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AN INTRODUCTION TO LIE GROUPS AND LIE ALGEBRAS, WITH APPLICATIONS. II: THE BASIC METHODS AND RESULTS OF REPRESENTATION THEORY*

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CONTENTS

1. Introduction and review.

1.1. Motivation. This paper, together with a third paper to follow, is a continuation of a survey [4] of Lie groups, Lie algebras, and their applications. In the previous paper, we were concerned primarily with the classification of Lie algebras and, in particular, of semisimple Lie algebras. Much of the applied literature makes use of representation theory, and in view of this we are writing the present paper to carry the exposition further to the point of classifying the finite-dimensional representations of semisimple Lie algebras. As in our first paper, our primary intention is to introduce the concepts used in the current applied literature in a form accessible to the nonspecialist. We have tried to indicate some of the many directions in which research has been carried out, but our purpose is not a comprehensive survey of all known applications. Much major research has had to be left out to keep the discussion within manageable dimensions. On the other hand, we have attempted to provide in some instances enough details that our paper could be directly useful. The reader will realize that in order to pursue any particular application seriously, one must go beyond the material presented here. Our bibliography will serve as a guide for collateral reading and amplification of points presented summarily here. So much has been written that a complete bibliography is out of the question. We apologize to those authors whose work we have left out.

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Although the representation theory of Lie groups and Lie algebras has long been a subject of intensive research activity in pure mathematics, a more widespread appreciation of the thoroughly practical value of the subject has become evident only rather recently. With this appreciation has come a brand new assault on the subject by applied mathematicians, having as its aim the further development of practical techniques, such as those for computing the Clebsch-Gordan and Racah coefficients of the classical Lie groups [6], [9]. In this effort, the use of automatic electronic digital computers has been essential. Concurrently, and largely independently, extensive simplifications and generalizations of the theory have been made by pure mathematicians [12], [13], [23], [24]. In view of this rapid development of the subject on all levels, it has become evident that there is a need to bridge the growing gap between pure and applied mathematicians studying the subject and to accelerate the diffusion of these important mathematical ideas to a more general public including applied mathematicians, physical scientists and research engineers. This has provided the motivation for writing these papers. We have tried to provide basic facts and a general orientation rather than an encyclopedic treatise.

In the Introduction of the present paper, we seek to orient the reader and to recall some of the ideas presented in our first paper. Following this, the main body of the paper is devoted to a discussion of the concepts of module and representation, the various operations for combining them, and their classification. In the third paper to follow, we give a detailed discussion of computational methods, including the explicit construction of the irreducible representations, and also deal with a few selected applications, chosen to illustrate the great diversity of possible topics. In that part, we have not attempted to cover any of these topics in depth, being content to provide a brief introduction to each subject to orient the nonspecialist.

1.2. Prerequisites and review of the structure of Lie algebras. While we generally assume familiarity with the material presented in the first paper, it seems appropriate to give here a brief sketch of some of the more important points, and especially to recall some of the main facts about the structure theory of semisimple Lie algebras.

In the previous paper we began by discussing the functorial relationship between Lie groups and Lie algebras, showing how various results about Lie groups could be translated into related results about their corresponding Lie algebras, and vice versa. In particular, we discussed the fact that analytic Lie group homomorphisms induce homomorphisms having corresponding kernels of the corresponding Lie algebras. We also discussed the fact that there is a unique simply connected Lie group for each Lie algebra, and that all other connected Lie groups corresponding to a given Lie algebra are quotients of this universal covering group modulo discrete normal subgroups. Having thus, in a sense, reduced Lie group theory to the theory of Lie algebras, we then proceeded to analyze the general structure of Lie algebras.

A Lie algebra L was defined to be a vector space in which there is defined a product rule ($L \times L \to L$) denoted by $[x, y]$ which satisfies three axioms: linearity in x and y, anticommutativity: $[x, y] = -[y, x]$, and the Jacobi identity: $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$. The Jacobi identity plays for Lie algebras the same role as the associative law plays for Lie groups, and the one law can be derived from the other. A subalgebra S of a Lie algebra L is defined to be a subspace which is closed under the bracket operation: $[S, S] \subset S$. An ideal S of a Lie algebra L is a subalgebra which satisfies $[L, S] \subset S$. A Lie algebra is Abelian if $[L, L] = 0$. A simple Lie algebra was defined as a Lie algebra L whose only ideals are the trivial ones 0 and L. A semisimple Lie algebra is a non-Abelian Lie algebra which also has no Abelian ideals other than 0. There are four main series of simple Lie algebras over the complex numbers corresponding to the unitary, orthogonal and symplectic Lie groups. The simple Lie algebra A_l corresponds to the Lie group $SU(l + 1)$, B_l corresponds to $SO(2l + 1)$, C_l to $Sp(l)$, and D_l to $SO(2l)$. The Levi decomposition theorem stated that any Lie algebra is the direct sum, as vector spaces, of its radical (the unique maximal solvable ideal) and a semisimple subalgebra, the latter in turn being a direct sum of its simple ideals. Thus it became clear that the structure of simple Lie algebras, besides being interesting in its own right, also plays a key role in the general theory of Lie algebras, and that for semisimple Lie algebras (those with zero radical), the structure of the simple ideals completely determines the structure of the algebra.

