \omega_{Fi}\, df^i$.

Thus, let $s\colon X \to P$ be a section defined in a neighbourhood of the point $b \in X$. We start with the case $q = 1$ (first-order structures). Then G is just the connected component of the identity in the group GL^0_n of real matrices of order $n = \dim X$. Theorem 3 yields a list of all possible stabilizers for the actions of G on a manifold Y with $m = \dim Y \leqslant n$. We consider all items in this list except for the cases 9) and 10), which will be studied in a future paper. Every case consists of one or several of the types stated below.

Type I (the action of G on Y is trivial). Then $D = Y$, and $w = s(b)$ determines a point of Y which does not depend on the choice of the coordinate system on X. Replacing Y by an appropriate neighbourhood of w, we may assume that Y is covered by an atlas of only one chart $Q = (Y, y^c)$. Choose a chart $F \ni b$ on X with $\operatorname{or}(F) = \operatorname{or}(X)$. The section s is given in the chart (Q, F) by m functions $y^c = s^c_F(f^i)$, where $1 \leqslant c \leqslant m,\ \ 1 \leqslant i \leqslant n$. We may write $s^c_F(f^i) = s^c(f^i)$ since the components s^c_F are independent of the choice of F for this type. A section s of this type is said to be sufficiently general if the Jacobian matrix $(\partial s^c/\partial f^i)$, $1 \leqslant c \leqslant m,\ \ 1 \leqslant i \leqslant n$, has rank m at the point b. If $m < n$, we can assume without loss of generality that $\det(\partial s^c/\partial f^i)(b) \neq 0$ for $1 \leqslant c \leqslant m,\ \ 1 \leqslant i \leqslant m$. Then the implicit function theorem enables us to choose a b-chart H on X in such a way that $s^i = \lambda^i + h^i$ for $1 \leqslant i \leqslant m$ and $f^i = h^i$ for $m+1 \leqslant i \leqslant n$. Here the $\lambda^i = s^i(b)$ are constants independent of h^i. This yields the canonical form for $m < n$. If $m = n$, then we also have another canonical form $s^i = \lambda^i + h^i$ for $i < n$, $s^n = \lambda^n - h^n$. These two canonical forms are inequivalent since the differential forms $dh^1 \wedge \dots \wedge dh^n$ and $-dh^1 \wedge \dots \wedge dh^n$ cannot be transformed to each other by an orientation-preserving diffeomorphism.

In what follows, we abbreviate the notation by using the term b-*sections* for sections that are sufficiently general at the point b.

Type II (the dimension of the G-orbits in Y equals 1). The stabilizer G^0_w of any point $w \in Y$ is equal to SL_n. By Lemma 10 we have a G-isomorphism

$$\nu\colon Y \to S \times \mathbb{R}^0. \tag{26}$$

It should be noted that $\mathrm{GL}_n^0/\mathrm{SL}_n = \mathbb{R}^0$. Here $\dim S = m-1$. We define a canonical coordinate y^0 on $\mathbb{R}^0$ by the condition

$$y^0(g \mod \mathrm{SL}_n) = \det g. \tag{27}$$

Choose a chart $F \ni b$ on X with $\mathrm{or}(F) = \mathrm{or}(X)$. Given a section $s\colon X \to P$, we consider the trivialization tr_F corresponding to the chart F and consider $\nu\big(\mathrm{tr}_F\big[s(b)\big]\big) = w_F \in S \times \mathbb{R}^0$. The projection w_S of w_F onto the factor S is independent of the choice of F. Replacing S by an appropriate neighbourhood of the point w_S, we may assume that S is covered by an atlas of only one chart with coordinates $(y^1, \dots, y^{m-1})$. Then Y is covered by one chart Q with the coordinate system $y^0, y^1, \dots, y^{m-1}$. The resulting chart on Y is said to be *adapted*. Given an adapted chart Q on Y, we say that the chart (Q, F) on P is *adapted to* F, or simply *adapted*. We easily see that the component $s_F^0(f^i)$ of the section s in the chart (Q, F) satisfies $s_F^0 > 0$. When F is changed, this component is transformed as a differential n-form on X while the components $s_F^1(f^i), \dots, s_F^{m-1}(f^i)$ are functions on X. Therefore we write $s_F = (\omega_F, \Psi^1, \dots, \Psi^{m-1})$, where $\omega_F = a_F\, df^1 \wedge \cdots \wedge df^n$, $a_F > 0$ and $\Psi^i = \Psi^i(f^1, \dots, f^n)$.

Definition 7. A section s is called a *b-section* (or is said to be sufficiently general at the point b) if $\mathrm{rk}(\partial\Psi^c/\partial f^i)(b) = m-1$, $1 \leqslant c \leqslant m-1$, $1 \leqslant i \leqslant n$. This definition is easily seen to be independent of the choice of an adapted chart Q on Y and a chart F on X.

Let us bring the b-section s to a canonical form. There is no loss of generality in assuming that $\det(\partial\Psi^c/\partial f^i)(b) \neq 0$ for $1 \leqslant c \leqslant m-1$, $1 \leqslant i \leqslant m-1$. Then we can find a chart H on X such that $\Psi^i = h^i$ for $1 \leqslant i \leqslant m-1$, $h^i = f^i$ for $m-1 \leqslant i \leqslant n-1$ and $h^n = \pm f^n$, where the sign of f^n is chosen to guarantee that $\mathrm{or}(H) = \mathrm{or}(X)$. We write

$$\omega_H = a_H\, dh^1 \wedge \cdots \wedge dh^n.$$

Clearly, $a_H > 0$. Now take a b-chart G on X such that $g^i = h^i - \lambda^i$ for $i < n$, $g^n = \Psi(h^1, \dots, h^n)$. Here Ψ is a function to be defined and the $\lambda^i = \Psi^i(b)$ are constants. Clearly, $J(G/H) = \partial\Psi/\partial h^n$. Choose Ψ using the condition $\partial\Psi/\partial h^n = a_H$. This yields the canonical form $s^i = g^i + \lambda^i$ for $i < m$, $s^0 = 1$.

We now construct the coordinates for all further types of Y using the same scheme as for Type II. The only difference is that we generally obtain an atlas on Y because some types of sample manifolds do not admit a global coordinate system. We start by defining a class of atlases on Y which are said to be *adapted* (for this type). If we choose an adapted atlas $\mathcal{Q}$ on Y, then we again obtain that every choice of a chart F on X determines a trivialization $\mathrm{tr}_F\colon P = P_U(Y) \to Y \times U$. Hence we get an atlas on P, which is said to be adapted to F (or simply adapted) and is denoted by $(\mathcal{Q}, F)$. Coordinates are said to be adapted if they are determined by an adapted atlas.

We consider the case when the dimension of G-orbits in Y equals $n-1$. There are two subcases: $\dim Y = n$ and $\dim Y = n-1$. We first consider the case when $\dim Y = n$.

Take a point $w \in Y$. In our case, $\dim S = 1$. We claim that the subgroups G_y^0 (with any $y \in Y$) are conjugate to the same subgroup $K \subset G$, which equals either H_1 or H_2 in Theorem 3. Indeed, we may assume that $y \in \gamma(S)$. Consider the Grassmannian manifold Γ of all vector subspaces of codimension $n-1$ in GL_n. Let Γ_1 (resp. Γ_2) be the connected subset of Γ consisting of all subalgebras conjugate to the Lie algebra $\mathcal{H}_1$ (resp. $\mathcal{H}_2$) of the group H_1 (resp. H_2). The intersection of Γ_1 and Γ_2 is empty. Assigning the Lie subgroup $G^0_{\gamma(z)}$ to every point $z \in S$, we see that the Lie algebras of $G^0_{\gamma(z)}$ form a connected curve in Γ, and the family $G^0_{\gamma(z)}$ is uniquely determined by this curve. However, if a connected curve is contained in $\Gamma_1 \cup \Gamma_2$, then it is actually contained in one of the sets Γ_i. Hence we encounter two cases, $K = H_1$ and $K = H_2$. Remark 10 enables us to assume that $Y = S \times \mathrm{GL}_n^0 / K$ in both cases.

Type III ($K = H_1$). In this case, $S_{n-1} = \mathrm{GL}_n^0 / K$ is the set of non-zero vectors (up to a positive scalar factor) in the space $\mathbb{R}^n$ of the representation ST_n (see § 2.1). One can identify S_{n-1} with a sphere. There is a natural projection $\mathbb{R}^n - \mathbf{0} \to S_{n-1}$ which sends each non-zero vector of $\mathbb{R}^n$ to the ray spanned by this vector. This projection commutes with the action of G and is denoted by μ. Elements of S_{n-1} are called *spherical vectors*. Consider the G-space $M = S \times (\mathbb{R}^n - \mathbf{0})$ with a trivial action of G on S. Let $\theta \colon M \to Y = S \times S_{n-1}$ be equal to the identity map on S and to μ on $\mathbb{R}^n - \mathbf{0}$. Then θ commutes with the action of G and, therefore, induces an epimorphism

$$\theta_X \colon L = P_X(M) \to P = P_X(Y)$$

of the corresponding bundles over X.

We denote the standard coordinates in $\mathbb{R}^n$ by $t^1, \dots, t^i, \dots, t^n$. Shrinking S, we can assume that S is endowed with a global coordinate v (since $\dim S = 1$). Regard (v, t^i) as an adapted coordinate system on M and let Q be the corresponding global chart. Then we can identify every section $s \colon X \to L$ with a pair $s = (\Psi, e)$, where Ψ is a scalar-valued function on X and e is a vector field without zeros on X. It follows that, for every chart $F = (f^i)$ on X, the section s is given in the chart (Q, F) (which is adapted to F) on L by the functions $v = \Psi(f)$, $t^i = e^i(f)$, where $e^i(f)$ are the coordinates of the vector field e in the base $\partial/\partial f^i$.

Let $\bar{s}$ be a section of the bundle $P \to X$. Since θ_X is epimorphic, there is a section $s \colon X \to L$ with $\theta_X(s) = \bar{s}$. Then we shall say that $\bar{s}$ *represents* s. The function Ψ in the identification $s = (\Psi, e)$ is uniquely determined by $\bar{s}$, and the vector field e is determined up to multiplication by any positive function. We define a *spherical vector field* $\bar{e}$ to be the set of all vector fields obtained from e by multiplication by any positive function, and we say that e *represents* $\bar{e}$.

Thus a section $\bar{s} \colon X \to P$ may be identified with a pair $\bar{s} = (\Psi, \bar{e})$, where Ψ is a scalar function and $\bar{e}$ is a spherical field on X. Let e be a vector field representing $\bar{e}$.

Definition 8. A section $\bar{s} = (\Psi, \bar{e}) \colon X \to P$ is called a *b-section* if

$$\frac{\partial \Psi}{\partial e(b)} \neq 0. \tag{28}$$

In other words, the derivative of Ψ at b in the direction of e is different from zero. (In particular, it follows that $e(b) \neq 0$.)

We easily see that this condition for $\bar{s}$ to be sufficiently general at b is independent of all choices involved: the choice of an adapted chart Q on L, the choice of any chart F on X and the choice of a representative e for $\bar{e}$.

Let $\bar{s} = (\Psi, \bar{e})$ be a b-section. We know that the vector field e is straightened in some chart $H \ni b$ on X. This means that the expression e_H of e in H has components $e_H^1 = 1$, $e_H^i = 0$ for $i > 1$. Replacing h^n by $-h^n$ if necessary, we may also assume that $\operatorname{or}(H) = \operatorname{or}(X)$.

Now we define a chart G on X as follows. We put $g^i = h^i - h^i(b)$ for $2 \leqslant i \leqslant n-1$, and $g^1 = \Psi - \Psi(b)$ (resp. $g^1 = -\Psi + \Psi(b)$) if $\partial\Psi/\partial e(b) > 0$ (resp. $\partial\Psi/\partial e(b) < 0$). Finally, we put $g^n = \pm\bigl(h^n - h^n(b)\bigr)$, where the sign is chosen so that $\operatorname{or}(H) = \operatorname{or}(G)$. Then H is a canonical b-chart for $\bar{s}$ (see the definition of a b-chart in § 2.1), and the section s takes the following form in the atlas $(\mathcal{Q}, H)$. If $\partial\Psi/\partial e(b) > 0$, then $v = g^1 + \lambda^1$, where $\lambda^1 = \Psi(b)$, and if $\partial\Psi/\partial e(b) < 0$, then $v = -g^1 + \lambda^1$; $t^2 = \dots = t^n = 0$, $t^1 > 0$. In § 6 below we shall construct an adapted atlas on Y and give a canonical form of the section $\bar{s}$ for this and further types of sample spaces up to Type XI.

Suppose that $K = H_2$. In this case we consider the space $\mathbb{R}^{n*}$ of the representation ST_n^*. Non-zero elements of $\mathbb{R}^{n*}$ are called *linear forms*. A *spherical linear form* is a linear form defined up to multiplication by a positive number. We denote the set of spherical linear forms by S_{n-1}^* and identify it with a sphere. We have $S_{n-1}^* = \mathrm{GL}_n^0/K$. As above, we define $M^* = S \times (\mathbb{R}^{n*} - \mathbf{0})$, $\mu^* \colon \mathbb{R}^{n*} - \mathbf{0} \to S_{n-1}^*$, $\theta^* \colon M^* \to Y$ and an epimorphism $\theta_X^* \colon L^* = P_X(M^*) \to P = P_X(Y)$. We define an adapted chart Q^* on Y^* consisting of the coordinate v on S and the standard coordinates $u^1, \dots, u^i, \dots, u^n$ in $\mathbb{R}^{n*}$. Any section $s \colon X \to L^*$ may be identified with a pair $s = (\Psi, \omega)$, where Ψ is a scalar function and ω is a differential 1-form without zeros on X. A section $\bar{s} \colon X \to P$ may be identified with the pair $\bar{s} = (\Psi, \overline{\omega})$, where Ψ is a scalar function and $\overline{\omega}$ is a 1-form without zeros determined up to multiplication by any positive function. We define a *spherical form* $\overline{\omega}$ to be the set of 1-forms obtained from ω by multiplication by any positive function and say that ω *represents* $\overline{\omega}$. We say that a section $s \colon X \to L^*$ *represents a section* $\bar{s} \colon X \to L^*$ if $\theta_X^*(s) = \bar{s}$. Writing $s = (\Psi, \omega)$ and $\bar{s} = (\Psi_1, \overline{\omega})$, we may rephrase this by saying that $\Psi_1 = \Psi$ and ω represents $\overline{\omega}$.

