Lie symmetries and invariants of the Lotka–Volterra system

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In this paper we use the Lie symmetry method for finding rational and transcendental symmetry transformations and invariants for the 3D Lotka–Volterra system. © *1995 American Institute of Physics.*

I. INTRODUCTION

In many branches of physics and applied mathematics we find systems described by coupled, nonlinear ordinary differential equations. If the dimension of the phase space is bigger than two, the overwhelming majority of these systems has domains in the phase space where the motion is chaotic in the sense that it depends sensitively on the choice of initial conditions. For instance, the trajectories in dissipative three dimensional dynamical systems can approach strange attractors. An important mathematical and physical problem is to find conditions for the absence of this chaotic behavior by looking for parameter values for which the dynamical system can be completely or partially integrated. If the system is an integrable one the solutions will be well behaved and we can get global informations on its long-term behavior. The notion of integrability is related to the existence of first integrals. So the questions above can be put in the following form: how can we identify the values of the parameters for which the equations of motion have first integrals? Several methods have been employed for studying the existence of first integrals and the integrability of dynamical systems. Some of them have been devised for Hamiltonian systems, such as the Ziglin–Yoshida analysis[1,2] or the method of Noether symmetries.[3–5] Other methods can be applied also for non-Hamiltonian systems: the direct method,[6–8] the singularity analysis,[9] the linear compatibility analysis method,[10] the use of Lax pairs,[11] the method of Lie symmetries,[12,13] the quasimonomial formalism,[14] the Carlemann embedding procedure,[15,16] etc.

The symmetry method, introduced by Lie,[17] consists of a systematic procedure for the determination of the continuous symmetry transformations of a system of differential equations. In the last two decades or so the application of this method has grown quickly, especially due to the utilization of computer algebra. It has been used, through the determination of symmetries and first integrals, for the identification of completely or partially integrable cases for several dynamical systems.[18–21] For example, Sen and Tabor made an extensive analysis of the Lie symmetries and integrable cases of the Lorenz model.[22] However all these works make the supposition of polynomial dependence of the symmetries on all dynamical variables. For the resolution of the so-called Lie conditions, which gives us the conditions for invariance of the system of equations, they suppose usually a polynomial dependence of the infinitesimal transformations on the velocities (for second order ODE) or on the variables (for first order ODE). This restriction limits the possible symmetry transformations to be found. In the usual physical situations, in mechanics for instance, the restriction is not serious because first integrals with physical meaning, as energy or angular momentum, have a polynomial dependence on the velocity and can be identified by related (Lie or Noether) symmetry transformations with a simple polynomial dependence. Our main purpose here is to exploit the possibility of determining Lie symmetry vector fields with a nonpolynomial (algebraic or transcendental) form on some dynamical variables.[23]

We take the Lotka–Volterra system (LV) as a paradigmatic example for our study. This model,

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widely used in applied mathematics, employs a first order nonlinear differential system with linear and quadratic interactions between the dynamical variables. It was introduced by Volterra[24] and Lotka[25] in the theory of biological populations. Furthermore it arises also in a variety of problems in physics: laser physics,[26] plasma physics,[27] convective instabilities,[28] neural networks,[29] etc. An important result by Brenig[30] shows that a large set of ordinary differential equations can be reduced to LV equations by quasimonomial transformations of variables. The possible existence of strange attractors in LV like systems has been discussed by Smale[31] and the occurrence of such attractors was proven in numerical experiments by Arneodo, Coullet and Tresser.[32] We developed also some numerical experiments for the 3D-LV system with the form given below; they suggested the occurrence of chaotic behavior for certain ranges of parameters. For the one dimensional continuous case of the LV system we have the logistic equation; in the discrete case we get the logistic map with a variety of periodic and chaotic behaviors. The integrability and the singularity structure of the 2D continuous case was studied recently by Sachdev and Ramanan.[33] A particular class of the 3D-LV equations, with a cyclic invariance of the coefficient matrix, was extensively analyzed by Gardini *et al.* in Ref. 34 where the dynamics of the continuous case is a simple one and the discrete version exhibits a very complex bifurcation structure, with a large variety of transitions from periodic to chaotic attractors. Grammaticos, Moulin-Ollagnier, Ramani, Strelcyn and Wojciechowski[35] did a thorough examination of the three dimensional case, with the form given below, through the singularity analysis, the linear compatibility method and the Jacobi last multiplier method. They found several situations where the system has time-independent first integrals (see also Refs. 36 and 37).

We find here the Lie symmetry vector fields for the Lotka–Volterra system and get invariants (first integrals) for many cases with specific values of parameters. Almost all time-independent invariants we get can be found in the very complete work by Grammaticos *et al.*[35] But we found two new cases with transcendental invariants, and we also considered invariants with an explicit time-dependence. The cases found here do not exhaust the situations where there are nontrivial Lie symmetries; in spite of the possibility of getting algebraic and transcendental invariants on two of the variables we make the assumption that the symmetry vector fields have a quadratic dependence on one of them. Some of the cases found in Ref. 35 cannot be obtained with this restriction. We think that similar analysis, where the use of the algebraic manipulation is essential, could be useful in studying other three dimensional dynamical systems.

II. LIE CONDITIONS FOR THE 3D LOTKA–VOLTERRA SYSTEM

The Lie method permits us the determination of the symmetry transformations of a set of differential equations.[12] By using these symmetries we can, in many cases, find first integrals (invariants) in a straightforward fashion and identify integrable systems. The symmetry vector fields can be obtained from the invariance of the system of first-order differential equations

$$\Delta_i(x_j,x_j')=0 \tag{1}$$

under infinitesimal transformations with the form

$$x_i \Rightarrow x_i+\varepsilon\,\eta_i(t,x_i), \quad t\Rightarrow t+\varepsilon\,\xi(t,x_i). \tag{2}$$

We will take $\xi=0$. The symmetry evolutionary vector field has the form

$$\mathbf{U}=\eta_i\partial_{x_i}. \tag{3}$$

The Lie conditions for invariance of the system (1) are

$$pr^{(1)}\mathbf{U}(\Delta_i)|_{\Delta_i=0}=[\eta_j\partial_{x_j}+(D_t\eta_j)\partial_{\dot{x}_j}](\Delta_i)|_{\Delta_i=0}=0. \tag{4}$$

If I is a time-dependent first integral for the system (1) then

$$D_t = \partial_t I + \mathbf{X}(I) = 0, \tag{5}$$

where $\mathbf{X}$ is the dynamical vector field.

A result that enables us to obtain first integrals, starting from the symmetries of the equations, is the following: given a set of first-order differential equations Δ_i, if $\mathbf{U_1}$ is a symmetry vector field of Δ_i then $\mathbf{U_2} = I\mathbf{U_1}$ is also a symmetry vector field of Δ_i, and only if I is a first integral of the system Δ_i. In particular, if a symmetry field $\mathbf{U_1}$ is not functionally independent of the dynamical vector field $\mathbf{X}$,

$$\mathbf{U_1} = F(t, x_i)\mathbf{X} \tag{6}$$

then F is a first integral of the system.