We now review briefly the pertinent facts about the structure of semisimple Lie algebras. Every semisimple Lie algebra L over the complex numbers has a Cartan subalgebra H. This is a maximal Abelian subalgebra having the property that, for every h in H, we have

$$(\mathrm{ad}\, h)^n x = [h, [h, \cdots [h, x] \cdots]] = 0$$

only if x is also in H. While the Cartan subalgebra is not unique, they all have the same dimension l, called the rank of the semisimple Lie algebra L. Given a particular Cartan subalgebra H, a root α is defined to be a linear form on H having the property that there exists e_α in L such that $[h, e_\alpha] = \alpha(h)\, e_\alpha$ for all h in H. Thus we see that $\alpha(h)$ is always an eigenvalue of the linear transformation ad h defined on L by $(\mathrm{ad}\, h)x = [h, x]$. For a semisimple Lie algebra over the complex numbers, the root space of α, consisting of all the vectors e_α which satisfy $(\mathrm{ad}\, h)e_\alpha = \alpha(h)e_\alpha$ for all $h \in H$, is a one-dimensional space $L_H{}^\alpha$ for each nonzero root α. The whole Lie algebra in this case is the direct sum of the Cartan subalgebra and all the root spaces corresponding to the nonzero roots.

The Killing form is a bilinear form defined on an arbitrary Lie algebra by $(x, y) = \mathrm{Tr}\,(\mathrm{ad}\, x)\,(\mathrm{ad}\, y)$, where Tr denotes the trace and ad x the linear transformation defined previously. The Killing form is nonsingular if and only if the Lie algebra is semisimple. The Killing form of a semisimple Lie algebra L is always nonsingular on any Cartan subalgebra, so it can be used to identify the Cartan subalgebra H with its dual H^* by letting every form α in H^* correspond to the unique vector h_α in H such that $\alpha(h) = (h_\alpha, h)$ for all h in H. With this identification, the bilinear form can equally well be regarded as being given on the dual space H^* with $(\alpha, \beta) = (h_\alpha, h_\beta)$ for all α, β in H^*. The extension of this bilinear form turns out to be real and positive definite on the space $H_R{}^*$ generated over the real numbers by the nonzero roots, thereby introducing an inner product, and hence a Euclidean geometry in that space.

1.3. An important example, the rotation group. While discussing representations in general, it seems appropriate to give a fairly thorough discussion of an important example, the ordinary rotation group $SO(3, R)$, in order to provide a concrete framework which can be kept in mind during the more general discussion. In our first paper, it was shown that the Lie algebra $so(3, R)$ of the rotation group is isomorphic to ordinary real three-dimensional space with the usual vector cross product as the Lie multiplication. Moreover, the simply connected group associated with this Lie algebra was shown to be the special unitary group $SU(2)$. The rotation group $SO(3, R)$ is isomorphic to $SU(2)$ modulo the discrete subgroup consisting of the identity and its negative, $\pm I$. The relationship between the groups $SO(3, R)$ and $SU(2)$ can be made explicit in the following manner. The rotations of $SO(3, R)$ correspond in a one-to-one fashion with the rotations of a sphere about its center. If the points of the sphere other than the "north pole" are identified with the complex plane by the usual projection from the "north pole" onto a plane tangent to the sphere at the "south pole," each rotation of the sphere induces a corresponding mapping of the plane onto itself—a complex function of a complex variable. The complex functions found in this manner obviously form a group isomorphic to the rotation group under composition and they are easily seen to be linear fractional transformations of the form

$$f(z) = \frac{\alpha z + \beta}{\gamma z + \delta}$$

with $\alpha\delta - \gamma\beta \neq 0$. Obviously $f(z)$ is unchanged if all the coefficients $\alpha, \beta, \gamma, \delta$ are multiplied by a common factor, so by introducing an appropriate factor we can always make $\alpha\delta - \gamma\beta = 1$. Since only the square of a common factor enters into the expression $\alpha\delta - \gamma\beta$, however, its negative will serve equally well to satisfy the condition $\alpha\delta - \gamma\beta = 1$. To each such $f(z)$, therefore, there correspond two matrices of coefficients,

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$$

and its negative, both with determinant 1. It is easily shown that the composition of two linear fractional transformations is a linear fractional transformation, and that the matrix of the composition is the product of the matrices of the composed transformations. We thus have a homomorphism of the group, $SL(2, C)$, of all 2×2 matrices of determinant 1 onto the group of linear fractional transformations of the complex plane, and the kernel of this homomorphism consists of the identity matrix and its negative, $\pm I$. Not every fractional linear transformation, however, corresponds to a rotation of a sphere. A straightforward computation shows that a fractional linear transformation will correspond to a rotation of the sphere if and only if its matrix is unitary; that is, it must have the form

$$\begin{bmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{bmatrix},$$

where the bars denote complex conjugation. Since the 2×2 unitary matrices with determinant 1 are precisely the subgroup $SU(2)$ of $SL(2, C)$, we have the desired homomorphism of $SU(2)$ onto $SO(3, R)$ with kernel $\pm I$. Writing $\alpha = x_1 + ix_2$, $\beta = x_3 + ix_4$, it is clear that the condition $\alpha\bar{\alpha} + \beta\bar{\beta} = 1$ is precisely the condition that the vector $x = (x_1, x_2, x_3, x_4)$ lie on the unit sphere in 4-space. Thus the group $SU(2)$ is homeomorphic with that sphere (see [8]).