We recall some notions introduced in [13], Appendix, § 1.

Let ω be a 1-form on an n-dimensional manifold X. For every positive integer $k \leqslant n$ we define a form

$$\omega^{(k)} = \begin{cases} \underbrace{d\omega \wedge \dots \wedge d\omega}_{c} & \text{for} \quad k = 2c, \\ \omega \wedge \underbrace{d\omega \wedge \dots \wedge d\omega}_{c} & \text{for} \quad k = 2c+1. \end{cases}$$

Here $\wedge$ means the exterior product of differential forms.

Consider the form $\lambda\omega$, where λ is a scalar function. It is easy to see that $d(\lambda\omega) = d\lambda \wedge \omega + \lambda\, d\omega$. Hence we have

$$(\lambda\omega)^{(k)} = \begin{cases} \lambda^c \omega^{(k)} + c\lambda^{c-1}\, d\lambda \wedge \omega^{(k-1)} & \text{for} \quad k = 2c, \qquad (29) \\ \lambda^{c+1}\omega^{(k)} & \text{for} \quad k = 2c+1. \qquad (30) \end{cases}$$

Lemma 11. *Suppose that $n = 2a$ is even and the 1-form ω is given by $\omega = \sum_{i=1}^a u^i\, dv^i$ in an appropriate coordinate system u^i, v^i on a neighbourhood U of a point $b \in X$. If $\omega(b) \neq 0$, then there is a new coordinate system r^i, s^i on a neighbourhood $V \subseteq U$ of b such that*

1) *ω is given in the new coordinates again by*

$$\omega = \sum_{i=1}^{a} r^i\, ds^i,$$

2) *one can arbitrarily prescribe the values $r^i(b)$, $s^i(b)$ of the new coordinates at the point b provided that the $r^i(b)$ are not all equal to zero,*

3) *the Jacobian of the transfer map to the new coordinate system equals* 1.

Now suppose that $n = 2a + 1$ is odd and the 1-form ω is given by $\omega = \sum_{i=1}^a u^i\, dv^i + dw$ in an appropriate coordinate system u^i, v^i, w on a neighbourhood U of a point $b \in X$. One can always find a new coordinate system r^i, s^i, z on a neighbourhood $V \subseteq U$ of b such that

1) *ω is given in the new coordinates again by*

$$\omega = \sum_{i=1}^{a} r^i\, ds^i + dz,$$

2) *one can arbitrarily prescribe the values $r^i(b)$, $s^i(b)$, $z(b)$ of the new coordinates at the point b,*

3) *the Jacobian of the transfer map to the new coordinate system equals* 1.

Proof. Let $n = 2a$ be even. Since $\omega(b) \neq 0$, there is j, $1 \leqslant j \leqslant a$, such that $u^j(b) \neq 0$. We make a linear transformation $v^i = b^i_k t^k$, $1 \leqslant i \leqslant a$, $1 \leqslant k \leqslant a$. Then ω takes the following form in the coordinates u, t:

$$\omega = \sum_{k=1}^{a} \left(\sum_{i=1}^{a} b^i_k u^i \right) dt^k.$$

We put $\sum_i b^i_k u^i = r^k$. Clearly, we may assume that the matrix $B = (b^i_k)$ is non-singular and the $r^i(b)$ have prescribed values, not all equal to zero. The form ω is given by $\omega = \sum_{i=1}^a r^i\, dt^i$ in the coordinates r, t. If we write the variables u, v, r, t as columns, then the transformation of coordinates may be written in matrix form as $v = Bt$, $r = B^T u$. It follows that the Jacobian of the transfer map from u, v to r, t equals 1. We shall look for the variables s^i in the form $s^i = t^i + q^i$, where the q^i are constants. Since we can choose the q^i arbitrarily, the case $n = 2a$ is proved.

Now let $n = 2a + 1$ be odd. We put $r^i = u^i - p^i$, $s^i = v^i - q^i$ for $1 \leqslant i \leqslant a$, and $z = w + \sum_{i=1}^a p^i s^i + d$. Then

$$\omega = \sum_{i=1}^{a} r^i\, ds^i + dz.$$

This proves the lemma since we can choose the constants p^i, q^i, d arbitrarily.

Darboux' theorem (see [13]) says that every 1-form μ on X with $\mu^{(n)}(b) \neq 0$ and $\mu(b) \neq 0$ (the second inequality follows from the first if n is odd) is locally reduced to the following canonical form in an appropriate chart $F \in b$:

$$\mu_F = \begin{cases} \sum_{i=1}^{a} f^{2i-1}\, df^{2i} & \text{for} \quad n = 2a, & (31) \\ \sum_{i=1}^{a} f^{2i-1}\, df^{2i} + df^{2a+1} & \text{for} \quad n = 2a+1. & (32) \end{cases}$$

When the manifold X is oriented, we must also know a canonical form for μ in a chart F with $\mathrm{or}(F) = \mathrm{or}(X)$. We easily see from the above results that the condition $\mathrm{or}(F) = \mathrm{or}(X)$ yields another canonical form,

$$\mu_F = \begin{cases} -f^1\, df^2 + \sum_{i=2}^{a} f^{2i-1}\, df^{2i} & \text{for} \quad n = 2a, & (33) \\ \sum_{i=1}^{a} f^{2i-1}\, df^{2i} - df^{2a+1} & \text{for} \quad n = 2a+1. & (34) \end{cases}$$

These cases differ by the sign of the form $\mu^{(n)}$.

Lemma 11 implies that there is a b-centred chart V on X such that the Jacobian of the transition map from F to V equals 1 and μ is given in V by

$$\mu_V = \begin{cases} (1+v^1)\, dv^2 + \sum_{i=2}^{a} v^{2i-1}\, dv^{2i} & \text{in case (31)}, & (35) \\ (1-v^1)\, dv^2 + \sum_{i=2}^{a} v^{2i-1}\, dv^{2i} & \text{in case (33)}, & (36) \end{cases}$$

$$\mu_V = \begin{cases} \sum_{i=1}^{a} v^{2i-1}\, dv^{2i} + dv^{2a+1} & \text{in case (32)}, & (37) \\ \sum_{i=1}^{a} v^{2i-1}\, dv^{2i} - dv^{2a+1} & \text{in case (34)}. & (38) \end{cases}$$

Now suppose that we are given a section $\bar{s}\colon X \to P = P_X(Y)$, and let $s\colon X \to L^* = P_X(M^*)$ be a section representing $\bar{s}$. We identify s with the pair $s = (\Psi, \omega)$.

Type IV. Here $K = H_2$ (the case $n = 2a$).

Definition 9. A section $\bar{s}$ is called a b-*section* if, at b,

$$d\Psi \wedge \omega^{(n-1)} \neq 0. \tag{39}$$

This condition is easily seen to be independent of the choice of ω.

Suppose that s satisfies (39). Then $\omega^{(n-1)}(b) \neq 0,\ \omega(b) \neq 0$. Therefore, multiplying ω by a positive function λ (if necessary) and using (29), we can always assume that the following condition holds at b:

$$0 < (\lambda\omega)^{(n)} = \lambda^a \omega^{(n)} + a\lambda^{a-1}\, d\lambda \wedge \omega^{(n-1)}.$$

We put $\varepsilon = \lambda\omega$ and assume that $\varepsilon^{(n)}(b) > 0$. Then ε takes the following form in some chart F oriented in the same way as X:

$$\varepsilon_F = \sum_{i=1}^{a} f^{2i-1}\, df^{2i}. \tag{40}$$

By Lemma 11 we can assume that

$$f^1(b) > 0, \qquad f^i(b) = 0 \quad \text{for} \quad i > 1. \tag{41}$$

Now we define a new chart G on X by $g^1 = \Psi(f^1, \dots, f^n)$, $g^{2i-1} = f^{2i-1}/f^1$ for $2 \leqslant i \leqslant a$, and $g^{2i} = f^{2i}$ for $1 \leqslant i \leqslant a$. Condition (39) guarantees that the transition map from F to G is non-degenerate. Indeed, we easily see that the determinant j of the transition matrix $J = J(G/F)$ from F to G remains unaltered if we omit all even rows and columns. The resulting matrix J' is given by

$$J' = \begin{pmatrix} \frac{\partial\Psi}{\partial f^1} & \frac{\partial\Psi}{\partial f^3} & \frac{\partial\Psi}{\partial f^5} & \frac{\partial\Psi}{\partial f^7} & \dots & \frac{\partial\Psi}{\partial f^{2a-1}} \\ -\frac{f^3}{(f^1)^2} & \frac{1}{f^1} & 0 & 0 & \dots & 0 \\ -\frac{f^3}{(f^1)^2} & 0 & \frac{1}{f^1} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ -\frac{f^{2a-1}}{(f^1)^2} & 0 & 0 & 0 & \dots & \frac{1}{f^1} \end{pmatrix}.$$

It is easy to calculate that the determinant of J equals

$$j = \frac{1}{(f^1)^a}\left(f^1\frac{\partial\Psi}{\partial f^1} + f^3\frac{\partial\Psi}{\partial f^3} + \dots + f^{2a-1}\frac{\partial\Psi}{\partial f^{2a-1}}\right).$$

On the other hand, $\varepsilon^{(n-1)} = \lambda^a\dot{\omega}^{(n-1)}$ by (30), whence (39) implies that $\big(\varepsilon^{(n-1)} \wedge d\Psi\big)(b) \neq 0$. Using (40), we can explicitly find the expression $\big(\varepsilon^{(n-1)} \wedge d\Psi\big)_F$ for $\varepsilon^{(n-1)} \wedge d\Psi$ in the chart F:

$$\big(\varepsilon^{(n-1)} \wedge d\Psi\big)_F = (a-1)!\left(f^1\frac{\partial\Psi}{\partial f^1} + f^3\frac{\partial\Psi}{\partial f^3} + \dots + f^{2a-1}\frac{\partial\Psi}{\partial f^{2a-1}}\right) df^1 \wedge df^2 \wedge \dots \wedge df^n.$$

It follows that $j(b) \neq 0$, so the transition to G is non-degenerate. The form ε is expressed in G as

$$\varepsilon_G = f^1\, dg^2 + \sum_{i=2}^{a} f^1 g^{2i-1}\, dg^{2i}.$$

Put $\lambda^1 = g^1(b) = \Psi(b)$. If $\operatorname{or}(G) = \operatorname{or}(X)$, then we define a chart H on X by $h^i = g^i - g^i(b)$. Then $\operatorname{or}(H) = \operatorname{or}(X)$ and H is a canonical b-chart for the section $\bar{s}$. The section s is given in H by

$$s_H = \left(\lambda^1 + h^1, f^1\, dh^2 + \sum_{i=2}^{a} f^1 h^{2i-1}\, dh^{2i}\right).$$

If the orientations of G and X are opposite, then we put $h^1 = g^1(b) - g^1$, $h^i = g^i - g^i(b)$ for $i > 1$. Then we again have $\operatorname{or}(H) = \operatorname{or}(X)$, and H is a canonical b-chart for $\bar{s}$. The section s is given in H by

$$s_H = \left(\lambda^1 - h^1, f^1\, dh^2 + \sum_{i=2}^{a} f^1 h^{2i-1}\, dh^{2i}\right).$$

The canonical form of $\bar{s}$ will be given in § 6.

Type V. Here $K = H_2$ $(n = 2a + 1)$.

Definition 10. A section $\bar{s}\colon X \to P$ is called a b-*section* if the following conditions hold at the point $b \in X$:

1) $\omega^{(2a+1)}(b) \neq 0$,
2) $(d\omega^{(2a-1)} \wedge d\Psi)(b) \neq 0$.

We easily see from (30) that these conditions do not depend on the choice of ω.

In what follows, the property of a section of being sufficiently general is always independent of the choice of coordinate system and the auxiliary constructions used to define this property. This fact will not be mentioned explicitly.

Let $\bar{s}$ be a b-section in the original chart G, which is oriented in the same way as X. To reduce $\bar{s}$ to canonical form, we recall the proof of Darboux' theorem [13]. This shows that the first step in finding a canonical chart F for a 1-form ε (originally defined in G) is to find the second coordinate function $f^2 = f(g^i)$ of this chart as a solution of the following linear partial differential equation of the first order:

$$df \wedge \varepsilon^{(2a)} = 0. \tag{42}$$

Let (Ψ, ω) be a pair representing $\bar{s}$. We can replace ω by $\varepsilon = \lambda\omega$ for any function $\lambda(g^i) > 0$. By (29), equation (42) for ε takes the form

$$\lambda^a\, df \wedge \omega^{(2a)} + a\, df \wedge d\lambda \wedge \omega^{(2a-1)} = 0. \tag{43}$$

Now we put $f = \Psi$ and regard (43) as an equation for the unknown function λ. This is a quasilinear partial differential equation of the first order. Condition 2) of Definition 10 implies that at least one partial derivative $\partial\lambda/\partial g^i$ occurs non-trivially in this equation. Therefore this equation has a local solution λ in a neighbourhood of b with $\lambda(b) > 0$.

Thus, when we start to prove Darboux's theorem for ε, we may take $f^2 = \Psi$.

Then we continue the proof without change, construct the other coordinate functions f^i and, as a result, obtain the coordinate system of a canonical chart F for ε such that

$$\varepsilon_F = \sum_{i=1}^{a} f^{2i-1}\, df^{2i} + df^{2a+1}.$$

We may assume that $f^i(b) = 0$ for $i \neq 2$ (for example, use the proof of Lemma 11). Now we put $\lambda^i = f^i(b)$. If the form $\omega^{(n)}$ is positive, we define a chart H by putting $h^i = f^i - \lambda^i$. Then H and X have the same orientation, H is a canonical b-chart for $\bar{s}$ and the section s is given in H by

$$s_H = \left(\lambda^2 + h^2, \sum_{i=1}^{a} h^{2i-1}\, dh^{2i} + dh^{2a+1}\right).$$

If the form $\omega^{(n)}$ is negative, then we define the chart H by putting $h^i = f^i - \lambda^i$ for $i < n$, $h^{2a+1} = -f^{2a+1}$. Then H and X have the same orientation, H is a canonical b-chart for $\bar{s}$ and the section s is given in H by

$$s_H = \left(\lambda^2 + h^2, \sum_{i=1}^{a} h^{2i-1}\, dh^{2i} - dh^{2a+1}\right).$$

Clearly, these two canonical forms are inequivalent, that is, they are not mapped to each other by an orientation-preserving change of coordinates. The canonical form of $\bar{s}$ is described in § 6.