The 3D Lotka–Volterra system is described by the following equations:

$$\dot{x} = x(cy + z + d_1), \quad \dot{y} = y(x + az + d_2), \quad \dot{z} = z(bx + y + d_3). \tag{7}$$

We applied the Lie conditions (4) on these equations, with the ansätz that the symmetry vector fields can be time-dependent and have quadratic dependence on z:

$$\begin{aligned} \eta_1 &= \eta_{10}(t,x,y) + \eta_{11}(t,x,y)z + \eta_{12}(t,x,y)z^2, \\ \eta_2 &= \eta_{10}(t,x,y) + \eta_{21}(t,x,y)z + \eta_{22}(t,x,y)z^2, \\ \eta_3 &= \eta_{30}(t,x,y) + \eta_{31}(t,x,y)z + \eta_{32}(t,x,y)z^2. \end{aligned} \tag{8}$$

This restriction is only a partial one because we admit that they are general functions of x and y. The system of equations to be solved, in the general case, is very complicated. The calculations of the symmetry vector fields were made by algebraic computation.

By applying the extended Lie operator to the Lotka–Volterra (7) we get the following system of differential equations to be solved:

$$0 = \partial_x(\eta_{12})(t,x,y)x - \eta_{12}(t,x,y) + \partial_y(\eta_{12})(t,x,y)ya,$$

$$\begin{aligned} 0 = {} & \partial_x(\eta_{11})(t,x,y)x + \partial_x(\eta_{12})(t,x,y)xcy + \partial_x(\eta_{12})(t,x,y)xd_1 + \partial_t(\eta_{12})(t,x,y) - \eta_{11}(t,x,y) \\ & - \eta_{12}(t,x,y)cy - \eta_{12}(t,x,y)d_1 + \partial_y(\eta_{11})(t,x,y)ya + \partial_y(\eta_{12})(t,x,y)yx + \partial_y(\eta_{12})(t,x,y)yd_2 \\ & - \eta_{22}(t,x,y)xc - \eta_{32}(t,x,y)x + 2\eta_{12}(t,x,y)bx + 2\eta_{12}(t,x,y)y + 2\eta_{12}(t,x,y)d_3, \end{aligned}$$

$$\begin{aligned} 0 = {} & \partial_x(\eta_{10})(t,x,y)x + \partial_x(\eta_{11})(t,x,y)xcy + \partial_x(\eta_{11})(t,x,y)xd_1 + \partial_t(\eta_{11})(t,x,y) - \eta_{31}(t,x,y)x \\ & - \eta_{10}(t,x,y) - \eta_{11}(t,x,y)cy - \eta_{11}(t,x,y)d_1 + \partial_y(\eta_{10})(t,x,y)ya + \partial_y(\eta_{11})(t,x,y)yx \\ & + \partial_y(\eta_{11})(t,x,y)yd_2 - \eta_{11}(t,x,y)xc + \eta_{11}(t,x,y)bx + \eta_{11}(t,x,y)y + \eta_{11}(t,x,y)d_3, \end{aligned}$$

$$\begin{aligned} 0 = {} & -\eta_{10}(t,x,y)cy - \eta_{10}(t,x,y)d_1 + \partial_x(\eta_{10})(t,x,y)xcy + \partial_x(\eta_{10})(t,x,y)xd_1 + \partial_t(\eta_{10})(t,x,y) \\ & - \eta_{30}(t,x,y)x - \eta_{20}(t,x,y)xc + \partial_y(\eta_{10})(t,x,y)yx + \partial_y(\eta_{10})(t,x,y)yd_2, \end{aligned}$$

$$0 = \partial_x(\eta_{22})(t,x,y)x - \eta_{22}(t,x,y)a + \partial_y(\eta_{22})(t,x,y)ya,$$

TABLE I. Lie symmetry vector fields of the L–V system ($a\neq 1$).