A representation of a group is a homomorphism onto a group of linear transformations of a vector space. The representations of $SO(3, R)$ are identical with those representations of $SU(2)$ whose kernels contain the negative of the identity, $-I$. There are no proper ideals in the Lie algebra $so(3, R) = su(2)$; such an ideal would be a proper subspace of 3-space, a line or plane passing through the origin in ordinary 3-dimensional space, such that the vector cross product of any vector in the subspace by any vector whatever lies in the subspace. Recalling that the cross product of two vectors is perpendicular to both (and is nonzero unless the two vectors are collinear or at least one is zero), it is easily seen that no line or plane can have this property. Since every closed normal subgroup of $SO(3, R)$ must have as its Lie algebra an ideal of $so(3, R)$, we can conclude that the only proper closed normal subgroups are those with trivial Lie algebras, namely, discrete subgroups. If a discrete subgroup is normal in $SO(3, R)$, however, and if g is one of its elements, then every element of the form $A^{-1}gA$ must also be in the discrete normal subgroup; that is, all the orthogonal similarity transformations of the rotation g must also be in the subgroup. But this is equivalent to saying that every rotation about any axis by an angle equal to that of the rotation g about its axis must be in the normal subgroup. By multiplying g by the inverse of an equivalent rotation about an axis which has been shifted a small amount, it is clear that we can obtain an element of the normal subgroup not the identity, but as close to the identity as we wish. (Formally, take A to be a very small rotation about an axis not that of g. Then $gA^{-1}g^{-1}A$ is close to the identity.) Thus no closed normal subgroup of $SO(3, R)$ can be discrete either, and we conclude that $SO(3, R)$ has no proper closed normal subgroups.

Since every finite-dimensional representation of $SO(3, R)$ is a homomorphism into the Lie group $GL(n, R)$ of all nonsingular linear transformations on a real n-dimensional vector space, for some n, its kernel must be a closed normal subgroup of $SO(3, R)$. By our previous remarks, the kernel must be all of $SO(3, R)$ or just the identity, and such a representation either maps all of $SO(3, R)$ onto the identity in $GL(n, R)$ (the trivial representation) or it is an isomorphism onto a subgroup of $GL(n, R)$. Moreover, any normal subgroup of $SU(2)$ would have as its image under the homomorphism of $SU(2)$ onto $SO(3, R)$

a normal subgroup of $SO(3, R)$, and thus the kernel $\{\pm I\}$ of the homomorphism is the only proper closed normal subgroup of $SU(2)$. Consequently, any finite-dimensional representation of $SU(2)$ which is not an isomorphism is induced by a representation of $SO(3, R)$.

Since the group $SU(2)$ is homeomorphic with the unit sphere in 4-space, it is clearly compact, and $SO(3, R)$, being a continuous image of $SU(2)$, is also compact. Compact Lie groups enjoy certain properties which simplify their representation theory substantially (see [23], [30]).

In general, when we speak of representations of a group, we admit the possibility that a representation may be a homomorphism of the given group to a group of linear transformations of an infinite-dimensional space. The theory of such infinite-dimensional representations is still a topic of active research interest and will not be discussed in detail in this paper. A survey of work on infinite-dimensional representations is given in [21], [22], [25]. It is usual, in discussing infinite-dimensional representations, to impose restrictions on the types of linear transformations which are allowed as representations. A specific type of topological structure is frequently demanded of the infinite-dimensional vector spaces, the most important case so far, for applications, being that in which the space is required to be a Hilbert space, or a subspace of a Hilbert space. The transformations considered are often required to behave suitably by being continuous, by preserving a metric, or by being otherwise well-behaved.

We recall that a Hilbert space is a complex vector space in which an inner product is defined and such that every Cauchy sequence of vectors in the space converges to a vector in the space [35]. If we have a representation of a compact group by linear transformations on a Hilbert space, then the inner product on the Hilbert space can always be redefined such that all the linear transformations are unitary, i.e., they are isometries and leave the inner product invariant (see [11, Chap. XI, §11] and [27, Chap. VI]). Thus only unitary representations, i.e., representations by unitary linear transformations, need be considered when one is dealing with compact groups. We say that a representation is decomposable when the space on which the linear transformations act is a direct sum of subspaces each of which is mapped into itself by all linear transformations in the image of the representation, and we say such subspaces are invariant. When a representation is decomposable, its linear transformations can be regarded as acting on each of the invariant subspaces independently, and we say that the representation is the direct sum of the representations obtained by restricting its transformations to each of the invariant subspaces involved in the decomposition. Every unitary representation of a compact group is a direct sum of indecomposable unitary representations acting on mutually orthogonal subspaces. Moreover, every indecomposable unitary representation of a compact group is finite-dimensional. Thus for a compact group it is only necessary to study the indecomposable unitary representations on finite-dimensional spaces in order to obtain the complete theory of representations on Hilbert spaces. Groups which are not compact may have nonunitary representations and infinite-dimensional indecomposable representations on Hilbert spaces. We shall be primarily concerned in this paper, however, with the finite-dimensional representations.