Now we consider the case when Y is a GL_n^0-orbit of dimension $n-1$.

Type VI ($K = H_1$). We have $Y = S_{n-1} = \mathrm{GL}_n^0/K$, and this case is even simpler than Type III since S is absent and the role of M is played by $\mathbb{R}^n - \mathbf{0}$.

We put $T(X) = P_X(\mathbb{R}^n - \mathbf{0})$ and let μ_X be the natural epimorphism $T(X) - \mathbf{0} \to P = P_X(Y)$ induced by the epimorphism μ. We choose the standard coordinates of $\mathbb{R}^n$ for the adapted coordinates in the space of representation ST_n. A section $\bar{e}\colon X \to P$ is a spherical vector field on X. It is called a *b-section* if $e(b) \neq 0$ for some genuine vector field e representing $\bar{e}$. If F is a chart that straightens e (that is, $e = \partial/\partial f^1$ and $\mathrm{or}(F) = \mathrm{or}(X)$), then the section $\bar{e}$ takes a canonical form in F (see § 6).

Let $K = H_2$. We have $Y = \mathrm{GL}_n^0/K$. We similarly define $T^*X = P_X(\mathbb{R}^n - \mathbf{0})$ and $\mu_X^*\colon T^*X - \mathbf{0} \to P = P_X(Y)$. We choose the standard coordinates of $\mathbb{R}^{n*}$ for the adapted coordinates in the space of representation ST_n^*. A section $\overline{\omega}\colon X \to P$ is a spherical form on X. Let ω be a 1-form representing $\overline{\omega}$.

Type VII ($n = 2a$). In this case, $\overline{\omega}$ is called a *b-section* if $\omega^{(n-1)}(b) \neq 0$. Arguing as above, we get a chart F (oriented in the same way as X) in which the form $\varepsilon = \lambda\omega$ is given by (40), and (41) holds. We define a b-chart H on X by putting $h^1 = f^1 - f^1(b)$, $h^{2i-1} = f^{2i-1}/f^1$ for $2 \leqslant i \leqslant a$, and $h^{2i} = f^{2i}$ for $1 \leqslant i \leqslant a$. The chart H is canonical for $\overline{\omega}$, and

$$\varepsilon_H = f^1\left(dh^2 + \sum_{i=2}^{a} h^{2i-1}\, dh^{2i}\right)$$

(see § 6 below).

Type VIII ($n = 2a+1$). In this case, $\overline{\omega}$ is called a *b-section* if $\omega^{(n)}(b) \neq 0$. The discussion of Type V shows that, up to proportionality, ω can be reduced to the canonical form

$$\omega = \sum_{i=1}^{a} f^{2i-1}\, df^{2i} + df^{2a+1} \qquad \text{or} \qquad \omega = \sum_{i=1}^{a} f^{2i-1}\, df^{2i} - df^{2a+1}$$

in an appropriate b-chart F. (We recall that a b-chart, by definition, is oriented in the same way as X.) Clearly, these two canonical forms cannot be transformed to each other because the signs of $\omega^{(n)}$ are different. This yields two canonical forms for the section $\overline{\omega}$ (see § 6).

It remains to consider the case $\dim \mathrm{GL}_n^0/K = n$ when Y is homogeneous and K is the stabilizer of a point $p \in Y$.

Type IX. Suppose that $K = T_1$ and $Y = \mathrm{GL}_n^0/K = Z^0 \times \mathrm{SL}_n/T_1 = Z^0 \times S_{n-1}$, where $Z^0 \cong \mathbb{R}^0$ (see § 3.6). Consider the G-manifold $B = \mathbb{R}^0 \times (\mathbb{R}^n - \mathbf{0})$, where the action of a matrix $g \in G$ on $\mathbb{R}^0$ is multiplication by the determinant of g, and the action of g on $\mathbb{R}^n$ is determined by the representation ST_n. Let $\xi\colon B \to Y$ be a map which equals the identity on $\mathbb{R}^0$ and equals μ on $\mathbb{R}^n - \mathbf{0}$. The map ξ induces an epimorphism $\xi_X\colon E = E_X(B) \to P = P_X(Y)$. We take the coordinate y^0 on $\mathbb{R}^0$

given by (27). By definition, the adapted coordinates on B are $(y^0, t^1, \dots, t^n)$, where the t^i are the standard coordinates in $\mathbb{R}^n$. It is easy to see that every section $s\colon X \to E$ may be identified with a pair (π, e), where π is an n-form on X and e is a vector field without zeros on X. A section $\bar{s}\colon X \to P$ may similarly be identified with a pair $(\pi, \bar{e})$, where π is an n-form on X and $\bar{e}$ is a spherical field on X. We say that a section $s = (\pi, e)$ represents $\bar{s} = (\pi, \bar{e})$ if $\xi_X(s) = \bar{s}$. This is equivalent to saying that e represents $\bar{e}$.

Suppose that $(\omega, \bar{e}) = \bar{s}\colon X \to P$ is a section and $s = (\pi, e)$ represents $\bar{s}$.

Definition 11. We say that $\bar{s}$ is a b-*section* if $e(b) \neq 0$.

Let s be a b-section. We choose a chart H on X such that $\mathrm{or}(H) = \mathrm{or}(X)$ and $e_H = \partial/\partial h^n$. ($H$ is a straightening chart for e.) Put $\pi_H = a_H\, dh^1 \wedge \cdots \wedge dh^n$. Clearly, $a_H > 0$. Now we pass to the chart G defined as in the discussion of Type II. In this chart, we have $s_G = (dg^1 \wedge \cdots \wedge dg^n, c(g)\partial/\partial g^n)$, where $c(g) > 0$. This yields a canonical form for the section $\bar{s}$ (see § 6).

Let $K = T_2$, $Y = \mathrm{GL}_n^0/K = Z^0 \times \mathrm{SL}_n/T_2 = Z^0 \times S^*_{n-1}$. We define $B^* = \mathbb{R}^0 \times (\mathbb{R}^{n*} - \mathbf{0})$, $\xi\colon B^* \to Y$, $\xi_X\colon E^* = E_X(B^*) \to P = P_X(Y)$. The adapted coordinates on E^* are defined as $(y^0, t^1, \dots, t^n)$, where the t^i are the standard coordinates in the space $\mathbb{R}^{n*}$ of the representation ST^*_n. We easily see that a section $s\colon X \to E^*$ may be identified with a pair (π, ω), where π and ω are forms of degree n and 1 (respectively) on X, $\pi > 0$ and ω never vanishes on X. One can similarly identify a section $\bar{s}\colon X \to P$ with a pair (π, ϖ), where π is a positive n-form on X and ϖ is a spherical form on X. We say that a section $s = (\pi, \omega)$ represents $\bar{s} = (\pi, \varpi)$ if $\xi^*_X(s) = \bar{s}$. This is equivalent to saying that ω represents ϖ.

Type X ($n = 2a$). Suppose that $(\pi, \varpi) = \bar{s}\colon X \to P$ is a section and $s = (\pi, \omega)$ represents $\bar{s}$.

Definition 12. A section $\bar{s}$ is called a b-*section* if

$$\omega^{(n-1)}(b) \neq 0. \tag{44}$$

Suppose that (44) holds. Arguing as in the discussion of Type IV, we get a chart F (oriented in the same way as X) in which $\varepsilon = \lambda\omega$ is given by (40), and (41) holds. Write

$$\pi_F = \Phi_F\, df^1 \wedge df^2 \wedge \cdots \wedge df^n,$$

where $\Phi_F > 0$. We define a new chart G on X by putting $g^1 = \Psi(f^1, \dots, f^n)$, $g^{2i-1} = f^{2i-1}/f^1$ for $2 \leqslant i \leqslant a$, $g^{2i} = f^{2i}$ for $1 \leqslant i \leqslant a$. The determinant j of the matrix $J(G/F)$ is given by

$$j = \frac{1}{(f^1)^a}\left(f^1\frac{\partial\Psi}{\partial f^1} + f^3\frac{\partial\Psi}{\partial f^3} + \cdots + f^{2a-1}\frac{\partial\Psi}{\partial f^{2a-1}}\right).$$

We can find the function Ψ from the equation $\Phi_F = (f^1\partial\Psi/\partial f^1 + f^3\partial\Psi/\partial f^3 + \cdots + f^{2a-1}\partial\Psi/\partial f^{2a-1})/(f^1)^a$. The form ω is expressed in the chart G as

$$\omega_G = f^1\left[dg^2 + \sum_{i=2}^{a} g^{2i-1}\, dg^{2i}\right],$$

and $\pi_G = dg^1 \wedge dg^2 \wedge \cdots \wedge dg^n$. This yields a canonical form for $\bar{s}$ (see § 6).

Type XI ($n = 2a+1$). Suppose that $(\pi, \overline{\omega}) = \bar{s}\colon X \to P$ is a section and $s = (\pi, \omega)$ represents $\bar{s}$.

Definition 13. A section $\bar{s}$ is called a b-*section* if

$$\omega^{(n)}(b) \neq 0. \tag{45}$$

Suppose that (45) holds. It follows from (30) that $\varepsilon = \lambda\omega$ satisfies the hypotheses of Darboux' theorem for any function $\lambda > 0$, and one can always choose λ in such a way that either $\varepsilon^{(n)} = a!\,\pi$ or $\varepsilon^{(n)} = -a!\,\pi$. We always have $\pi > 0$. If $\varepsilon^{(n)} = a!\,\pi$, then there is a canonical b-chart H for ε with

$$\varepsilon_H = \sum_{i=1}^{a} h^{2i-1}\,dh^{2i} + dh^{2a+1}.$$

If $\varepsilon^{(n)} = -a!\,\pi$, then there is a canonical b-chart H for ε with

$$\varepsilon_H = \sum_{i=1}^{a} h^{2i-1}\,dh^{2i} - dh^{2a+1}.$$

In both cases, $\pi_H = dh^1 \wedge dh^2 \wedge \cdots \wedge dh^n$. This yields two canonical forms for $\bar{s}$ (see § 6).

Type XII. Here $K = H_{1d}$. In this case, $Y = \mathrm{GL}^0_n / K = \mathbb{R}^n_d$ (see § 4.3). An element g of the group $\mathrm{GL}^0_n = G$ acts on a column vector $x \in \mathbb{R}^n - \mathbf{0} \cong \mathbb{R}^n_d$ by $g(x) = v$, where $v = \det(g^i_j)^{(d-1)/n} gx$. The standard coordinate system of $\mathbb{R}^n$ is declared to be the adapted coordinate system on Y. This choice of an adapted coordinate system enables one to identify sections $s\colon X \to P = P_X(Y)$ with non-zero vector fields e' having a modified transformation rule under coordinate changes: passing from a chart F to a chart H with the Jacobian matrix $J = J(H/F)$ and determinant $j = \det J > 0$, we have

$$e'_H = j^{(d-1)/n} J e'_F. \tag{46}$$

We shall use the term *quasifield* for the geometric structure e'. Choose a chart F on X and consider a genuine vector field e on X such that $e_F = e'_F$.

Definition 14. For $d \neq 1 - n$, a quasifield e' is called a b-*quasifield* if $e(b) \neq 0$.

Let e' be a b-quasifield. We choose a chart $V \ni b$ (oriented in the same way as X) such that $e = \partial/\partial v^1$. Clearly, e' is expressed in V as $e'_V = (R, 0, \ldots, 0)$, where $R = R(v) > 0$. We define a chart H by putting $h^1 = \Psi(v)$, $h^i = v^i$ for $i > 1$. Write $J = J(H/V)$ and $j = \det J$. Then it is clear that $j = \partial\Psi/\partial v^1$ and $Je'_V = \partial\Psi/\partial v^1 (e'_V)$, whence $e'_H = (\partial\Psi/\partial v^1)^{(n+d-1)/n} R$. If $d \neq 1 - n$, then we can always choose Ψ in such a way that $e'_H = (1, 0, \ldots, 0)$: it suffices to find Ψ from the equation $(\partial\Psi/\partial v^1) = R^{n/(n+d-1)}$. This yields a canonical form for the section e'.

If $d = 1 - n$, then (46) takes the form

$$e'_H = j^{-1} J e'_F. \tag{47}$$

Definition 15. Suppose that $d = 1 - n$ and fix a quasifield e' with $e'(b) \neq 0$. A chart W is said to be *special* (for the quasifield e') if $\operatorname{or}(W) = \operatorname{or}(X)$, $W \ni b$ and all the components of e'_W vanish in the chart W except for the first: $e'_W = (S, 0, \dots, 0)$ and, moreover, $S > 0$. It is clear that special charts exist. We denote the first component of the quasifield e' in a special chart F by S_F instead of e'^1_F. A quasifield e' is called a *b-quasifield* if $e'(b) \neq 0$ and there is a special chart V such that $[\partial S_V/\partial v^1](b) \neq 0$.

Remark 11. If e' is a b-quasifield, then $[\partial S_F/\partial f^1](b) \neq 0$ for any special chart F.

Indeed, for every such chart F, the Jacobian matrix $J = J(F/V)$ satisfies $\partial f^2/\partial v^1 = 0, \dots, \partial f^n/\partial v^1 = 0$. Therefore $j = \det J = (\partial f^1/\partial v^1)t$, where $t = \det(\partial f^i/\partial v^j)$, $2 \leqslant i, j \leqslant n$, and we get $S_F = t^{-1} S_V$ by formula (47). Since t is independent of v^1, differentiation of the last equation with respect to v^1 yields that

$$\frac{\partial S_F}{\partial f^1}\frac{\partial f^1}{\partial v^1} = t^{-1}\frac{\partial S_V}{\partial v^1}\,.$$

This proves the remark.

Suppose that e' is a b-quasifield and V is a special chart. We define a special b-chart F by putting $f^2 = S_V(b)\big(v^2 - v^2(b)\big)$, $f^i = v^i - v^i(b)$ for $3 \leqslant i \leqslant n$ $f^1 = S_V(v)/S_V(b) - 1$ if $[\partial S_V/\partial v^1](b) > 0$, and $f^1 = 1 - S_V(v)/S_V(b)$ if $[\partial S_V/\partial v_1](b) < 0$. Thus we get two inequivalent canonical forms, $e'_F = (1 \pm f^1, 0, \dots, 0)$.