N°	Parameters	Symmetry vector fields
1	a,b,c $d_1=d_2=d_3$	$\mathbf{U}_1=(xz+xcy+xd_1)\partial_x+(yd_1+yaz+yx)\partial_y+(zy+zd_1+zbx)\partial_z$ $\mathbf{U}_2=e^{-d_1t}((xcy+xz)\partial_x+(yaz+yx)\partial_y+(zy+zbx)\partial_z)$
2	$a,b,c=-\frac{1}{ab}$ d_1,d_2,d_3	$\mathbf{U}_1=(d_1x+xz+cxy)\partial_x+(xy+d_2y+ayz)\partial_y+(zy+zd_3+zbx)\partial_z$ $\mathbf{U}_2=zx^{ab}y^{-b}e^{-t(d_3-d_2b+d_1ab)}\mathbf{U}_1$
3	$a,b,c=-\frac{1}{ab}$ $d_1=d_2=d_3$	$\mathbf{U}_1=y(x+d_1+az)\partial_y-\frac{x(-d_1ba-zba+y)\partial_x}{ab}+z(y+d_1+bx)\partial_z$ $\mathbf{U}_2=\frac{e^{-d_1t}((zxba-xy)\partial_x+(yxba+za^2yb)\partial_y+(b^2zxa+zyba)\partial_z)}{ab}$ $\mathbf{U}_3=-\frac{e^{-d_1t}}{a^2b}(((b^2d_1a^2+b^2za^2-yab)x^2$ $+(((a+ab)z-d_1a+d_1ba)y-y^2-z^2a^2b)x)\partial_x$ $+(b^2yx^2a^2+(y^2ab+((a^3b^2-a^2b)z+d_1a^2b+b^2d_1a^2)y)x$ $+(za^2b+d_1ba)y^2+((a^3d_1b-d_1a^2b)z-a^3z^2b)y)\partial_y$ $+(b^3zx^2a^2+((b^2a^2+b^2a)zy-b^2z^2a^2+2b^2zd_1a^2)x+zy^2ab$ $+((d_1ba+d_1a^2b)z-z^2a^2b)y-z^2d_1a^2b)\partial_z)$ $\mathbf{U}_4=zx^{ab}y^{-b}e^{-(ab+1-b)d_1t}\mathbf{U}_2,\quad \mathbf{U}_5=zx^{ab}y^{-b}e^{-(ab+1-b)d_1t}\mathbf{U}_1$ $\mathbf{U}_6=-\frac{(y-az+bxa)e^{-d_1t}}{a}\mathbf{U}_2$
4	$a=1-\frac{1}{b},b$ $c=-\frac{1}{ab}$ d_1	$\mathbf{U}_1=\frac{1}{b(b-1)}((-bxz-bxy-bxd_1+b^2xd_1+b^2xz)\partial_x$ $+(-byd_1-bxy+b^2yd_1+zy+b^2zy+b^2xy-2byz)\partial_y$ $+(-bzd_1-b^2xz+zb^2d_1-byz+b^2zy+b^3xz)\partial_z)$ $\mathbf{U}_2=\frac{e^{-d_1t}}{b(b-1)^2}((-bxz-bxy-bxd_1+b^2xd_1+b^2xz)\partial_x$ $+(-byd_1-bxy+b^2yd_1+zy+b^2zy+b^2xy-2byz)\partial_y$ $+(-bzd_1-b^2xz+zb^2d_1-byz+b^2zy+b^3xz)\partial_z)$ $\mathbf{U}_3=\frac{xe^{-d_1t}}{b-1}((bz-z-by)\partial_x+(b^2y-by)\partial_y+(zb^2-bz)\partial_z)$ $\mathbf{U}_4=\frac{ye^{-d_1t}}{b-1}(bx\partial_x-(xb^2-bz-bx+z)\partial_y+zb\partial_z)$ $\mathbf{U}_5=\frac{x^{1-b}y^be^{-d_1t}}{b-1}((bx-x)\partial_x+(by-y)\partial_y+(xb^2+by-bx)\partial_z)$ $\mathbf{U}_6=x^{b-1}zy^{-b}\mathbf{U}_3,\quad \mathbf{U}_7=x^{b-1}zy^{-b}\mathbf{U}_1$ $\mathbf{U}_8=x^{b-1}zy^{-b}\mathbf{U}_4,\quad \mathbf{U}_9=-\frac{e^{-d_1t}(xb^2+by-bx-bz+z)\mathbf{U}_3}{b-1}$ $\mathbf{U}_{10}=-\frac{e^{-d_1t}(xb^2+by-bx-bz+z)\mathbf{U}_4}{b-1},\quad \mathbf{U}_{11}=\frac{(xb^2+by-bx-bz+z)^2e^{-2d_1t}}{(b-1)^2}\mathbf{U}_5$ $\mathbf{U}_{12}=-\frac{e^{-d_1t}(xb^2+by-bx-bz+z)}{b-1}\mathbf{U}_5$
5	$a=-1$ $b=c=1$ $d_1=d_2=d_3$	$\mathbf{U}_1=x(y+d_1+z)\partial_x+y(d_1+x-z)\partial_y+z(x+d_1+y)\partial_z$ $\mathbf{U}_2=e^{-d_1t}(-(x^2y+x^2d_1-2xzy-xz^2-xy^2+x^2z)\partial_x$ $-(y^2z-xy^2-2yzd_1+2yd_1x-2xzy-y^2d_1+yz^2+x^2y)\partial_y$ $-(-y^2z-yz^2+x^2z-xz^2-z^2d_1+2zd_1x)\partial_z)$ $\mathbf{U}_3=e^{-d_1t}((xy+zx)\partial_x+(xy-zy)\partial_y+(zx+zy)\partial_z)$ $\mathbf{U}_5=e^{-d_1t}(-y(x-z)(\partial_y-\partial_z)),\quad \mathbf{U}_4=z(\partial_x+2\partial_z-\partial_y)$ $\mathbf{U}_6=\frac{z(\partial_x+\partial_z)(y+z)}{y},\quad \mathbf{U}_7=\frac{e^{d_1t}z}{xy}\mathbf{U}_1,\quad \mathbf{U}_8=-e^{-d_1t}(x-z-y)\mathbf{U}_3$ $\mathbf{U}_9=\frac{e^{d_1t}z}{xy}\mathbf{U}_5\quad \mathbf{U}_{10}=-\frac{e^{-d_1t}(x-z-y)}{2}\mathbf{U}_5$

TABLE I. (*Continued.*)

6	$a,b=1$ $c=0$ $d_1=d_2=d_3$	$\mathbf{U}_1=x(z+d_1)\partial_x+y(x+za+d_1)\partial_y+z(x+d_1+y)\partial_z$ $\mathbf{U}_2=\frac{x^a(za-y)e^{-(a+1)d_1t}}{ya}(xz\partial_x+(zya+xy)\partial_y+(zx+zy)\partial_z)$ $\mathbf{U}_3=\frac{x^a(za-y)e^{-d_1at}}{ya}\mathbf{U}_1$
7	$a=-1-\frac{1}{b},b$ $c=\frac{1}{ab}$ $d_1=d_2=d_3$	$\mathbf{U}_1=-\frac{x(y-bz-d_1-bd_1-z)}{1+b}\partial_x+\frac{(bd_1-bz+bx-z)y}{b}\partial_y+(bx+y+d_1)z\partial_z$ $\mathbf{U}_2=\frac{xe^{-d_1t}}{1+b}((by+z+bz)\partial_x+(-yb^2-by)\partial_y+(b^2z+bz)\partial_z)$ $\mathbf{U}_3=-\frac{ye^{-d_1t}}{1+b}((b^2x-bz+bx-z)\partial_y-bx\partial_x+z\partial_zb)$
8	$a,b=1$ $c=0$ d_1,d_2,d_3	$\mathbf{U}_1=x(d_1+z)\partial_x+y(x+d_2+za)\partial_y+z(y+d_3+x)\partial_z$ $\mathbf{U}_2=\frac{x^ae^{-t(d_3-d_2+d_1a)}}{ya^2}((xd_3ya-xd_2ya-xzya+xzd_1a^2+xz^2a^2$ $-xd_1ya)\partial_x+(a^3zd_3y-xy^2a+xzya^2+zd_3ya^2+a^3z^2y$ $-y^2za^2-d_3y^2a-a^3zd_2y)\partial_y+(d_2^2y+y^2d_2-xzya+z^2d_3a^2$ $+d_2^2ya+z^2ya^2-d_3d_2y+d_3^2ya+zd_3ya^2-y^2d_3+xz^2a^2$ $-y^2za-2d_3d_2ya-zd_2ya-zd_2ya^2)\partial_z)$
9	$a,b=\frac{(a+1)}{(a-1)}$ $c=-\frac{(a-1)}{(a+1)}$ $d_1=d_2,d_3$	$\mathbf{U}_1=\frac{x(d_1a+d_1+za+z-ya+y)\partial_x}{a+1}+y(x+d_2+za)\partial_y$ $+\frac{z(ya-y+d_3a-d_3+xa+x)\partial_z}{a-1}$ $\mathbf{U}_2=\frac{x^{2a/(a-1)}zy^{-2/(a-1)}e^{-t(d_3+d_1)}}{(a-1)}((a-1)\partial_x+(-a-1)\partial_y)$

$$0=\partial_t(\eta_{22})(t,x,y)-\eta_{12}(t,x,y)y+\partial_y(\eta_{21})(t,x,y)ya+\partial_y(\eta_{22})(t,x,y)yx+\partial_y(\eta_{22})(t,x,y)yd_2$$
$$+\partial_x(\eta_{21})(t,x,y)x+\partial_x(\eta_{22})(t,x,y)xcy+\partial_x(\eta_{22})(t,x,y)xd_1-\eta_{21}(t,x,y)a-\eta_{22}(t,x,y)x$$
$$-\eta_{22}(t,x,y)d_2-\eta_{32}(t,x,y)ya+2\eta_{22}(t,x,y)bx+2\eta_{22}(t,x,y)y+2\eta_{22}(t,x,y)d_3,$$