Returning to the case of the rotation group $SO(3, R)$ and its universal covering group $SU(2)$ we see that we need only consider the finite-dimensional unitary representations of these groups. We shall return repeatedly to this example to illustrate various points in our later discussion.

1.4. Complexification of real simple Lie algebras. In [4] we classified all the simple Lie algebras over the complex numbers, finding the four general sequences of simple Lie algebras: A_l, B_l, C_l and D_l and the exceptional simple Lie algebras G_2, F_4, E_6, E_7 and E_8. (Of course, for $l < 4$, some of the general sequence Lie algebras are not defined—they collapse into one another—so we have only A_1, A_2, B_2, A_3, B_3, C_3.) It is apparent that real Lie algebras, such as our example $so(3, R)$, are quite important for many applications, but classifying the real simple Lie algebras is only slightly more complicated than classifying the complex ones. To see why this is the case, consider a real r-dimensional Lie algebra with basis $e_1, e_2, \cdots, e_r$. By linearity, the Lie product is defined when the product of basis vectors is given:

$$[e_i, e_k] = \sum_{j=1}^{r} C_{ik}^{j} e_j,$$

i.e., it is determined by the structure constants C_{ik}^{j}. Since the definition of a Lie algebra essentially amounts to imposing certain restrictions on the structure constants (see [4, p. 16]), we can obviously extend L to a complex Lie algebra having the same basis and the same structure constants. This complex Lie algebra is called the complexification of L. Conversely, if we are given a complex Lie algebra L, it can always be regarded as a real Lie algebra L^R, since multiplication by reals is appropriately defined a fortiori. If this resulting real Lie algebra L^R can be written as the direct sum $L_0 \oplus iL_0$ for some real subalgebra L_0, then L_0 is called a real form of L, and obviously L is isomorphic to the complexification of L_0. It can be shown that the simple real Lie algebras are either simple complex Lie algebras being written as real Lie algebras or real forms of simple complex Lie algebras (but not both).

The complication of the classification problem for simple real Lie algebras arises from the fact that a given simple complex Lie algebra may have several distinct real forms (or, putting it the other way, distinct simple real Lie algebras may have isomorphic complexifications). For instance, the real Lie algebra of the rotation group and that of the plane Lorentz group (the set of all linear transformations on (x, y, t) leaving $x^2 + y^2 - t^2$ invariant) have isomorphic complexifications, namely A_1, without being themselves isomorphic.

Of all the real forms of a given simple complex Lie algebra, there is precisely one which is the real Lie algebra of a compact Lie group. We call this the compact real form, and in general a real Lie algebra which is the Lie algebra of some compact group is called compact. A less intuitive, but sometimes more useful characterization of compact semisimple Lie algebras arises from the fact that a real semisimple Lie algebra is compact if and only if its Killing form is negative definite. All of the noncompact real forms of a simple complex Lie algebra can be obtained from the compact one [16], [17], [20], [33].

Given a simple complex Lie algebra L, recall that we can find a basis of the form $h_1, h_2, \cdots, h_l, e_\alpha, e_{-\alpha}, e_\beta, e_{-\beta}, \cdots$, where the $\{h_i\}$ form a basis for a Cartan subalgebra, l is the rank of L, and the e's are root-vectors, i.e.,

$$[h, e_\alpha] = \alpha(h)e_\alpha,$$
$$[h, e_{-\alpha}] = -\alpha(h)e_{-\alpha},$$

for each h in the Cartan subalgebra, and α ranges over all positive roots (cf. [4, §3.3]). We may normalize the e's so that $(e_\alpha, e_{-\alpha}) = 2$. The vectors ih_1, $ih_2, \cdots, ih_l, i(e_\alpha + e_{-\alpha}), i(e_\beta + e_{-\beta}), \cdots, (e_\alpha - e_{-\alpha}), (e_\beta - e_{-\beta}), \cdots$ then form a basis for the compact real form L_0 of the simple complex Lie algebra L. (These vectors are, of course, only one of many bases for the compact real form. This choice makes it particularly easy to compute that the Killing form is negative definite over the real space generated by them.)