Suppose that $K = H_{2d}$. In this case, $Y = \mathrm{GL}^0_n / K$. We write $Y = \mathbb{R}^{n*}_d$ (see § 4.4). We can assume that $\mathbb{R}^{n*}_d$ is the space $\mathbb{R}^{n*} - \mathbf{0}$ and the action of a matrix $g \in \mathrm{GL}^0_n = G$ on a row vector $x \in \mathbb{R}^{n*}$ is given by $g(x) = v$, where $v = xg^{-1}\det(g^i_j)^{(d-1)/n}$. The standard coordinate system of $\mathbb{R}^{n*}$ is declared to be the adapted coordinate system on Y. It follows easily that one can identify sections $s\colon X \to P = P_X(Y)$ with non-zero differential 1-forms μ on X with a modified transformation rule under coordinate changes: passing from a chart F to a chart H with Jacobian matrix $J = J(H/F)$ and determinant $j = \det J > 0$, we have

$$\mu_H = j^{(d-1)/n}\big(J^T\big)^{-1}\mu_F. \tag{48}$$

1. Suppose that $d = 1$. Then μ is an ordinary 1-form, and we have already seen its canonical form provided that $\mu^{(n)}(b) \neq 0$ and $\mu(b) \neq 0$ (that is, μ is sufficiently general at b). Namely, if $n = 2a$, then μ reduces in an appropriate b-chart V to one of the two inequivalent canonical forms (35), (36). If $n = 2a+1$, then μ reduces in an appropriate b-chart V to one of the two inequivalent canonical forms (37), (38). These cases differ by the sign of $\mu^{(n)}$.

2. Suppose that $d \neq 1$ and write $(d-1)/n = u \neq 0$. We use the term *quasiform* for the geometric structure μ. Let $F \ni b$ be a chart (oriented in the same way as X) and let ω be an ordinary 1-form on X such that $\omega_F = \mu_F$. We put $\mu^{(k)}_{(F)} = \omega^{(k)}$. One can assume that $\mu^{(k)}_{(F)}$ is an ordinary k-form depending on the chart F used in its definition: passing from F to a chart H with Jacobian matrix $J = J(H/F)$,

determinant $j = \det J > 0$ and $\lambda = j^u$, we have

$$\mu_{(H)}^{(k)} = \begin{cases} \lambda^c \mu_{(F)}^{(k)} + c\lambda^{c-1}\, d\lambda \wedge \mu_{(F)}^{(k-1)} & \text{for} \quad k = 2c, \\ \lambda^{c+1} \mu_{(F)}^{(k)} & \text{for} \quad k = 2c+1. \end{cases} \tag{49}$$

Although the definition of $\mu_{(F)}^{(k)}$ depends on the choice of F, the property $\mu_{(F)}^{(k)}(b) \neq 0$ is independent of this choice if k is odd. Since X is oriented, the sign of the form $\mu_{(F)}^{(n)}(b)$ is also independent of F (if $n = \dim X$ is odd) and is preserved under orientation-preserving coordinate changes. Hence the conditions $\mu^{(k)}(b) \neq 0,\ \ \mu^{(n)}(b) > 0$ make sense for odd k and n.

Type XIII ($n = 2a$). We begin with a definition.

Definition 16. A section μ is called a b-*section* if $\mu^{(n-1)}(b) \neq 0$.

Remark 12. Since $n-1$ is odd, the condition $\mu^{(n-1)}(b) \neq 0$ implies that $\mu(b) \neq 0$.

Let μ be a b-section. We choose a chart $F \ni b$ (oriented in the same way as X) and consider $\mu_{(F)}^{(k)}$. We claim that there is a chart H (oriented in the same way as X) such that $\mu_{(H)}^{(n)}(b) > 0$. Indeed, let H be given by $h^1 = \Phi(f),\ \ h^i = f^i$ for $i > 1$. Then $j = \det\bigl[J(H/F)\bigr] = \partial\Phi/\partial f^1$. Formula (49) with $k = n$ yields that

$$\mu_{(H)}^{(n)} = \lambda^{a-1}\bigl(\lambda\mu_{(F)}^{(n)} + a\, d\lambda \wedge \mu_{(F)}^{(n-1)}\bigr).$$

Since $\mu_{(F)}^{(n-1)}(b) \neq 0$, one can choose a function $\lambda > 0$ with $\bigl[\lambda\mu_{(F)}^{(n)} + a\, d\lambda \wedge \mu_{(F)}^{(n-1)}\bigr](b) > 0$ and then find Φ from the equation $\partial\Phi/\partial f^1 = \lambda^{1/u}$. This guarantees that $\mu_{(H)}^{(n)}(b) > 0$.

We put $\varepsilon = \mu_{(H)}$. Since $\varepsilon^{(n)}(b) > 0$ and $\varepsilon(b) \neq 0$, Darboux' theorem yields a chart V (oriented in the same way as X) in which ε takes the canonical form $\varepsilon_V = \sum_{i=1}^a v^{2i-1}\, dv^{2i}$. By (48) we have $\mu_V = \Psi\varepsilon_V$, where Ψ is a scalar function with $\Psi(b) > 0$. The explicit form of Ψ is irrelevant. We look for the chart W in the form $w^{2i-1} = \Theta(v)v^{2i-1},\ \ w^{2i} = v^{2i}$ for $1 \leqslant i \leqslant a$. The determinant j of the Jacobian matrix $J = J(W/V)$ is easily calculated:

$$j = \Theta^{a-1}\left(\Theta + v^1\frac{\partial\Theta}{\partial v^1} + \cdots + v^{2a-1}\frac{\partial\Theta}{\partial v^{2a-1}}\right).$$

Making a change of variables, we get

$$\varepsilon_W = \Theta^{-1}\left(\sum_{i=1}^a w^{2i-1}\, dw^{2i}\right).$$

Therefore $\mu_W = j^u J\mu_V = j^u\Psi J\varepsilon_V = j^u\Psi\varepsilon_W$. We get Θ from the condition $j^u\Psi = \Theta$, which is equivalent to the quasilinear partial differential equation

$$\Theta^{a-1}\left(\Theta + v^1\frac{\partial\Theta}{\partial v^1} + \cdots + v^{2a-1}\frac{\partial\Theta}{\partial v^{2a-1}}\right) = \left(\frac{\Theta}{\Psi}\right)^{1/u}$$

provided that $\Theta > 0$. This equation has a solution Θ with $\Theta(b) > 0$. Hence $\operatorname{or}(W) = \operatorname{or}(X)$ and the quasiform μ takes the canonical form $\mu_W = \sum_{i=1}^{a} w^{2i-1}\, dw^{2i}$ in the chart W.

To find the canonical form for μ, we consider a differential 1-form whose expression in W is $\sum_{i=1}^{a} w^{2i-1}\, dw^{2i}$. Lemma 11 yields a b-chart Z in which this form is given by $(1+z^1)\, dz^2 + \sum_{i=2}^{a} z^{2i-1}\, dz^{2i}$, and the determinant of the Jacobian matrix $J(Z/W)$ equals 1. Clearly,

$$\mu_Z = (1+z^1)\, dz^2 + \sum_{i=2}^{a} z^{2i-1}\, dz^{2i}.$$

This yields the canonical form for the section μ.

Type XIV ($n = 2a+1$). We begin with a definition.

Definition 17. Suppose that $u(a+1) \neq 1$. A quasiform $\mu\colon X \to P = P_X(Y)$ is called a b-*section* if $\mu^{(n)}(b) \neq 0$.

Let μ be a b-section. We choose a chart $F \ni b$ oriented in the same way as X. Let ε be a 1-form such that $\varepsilon_F = \mu_F$. We write $\varepsilon^{(n)} = \Phi\, df^1 \wedge df^2 \wedge \dots \wedge df^n$ and suppose that $\Phi(b) > 0$. Let Ψ be a function (to be specified later) on X with $\Psi(b) > 0$. Darboux' theorem yields a chart V (oriented in the same way as X) which reduces $\Psi\varepsilon$ to the canonical form

$$(\Psi\varepsilon)_V = \sum_{i=1}^{a} v^{2i-1}\, dv^{2i} + dv^{2a+1}.$$

Then we have

$$(\Psi\varepsilon)_V^{(n)} = a!\, dv^1 \wedge dv^2 \wedge \dots \wedge dv^{2a+1}. \tag{50}$$

Clearly,

$$(\Psi\varepsilon)_F^{(n)} = \Psi^{a+1}\varepsilon_F^{(n)} = \Psi^{a+1}\Phi\, df^1 \wedge df^2 \wedge \dots \wedge df^{2a+1}. \tag{51}$$

We put $J = J(V/F)$ and $j = \det J$. Comparing (51) and (50), we get $a!\, j = \Psi^{a+1}\Phi$. We have

$$\begin{aligned}\mu_V &= j^u \bigl(J^T\bigr)^{-1} \varepsilon_F = \Psi^{-1}\bigl[(\Psi^{a+1}\Phi)(a!)^{-1}\bigr]^u \bigl(J^T\bigr)^{-1} (\Psi\varepsilon)_F \\ &= (a!)^{-u}\Psi^{u(a+1)-1}\Phi^u (\Psi\varepsilon)_V.\end{aligned}$$

We want to find the function $\Psi > 0$ from the condition $\Psi^{u(a+1)-1} = \Phi^{-u}(a!)^u$. This can be done if $u(a+1)-1 \neq 0$. In this case we have $\operatorname{or}(V) = \operatorname{or}(X)$ and the quasiform μ takes the canonical form $\mu_V = \sum_{i=1}^{a} v^{2i-1}\, dv^{2i} + dv^{2a+1}$ in the chart V.

Now suppose that $\Phi(b) < 0$. Using Darboux' theorem, we get a chart V (oriented in the same way as X) such that $(\Psi\varepsilon)_V = \sum_{i=1}^{a} v^{2i-1}\, dv^{2i} - dv^{2a+1}$. Then

$$(\Psi\varepsilon)_V^{(n)} = -a!\, dv^1 \wedge dv^2 \wedge \dots \wedge dv^{2a+1}.$$

As above, we get the equation $a!\, j = -\Psi^{a+1}\Phi$ and find the function $\Psi > 0$ from the condition $\Psi^{u(a+1)-1} = (-\Phi)^{-u}(a!)^u$. Then $\operatorname{or}(V) = \operatorname{or}(F)$ and the quasiform

μ has the canonical form $\mu_V = \sum_{i=1}^{a} v^{2i-1}\, dv^{2i} - dv^{2a+1}$ in the chart V. The two canonical forms cannot be transformed to each other because of the different signs of $\mu^{(n)}$. Applying Lemma 11 as above, we may assume that the canonical chart V is centred at b.

It remains to consider the case $u(a+1) - 1 = 0$. Then (48) takes the form

$$\mu_H = j^{1/(a+1)} \big(J^T\big)^{-1} \mu_F. \tag{52}$$

Let $\mu\colon X \to P = P_X(Y)$ be a quasiform with $\mu^{(n)}(b) \neq 0$. We choose a chart $F \ni b$ (oriented in the same way as X) and consider a 1-form ω such that $\omega_F = \mu_F$. Write

$$\omega^{(n)} = \Omega(f^1, \dots, f^n)\, df^1 \wedge df^2 \wedge \dots \wedge df^{2a+1}. \tag{53}$$

We claim that the scalar function $\Omega = \Omega(f^1, \dots, f^n)$ is a differential invariant (of order 1) if $u(a+1) = 1$. Indeed, consider another chart H (oriented in the same way as X) and a 1-form α with $\alpha_H = \mu_H$. We see from (52) that $\alpha_H = \mu_H = j^{1/(1+a)}\omega_H$. Hence (30) implies that $\alpha^{(n)} = j\omega^{(n)}$. However, (53) yields that

$$\omega^{(n)} = \Omega j^{-1}\, dh^1 \wedge dh^2 \wedge \dots \wedge dh^{2a+1}.$$

Hence $\alpha^{(n)} = \Omega\, dh^1 \wedge dh^2 \wedge \dots \wedge dh^{2a+1}$, as required.

The differential invariant constructed from the quasiform μ will be denoted by Ω_μ (or simply by Ω if μ is clear from the context).

Definition 18. Suppose that $u(a+1) = 1$. A section μ is called a b-*section* if

1) $\mu^{(n)}(b) \neq 0$,
2) $\Omega(b) \neq 0$,
3) $\big[d\Omega \wedge \mu^{(n-2)}\big](b) \neq 0$.

Let μ be a b-section. We choose a chart $V \ni b$ (oriented in the same way as X) and consider a 1-form ω such that $\omega_V = \mu_V$. There is a b-chart F on X in which ω has a canonical form. Suppose that

$$\mu_F = \beta\omega_F = \beta\bigg(\sum_{i=1}^{a} f^{2i-1}\, df^{2i} + df^{2a+1}\bigg),$$

where $\beta > 0$ is a scalar function. Then

$$[\beta\omega]_F^{(n)} = \beta^{a+1} a!\, df^1 \wedge df^2 \wedge \dots \wedge df^{2a+1},$$

and, therefore, $\Omega = a!\, \beta^{a+1}$. Put $\beta\omega = \varepsilon$. We consider a scalar function $\alpha > 0$ (to be specified later) and look for a canonical chart H on X for the 1-form $\alpha\varepsilon$. The second coordinate function h^2 of H is found from the relation $dh^2 \wedge (\alpha\varepsilon)^{(2a)} = 0$, which may be rewritten as

$$\alpha\, dh^2 \wedge \varepsilon^{(2a)} - a\, dh^2 \wedge d\alpha \wedge \varepsilon^{(2a-1)} = 0.$$

We look for a function α such that $h^2 = \beta$. The desired function α is a solution of the equation

$$\alpha\, d\beta \wedge \varepsilon^{(2a)} - a\, d\beta \wedge d\alpha \wedge \varepsilon^{(2a-1)} = 0. \tag{54}$$

This is a quasilinear partial differential equation of the first order with respect to α. By a trivial calculation, condition 3) of Definition 18 implies that $d\beta \wedge \varepsilon^{(2a-1)} \neq 0$, whence at least one partial derivative $\partial\alpha/\partial f^i$ occurs non-trivially in equation (54). Hence the desired function $\alpha > 0$ exists. Then, following the proof of Darboux' theorem (see [13]), we construct a coordinate system $H = (h^1, \dots, h^n)$ such that

1) $h^2 = \beta$,

2) $(\alpha\varepsilon)_H = (\alpha\beta\omega)_H = \sum_{i=1}^{a} h^{2i-1}\, dh^{2i} + dh^{2a+1}$.