$$0=\partial_x(\eta_{20})(t,x,y)x-\eta_{21}(t,x,y)y+\partial_x(\eta_{21})(t,x,y)xcy+\partial_x(\eta_{21})(t,x,y)xd_1-\eta_{31}(t,x,y)ya$$
$$+\partial_y(\eta_{20})(t,x,y)ya+\partial_y(\eta_{21})(t,x,y)yx+\partial_y(\eta_{21})(t,x,y)yd_2+\partial_t(\eta_{21})(t,x,y)-\eta_{20}(t,x,y)a$$
$$-\eta_{21}(t,x,y)x-\eta_{21}(t,x,y)d_2+\eta_{21}(t,x,y)bx+\eta_{21}(t,x,y)y+\eta_{21}(t,x,y)d_3,$$

$$0=\partial_x(\eta_{20})(t,x,y)xcy-\eta_{10}(t,x,y)y+\partial_x(\eta_{20})(t,x,y)xd_1+\partial_t(\eta_{20})(t,x,y)-\eta_{20}(t,x,y)x$$
$$-\eta_{20}(t,x,y)d_2+\partial_y(\eta_{20})(t,x,y)yx+\partial_y(\eta_{20})(t,x,y)yd_2-\eta_{30}(t,x,y)ya,$$

$$0=\partial_x(\eta_{32})(t,x,y)x-\eta_{12}(t,x,y)b-\eta_{22}(t,x,y)+\partial_y(\eta_{32})(t,x,y)ya,$$

$$0=\partial_x(\eta_{31})(t,x,y)x-\eta_{11}(t,x,y)b+\partial_x(\eta_{32})(t,x,y)xcy+\partial_x(\eta_{32})(t,x,y)xd_1+\partial_t(\eta_{32})(t,x,y)$$
$$+\partial_y(\eta_{31})(t,x,y)ya+\partial_y(\eta_{32})(t,x,y)yx+\partial_y(\eta_{32})(t,x,y)yd_2-\eta_{21}(t,x,y)+\eta_{32}(t,x,y)bx$$
$$+\eta_{32}(t,x,y)y+\eta_{32}(t,x,y)d_3,$$

TABLE II. Invariants of the L–V system ($a\neq 1$).

N°	Parameters	Invariants
1	$a,b,c,\ d_1=d_2=d_3$	
2	$a,b,c=-\frac{1}{ab},\ d_1,d_2,d_3$	$I_1=zx^{ab}y^{-b}e^{-t(d_3-d_2b+d_1ab)}$
3	$a,b,c=-\frac{1}{ab}$ $d_1=d_2=d_3$	$I_1=-\frac{(y-az+bxa)e^{-d_1t}}{a}$ $I_2=\frac{x^{-ab}y^be^{(ab+1-b)d_1t}}{z}$
4	$a=1-\frac{1}{b},b,\ c=-\frac{1}{ab}$ $d_1=d_2=d_3$	$I_1=\frac{x^{1-b}y^b}{z}$ $I_2=-\frac{e^{-d_1t}(xb^2+by-bx-bz+z)}{b-1}$
5	$a=-1,\ b=c=1$ $d_1=d_2=d_3$	$I_1=-e^{-d_1t}(x-z-y)$ $I_2=\frac{ze^{d_1t}}{xy}$
6	$a,\ b=1,\ c=0,\ d_1=d_2=d_3$	$I_1=\frac{x^{-a}yae^{d_1at}}{za-y}$
7	$a=-1-\frac{1}{b},b,\ c=\frac{1}{ab},\ d_1=d_2=d_3$	
8	$a,\ b=1,\ c=0,\ d_1,d_2,d_3$	
9	$a,b=\frac{(a+1)}{(a-1)},\ c=-\frac{(a-1)}{(a+1)},\ d_1=d_2,d_3$	

$$0=\partial_x(\eta_{30})(t,x,y)x-\eta_{10}(t,x,y)b+\partial_x(\eta_{31})(t,x,y)xcy+\partial_x(\eta_{31})(t,x,y)xd_1+\partial_t(\eta_{31})(t,x,y)$$
$$+\partial_y(\eta_{30})(t,x,y)ya+\partial_y(\eta_{31})(t,x,y)yx+\partial_y(\eta_{31})(t,x,y)yd_2-\eta_{20}(t,x,y),$$
$$0=\partial_x(\eta_{30})(t,x,y)xcy+\partial_x(\eta_{30})(t,x,y)xd_1+\partial_t(\eta_{30})(t,x,y)-\eta_{30}(t,x,y)bx-\eta_{30}(t,x,y)y$$
$$-\eta_{30}(t,x,y)d_3+\partial_y(\eta_{30})(t,x,y)yx+\partial_y(\eta_{30})(t,x,y)yd_2. \tag{9}$$

III. LIE SYMMETRIES AND INVARIANTS OF 3D LOTKA–VOLTERRA EQUATIONS

The power series solutions of equations (9), in the general case where $a\neq 1$, have the form:

$$\eta_{12}(t,x,y)=\sum_m r_{1,m}(t)x^{1-am}y^m,\quad \eta_{22}(t,x,y)=\sum_m r_{2,m}(t)y^{1+m}x^{-am},$$

$$\eta_{32}(t,x,y)=\sum_m x^{-am}y^m(r_{3,m}(t)+r_{4,m}(t)x+yr_{5,m}(t)),$$

$$\eta_{11}(t,x,y)=\sum_m x^{-am}y^m(r_{6,m}(t)x^2+r_{7,m}(t)xy+r_{8,m}(t)x+r_{9,m}(t)+r_{10,m}(t)y+r_{11,m}(t)y^2),$$

TABLE III. Lie symmetry vector fields of the L–V system ($a=1$).