It must be understood that if we take a complex Lie algebra, L, with basis e_j and choose to regard it as a real Lie algebra, L^R, with basis $\{e_j, ie_j\}$, the imaginary unit i loses its intrinsic meaning except as it is involved in defining the Lie products of basis elements,

$$[e_j, ie_k] = i[e_j, e_k],$$
$$[ie_j, ie_k] = -[e_j, e_k].$$

If we are to find a real form of L, we must introduce some operation J corresponding to the desired multiplication by i and write the direct sum $L^R = L_0 \oplus JL_0$. All that is required of J is that it be a linear operator on L^R having its square the negative of the identity, $J^2 = -I$, and that $[x, Jy] = J[x, y]$. The fact that there can be several real forms results from the fact that there can be several distinct suitable operators J and corresponding L_0's such that $L^R = L_0 \oplus JL_0$. If we have a real form L_0, and the corresponding operator J, we can define a Lie algebra "conjugation" S, an automorphism on L^R, by $S(x + Jy) = x - Jy$. Clearly, $S^2 = I$, and moreover, S is the identity when restricted to L_0, but its negative when restricted to the complement JL_0. Thus L_0 is the eigenspace of the linear operator S corresponding to the eigenvalue 1, and JL_0 is that corresponding to the eigenvalue -1.

Suppose we start with the compact real form L_0 of a simple complex Lie algebra L, and we find an automorphism S_0 of L_0 satisfying $S_0^2 = I$. Let us denote the eigenspace of S_0 corresponding to the eigenvalue $+1$ by L_0^+ and that corresponding to -1 by L_0^- so $L_0 = L_0^+ \oplus L_0^-$. Let $e_1, \cdots, e_k$ be a basis for L_0^+ and $e_{k+1}, e_{k+2}, \cdots, e_r$ a basis for L_0^-. Since $L^R = L_0 + iL_0$, we can extend S_0 to an automorphism of the real Lie algebra L^R by defining

$$S(L_0) = S_0(L_0),$$
$$S(iL_0) = -iS_0(L_0).$$

Defining $L_1 = L_0^+ \oplus iL_0^-$ we find that the vectors of L_1 are left fixed by S and those of $iL_1 = iL_0^+ \oplus L_0^-$ are carried to their negatives. Since $L^R = L_1 \oplus iL_1$, S is an operator of order two on L^R (i.e., $S^2 = I$) and $S(\alpha x) = \bar{\alpha}S(x)$ for any complex α. A straightforward computation (using the fact that S is an automorphism of L_0, so $S_0([x, y]) = [S_0(x), S_0(y)]$ for x, y in L_0) shows that $S([x, y]) = [S(x), S(y)]$ in L^R. Since L_1 is the set of vectors left fixed by S, we see that if x, y are in L_1, then so is $[x, y]$, for

$$S[x, y] = [Sx, Sy] = [x, y],$$

so L_1 is a subalgebra of L^R and hence a real form of L. Since every real form of L induces a conjugation operation which has the properties we required of S_0 on L_0, every real form of L can be obtained in this manner from some automorphism S_0 on L_0.

Because of the correspondence demonstrated here, we shall deal in general only with complex Lie algebras in the rest of this paper. It can be assumed that a complex Lie algebra is meant in any further general discussion unless a specific exception is made.

1.5. Complexification of the Lie algebra of the ordinary rotation group. The process of complexification may be illuminated by consideration of the real Lie algebra $su(2)$ of the rotation group, the three-dimensional space with the usual vector cross product. We can take as basis elements e_1, e_2, e_3 with the Lie product defined by $[e_1, e_2] = e_3$, $[e_2, e_3] = e_1$, $[e_3, e_1] = e_2$. The complexification of $su(2)$ then has the same basis and same Lie products over the complex numbers. In angular momentum theory it is conventional not to use the elements e_1, e_2, e_3, but to replace them with an obviously equivalent set $j_1 = ie_1$, $j_2 = ie_2$, $j_3 = ie_3$. Any one of these vectors forms a basis for a Cartan subalgebra. It is conventional to choose $h = j_3 = ie_3$ as the basis of a Cartan subalgebra. The roots are zero and the linear forms $\pm\alpha$ on the Cartan subalgebra, with $\pm\alpha$ defined by $\alpha(\lambda h) = \lambda$ and $-\alpha(\lambda h) = -\lambda$ for all complex numbers λ. The corresponding root vectors are $e_\alpha = j_+ = j_1 + ij_2$ and $e_{-\alpha} = j_- = j_1 - ij_2$. We have

$$[h, e_\alpha] = e_\alpha,$$

$$[h, e_{-\alpha}] = -e_{-\alpha},$$

$$[e_\alpha, e_{-\alpha}] = 2h.$$

The Lie algebra has three complex dimensions, and h, e_α and $e_{-\alpha}$ form a basis. This complex Lie algebra is, of course, the simple Lie algebra A_1.

The compact real form of A_1 is the original Lie algebra $su(2)$, since the groups $SU(2)$ and $SO(3, R)$ are compact. According to the formula of the preceding section, we should have $\{ih = -e_3, i(e_\alpha + e_{-\alpha}) = -2e_1, (e_\alpha - e_{-\alpha}) = -2e_2\}$ as a basis for $su(2)$, which it obviously is.