We put $J = J(H/F)$ and $j = \det J$. We have

$$(\alpha\beta\omega)_F^{(n)} = a!\, \alpha^{a+1}\beta^{a+1}\omega_F^{(n)} = a!\, \alpha^{a+1}\beta^{a+1}\, df^1 \wedge df^2 \wedge \dots \wedge df^{2a+1}.$$

Since the expression $(\alpha\beta\omega)_H$ is canonical, we get

$$(\alpha\beta\omega)_H^{(n)} = a!\, dh^1 \wedge dh^2 \wedge \dots \wedge dh^{2a+1}.$$

Hence $j = \alpha^{a+1}\beta^{a+1}$. Moreover,

$$\begin{aligned} \mu_H &= j^{1/(a+1)} \big(J^T\big)^{-1} \mu_F = j^{1/(a+1)} \big(J^T\big)^{-1} \varepsilon_F \\ &= \alpha^{-1} j^{1/(a+1)} \big(J^T\big)^{-1} (\alpha\varepsilon)_F = \alpha^{-1} j^{1/(a+1)} (\alpha\varepsilon)_H. \end{aligned}$$

Substituting $j = \alpha^{a+1}\beta^{a+1}$ and $(\alpha\varepsilon)_H = \sum_{i=1}^{a} h^{2i-1}\, dh^{2i} + dh^{2a+1}$, we get

$$\mu_H = \alpha^{-1}\alpha\beta\bigg(\sum_{i=1}^{a} h^{2i-1}\, dh^{2i} + dh^{2a+1}\bigg).$$

Clearly, $\mathrm{or}(H) = \mathrm{or}(X)$. Since $\beta = h^2$, this yields the first canonical form

$$\mu_H = h^2\bigg(\sum_{i=1}^{a} h^{2i-1}\, dh^{2i} + dh^{2a+1}\bigg)$$

for the section μ.

It remains to consider the case when

$$\mu_F = \beta\omega_F = \beta\bigg(\sum_{i=1}^{a} f^{2i-1}\, df^{2i} - df^{2a+1}\bigg), \qquad \beta > 0.$$

In this case, $\Omega = -a!\, \beta^{a+1}$. We again put $\beta\omega = \varepsilon$ and look for a coordinate system $H = (h^1, \dots, h^n)$ such that $\mathrm{or}(H) = \mathrm{or}(X)$ and

1) $h^2 = \beta$,

2) $(\alpha\varepsilon)_H = (\alpha\beta\omega)_H = \sum_{i=1}^{a} h^{2i-1}\, dh^{2i} - dh^{2a+1}$.

Arguing as in the previous case, we get the second canonical form

$$\mu_H = h^2\bigg(\sum_{i=1}^{a} h^{2i-1}\, dh^{2i} - dh^{2a+1}\bigg)$$

for the section μ. The two canonical forms are inequivalent because the invariant Ω equals $a!\,(h^2)^{a+1}$ in the first case and $-a!\,(h^2)^{a+1}$ in the second. We may pass to a centred chart V by putting $\lambda^i = h^i(b)$ for $1 \leqslant i \leqslant 2a+1$ and defining the coordinates as follows.

1) If $\mu_H = h^2\bigl(\sum_{i=1}^a h^{2i-1}\,dh^{2i} + dh^{2a+1}\bigr)$, then $v^i = h^i - \lambda^i$ for $1 \leqslant i \leqslant 2a$ and

$$v^{2a+1} = h^{2a+1} + \sum_{i=1}^{a} \lambda^{2i-1} h^{2i} - \lambda^{2a+1} - \sum_{i=1}^{a} \lambda^{2i-1}\lambda^{2i}.$$

2) If $\mu_H = h^2\bigl(\sum_{i=1}^a h^{2i-1}\,dh^{2i} - dh^{2a+1}\bigr)$, then $v^i = h^i - \lambda^i$ for $1 \leqslant i \leqslant 2a$ and

$$v^{2a+1} = h^{2a+1} - \sum_{i=1}^{a} \lambda^{2i-1} h^{2i} - \lambda^{2a+1} + \sum_{i=1}^{a} \lambda^{2i-1}\lambda^{2i}.$$

The canonical forms are given in the chart V by

$$\mu_V = (v^2+\lambda^2)\biggl(\sum_{i=1}^{a} v^{2i-1}\,dv^{2i} \pm dv^{2a+1}\biggr),$$

and the invariants Ω are equal to $\pm a!\,(v^2+\lambda^2)^{a+1}$ respectively.

Having found the canonical forms for special structures of the first order, we pass to structures of higher order. By Theorem 4 there are no special bundles of geometric structures of order exceeding 2, and there are two non-isomorphic special bundles of geometric structures of order 2. Both bundles are determined by the subgroups $H_1', H_2' \subset \mathrm{Af}_n$ (see § 4.8). To describe them we first put $G = G^2(n)^0$.

We define a homomorphism $\varepsilon\colon \mathrm{Af}_n \to \mathrm{GL}_{n+1}$ by sending every $p = (A, a) \in \mathrm{Af}_n$ to the matrix

$$\varepsilon(p) = \begin{pmatrix} A & 0 \\ a & 1 \end{pmatrix} \tag{55}$$

of order $n+1$. The embedding ε is an isomorphism of Af_n onto the subgroup $B_n \subset \mathrm{GL}_{n+1}$ of matrices whose last column consists of zeros except for the lowest element of the principal diagonal, which equals 1 (the other entries are arbitrary). The group GL_{n+1} acts on $\mathbb{R}^{n+1}$ by the identity representation ST_{n+1}.

Type XV. We can identify the sphere $S_n \subset \mathbb{R}^{n+1}$ with the set of rays in $\mathbb{R}^{n+1}$. Hence the restriction of ST_{n+1} to B_n induces an action of Af_n on S_n via the embedding ε. This action has fixed points $Q_1 = (0,\dots,1)$ and $Q_2 = (0,\dots,-1)$ and is transitive on the complement $Y = S_n - Q_1 - Q_2$. Let K be the stabilizer of $(1,\dots,0)$ under this action. The embedding ε sends K to the subgroup of B_n whose elements are matrices $\left(\begin{smallmatrix} A & 0 \\ a & 1 \end{smallmatrix}\right)$ such that A preserves the plane $x^1 = 0$. In other words, all the elements in the first column of A vanish except for a_1^1, and the first coordinate a_1 of the vector $a = (a_i)$ also vanishes. Hence K preserves the plane $x^1 = 0$ under the action (22) of Af_n on $\mathbb{R}^{n*}$. It follows by § 4.8 that K coincides with H_2'. Thus $Y = \mathrm{Af}_n/H_2'$ is of type XV, and the epimorphism $\phi\colon G \to \mathrm{Af}_n$ from § 4.6 induces a ϕ-compatible isomorphism $\mathrm{Af}_n/H_2' \approx G/H_2$ of homogeneous spaces.

Let us describe the action of G on Y. We identify S_n with the unit sphere in $\mathbb{R}^{n+1}$. Let $t^1,\dots,t^{n+1}$ be the standard coordinates in $\mathbb{R}^{n+1}$. This provides $n+1$ coordinates $z^1,\dots,z^{n+1}$ for every point $y \in Y$ (regarded as a point of S_n). We introduce coordinates on Y by putting $y^i = z^i/(1-z^{n+1})$ for $1 \leqslant i \leqslant n$

(the stereographic projection of the sphere from the point Q_1) and declare them to be adapted. An element $p=(A,a)\in \mathrm{Af}_n$ sends a point $y\in Y$ with coordinates $(y^1,\dots,y^n)$ to the point $py=v\in Y$ whose adapted coordinates v^i are determined as follows. Let z be the same point y regarded as a point of the unit sphere S_n in $\mathbb{R}^{n+1}$ and let $z^1,\dots,z^{n+1}$ be its coordinates. The image $w=pz=(w^1,\dots,w^{n+1})$ of z under the action of $p\in\mathrm{Af}_n$ has coordinates $w^i=h^i_jz^j$ for $1\leqslant i\leqslant n$ and $w^{n+1}=z^{n+1}+a_jz^j$, where $A=(h^i_j)$. Let $u=(u^i)$ be the point of intersection of S_n with the ray through w. Then $u^i=w^i/r$, where $r=\sqrt{(w^1)^2+\dots+(w^{n+1})^2}$ and $v^i=u^i/(1-u^{n+1})$ for $1\leqslant i\leqslant n$. Thus we see that the coordinates v^i of the image v of y are given by

$$v^i=\bigl(h^i_jy^j\bigr)R(y^1,\dots,y^n;p). \tag{56}$$

Here R is a function of y^i and p whose explicit form is irrelevant.

Thus we see that the adapted coordinates of any point $y\in Y$

1) are finite and not all equal to zero,

2) are transformed under the action of G as the coordinates of a vector, up to a scalar factor.

Therefore a section $s\colon X\to P=P_X(Y)$ may be regarded as a vector field on X with the modified transformation rule (56) under coordinate changes. We call such a geometric object a *pseudofield*.

Let $s\colon X\to P$ be a pseudofield defined in a neighbourhood of $b\in X$. We choose a chart $N\ni b$ (oriented in the same way as X) and consider a vector field e on X such that $e_N=s_N$. As already mentioned, we have $s_N\neq 0$, whence $e_N\neq 0$. Thus there is a chart Q such that $\mathrm{or}(Q)=\mathrm{or}(X)$ and $e_Q=\partial/\partial q^1$. Therefore $s_Q=\bigl(\Psi(q),0,\dots,0\bigr)$, $\Psi(q)\neq 0$. We define a chart F by the conditions $\mathrm{or}(F)=\mathrm{or}(X)$, $f^1=\Phi(q)$ and $f^i=q^i$ for $i>1$. The function Φ will be specified in a moment.

The transition from Q to F determines an element $h\in G$ by $h=(h^i_j,h^i_{jk})$, where

$$(h^i_j)=\begin{pmatrix} h^1_1 & h^1_2 & h^1_3 & \dots & h^1_{n-1} & h^1_n\\ 0 & 1 & 0 & \dots & 0 & 0\\ 0 & 0 & 1 & \dots & 0 & 0\\ \dots & \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & 0 & \dots & 0 & 1\end{pmatrix},$$

$$h^1_j=\frac{\partial\Phi}{\partial q^j}\,,\qquad h^1_{jk}=\frac{\partial^2\Phi}{\partial q^j\partial q^k}\,, \tag{57}$$

$h^i_j=\delta^i_j$ for $i>1$, and $h^i_{jk}=0$ for $i>1$.

The inverse matrix $D=(d^i_j)$ of the matrix $A=(h^i_j)$ is given by

$$D=\begin{pmatrix} d^1_1 & \dots & d^1_n\\ \dots & \dots & \dots\\ d^n_1 & \dots & d^n_n\end{pmatrix}=\begin{pmatrix} (h^1_1)^{-1} & -(h^1_1)^{-1}h^1_2 & -(h^1_1)^{-1}h^1_3 & \dots & -(h^1_1)^{-1}h^1_n\\ 0 & 1 & 0 & \dots & 0\\ 0 & 0 & 1 & \dots & 0\\ \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & 0 & \dots & 1\end{pmatrix}.$$

To calculate $p=(A,a)=\phi(h)\in\mathrm{Af}_n$, we must find $a=(a_j)=\bigl(\sum_{s,u}d^u_sh^s_{ju}\bigr)$ from formula (20). Clearly, $a_j=d^1_1h^1_{j1}=(h^1_1)^{-1}h^1_{1j}$. Then we must find the coordinates v^i

of the element $v = py$ for $y = (y^i)$, where $y^1 = \Psi(q)$ and $y^i = 0$ for $i > 1$. To do this, we first determine $z = (z^i)$. Clearly, $z^i = 0$ for $i > 1$. The conditions $(z^1)^2 + (z^{n+1})^2 = 1$ and $z^1 = y^1(1 - z^{n+1})$ imply that

$$z^1 = \frac{2y^1}{(y^1)^2 + 1}, \qquad z^{n+1} = 1 - \frac{2}{(y^1)^2 + 1}. \tag{58}$$

Now let us find the coordinates of $w = pz = (w^i)$. Clearly, $w^1 = z^1 h_1^1$, $w^i = v^i = 0$ for $2 \leqslant i \leqslant n$, and

$$w^{n+1} = z^{n+1} + a_1 z^1 = z^{n+1} + z^1 (h_1^1)^{-1} h_{11}^1. \tag{59}$$

Now let us determine the function $\Phi(q)$ from the condition $w^{n+1} \equiv 0$. Substituting (57) and (58) in (59), we get

$$1 - 2\big[(y^1)^2 + 1\big]^{-1} + (h_1^1)^{-1} h_{11}^1 \cdot 2y^1 \big[(y^1)^2 + 1\big]^{-1} = 0,$$

which yields an ordinary differential equation for $h_1^1(q) = \frac{\partial \Phi}{\partial q^1}$:

$$h_1^1 \big[\big(\Psi(q)\big)^2 - 1\big] + 2\Psi(q) \frac{\partial h_1^1}{\partial q^1} = 0.$$

Since $\Psi(q) \neq 0$, we can find a solution $h_1^1(q)$ of this equation with $h_1^1(b) > 0$. Given this solution, we find the function $\Phi(q)$ from the condition $h_1^1(q) = \frac{\partial \Phi}{\partial q^1}$. Then the transition to F preserves the orientation. The condition $w^{n+1} \equiv 0$ implies that $u^{n+1} \equiv 0$ and $v^1(q) \equiv 1$. Thus the section s has the canonical form $s = (1, 0, \dots, 0)$ in the chart F. This completes the discussion of Type XV.

To consider the remaining types, we note that the second necessary homogeneous space $Y = G/H_2 = \mathrm{Af}_n / H_1'$ has already been constructed in § 4.8. Indeed, the space $\mathbb{R}^{n*}$ with the action (22) of Af_n may be taken for Y because this action is transitive and the stabilizer of $\mathbf{0} \in \mathbb{R}^{n*}$ coincides with $H_1' = (\mathrm{GL}_n^0, 0)$. We take the standard coordinate system of $\mathbb{R}^{n*}$ for the adapted coordinate system on Y.