Nº	Parameters	Symmetry vector fields
1	$a=1,b$ $d_1=d_2=d_3$ $c\neq 0$	$\mathbf{U}_1=x(cy+z+d_1)\partial_x+y(x+z+d_1)\partial_y+z(bx+y+d_1)\partial_z$ $\mathbf{U}_2=e^{-td_1}((xcy+xz)\partial_x+(yx+yz)\partial_y+(yz+zbx)\partial_z)$ $\mathbf{U}_3=-\frac{z(x-cy)^{\frac{bc+1}{c}}e^{-td_1}}{x^{\frac{1}{c}}y^b}\mathbf{U}_1$
2	$a=1,b=1$ $d_1=d_2=d_3$ $c\neq 0$	$\mathbf{U}_1=x(cy+z+d_1)\partial_x+y(x+z+d_1)\partial_y+z(x+y+d_1)\partial_z$ $\mathbf{U}_2=-\frac{e^{-td_1}}{x^{\frac{1}{c}}(c+1)y}((x-cy)^{\frac{1}{c}}z(x-cy)(x(cy+z+d_1)\partial_x$ $+y(x+z+d_1)\partial_y+z(x+y+d_1)\partial_z)+((x(yx+zcy+yz$ $-xz)\partial_x+y(2yx-y^2c+yz+zcy-xz)\partial_y+z(zcy-xz-y^2c$ $+y^2+2yx)\partial_z)d_1+x^{\frac{1}{c}}(y-z)(x-cy)((cy+z)x\partial_x$ $+y(x+z)\partial_y+z(x+y)\partial_z)))$ $\mathbf{U}_3=-\frac{1}{y}e^{-td_1}((y-z)(x-cy)((cy+z)x\partial_x+y(x+z)\partial_y$ $+z(x+y)\partial_z)+(x(yx-xz-y^2c^2-y^2c)\partial_x-y(y^2c+xcy$ $+xz-yx)\partial_y-z(xz-zcy-yx+2y^2c+xcy)\partial_z)d_1)$ $\mathbf{U}_4=e^{-td_1}((cy+z)x\partial_x+y(x+z)\partial_y+z(x+y)\partial_z)$ $\mathbf{U}_5=-\frac{(y-z)(x-cy)e^{-td_1}}{y}\mathbf{U}_4$ $\mathbf{U}_6=\frac{((y-z)x^{\frac{1}{c}}+z(x-cy)^{\frac{1}{c}})}{x^{\frac{1}{c}}(y-z)}\mathbf{U}_5$
3	$a=1,b=2$ $d_1=d_2=d_3$ $c\neq 0$	$\mathbf{U}_1=x(cy+z+d_1)\partial_x+y(x+z+d_1)\partial_y+z(2x+y+d_1)\partial_z$ $\mathbf{U}_2=e^{-td_1}((xcy+xz)\partial_x+(yx+yz)\partial_y+(2xz+yz)\partial_z)$ $\mathbf{U}_3=e^{-td_1}\left(\left(\frac{2xz(x-cy)^{\frac{1}{c}}(x-cy)^2(cy+z+d_1)}{x^{\frac{1}{c}}}\right.\right.$ $+(2y(-cy-y+2zc+z)x^2-2zx^3-(c+1)y^2(2zc-cy+z)x)d_1$ $\left.-2x(cy+z)(x-cy)(cy^2+y^2-zcy-yz+xz)\frac{1}{y^2}\right)\partial_x$ $+\left(\frac{2z(x-cy)^{\frac{1}{c}}(x-cy)^2(x+z+d_1)}{x^{\frac{1}{c}}}+(y(-3y-3cy+2z+4zc)x\right.$ $-2x^2z+2c(c+1)y^3-(c+1)(2c+1)zy^2)d_1-2(x-cy)$ $\left.\times(x+z)(cy^2+y^2-zcy-yz+xz)\right)\frac{1}{y}\partial_y$ $\times\left(z\left(\frac{2z(x-cy)^{\frac{1}{c}}(x-cy)^2(2x+y+d_1)}{x^{\frac{1}{c}}}+(2y(-2cy-2y+2zc+z)x\right.\right.$ $-2x^2z+(c+1)(2c-1)y^3-2c(c+1)zy^2)d_1-2(2x+y)$ $\left.\left.\left.\times(x-cy)(cy^2+y^2-zcy-yz+xz)\right)\right)\frac{1}{y^2}\partial_z\right)$

TABLE III. (*Continued.*)

N°	Parameters	Symmetry vector fields
		$\mathbf{U}_4=e^{-td_1}\Big(\Big(x((y(-cy-y+2zc+z)x-x^2z+c(c+1)^2y^3)d_1$ $-(cy+z)(x-cy)(cy^2+y^2-zcy-yz+xz))\frac{1}{y^2}\Big)\partial_x$ $+\Big((y(c^2y-y+z+2zc)x-x^2z+c(c+1)y^3)d_1$ $-(x-cy)(x+z)(cy^2+y^2-zcy-yz+xz)\frac{1}{y}\Big)\partial_y$ $+\Big(z\Big((y(2c^2y-y+cy+z+2zc)x-x^2z+2c(c+1)y^3-c(c+1)zy^2)d_1$ $-(2x+y)(x-cy)(cy^2+y^2-zcy-yz+xz))\frac{1}{y^2}\Big)\partial_z\Big)$ $\mathbf{U}_5=-\frac{(x-cy)(cy^2+y^2-zcy-yz+xz)e^{-td_1}}{y^2}\mathbf{U}_2$ $\mathbf{U}_6=\left(1-\frac{z(x-cy)^{\frac{1}{c}}(x-cy)}{(cy^2+y^2-zcy-yz+xz)x^{\frac{1}{c}}}\right)\mathbf{U}_5$
4	$a=1$ $c=-\frac{1}{b+1}$ $b\neq-1$ $d_1=d_2=d_3$	$\mathbf{U}_1=-\frac{x(-zb-z-bd_1-d_1+y)}{b+1}\partial_x+y(x+z+d_1)\partial_y+z(bx+y+d_1)\partial_z$ $\mathbf{U}_2=\frac{e^{-td_1}}{(y+bx+x)}((xd_1(b+1)(b(b+1)^2x^2+(b+1)(2b-1)yx$ $+(b-1)y^2+yz)-x(y-z-zb)(by^2+2yb^2x+2ybx-zyb$ $-zxb^2+bx^2-xz+b^3x^2+2b^2x^2-2zbx))\partial_x+(yd_1(b+1)$ $\times((b+1)^3x^2+(b+1)(2b+1)yx+yz+by^2)+y(b+1)(x+z)$ $\times(b(b+1)^2x^2+(b+1)(2yb-zb-z)x+by^2-zyb))\partial_y+(z(b+1)$ $\times(bx+y)(b(b+1)^2x^2+(b+1)(2yb-zb-z)x+by^2-zyb)$ $+z(b+1)d_1(2b(b+1)^2x^2+(b+1)$ $\times(4yb-zb-z+y)x+(2b+1)y^2-zyb))\partial_z)$ $\mathbf{U}_3=\frac{e^{-td_1}}{(y+bx+x)}((xd_1(b(b+1)^3x^2+(b+1)(2yb^2+yb-zb-z)x$ $+b^2y^2)-x(y-z-zb)(b(b+1)^2x^2+(b+1)(2yb-zb-z)x$ $+by^2-zyb))\partial_x+(y(b+1)(x+z)(b(b+1)^2x^2+(b+1)(2yb$ $-zb-z)x+by^2-zyb)+y(b+1)d_1(b(b+2)(b+1)x^2+(2yb^2$ $-zb+3yb-z)x+by^2))\partial_y+(z(b+1)(bx+y)(b(b+1)^2x^2$ $+(b+1)(2yb-zb-z)x+by^2-zyb)+z(b+1)d_1(b(b+1)$ $\times(2b+1)x^2+(3yb-zb^2-2zb+4yb^2-z)x-zyb+2by^2))\partial_z)$ $\mathbf{U}_4=e^{-td_1}((zbx-yx+xz)\partial_x+(zyb+yz+ybx+yx)\partial_y$ $+(zyb+yz+zxb^2+zbx)\partial_z)$ $\mathbf{U}_5=\frac{(by^2+2yb^2x+2ybx-zyb-zxb^2+bx^2-xz+b^3x^2+2b^2x^2-2zbx)e^{-td_1}}{y+bx+x}\mathbf{U}_4$
5	$a=1$ $b=0$ $c\neq0$ $d_1=d_2=d_3$	$\mathbf{U}_1=x(cy+z+d_1)\partial_x+y(x+z+d_1)\partial_y+z(y+d_1)\partial_z$ $\mathbf{U}_2=-\frac{e^{-td_1}}{x^{\frac{1}{c}}}(x^{\frac{1}{c}}d_1(x(cy+z)\partial_x+y(z+x)\partial_y+zy\partial_z)$ $-z(x-cy)^{\frac{1}{c}}(x(cy+z+d_1)\partial_x+y(x+z+d_1)\partial_y+z(y+d_1)\partial_z))$ $\mathbf{U}_3=\frac{e^{-2td_1}}{x^{\frac{1}{c}}}(x-cy)^{\frac{1}{c}}z(x(cy+z)\partial_x+y(z+x)\partial_y+zy\partial_z)$ $\mathbf{U}_4=\frac{e^{td_1}x^{\frac{1}{c}}}{(x-cy)^{\frac{1}{c}}z}\mathbf{U}_3$