Suppose we had an automorphism S_0 of $su(2)$ satisfying the condition $S_0^2 = I$. From this condition it follows that S_0 can have only ± 1 as eigenvalues. If all its eigenvalues were $+1$, then S_0 would be the identity, and the real form generated by it would just be $su(2)$ again. The condition that S_0 be a Lie algebra automorphism is

$$S_0([x, y]) = [S_0(x), S_0(y)]$$

for all x and y in the Lie algebra. If all the eigenvalues of S_0 were -1, then S_0 would be $-I$, and we would have

$$[S_0(e_1), S_0(e_2)] = [-e_1, -e_2] = e_3,$$

while

$$S_0([e_1, e_2]) = S_0(e_3) = -e_3$$

so that S_0 would not be an automorphism. Thus any nontrivial automorphism must have at least one eigenvalue $+1$ and one eigenvalue -1. Letting x and y be corresponding eigenvectors, we have

$$[S_0x, S_0y] = [x, -y] = -[x, y],$$

so if S_0 is to be an automorphism we must have

$$S_0([x, y]) = -[x, y],$$

and $[x, y]$ is also an eigenvector corresponding to the eigenvalue -1. Since the Lie product we are dealing with is just the familiar vector cross product, we know that $[x, y]$ is a nonzero vector orthogonal to both x and y. The vectors y and $[x, y]$ together form an orthogonal basis for the eigenspace corresponding to the eigenvalue -1 and, moreover,

$$\begin{aligned} S_0([y, [x, y]]) &= [S_0y, S_0[x, y]] \\ &= [-y, -[x, y]] = [y, [x, y]], \end{aligned}$$

so $[y, [x, y]]$ is an eigenvector of S_0 corresponding to the eigenvalue $+1$. Since this eigenspace can only be one-dimensional, $[y, [x, y]]$ is a multiple of x. But we know that $[y, [x, y]]$ is perpendicular to y and $[x, y]$, so x must be also. Thus x, y and $[x, y]$ are a mutually orthogonal set of vectors. If x and y are taken as unit vectors, then $[x, y]$ will also be one. Now we know that in the formulation of the vector cross product algebra the vectors e_1, e_2, e_3 can be chosen as three orthogonal unit vectors such that $[e_1, e_2] = e_3$, different choices of these vectors leading to isomorphic Lie algebras. We can, therefore, assume $e_1 = x$, $e_2 = y$, $e_3 = [x, y]$, without loss of generality. Thus there is, up to isomorphism, only one noncompact real form of A_1, the complexification of the compact real algebra $L_0 = su(2)$, corresponding to

$$S_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Since L_0^+ is spanned by e_1, and L_0^- is spanned by e_2 and e_3, a basis for the noncompact real form $L_1 = L_0^+ \oplus iL_0^-$ is given by the vectors e_1, ie_2 and ie_3. The Lie products of these basic elements are of course given by

$$[e_1, ie_2] = ie_3,$$
$$[ie_2, ie_3] = -e_1,$$
$$[ie_3, e_1] = ie_2.$$

The real Lie algebra so obtained is the real Lie algebra of several Lie groups, including the three-dimensional Lorentz group $SO(2, 1; R)$ leaving invariant the form $x^2 + y^2 - t^2$ in (x, y, t) space, the real unimodular group $SL(2, R)$, the real symplectic group $Sp(1, R)$, and the pseudounitary group $SU(1, 1)$. All of these Lie groups are locally isomorphic since their Lie algebras are isomorphic. The representations of these noncompact groups have been studied extensively [2], [5], [31].

If the complex Lie algebra A_1 with three complex dimensions is regarded as a real Lie algebra A_1^R with six real dimensions, a basis being given by e_1, e_2, e_3, $ie_1 = j_1$, $ie_2 = j_2$, $ie_3 = j_3$, we obtain the real Lie algebra corresponding to several more well-known Lie groups, including the ordinary Lorentz group $SO(3, 1; R)$ leaving invariant the form $x^2 + y^2 + z^2 - t^2$ in (x, y, z, t) space, the complex unimodular group $SL(2, C)$, and the complex orthogonal group $SO(3, C)$, all of these Lie groups being locally isomorphic. The complexification of A_1^R is $A_1 \oplus A_1$. Note also that $SO(3, R) \times SO(3, R)$ and $SO(4, R)$ have real Lie algebras isomorphic to $su(2) \oplus su(2)$, whose complexification also yields $A_1 \oplus A_1$. In other words, both A_1^R and $su(2) \oplus su(2)$ are real forms of the semisimple complex Lie algebra $A_1 \oplus A_1$.

2. **Modules and representations.** One of the main tools in applications of Lie group theory is the concept of a representation. Often in applications it is sufficient to treat this concept in a fairly loose manner; one speaks of vectors, tensors, pseudoscalars, spinors and the like as being geometrical objects such as directed arrows, ellipsoids, and the like [34]. Representations of the rotation group crop up throughout physics in the form of spherical harmonics and Legendre functions, multipole expansions, etc. The actual representation concept is often held in the background. For our purposes, however, such an intuitive conception is not sufficiently precise, because we want to discuss some of the deeper results, and also because we want our discussion to apply to all Lie algebras, not just the rotation group.