Sections $s\colon X \to P = P_X(Y)$ are referred to as *pseudoforms*. Let $F \ni b$ and $V \ni b$ be any charts oriented in the same way as X. By (22), to find a formula for the transformation of $s_F = (s_{F\,1}, \dots, s_{F\,n})$ to $s_V = (s_{V1}, \dots, s_{Vn})$, one must find the element $h \in G$ corresponding to the transition from F to V, take the element $p = (A, a) = \phi(h) \in \mathrm{Af}_n$ and take the element $w = p^{-1} = (A^{-1}, -aA^{-1})$. Then we get

$$s_V = s_F w = s_F D + d, \qquad D = A^{-1}, \quad d = -aA^{-1}. \tag{60}$$

Put $h^{-1} = f$. Clearly, $\phi(f) = w$. Write $f = (f_j^i, f_{jk}^i) \in G$, where $f_j^i = \frac{\partial f^i}{\partial v^j}$ and $f_{jk}^i = \frac{\partial^2 f^i}{\partial v^j \partial v^k}$. Let $A = (v_j^i)$ be the inverse matrix of $D = (f_j^i)$. By (20), we have $w = (D, d)$, where $d = (d_j)$ and

$$d_j = \sum_{s,u} v_s^u f_{ju}^s. \tag{61}$$

Lemma 12. *Let* $\Psi(v) = \ln\bigl[\det(D)\bigr]$. *Then* $d_j = \frac{\partial \Psi}{\partial v^j}$.

Proof. We put $R = \det(D)$. Then $\frac{\partial \Psi}{\partial v^j} = (R^{-1})\frac{\partial R}{\partial v^j}$. Differentiating the determinant, we get

$$\frac{\partial R}{\partial v^j} = \begin{vmatrix} f^1_{1j} & \dots & f^1_{nj} \\ f^2_1 & \dots & f^2_n \\ \dots & \dots & \dots \\ f^n_1 & \dots & f^n_n \end{vmatrix} + \begin{vmatrix} f^1_1 & \dots & f^1_n \\ f^2_{1j} & \dots & f^2_{nj} \\ \dots & \dots & \dots \\ f^n_1 & \dots & f^n_n \end{vmatrix} + \dots + \begin{vmatrix} f^1_1 & \dots & f^1_n \\ f^2_1 & \dots & f^2_n \\ \dots & \dots & \dots \\ f^n_{1j} & \dots & f^n_{nj} \end{vmatrix}.$$

Expand the sth term with respect to the sth row. Since the cofactor D^s_u of the entry f^s_u equals Rv^u_s, we get

$$\frac{\partial R}{\partial v^j} = R \sum_{s,u} v^u_s f^s_{ju}.$$

The lemma is proved.

Let $s\colon X \to P = P_X(Y)$ be a pseudoform and let $F \ni b$ be any chart oriented in the same way as X. We denote by ω a 1-form on X such that $s_F = \omega_F$. As usual, s_F and ω_F are regarded as row vectors. We recall that $\omega_V = \omega_F D$. Using formulae (60), we see from Lemma 12 that

$$s_V = \omega_V + d\Psi. \tag{62}$$

Type XVI ($n = 2a$). Let s, F, ω be as just defined and put $s^{(n)} = \omega^{(n)}$. Note that $s^{(n)}$ is a well-defined n-form on X. For if we choose another chart V on X and a 1-form ε on X with $s_V = \varepsilon_V$, then (62) implies that $\varepsilon_V = \omega_V + d\Psi$, whence $\varepsilon^{(n)} = \omega^{(n)}$.

Definition 19. A pseudoform s is called a b-*section* if $s^{(n)}(b) \neq 0$ and s satisfies an additional condition (to be stated later).

Suppose that s is a b-section, F and ω are as above and V is a canonical chart for ω oriented in the same way as X. Then $s_V = \omega_V + d\Psi$. We first assume that $\omega_V = \sum_{i=1}^a v^{2i-1}\, dv^{2i}$. Denote the 1-form $\omega + d\Psi$ by ε. Clearly, $\omega^{(n)}(b) = \varepsilon^{(n)}(b) \neq 0$. Hence Darboux' theorem yields a chart H in which ε has the canonical form $\varepsilon_H = \sum_{i=1}^a h^{2i-1}\, dh^{2i}$. The determinant j of the Jacobian matrix $J(H/V)$ equals 1. Indeed,

$$\varepsilon^{(n)}_V = (a!)\, dv^1 \wedge \dots \wedge dv^{2a}, \qquad \varepsilon^{(n)}_H = (a!)\, dh^1 \wedge \dots \wedge dh^{2a},$$

whence $j = 1$. Now Lemma 12 implies that $s_H = \varepsilon_H = \sum_{i=1}^a h^{2i-1}\, dh^{2i}$ also has the canonical form.

It remains to consider the case when $\omega_V = -v^1\, dv^2 + \sum_{i=2}^a v^{2i-1}\, dv^{2i}$. We again put $\varepsilon = \omega + d\Psi$ and pass to a chart H in which ε has the canonical form $\varepsilon_H = -h^1\, dh^2 + \sum_{i=2}^a h^{2i-1}\, dh^{2i}$. We similarly get the canonical form $s_H = -h^1\, dh^2 + \sum_{i=2}^a h^{2i-1}\, dh^{2i}$. It is clear that the resulting canonical forms of s are inequivalent: they differ by the sign of $s^{(n)}$.

We can now state the additional condition occurring in Definition 19. This condition requires that $s_F(b) \neq 0$ in some canonical chart F. Then the same holds in any other canonical chart W because the determinant j of the transition matrix from W to F is identically equal to 1. Indeed, we have $s_F^{(n)} = a_F\, df^1 \wedge \cdots \wedge df^{2a}$, $s_W^{(n)} = a_W\, dw^1 \wedge \cdots \wedge dw^{2a}$ and $ja_F = a_W$. Since $a_F = a_W = \pm 1$ for a canonical chart, we see that $j = 1$. It follows that the pseudoform s is transformed as a differential 1-form when we pass from F to W. Hence the condition $s_F(b) \neq 0$ implies that $s_W(b) \neq 0$ after passage to W.

Thus, for every b-section there is a canonical chart F with $s_F(b) \neq 0$. Hence there is a canonical b-chart H (since the determinant of the transition from F to H is identically equal to 1) in which the canonical forms of s are equal to $(1 \pm h^1)\, dh^2 + \sum_{i=2}^a h^{2i-1}\, dh^{2i}$.

Type XVII ($n = 2a+1$). Let $s\colon X \to P = P_X(Y)$ be a pseudoform and let $F \ni b$ be any chart oriented in the same way as X. We denote by ω a 1-form on X such that $s_F = \omega_F$. As in the discussion of Type XVI, we see that $s^{(n-1)} = \omega^{(n-1)}$ is a well-defined form of degree $n-1 = 2a$ on X.

Definition 20. A pseudoform s is called a b-*section* if $s^{(n-1)}(b) \neq 0$.

Let s be a b-section and let F and ω be as above. We can always assume that $\omega_F^{(n)}(b) > 0$. For otherwise let V be a chart which is tangent to F up to order 2 at b and let μ be a differential 1-form on X such that $\mu_V = s_V$. It follows from (62) that $\mu = \omega + d\Psi$, where

$$\Psi(v) = \ln\bigl[\det(f^i_j)\bigr], \qquad f^i_j = \frac{\partial f^i}{\partial v^j}\,.$$

Clearly, $\mu^{(n)} = \mu^{(n-1)} \wedge \mu = \omega^{(n-1)} \wedge (\omega + d\Psi)$.

We write

$$\omega_V^{(n-1)} = \sum_{i=1}^n w_i\, dv^1 \wedge \cdots \wedge d\hat{v}^i \wedge \cdots \wedge dv^n, \qquad d\Psi = \sum_{i=1}^n d_i\, dv^i, \qquad \omega_V = \sum_{i=1}^n t_i\, dv^i.$$

Here and in what follows, the symbol $\hat{}$ over a term means that this term is omitted. We note that the values $w_i(b)$, $t_i(b)$ of the components of the forms $\omega^{(n-1)}$, ω in the chart V coincide with the values of the corresponding components in the chart F. Then

$$\omega_V^{(n)} = \mu_V^{(n)} = m\, dv^1 \wedge \cdots \wedge dv^i \wedge \cdots \wedge dv^n, \qquad m = \sum_{i=1}^n (-1)^{i-n} w_i (d_i + t_i).$$

Since $\omega^{(n-1)}(b) = s^{(n-1)}(b) \neq 0$, the w_i are not all equal to zero. Using the explicit form (61) of d_j, it is easy to attach either sign to $m(b)$ by choosing the chart V and changing $f^s_{ju}(b) = \bigl[\frac{\partial^2 f^s}{\partial v^j \partial v^u}\bigr](b)$. Hence we can modify our notation and assume that $\omega_F^{(n)}(b) > 0$ in the original chart F. Darboux' theorem yields a chart H such that $\omega_H = \sum_{i=1}^a h^{2i-1}\, dh^{2i} + dh^{2a+1}$. It is clear that the transition map from F to H preserves orientation. We know that $s_H = \omega_H + d\Psi_H$. We look for a canonical chart V given by $v^i = h^i$ for $1 \leqslant i \leqslant 2a$, $v^{2a+1} = h^{2a+1} + \Phi(h)$.

Then $s_V = \omega_V + d\Psi_V + d\Theta_V$, where $\Theta = -\ln[\det J]$ and $J = J(V/H)$ is the Jacobian matrix of the transition from H to V. We have

$$J = \begin{pmatrix} & E_{n-1} & & & 0 \\ \dfrac{\partial\Phi}{\partial h^1} & \dfrac{\partial\Phi}{\partial h^2} & \dots & \dfrac{\partial\Phi}{\partial h^{n-1}} & 1+\dfrac{\partial\Phi}{\partial h^n} \end{pmatrix}$$

and, therefore, $\Theta = -\ln\bigl(1+\frac{\partial\Phi}{\partial h^n}\bigr)$. It is also clear that $\omega_V = \sum_{i=1}^{a} v^{2i-1}\,dv^{2i} + dv^{2a+1} - d\Phi_V$. Hence, if the function Θ satisfies

$$d\Psi - d\ln\biggl(1+\frac{\partial\Phi}{\partial h^n}\biggr) - d\Phi = 0,$$

then the section s has the canonical form $\omega_V = \sum_{i=1}^{a} v^{2i-1}\,dv^{2i} + dv^{2a+1}$ in the chart V.

Thus it suffices to find the function Φ from the relation $\Psi - \ln\bigl(1+\frac{\partial\Phi}{\partial h^n}\bigr) - \Phi = 0$. This equation is equivalent to the equation $e^{\Psi-\Phi} = 1+\frac{\partial\Phi}{\partial h^n}$, which obviously has solutions. It is clear that the chart V is oriented in the same way as X. Finally, Lemma 11, 3) shows that we can replace V by a chart centred at b.

5.4. We earlier defined the notion of an S-*unfolding* for every regular point w of a special manifold D (the manifold Y constructed in § 5.2). We have shown that there are only 19 types of non-isomorphic manifold Y (after an appropriate shrinking of S). For the convenience of the reader, we give a list of these G-manifolds in § 6. They are called *sample manifolds*, and the corresponding bundles of geometric structures are called *sample bundles*. We recall that all objects (functions, manifolds and so on) and maps are assumed to be real and infinitely differentiable, and a bundle of geometric structures is said to be special if the dimension of its generic fibre does not exceed $n = \dim X$.

If $\mathcal{Q}$ is an atlas on the G-manifold Y, then any choice of a chart F on X determines a natural trivialization $\mathrm{tr}_F\colon P = P(Y) \to Y \times X$ and thus an atlas on P. This atlas is denoted by $(\mathcal{Q}, F)$. For every type I–XVII of sample manifold Y_i, the list in § 6 indicates an atlas $\mathcal{Q}_i$ (called the *adapted atlas*) for which the assertion of Theorem 5 holds, along with the canonical forms for b-sections (see Theorem 5).

§ 6. The list of sample manifolds and corresponding canonical forms

6.1. Consider the case when $q = 1$. Put $G = G^1(n)^0$. Let $y^i = \varphi^i(x^1,\dots,x^n)$ be the 1-jet of a germ of an orientation-preserving map $\Phi\colon \mathbb{R}^n \to \mathbb{R}^n$ sending $\mathbf{0}$ to $\mathbf{0}$. We consider the matrix $g(\Phi) = (g^i_j) = \partial\varphi^i/\partial x^j|_{x=0}$, where i (resp. j) is the row (resp. column) number of the matrix $g(\Phi)$. This yields an identification $G = \mathrm{GL}^0_n$, which is often used below.

Type I. The action of G on Y_1 is trivial. We may assume that Y_1 is the open ball with centre $\mathbf{0}$ and radius 1 in the m-dimensional vector space $\mathbb{R}^m$, and the standard coordinates $y^1,\dots,y^m$ of this space define the adapted atlas $\mathcal{Q}_1$ on Y_1. For every b-section s there is a b-chart F such that s takes the form $y^1 = f^1+\lambda^1,\dots$, $y^m = f^m+\lambda^m$ in the atlas $(\mathcal{Q}_1, F)$. For $m = n$ there is another canonical form: $y^1 = f^1+\lambda^1,\dots,\ y^{m-1} = f^{m-1}+\lambda^{m-1},\ y^m = \lambda^m - f^m$. The functions y^i are invariants.

(Here and in what follows, an invariant means a local invariant with respect to the action of $\mathscr{V}$ on $P_i = P(Y_i)$, where $\mathscr{V}$ is the group of orientation-preserving diffeomorphisms of X.)

Type II. Here the sample manifold is $Y_2 = S \times \mathbb{R}^0$. The action of G on S is trivial and the action of $g \in G = \mathrm{GL}_n^0$ on the set $\mathbb{R}^0$ of positive numbers is multiplication by $\det g$. Let y^0 be the natural coordinate on $\mathbb{R}^0$ defined by $y^0(g \bmod \mathrm{SL}_n) = \det g$. We may assume that S is an open ball with centre $\mathbf{0}$ and radius 1 in the vector space $\mathbb{R}^{m-1}$ of dimension $m-1$ with standard coordinates $y^1, \dots, y^{m-1}$, and the adapted atlas $\mathcal{Q}_2$ on Y_2 is determined by the coordinates $y^0, y^1, \dots, y^{m-1}$. The canonical form of a b-section is given by $y^i = f^i + \lambda^i$ for $1 \leqslant i \leqslant m-1$, $y^0 = 1$. The functions $y^1, \dots, y^{m-1}$ are invariants.