TABLE III. (*Continued.*)

N°	Parameters	Symmetry vector fields
6	$a=1$ $b=1$ $c=0$ $d_1=d_2=d_3$	$\mathbf{U}_1=x(z+d_1)\partial_x+y(x+z+d_1)\partial_y+z(x+y+d_1)\partial_z$ $\mathbf{U}_2=e^{-2d_1t}\Big(-x\Big(yxz-z^2x+yzd_1+xzd_1e^{-\frac{y}{x}}+yd_1x-xzd_1+xz^2+\frac{e^{-\frac{y}{x}}}{y}\Big)\partial_x$ $-(yzd_1+xzd_1e^{-\frac{y}{x}}-z^2x-x^2z+2yd_1x+yxz-xzd_1+zx^2e^{-\frac{y}{x}}+xz^2e^{-\frac{y}{x}}+yx^2)\partial_y$ $-\frac{z}{y}(2yd_1x+y^2d_1-yxz-x^2z+zx^2e^{-\frac{y}{x}}+y^2x+xzd_1e^{-\frac{y}{x}}+xze^{-\frac{y}{x}}y+yx^2-xzd_1)\partial_z\Big)$ $\mathbf{U}_3=\frac{e^{-2d_1t}x(y-z)}{y}(-(xz\partial_x+y(z+x)\partial_y+z(y+x)\partial_z))$ $\mathbf{U}_4=e^{-d_1t}(xz\partial_x+y(z+x)\partial_y+z(y+x)\partial_z)$ $\mathbf{U}_5=-\frac{e^{-d_1t}x(y-z)}{y}\mathbf{U}_1,\quad \mathbf{U}_6=\frac{(y+ze^{-\frac{y}{x}}-z)}{(y-z)}\mathbf{U}_3$
7	$a=1$ $b=2$ $c=\frac{1}{2}$ $d_1=d_2=d_3$	$\mathbf{U}_1=y(x+z+d_1)\partial_y-\frac{x(y-2z-2d_1)}{2}\partial_x+z(2x+y+d_1)\partial_z$ $\mathbf{U}_2=\frac{z^2e^{-2d_1t}x^5}{y^3(y+2x)}(\partial_x-2\partial_y)$ $\mathbf{U}_3=\frac{(y+2x)(y^2+2xz-yz)}{2zx^2}\mathbf{U}_2$ $\mathbf{U}_4=e^{-2d_1t}\Big(\frac{x(y+2x)(4xz^2-6zyx+zy^2-y^3)}{4y^2}\partial_x+\frac{(y+2x)}{2y}(6zx^2+y^2x$ $-3zyx-zy^2+z^2y)\partial_y+\frac{z(zy^3-4xy^3+2zy^2x+4zyx^2-4y^2x^2-y^4+8zx^3)}{2y^2}\partial_z\Big)$ $\mathbf{U}_5=e^{-2d_1t}\Big(-\frac{x(2z-y)(2x-z+y)}{2}\partial_x-y(z+x)(2x-z+y)\partial_y-z(y+2x)(2x-z+y)\partial_z\Big)$ $\mathbf{U}_6=e^{-2d_1t}\Big(\frac{y(y+2x)(2x-z+y)^2}{2x}\partial_x-\frac{y(y+2x)(x-y)(2x-z+y)^2}{x^2}\partial_y$ $+\frac{y^2(y+2x)(2x-z+y)^2}{x^2}\partial_z\Big)$ $\mathbf{U}_7=\frac{ze^{-d_1t}x^2}{y^2}\mathbf{U}_1,\quad \mathbf{U}_8=-\frac{e^{d_1t}}{(2x-z+y)}\mathbf{U}_5$ $\mathbf{U}_9=e^{-d_1t}\Big(\frac{x(y^2+4xz)}{4y}\partial_x-\frac{(y^2x+4zx^2-zy^2-2zyx)}{2y}\partial_y+\Big(\frac{yz}{2}+xz\Big)\partial_z\Big)$ $\mathbf{U}_{10}=-\frac{2e^{d_1t}}{2x-z+y}\mathbf{U}_6$
8	$a=1$ $b=-2$ $c=\frac{1}{2}$ $d_1=d_2=d_3$	$\mathbf{U}_1=\frac{x(2z+y+2d_1)}{2}\partial_x+y(x+z+d_1)\partial_y-z(2x-y-d_1)\partial_z$ $\mathbf{U}_2=\frac{ze^{-2d_1t}y^2}{2x^2}((2z+y)x\partial_x+2y(z+x)\partial_y-2z(2x-y)\partial_z)$ $\mathbf{U}_3=\frac{e^{-d_1t}}{4x}(y(4x^2+yz+2xz)(\partial_x+2\partial_y))$ $\mathbf{U}_4=\frac{e^{-d_1t}}{2}((2x-y+z)((2z+y)x\partial_x+2y(z+x)\partial_y$ $-2z(2x-y)\partial_z)+d_1(x(4x-3y)\partial_x+2y(x-y)\partial_y$ $+2z(z+4x-2y)\partial_z))$ $\mathbf{U}_5=\frac{(2x-y+z)e^{-2d_1t}}{4y^2}((32x^4+(16z-24y)x^3-4zx^2y+3xy^3)\partial_x$ $+(16yx^3-8y(2y-z)x^2+2y^2(3y-2z)x+2zy^3)\partial_y$ $+(32(2y-z)x^3-64x^4-16y(y-z)x^2-4zxy^2+2zy^3)\partial_z)$