To be precise, a representation of a Lie algebra L consists of a vector space V and a homomorphism f from L into the Lie algebra $E(V)$ of all linear transformations on V. (The notation $E(V)$ derives from the use of the term "endomorphism" to denote a linear transformation from a vector space V back into V itself.) If a, b are in $E(V)$, their Lie product $[a, b]$ in $E(V)$ is defined as $ab - ba$. The term "representation" strictly refers to the pair (V, f), although colloquially it is often used to refer just to the homomorphism $f: L \to E(V)$ alone. We also note that the requirement that f be a homomorphism means that if x is in L, then $f(x)$ is a linear transformation on V, and $f(x)$ depends on x linearly: $f(c_1x_1 + c_2x_2) = c_1f(x_1) + c_2f(x_2)$ for all x_1, x_2 in L and complex numbers c_1 and c_2. Also we have

$$f([x_1, x_2]) = f(x_1)f(x_2) - f(x_2)f(x_1).$$

Before proceeding, let us give two simple examples of representations of Lie algebras. In our first example we consider the simple complex Lie algebra A_1 obtained by complexifying the real Lie algebra of the ordinary three-dimensional rotation group $SO(3, R)$. A basis for A_1 is given by j_1, j_2, j_3 as explained earlier. These elements generate the rotations about the x, y, z axes, respectively. Let V be a two-dimensional vector space over the complex numbers, which we shall call the spinor space. Let a basis for V be selected. Then the Pauli matrices

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

define linear transformations in the spinor space V. If we let $f: A_1 \to E(V)$ denote the linear mapping which takes j_a into $\frac{1}{2}\sigma_a$, $a = 1, 2, 3$, then (V, f) is a representation of the Lie algebra A_1. It is called the spinor representation, or the spin half representation [29].

For our second example, we consider an arbitrary Lie algebra L, and we recall that for any element x in L, the adjoint operator $\operatorname{ad} x$ in $E(L)$ is defined by $(\operatorname{ad} x)y = [x, y]$ for all y in L. The mapping $\operatorname{ad}: L \to E(L)$ which takes x into $\operatorname{ad} x$ is a homomorphism from L to $E(L)$. The pair (L, ad) is then a representation of the Lie algebra L. It is called the adjoint representation of the Lie algebra.

2.1. Modules over Lie algebras. In some respects the concept of a representation is unnecessarily complicated; the object being represented (e.g., a group or algebra) is first mapped by a homomorphism into a similar object composed of linear transformations, acting in turn on a vector space. There is no loss of generality, of course, if we regard the elements of the original object as affecting the linear transformations on the vector space directly, disregarding the intermediate homomorphism. When we take this viewpoint and the object being represented is a group, we speak of the vector space as a module over the group. When we are dealing with an algebra, we refer to the vector space as a module over the algebra. The study of modules over an algebra is equivalent to the study of the representations. The only difference is a point of emphasis. In talking about modules we emphasize the vector spaces involved, whereas in talking about representations, we emphasize the homomorphisms. Generally speaking, the module formulation is the easier to work with.

Let us now give a precise definition. We deal here and in the sequel with finite-dimensional Lie algebras over the complex numbers, although obviously some of our basic definitions can be formulated more generally. A module over a Lie algebra L is a vector space M with a product rule $L \times M \to M$ such that if v is a vector in M and x is an element of the Lie algebra, then xv in M satisfies the following laws:

$$x(v_1 + v_2) = xv_1 + xv_2,$$
$$(x_1 + x_2)v = x_1v + x_2v,$$
$$(\alpha x)v = x(\alpha v) = \alpha xv,$$
$$[x_1, x_2]v = x_1(x_2v) - x_2(x_1v),$$

where α is any complex number.

If M is a module over a Lie algebra L, then for any x in L, let $f(x)$ in $E(M)$ be the linear transformation which maps v in M into xv. Then (M, f) is a representation of the Lie algebra L. Conversely, given a representation (M, f) we can make the vector space M into a module by defining the product xv of x in L and v in M to be $f(x)v$. Thus the concepts of module and representation are equivalent.

A homomorphism h of one module M over a Lie algebra L into another module N over L is a linear transformation $h:M \to N$ which preserves the multiplication by elements of L, that is, $h(xv) = xh(v)$. If the homomorphism h is one-to-one and onto, and hence is invertible, then we call it an isomorphism, and in this case we also say that the two modules M and N are isomorphic. If f is the representation of L associated with the module M and if f' is the representation of L associated with the isomorphic module N, then we have $hf(x) = f'(x)h$, and h being invertible, we may write this as $f'(x) = hf(x)h^{-1}$. Thus the linear transformations $f(x)$ and $f'(x)$ are related by a similarity transformation which is independent of x. Such representations, corresponding to isomorphic modules, are called equivalent.