For types III–XI of sample manifolds, we use the identity representation ST_n of G and the dual representation. We recall that ST_n acts on $\mathbb{R}^n$. The set of rays in $\mathbb{R}^n$ is identified with the sphere S_{n-1}, which is thus acted on by G. We consider the standard coordinate system $t^1, \dots, t^i, \dots, t^n$ in $\mathbb{R}^n$ and use it to construct an atlas on S_{n-1}. Let $S_{(i)+}$ (resp. $S_{(i)-}$) be the hemisphere of S_{n-1} defined by $t^i > 0$ (resp. $t^i < 0$). We define coordinates $y^1, \dots, \hat{y}^i, \dots, y^n$ on $S_{(i)\pm}$ by putting $y^j = t^j/t^i$. The charts $S_{(i)\pm}$ form an atlas $\mathcal{S}$ on S_{n-1}. In a similar way, ST_n^* acts on $\mathbb{R}^{n*}$, and the set of rays in $\mathbb{R}^{n*}$ is identified with the sphere S_{n-1}^*, which thus admits the induced action of G. We use the coordinate system $u^1, \dots, u^i, \dots, u^n$ in $\mathbb{R}^{n*}$ to construct an atlas on S_{n-1}^*. Let $S_{(i)+}^*$ (resp. $S_{(i)-}^*$) be the hemisphere in S_{n-1}^* defined by $u^i > 0$ (resp. $u^i < 0$). We define coordinates $y^1, \dots, \hat{y}^i, \dots, y^n$ on $S_{(i)\pm}^*$ by putting $y^j = u^j/u^i$. The charts $S_{(i)\pm}^*$ form an atlas $\mathcal{S}^*$ on S_{n-1}^*.

Let S be the interval $(-1, +1)$ with a trivial action of G, and let v be the natural coordinate on S.

Type III. Here the sample manifold is $Y_3 = S \times S_{n-1}$. The adapted atlas $\mathcal{Q}_3$ on Y_3 consists of the charts $S \times S_{(i)\pm}$ with local coordinates $(v, y^1, \dots, \hat{y}^i, \dots, y^n)$. For every b-section $s\colon X \to P_3$ there is a b-chart F on X such that s is representable in the chart $S \times S_{(1)+}$ of the atlas $\mathcal{Q}_3$ on Y_3 and has the canonical form $v = \lambda \pm f^1$, $y^2 \equiv \dots \equiv y^n \equiv 0$ in the chart $(F, S \times S_{(1)+})$ of the atlas $(F, \mathcal{Q}_3)$ on P_3. The function v is an invariant.

For types IV and V we have $Y_{4,5} = S \times S_{n-1}^*$. The adapted atlas $\mathcal{Q}_{4,5}^*$ consists of the charts $S \times S_{(i)\pm}^*$ with local coordinates $(v, y^1, \dots, \hat{y}^i, \dots, y^n)$.

Type IV ($n = 2a$). For every b-section $s\colon X \to P_4$ there is a b-chart F such that s is representable in the chart $S \times S_{(2)+}^*$ and has the canonical form $v = \lambda \pm f^1$, $y^{2i-1} \equiv 0$ for $1 \leqslant i \leqslant a$, $y^{2i} = f^{2i-1}$ for $2 \leqslant i \leqslant a$. The function v is an invariant.

Type V ($n = 2a+1$). In this case, every b-section s in the canonical b-chart F is either representable in $S \times S_{(1+2a)+}^*$ and has the canonical form $v = \lambda + f^2$, $y^{2i-1} \equiv 0$, $y^{2i} = f^{2i-1}$ for $1 \leqslant i \leqslant a$, or representable in $S \times S_{(1+2a)-}^*$ and has the canonical form $v = \lambda + f^2$, $y^{2i-1} \equiv 0$, $y^{2i} = -f^{2i-1}$ for $1 \leqslant i \leqslant a$. The function v is an invariant.

Type VI. The sample manifold is $Y_6 = S_{n-1}$. The adapted atlas $\mathcal{Q}_6$ equals $\mathcal{S}$. Every b-section $s\colon X \to P_6$ in the canonical b-chart F is representable in the chart $S_{(1)+}$ and takes the form $y^2 \equiv \dots \equiv y^n \equiv 0$.

For types VII and VIII, we have $Y_{7,8} = S^*_{n-1}$ and the adapted atlas $\mathcal{Q}_{7,8}$ equals $\mathcal{S}^*$.

Type VII ($n = 2a$). For every b-section $s\colon X \to P_7$ there is a b-chart F such that s is representable in $S^*_{(2)+}$ and has the canonical form $y^{2i-1} \equiv 0$ for $1 \leqslant i \leqslant a$, $y^{2i} = f^{2i-1}$ for $2 \leqslant i \leqslant a$.

Type VIII ($n = 2a+1$). In this case, every b-section s in the canonical b-chart F is either representable in $S^*_{(1+2a)+}$ and has the canonical form $y^{2i-1} \equiv 0,\ y^{2i} = f^{2i-1}$ for $1 \leqslant i \leqslant a$, or representable in $S^*_{(1+2a)-}$ and has the canonical form $y^{2i-1} \equiv 0$, $y^{2i} = -f^{2i-1}$ for $1 \leqslant i \leqslant a$.

Type IX. The sample manifold is $Y_9 = \mathbb{R}^0 \times S_{n-1}$. The adapted atlas $\mathcal{Q}_9$ on Y_9 consists of the charts $\mathbb{R}^0 \times S_{(i)\pm}$ with local coordinates $(y, y^1, \dots, \hat{y}^i, \dots, y^n)$. In this case, every b-section s in a canonical b-chart F is representable in $\mathbb{R}^0 \times S_{(n)+}$ and has the canonical form $y \equiv 1,\ y^1 \equiv \dots \equiv y^{n-1} \equiv 0$.

For types X and XI we have $Y_{10,11} = \mathbb{R}^0 \times S^*_{n-1}$. The adapted atlas $\mathcal{Q}_{10,11}$ consists of the charts $\mathbb{R}^0 \times S^*_{(i)\pm}$ with local coordinates $(y, y^1, \dots, \hat{y}^i, \dots, y^n)$.

Type X ($n = 2a$). For every b-section $s\colon X \to P_{10}$ there is a b-chart F such that s is representable in $\mathbb{R}^0 \times S^*_{(2)+}$ and has the canonical form $y = 1,\ y^{2i-1} \equiv 0$ for $1 \leqslant i \leqslant a,\ y^{2i} = f^{2i-1}$ for $2 \leqslant i \leqslant a$.

Type XI ($n = 2a + 1$). Every b-section s in a canonical b-chart F is either representable in $\mathbb{R}^0 \times S^*_{(1+2a)+}$ and has the canonical form $y \equiv 1,\ y^{2i-1} \equiv 0,\ y^{2i} = f^i$ for $1 \leqslant i \leqslant a$, or representable in $\mathbb{R}^0 \times S^*_{(1+2a)-}$ and has the canonical form $y \equiv 1$, $y^{2i-1} \equiv 0,\ y^{2i} = -f^{2i-1}$ for $1 \leqslant i \leqslant a$.

Type XII. In this case, Y_{12} equals $\mathbb{R}^n - \mathbf{0}$ and the adapted coordinate system in Y_{12} is the standard coordinate system on $\mathbb{R}^n$. However, the action of G on $\mathbb{R}^n$ depends on a real parameter d, so we write $Y_{12} = \mathbb{R}^n_d$. The action of a matrix $g \in G$ on a column vector $x \in \mathbb{R}^n$ is given by $g(x) = \det(g)^{(d-1)/n} gx$.

a) Suppose that $d \neq 1 - n$. Then every b-section s reduces to the canonical form $y_1 \equiv 1,\ y_i \equiv 0$ for $i \geqslant 2$ in an appropriate b-chart F.

b) Suppose that $d = 1 - n$. Then every b-section s reduces to the canonical form $y_1 \equiv 1 \pm f^1,\ y_i \equiv 0$ for $i \geqslant 2$ in an appropriate b-chart F.

For types XIII and XIV, the sample manifolds $Y_{13,14}$ are equal to $\mathbb{R}^{n*} - \mathbf{0}$ and the adapted coordinate system is the standard coordinate system on $\mathbb{R}^{n*}$. The action of a matrix $g \in G$ on a row vector $x \in \mathbb{R}^{n*}$ is given by $g(x) = \det(g)^{(d-1)/n} x g^{-1}$. This action also depends on the real parameter d, so we write $Y_{13,14} = \mathbb{R}^{n*}_d$.

Type XIII ($n = 2a$). Two cases may occur.

a) If $d \neq 1$, then every b-section s reduces in an appropriate b-chart F to the canonical form $y^{2i-1} \equiv 0$ for $1 \leqslant i \leqslant a,\ y^2 = 1 + f^1,\ y^{2i} = f^{2i-1}$ for $2 \leqslant i \leqslant a$.

b) If $d = 1$, then every b-section s reduces in an appropriate b-chart F to the canonical form $y^{2i-1} \equiv 0$ for $1 \leqslant i \leqslant a,\ y^2 = 1 \pm f^1,\ y^{2i} = f^{2i-1}$ for $2 \leqslant i \leqslant a$.

Type XIV ($n = 2a + 1$). Two cases may occur.

a) If $(d - 1)/n \neq 1/(1 + a)$, then every b-section s reduces in an appropriate b-chart F to the canonical form $y^{2i-1} \equiv 0,\ y^{2i} = f^{2i-1}$ for $1 \leqslant i \leqslant a,\ y^{2a+1} = \pm 1$.

b) If $(d - 1)/n = 1/(1 + a)$, then every b-section s reduces in an appropriate b-chart F to the canonical form $y^{2i-1} \equiv 0,\ \ y^{2i} = (\lambda + f^2) f^{2i-1}$ for $1 \leqslant i \leqslant a$,

$y^{2a+1} = \pm(\lambda + f^2)$. There is a differential invariant Ω of the first order (that is, it is expressible in terms of the components of s and their first derivatives). We have $\Omega = \pm(\lambda + f^2)^{a+1}$ in the chart F.

We have listed all the non-exceptional types of sample manifolds for $q = 1$.

6.2. There are three types of sample manifolds for $q = 2$. To describe them, we put $G = G^2(n)^0$. Let $\psi\colon \mathbb{R}^n \to \mathbb{R}^n$ be an orientation-preserving diffeomorphism with $\psi(\mathbf{0}) = \mathbf{0}$. It can be written as $y^i = \psi^i(x^1, \dots, x^n)$. Its 2-jet at $x = \mathbf{0}$ is determined by the matrix $g = (g^i_j)$, which corresponds to the 1-jet, and the tensor $g^i_{jk} = \partial^2\psi^i/\partial x^k x^j|_{x=0}$. The elements of G are identified with pairs (g^i_j, g^i_{jk}), and multiplication is induced by the rule $(\varphi\psi)(x) = \varphi\bigl(\psi(x)\bigr)$, where φ and ψ map $\mathbb{R}^n$ to $\mathbb{R}^n$ and satisfy $\varphi(\mathbf{0}) = \psi(\mathbf{0}) = \mathbf{0}$. It follows that $(h^i_j, h^i_{jk})(g^i_j, g^i_{jk}) = (t^i_j, t^i_{jk})$, where $t^i_j = h^i_s g^s_j$ and $t^i_{jk} = h^i_s g^s_{jk} + h^i_{sr} g^s_j g^r_k$.

We recall that Af_n stands for the affine group of $\mathbb{R}^n$. It is identified with the set of pairs (A, a), where A is a non-singular matrix of order n and a is a column vector in $\mathbb{R}^{n*}$. The group operation in Af_n is given by $(A_1, a_1)(A_2, a_2) = (A_1A_2, a_1A_2 + a_2)$. The group Af_n acts on $\mathbb{R}^{n*}$ on the right: the action of $p = (A, a) \in \mathrm{Af}_n$ on a row vector $t = (t_i) \in \mathbb{R}^{n*}$ is equal to $tp = tA + a$ (see (21)). We also use the corresponding left action $pt = tp^{-1} = tA^{-1} - aA^{-1}$ (see (22)). To define an epimorphism $\phi\colon G \to \mathrm{Af}_n$, we write $(A, g^i_{jk}) = g \in G$, let $D = (d^i_j)$ be the matrix inverse to A and put $t^i_{jk} = d^i_s g^s_{jk}$ and $a_j = t^u_{ju}$. Then $\phi(g) = (A, a) \in \mathrm{Af}_n$, where $a = (a_j)$ (see (20)). We also have an embedding $\varepsilon\colon \mathrm{Af}_n \to \mathrm{GL}_{n+1}$ defined by (55). The image of ε is a subgroup of GL_{n+1}, and we denote this subgroup by B_n. The group GL_{n+1} acts on $\mathbb{R}^{n+1}$ by the identity representation ST_{n+1}.

The manifolds Y_{15}–Y_{17} and their adapted atlases are described above.

Type XV. Every section s reduces to the canonical form $s_F = (1, 0, \dots, 0)$ in an appropriate b-chart F.

Type XVI ($n = 2a$). Every b-section s reduces in an appropriate b-chart F to the canonical form $y^{2i-1} \equiv 0$ for $1 \leqslant i \leqslant a$, $y^2 = 1 \pm f^1$, $y^{2i} = f^{2i-1}$ for $2 \leqslant i \leqslant a$.

Type XVII ($n = 2a + 1$). Every b-section s reduces in an appropriate b-chart F to the canonical form $y^{2i-1} \equiv 0$, $y^{2i} = f^{2i-1}$ for $1 \leqslant i \leqslant a$, $y^{2a+1} \equiv 1$.

Type XVIII ($n = 4$). The sample manifold is $Y_{18} = Y_{\mathrm{gr}}$, the Grassmannian manifold of 2-dimensional planes in 4-dimensional space. It is endowed with the natural action of GL_4.