TABLE III. (*Continued.*)

N°	Parameters	Symmetry vector fields
		$\mathbf{U}_6=\frac{e^{-2d_1t}y}{4x^2}((16x^4+(16z-8y)x^3+4z^2x^2-3zxy^2)\partial_x$ $+(32x^4-16(y-2z)x^3+8z^2x^2-2yz(3y-2z)x-2z^2y^2)\partial_y$ $+2z^2y(2x-y)\partial_z)$ $\mathbf{U}_7=-\frac{e^{-d_1t}}{4x}(((2y-4z)x^2+2zxy+zy^2)\partial_x+2y(2x^2+yz)\partial_y$ $+4z(2x-y)x\partial_z)$ $\mathbf{U}_8=-\frac{e^{-d_1t}}{8y^2x}((64x^5-16(3y-2z)x^4-8zx^3y+2y^3x^2-2zy^3x$ $-zy^4)\partial_x+(32x^4y-16y(2y-z)x^3+4y^2(y-2z)x^2-2zy^4)\partial_y$ $-4x(2x-y)(16x^3+(8z-8y)x^2+zy^2)\partial_z)$ $\mathbf{U}_9=-\frac{x(2x-y)}{y^2}((y+4x)\partial_x+2y\partial_y-8x\partial_z),\quad \mathbf{U}_{10}=\frac{zy^2e^{-d_1t}}{x^2}\mathbf{U}_1$ $\mathbf{U}_{11}=\frac{x^2(2x-y+z)}{zy^2}\mathbf{U}_2,\quad \mathbf{U}_{12}=\frac{2zy^2}{e^{d_1t}x^2}\mathbf{U}_3$ $\mathbf{U}_{13}=\frac{16y^4z^2e^{-d_1t}}{x^2(4x^2+yz+2xz)(2x-y)}\mathbf{U}_3,\quad \mathbf{U}_{14}=\frac{8e^{d_1t}x^2}{(2x-y)(4x^2+yz+2xz)}\mathbf{U}_3$
9	$a=1,b$ $c\neq 0,d_1=d_2$ $d_3=0$	$\mathbf{U}_1=x(cy+z+d_1)\partial_x+y(x+z+d_1)\partial_y+z(bx+y)\partial_z$ $\mathbf{U}_2=\frac{z(x-cy)^{(1+cb)/c}}{x^{1/c}y^b}\mathbf{U}_1$
10	$a=1$ $b=1$ $d_1=d_2$ $d_3=0$	$\mathbf{U}_1=x(cy+z+d_1)\partial_x+y(x+z+d_1)\partial_y+z(x+y)\partial_z$ $\mathbf{U}_2=\frac{(d_1cy-d_1y\ln(\lvert x\rvert)+d_1y\ln(\lvert y\rvert)+d_1y\ln(\lvert c\rvert)-yx+xz+cy^2-zcy)}{y}\mathbf{U}_1$ $\mathbf{U}_3=-\frac{\left(-x^{\frac{c-1}{c}}z(x-cy)^{\frac{1}{c}}-d_1cy-d_1y+\frac{zc}{x^{\frac{1}{c}}}(x-cy)^{\frac{1}{c}}y\right)}{y}\mathbf{U}_1$
11	$a=1$ $b\neq -1$ $c=-\frac{1}{b+1}$ $d_1=d_2$ $d_3=0$	$\mathbf{U}_1=\frac{x(d_1+d_1b-y+zb+z)\partial_x}{b+1}+y(x+z+d_1)\partial_y+z(bx+y)\partial_z$ $\mathbf{U}_2=\left(bd_1\ln\left(\frac{\lvert y\rvert}{\lvert x\rvert}\right)-bx-\frac{yb}{b+1}+\frac{zb}{b+1}+\frac{xz}{bx+y+x}\right)\mathbf{U}_1$

$$\eta_{21}(t,x,y)=\sum_m x^{-am}y^m(r_{12,m}(t)x^2+r_{13,m}(t)xy+r_{14,m}(t)x+r_{15,m}(t)+r_{16,m}(t)y+r_{17,m}(t)y^2),$$

$$\eta_{31}(t,x,y)=\sum_m x^{-am}y^m(r_{18,m}(t)x^2+r_{19,m}(t)xy+r_{20,m}(t)x+r_{21,m}(t)+r_{22,m}(t)y+r_{23,m}(t)y^2),$$

$$\eta_{10}(t,x,y)=\sum_m x^{-am}y^m(r_{24,m}(t)x^3+r_{25,m}(t)x^2y+r_{26,m}(t)xy^2+r_{27,m}(t)y^3+r_{28,m}(t)x^2$$
$$+r_{29,m}(t)xy+r_{30,m}(t)x+r_{31,m}(t)+r_{32,m}(t)y+r_{33,m}(t)y^2),$$

$$\eta_{20}(t,x,y)=\sum_m x^{-am}y^m(r_{34,m}(t)x^3+r_{35,m}(t)x^2y+r_{36,m}(t)xy^2+r_{37,m}(t)y^3+r_{38,m}(t)x^2$$
$$+r_{39,m}(t)xy+r_{40,m}(t)x+r_{41,m}(t)+r_{42,m}(t)y+r_{43,m}(t)y^2),$$

$$\eta_{30}(t,x,y)=\sum_m x^{-am}y^m(r_{44,m}(t)x^3+r_{45,m}(t)x^2y+r_{46,m}(t)xy^2+r_{47,m}(t)y^3+r_{48,m}(t)x^2$$
$$+r_{49,m}(t)xy+r_{50,m}(t)x+r_{51,m}(t)+r_{52,m}(t)y+r_{53,m}(t)y^2), \tag{10}$$

where the $r_{i,m}(t)$ must satisfy 95 ordinary differential equations. Solving this system we find several cases with nontrivial Lie symmetry vector fields (Table I). In Table II we list the invariants that we can find by using the method discussed in Sec. II. Let us stress that this system is invariant under a simultaneous cyclic permutation of a,b,c, of d_1,d_2,d_3 and x,y,z. Therefore, if we know a symmetry vector field (first integral) for some values of the parameters it is transformed into a new symmetry vector field (first integral) of transformed system.

If $a=1$, then η_{ij} can be written in the following form:

$$\eta_{12}(t,x,y)=\sum_m r_{1,m}(t)x^{1-m}y^m, \quad \eta_{22}(t,x,y)=\sum_m r_{2,m}(t)x^{1-m}y^m,$$

$$\eta_{32}(t,x,y)=\sum_m x^{-m}y^m(r_{3,m}(t)+r_{4,m}(t)x),$$

$$\eta_{11}(t,x,y)=\sum_m x^{-m}y^m(r_{5,m}(t)x^2+r_{6,m}(t)x+r_{7,m}(t)),$$

$$\eta_{21}(t,x,y)=\sum_m x^{-m}y^m(r_{8,m}(t)x^2+r_{9,m}(t)x+r_{10,m}(t)),$$

$$\eta_{31}(t,x,y)=\sum_m x^{-m}y^m(r_{11,m}(t)x^2+r_{12,m}(t)x+r_{13,m}(t)),$$

$$\eta_{10}(t,x,y)=\sum_m x^{-m}y^m(r_{14,m}(t)x^3+r_{15,m}(t)x^2+r_{16,m}(t)x+r_{17,m}(t)),$$

$$\eta_{20}(t,x,y)=\sum_m x^{-m}y^m(r_{18,m}(t)x^3+r_{19,m}(t)x^2+r_{20,m}(t)x+r_{21,m}(t)),$$

$$\eta_{30}(t,x,y)=\sum_m x^{-m}y^m(r_{22,m}(t)x^3+r_{23,m}(t)x^2+r_{24,m}(t)x+r_{25,m}(t)), \tag{11}$$

where the $r_{i,m}(t)$ must satisfy 35 ordinary differential equations. We do not claim to have found all solutions of this system. The symmetry vector fields and invariants for several representative cases are showed in Table III and Table IV, respectively.

TABLE IV. Invariants of the L–V system ($a=1$).

N°	Parameters	Invariants
1	$a=1,b$ $d_1=d_2=d_3$ $c\neq 0$	$I_1=\frac{z(x-cy)^{\frac{bc+1}{c}}e^{-td_1}}{x^{\frac{1}{c}}y^b}$
2	$a=1,b=1$ $d_1=d_2=d_3$ $c\neq 0$	$I_1=\frac{z(x-cy)^{\frac{1}{c}}}{(z-y)x^{\frac{1}{c}}}$ $I_2=\frac{(y-z)(x-cy)e^{-td_1}}{y}$
3	$a=1,b=2$ $d_1=d_2=d_3;\ c\neq 0$	$I_1=-\frac{(x-cy)(cy^2+y^2-zcy-yz+xz)e^{-td_1}}{y^2}$ $I_2=\frac{z(x-cy)^{\frac{1}{c}}(x-cy)}{(cy^2+y^2-zcy-yz+xz)x^{\frac{1}{c}}}$
4	$a=1,\quad c=-\frac{1}{b+1}$ $b\neq -1,\quad d_1=d_2=d_3$	$I_1=-\frac{(by^2+2yb^2x+2ybx-zyb-zxb^2+bx^2-xz+b^3x^2+2b^2x^2-2zbx)e^{-td_1}}{y+bx+x}$
5	$a=1,\quad b=0,\quad c\neq 0$ $d_1=d_2=d_3$	$I_1=\frac{x^{\frac{1}{c}}e^{td_1}}{z(x-cy)^{\frac{1}{c}}}$
6	$a=1$ $b=1$ $c=0$ $d_1=d_2=d_3$	$I_1=-\frac{e^{-d_1t}x(y-z)}{y}$ $I_2=\frac{(ze^{-\frac{y}{x}})}{(y-z)}$
7	$a=1$ $b=2$ $c=-\frac{1}{2},\ d_1=d_2=d_3$	$I_1=-\frac{e^{-d_1t}(2x-z+y)}{2}$ $I_2=\frac{(y+2x)(y^2+2xz-yz)}{2zx^2}$ $I_3=\frac{ze^{-d_1t}x^2}{y^2}$
8	$a=1$ $b=-2$ $c=\frac{1}{2},\ d_1=d_2=d_3$	$I_1=\frac{x^2(2x-y+z)}{zy^2}$ $I_2=\frac{(2x-y)(4x^2+yz+2xz)}{8zy^2}$ $I_3=\frac{2zy^2e^{-d_1t}}{x^2}$
9	$a=1,b,\quad c\neq 0$ $d_1=d_2,\quad d_3=0$	$I_1=\frac{z(x-cy)^{\frac{1+cb}{c}}}{x^{\frac{1}{c}}y^b}$
10	$a=1,\quad b=1$ $d_1=d_2,\quad d_3=0$	$I_1=d_1\ln(\lvert y\rvert)-d_1\ln(\lvert x\rvert)-\frac{(y-z)(x-cy)}{y}$ $I_2=\frac{z(x-cy)^{\frac{1}{c}}(x-cy)}{yx^{\frac{1}{c}}}$
11	$a=1,\quad c=-\frac{1}{b+1}$ $d_1=d_2,\quad d_3=0$	$I_1=bd_1\ln\left(\frac{\lvert y\rvert}{\lvert x\rvert}\right)-bx-\frac{yb}{b+1}+\frac{zb}{b+1}+\frac{xz}{bx+y+x}y$

The invariant I_1 in the case (9) of Table IV is a generalization of an invariant found by Grammaticos *et al.*[35] for the particular case where $b=1$. We note that, as remarked in Ref. 35 the Lotka–Volterra system possesses an invariant measure density, i.e., there exists a function $M(x,y,z)$ satisfying

$$\partial_x(M\dot{x})+\partial_y(M\dot{y})+\partial(M\dot{z})=0, \tag{12}$$

where $\dot{x},\dot{y}$ and $\dot{z}$ are given by the dynamical equations (7). The Jacobi last multiplier method allows us, in this case, to find a second invariant I_2 starting from the initial one I_1. This method applied in the case (9) leads to the invariant:

$$I_2=(\ln(x)-\ln(y))d_1+x-cy+I_1\int^{x/y}(u-c)^{-(1+cb)/c}u^{(1-c)/c}du. \tag{13}$$

Therefore, we get a new integrable case of the 3D Lotka–Volterra system. We found also the following expressions for the case $a=1$, $c \neq 0$, $d_1=d_2=d_3$ and $b=integer$:

$$I_1=\frac{(x-cy)e^{-d_1t}f(b)}{y^b},$$

$$I_2=\frac{(x-cy)^{1/c}z(x-cy)^{b-1}}{x^{1/c}f(b)}, \tag{14}$$

where $f(b)$ is given by

$$f(b)=(-1)^b\prod_{j=1}^{b-1}(jc+1)y^b+\frac{z(\Sigma_{j_2=1}^{b}(-y)^{b-j_2}x^{j_2-1}\Pi_{j_1=j_2}^{b}(cj_1+1)\Pi_{j=1}^{j_2-1}(b-j))}{bc+1}. \tag{15}$$

We proved by direct computation that I_1 and I_2 in (15) are actually invariants for the following values of b: 1, 2, 3, and 4. We make the conjecture that I_1 and I_2 in (15) are in fact invariants for any integer value of parameter b.

IV. CONCLUSION

We studied here the Lie symmetries of the 3D Lotka–Volterra system and we found several parameter values at which one or more first integrals exist. The main purpose of this paper was to show that the determination of rational and transcendental Lie symmetry transformations can be useful for identification of nonpolynomial first integrals for systems of ordinary differential equations.

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