Type XIX ($n = 3$). The sample manifold is $Y_{19} = Y_{\mathrm{fl}}$, the 3-dimensional manifold of flags of type $(1, 2)$ in 3-dimensional space. It is endowed with the natural action of GL_3. The stabilizers of this action are Borel subgroups of GL_3.

6.3. The following theorem combines the results obtained above. We assume that $n > 2$. The analogues for $n \leqslant 2$ are obtained by an easy modification (and hence omitted).

Theorem 5. *Let X be an oriented n-dimensional manifold, $G = G^q(n)^0$ the connected component of the identity in the differential group of order q and dimension n, and D a special G-manifold, that is, a manifold of dimension at most n acted on by the Lie group G, $n > 2$. Then there are 19 types of special G-manifold Y_i*

(called sample manifolds) as listed above. For every regular point w of any special G-manifold there is a local G-isomorphism ε (that is, an isomorphism commuting with the local action of G) from an appropriate neighbourhood of w onto an open subset of a sample manifold Y_i. This sample manifold Y_i is uniquely determined. Let $\mathcal{Q}_i$ be an adapted atlas on Y_i. The exact order q of the sample manifolds equals 1 or 2.

Fix any point $b \in X$ and consider a section $s\colon X \to P_i = P(Y_i)$ which is defined in a neighbourhood of b and is sufficiently general at b. We recall that such sections are called b-sections. The notion of a b-section is defined separately for each type Y_i. For types I–XVII, each b-section s admits a b-centered chart F oriented in the same way as X (a b-chart for brevity) with coordinate functions $f^1, \dots, f^n$ such that the expression of s in the atlas $(\mathcal{Q}_i, F)$ of the manifold P_i takes the canonical form stated in the list. All the canonical forms in the list are inequivalent.

The notion of a b-section and canonical forms can also be defined for types XVIII, XIX. This will be done in a future paper. Types XVIII and XIX are said to be *exceptional* and types I–XVII *non-exceptional*.

The proof of Theorem 5 follows from the previous content of the paper. Indeed, applying the construction of § 5.2, we get an S-unfolding Y and a point $y \in Y$ such that the actions near w and y are locally isomorphic. By construction, all points of Y are regular and the stabilizer G_s of any point $s \in Y$ is a closed connected subgroup whose codimension is independent of s and does not exceed n. The list of all possible stabilizers G_s is given in Theorem 3 for $q = 1$ and in Theorem 4 for $q \geqslant 2$. If $q = 1$, then $G^q(n)^0 = \mathrm{GL}_n^0$ and hence G_y is a subgroup of GL_n^0. All possible cases are discussed above. For example, if $G_y = \mathrm{GL}_n^0$, then we have $G_s = \mathrm{GL}_n^0$ for all $s \in Y$. Hence the action on Y is trivial, and we get type I, which is discussed in § 5.3. If $G_y = \mathrm{SL}_n$, then we have $G_s = \mathrm{SL}_n$ for all $s \in Y$. Hence all orbits are 1-dimensional, and we get type II, which is also discussed above. The other cases are studied in a similar way.

§ 7. Subgroups of small codimension in SL_n

Here we prove Assertion 1 in § 4.1. We follow a plan suggested by Onishchik and apply the arguments of [14], Proposition 5, where a similar problem is solved. We use well-known results from Lie group theory (see, for example, [14]–[18]).

Proof of Assertion 1. Let H be a proper connected subgroup of SL_n whose codimension does not exceed $n > 2$. We regard the embedding of H in SL_n as a faithful representation ρ of H on $\mathbb{R}^n$. The assertion will be proved if we can show that ρ is reducible over $\mathbb{R}$. For if a subgroup of SL_n has an invariant subspace of dimension m in $\mathbb{R}^n$, then its codimension in SL_n cannot be less than $m(n-m)$. If the codimension of such a subgroup does not exceed n, then $n \geqslant m(n-m)$. For every $n \geqslant 2$, this inequality holds only when $m = 1$ or $m = n-1$, except for the case $n = 4,\ m = 2$. If $m = 1$ (resp. $m = n-1$), then H is conjugate to a subgroup of T_1 (resp. of T_2). If $n = 4$ and $m = 2$, then H is conjugate to the group P_{24}. Hence the assertion follows from Lemma 4.

Thus we assume that ρ is irreducible over $\mathbb{R}$ and seek a contradiction.

Lemma 13. *The representation ρ is irreducible over $\mathbb{C}$.*

Proof. Let $L \subset \mathbb{C}^n$ be a proper non-zero irreducible invariant subspace of ρ. We denote by L_1 the subspace complex-conjugate to L. Then L_1 is also invariant with respect to ρ. We put $\dim L_1 = \dim L = m$. If $m < n$, then the space $L_2 = L + L_1$ is defined over $\mathbb{R}$ and is invariant with respect to ρ, a contradiction. If $m \geqslant n$, then either $L_3 = L \cap L_1$ is non-zero or L_3 is the zero subspace. In the first case, L_3 is defined over $\mathbb{R}$ and is invariant with respect to ρ, a contradiction. In the second case we have $n = 2m,\ \mathbb{C}^n = L + L_1$ and the projection of $\mathbb{C}^n$ onto L along L_1 determines an embedding of H in the group $\mathrm{GL}_m(\mathbb{C})$ of all complex non-singular matrices. The real dimension of this group equals $2m^2$. However, $\dim H \geqslant n^2 - n - 1 > 2m^2$ for $n > 2$. The lemma is proved.

We put $\mathcal{H} = \mathrm{Lie}(H)$ and continue the argument in terms of Lie algebras. Extend the field of scalars of the Lie algebra $\mathcal{H}$ to $\mathbb{C}$ and denote the resulting algebra by $\mathcal{H}_{\mathbb{C}}$. Then ρ determines a representation (again denoted by ρ) of the algebra $\mathcal{H}$ in SL_n. In what follows we regard $\mathbb{C}$ as the base field. All algebras, decompositions, dimensions and so on will be considered over $\mathbb{C}$. We write $GL_n(\mathbb{C})$ and $SL_n(\mathbb{C})$ respectively for the Lie algebra of all complex matrices of order n and the subalgebra of traceless matrices. The extension of ρ to $\mathcal{H}_{\mathbb{C}}$ is denoted by $\rho_{\mathbb{C}}\colon \mathcal{H}_{\mathbb{C}} \to SL_n(\mathbb{C})$. Clearly, the (complex) codimension of $\rho_{\mathbb{C}}(\mathcal{H}_{\mathbb{C}})$ in $SL_n(\mathbb{C})$ does not exceed n.

Lemma 14. *The algebra $\mathcal{H}_{\mathbb{C}}$ is semisimple.*

Proof. Let $\mathcal{R}$ be the soluble radical of the Lie algebra $\mathcal{H}_{\mathbb{C}}$. If $\mathcal{R}$ is not the zero subalgebra, then the representation ρ of $\mathcal{R}$ has a non-zero eigenvector. Consider the weight subspace L of this vector. Since $\mathcal{R}$ is an ideal in $\mathcal{H}_{\mathbb{C}}$, we easily see that L is an eigenspace for ρ. Hence $L = \mathbb{C}^n$ and the action of any element of $\mathcal{R}$ on $\mathbb{C}^n$ is multiplication by a scalar matrix. Since $\rho(\mathcal{H}_{\mathbb{C}}) \subseteq SL_n(\mathbb{C})$, it follows that $\mathcal{R}$ is the zero subalgebra. The lemma is proved.

Put $m = \dim \mathcal{H}_{\mathbb{C}}$. Let $\mathcal{H}_{\mathbb{C}} = \mathcal{H}_1 + \dots + \mathcal{H}_r$ be a decomposition of the Lie algebra $\mathcal{H}_{\mathbb{C}}$ into a direct sum of simple complex algebras. Since $\rho_{\mathbb{C}}$ is an irreducible complex representation of a semisimple algebra, it splits into the tensor product (over $\mathbb{C}$) of the corresponding irreducible complex representations ρ_j of the algebras $\mathcal{H}_j$. We denote the dimension of a representation π by $\dim \pi$. Note that $\dim \rho_{\mathbb{C}} = n$, $\dim \mathcal{H}_{\mathbb{C}} = m$ and $m \geqslant n^2 - n - 1$.

Lemma 15. *The algebra $\mathcal{H}_{\mathbb{C}}$ is simple.*

Proof. Suppose that $\mathcal{H}_1$ and $\mathcal{H}_2$ are semisimple Lie algebras, $m_1 = \dim \mathcal{H}_1,\ m_2 = \dim \mathcal{H}_2,\ \rho_1$ and ρ_2 are faithful irreducible representations of them, $\mathcal{H}_{\mathbb{C}} = \mathcal{H}_1 + \mathcal{H}_2$, and $\rho_{\mathbb{C}} = \rho_1 \otimes \rho_2,\ n_1 = \dim \rho_1$ and $n_2 = \dim \rho_2$. Then $n = n_1 n_2$ and $m = m_1 + m_2$. We have

$$m_1 + m_2 \geqslant (n_1 n_2)^2 - (n_1 n_2) - 1.$$

Clearly, $n_1^2 \geqslant m_1$ and $n_2^2 \geqslant m_2$. It follows that

$$n_1^2 + n_2^2 \geqslant (n_1 n_2)^2 - (n_1 n_2) - 1.$$

This is impossible for $n_1 \geqslant 2,\ \ n_2 \geqslant 2$. The lemma is proved.

Thus we may assume that $\mathcal{H}_{\mathbb{C}}$ is a simple Lie algebra and $\rho_{\mathbb{C}}$ is an irreducible representation of $\mathcal{H}_{\mathbb{C}}$. It is known that every irreducible representation of a semisimple (in particular, simple) Lie algebra is uniquely determined by its dominant character, which is the highest weight of the representation. Let $\rho(\Lambda)$ be the representation with highest weight Λ.

Lemma 16. *If* Λ, M *are dominant characters and* $M > \Lambda$, *then* $\dim \rho(M) > \dim \rho(\Lambda)$.

For a proof, see [14], Ch. 1, § 3.8, Proposition 4.

We can now complete the proof of Assertion 1 by exhausting all types of simple Lie algebra. The following assertions on the dimensions of the fundamental representations of simple Lie algebras are based on Table 2 in [17], Russian p. 272.

We start with the exceptional Lie algebras. The minimal dimensions k of fundamental representations for algebras of type E_6, E_7, E_8, F_4, G_2 are equal to 27, 56, 248, 26, 7 respectively, and the dimensions m of these algebras are equal to 78, 133, 248, 52, 14 respectively. We see that k^2-k-1 exceeds m for all these algebras.

Now we consider the classical Lie algebras.

Series D_s, $s \geqslant 4$. The dimension of the algebra D_s of rank s is $m = s(2s-1)$. The minimal dimension of the fundamental representation of D_s is $k = 2s$. Hence $k^2 - k - 1$ always exceeds m.

Series B_s, $s \geqslant 2$. The dimension of the algebra B_s of rank s is $m = s(2s+1)$. The minimal dimension of the fundamental representation of B_s is $k = 2s+1$ for $s \geqslant 3$ and $k = 4$ for $s = 2$. Hence $k^2 - k - 1$ always exceeds m.

Series C_s, $s \geqslant 3$. The dimension of the algebra C_s of rank s is $m = s(2s+1)$. The dimensions of the fundamental representations of C_s are equal to $\dim \rho(\pi_k) = C_{2s}^k - C_{2s}^{k-2}$, $1 \leqslant k \leqslant s$. The following estimate is proved in Proposition 5 of [14] for $s \geqslant k \geqslant 3$:

$$\dim \rho(\pi_k) > r(s) = \frac{4s(2s-1)(s-1)}{3(s+2)}\,.$$

The dimensions $\dim \rho(\pi_1)$ and $\dim \rho(\pi_2)$ of the fundamental representations π_1 and π_2 are equal to $2s$ and $s(2s-1)-1$ respectively. We easily see that $r(s) \geqslant 2s$ and $s(2s-1)-1 \geqslant 2s$. Hence the minimal dimension of a fundamental representation is $k = 2s$. It is easy to see that $k^2 - k - 1$ always exceeds m.

Thus the minimal dimension k of a fundamental representation and the dimension m of the algebra satisfy $k^2 - k - 1 > m$ for all the algebras considered above. Since every dominant weight dominates some fundamental weight, the dimension n of any irreducible representation satisfies $n \geqslant k$ by Lemma 16. Since $k > 1$, it follows that $n^2 - n - 1 \geqslant k^2 - k - 1$. This enables us to exclude all types except for A_s.

Series A_s, $s \geqslant 1$. The dimension of the algebra A_s of rank s is $m = s(s+2)$. The fundamental representations $\rho(\pi_1)$ and $\rho(\pi_s)$ have dimension $s+1$, and the images of A_s under these representations coincide with the whole of $SL_n(\mathbb{C})$. Therefore we can assume that the highest weight Λ of the representation ρ is not equal to π_1 or π_s. The following dimensions are calculated in [14] for $s \geqslant 2$:

$$n_1 = \dim \rho(2\pi_s) = \dim \rho(2\pi_1) = \frac{(s+1)(s-2)}{2} + 2(s+1),$$
$$n_2 = \dim \rho(\pi_1 + \pi_s) = s^2 + 2s.$$

If $s > 2$, then A_s also has fundamental representations $\rho(\pi_2), \dots, \rho(\pi_{s-1})$ of dimensions $\dim \rho(\pi_j) = C^j_{s+1}$. The minimal dimension is $n_3 = C^2_{s+1}$. Clearly, the highest weight Λ dominates either $2\pi_1$, $2\pi_s$, $\pi_1 + \pi_s$ or π_j for $1 < j < s$. Hence the number $n = \dim \rho(\Lambda)$ always satisfies $n \geqslant n_4 = \min(n_1, n_2) > 1$ for $s = 2$ and $n \geqslant n_4 = \min(n_1, n_2, n_3) > 1$ for $s > 2$.

A direct examination shows that $n_4^2 - n_4 - 1 > m$. Hence $n^2 - n - 1 > m$ by Lemma 16. This excludes the case $s \geqslant 2$.

It is also easy to exclude the remaining case, the algebra A_1 of rank $s = 1$. Indeed, it is well known that, for every positive integer k, this algebra has a unique irreducible representation, which has dimension $k+1$ and highest weight $k\pi_1$. These dimensions are always greater than or equal to 3 for $k \geqslant 2$. Since $3^2 - 3 - 1 > 3$, the assertion is proved.

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