If a basis $v_1, \cdots, v_n$ is selected in the vector space M, then the linear transformations $f(x)$ correspond to matrices $((f_{ij}(x)))$, defined by

$$f(x)v_j = \sum_{i=1}^{n} v_i f_{ij}(x).$$

The mapping which assigns the matrix $((f_{ij}(x)))$ to x is called a matrix representation of the Lie algebra L. If a different basis is selected for M, of course, the matrices of the matrix representation will all be subjected to a common similarity transformation. Thus again, matrix representations related by a common similarity transformation are considered to be equivalent or isomorphic. It seems clear that equivalent representations or isomorphic modules will not differ in any important respect. It is easy to show that if M and N are isomorphic modules, then there exists a choice of bases for both of these modules such that the representation matrices obtained will be the same matrices for both modules.

If a module M over a Lie algebra L has a subspace S which remains invariant under the action of L, symbolically $LS \subset S$, then S itself can be regarded as a module over L, and we call it a submodule of the module M. That is, a subspace S of a module M is a submodule if xv is in S for all x in L and all v in S. A module which has no submodules other than zero and itself is called an irreducible module. The corresponding representation is then also called irreducible.

Since a module M is a fortiori an Abelian group, and a submodule N is a normal subgroup, we can form the quotient group M/N whose elements are cosets of the form $v + N$, with v in M. Since N is a vector space, multiplication by complex numbers is well-defined on M/N, and hence M/N is a vector space; similarly, since N is a module, multiplication by elements of L is well-defined on M/N, and hence M/N may be considered a module over the Lie algebra. It is easily shown that if h is a homomorphism of a module M onto a module N, then the kernel K of the homomorphism is a submodule of M, and the quotient module M/K is isomorphic with the image N of the homomorphism.

Representations and modules can be defined for Lie groups as well as for Lie algebras. If G is a Lie group, then an analytic representation of G is a pair (M, f), where M is a vector space (module), and f now is an analytic homomorphism of G into the Lie group $GL(M)$ of all automorphisms in M, that is, invertible linear transformations of M onto itself. If n is the dimension of M, then $GL(M)$ is isomorphic to the Lie group $GL(n, C)$ of nonsingular $n \times n$ matrices over C. We see therefore that a representation of a Lie group is equivalent to a matrix representation, a homomorphism D of the Lie group G into the matrix group $GL(n, C)$. Then $D(g)$ is an $n \times n$ matrix representing the element g in G. The notation $D(g)$ arises from the German "Darstellung" for "representation."

Equivalence of group representations is defined just as before, two matrix representations being equivalent or isomorphic if they are related by a common similarity transformation.

2.2. Module operations. There are several useful ways of combining modules to obtain new ones. The most immediate of these is the direct sum. The direct sum can actually be defined in two very slightly different ways which are essentially equivalent; these are called the internal and external direct sums. Given two modules M_1 and M_2 over a Lie algebra L, their external direct sum as vector spaces is the vector space consisting of ordered pairs (v_1, v_2), where v_1 is in M_1 and v_2 in M_2, with addition and multiplication by complex numbers defined componentwise. If multiplication by elements of the Lie algebra L is also defined componentwise,

$$x(v_1, v_2) = (xv_1, xv_2),$$

we again have a module over L called the external direct sum of M_1 and M_2, denoted by $M_1 \oplus M_2$. When a module M has two submodules M_1 and M_2 such that every element in M has a unique expression as a sum of an element of M_1 and an element of M_2, then we say that M is the internal direct sum of its submodules M_1 and M_2. In this case it is also true that M is isomorphic to the external direct sum of M_1 and M_2. More generally, the sum of two subspaces of a vector space is the set of all linear combinations of vectors in both of the subspaces. In other words, the sum of two subspaces is the linear subspace generated by their union. The sum of two subspaces is a direct sum when the intersection of the two subspaces consists of the zero vector. Note that the internal direct sum can only be defined when both vector spaces are subspaces of some common vector space and have zero intersection, whereas the external direct sum can be defined for an arbitrary pair of vector spaces. We shall use the same notation $\oplus$ for both internal and external direct sums; the distinction between the two can usually be understood from context.

If a module M is the direct sum of two nonzero submodules, we say that M is decomposable, and the corresponding representation is also said to be decomposable. Obviously an irreducible module cannot be decomposable, but the converse need not be true. Every finite-dimensional module which is decomposable is a direct sum of indecomposable modules.

A module having the property either of being itself irreducible or of being the direct sum of irreducible submodules is called completely reducible. An obvious necessary and sufficient condition for a finite-dimensional module to be completely reducible is that every submodule should be a direct summand, that is, for every submodule N, there is another submodule N' such that $M = N \oplus N'$.

Every finite-dimensional module over a semisimple Lie algebra is completely reducible. Thus the irreducible modules of a semisimple Lie algebra form the building blocks out of which any other module over the Lie algebra can be constructed, and our attention can therefore be directed in this case to a study of the irreducible modules. This is not true however for a general Lie algebra. For general Lie algebras it suffices to consider the indecomposable modules, but these need not be irreducible.

A second important way of combining modules is by means of the tensor product. There are several ways to define the tensor product. The best way is to give a universal definition, defining the tensor product by means of its properties. Another way, which assures us of the existence of the object being defined, is to give a constructive definition, which is what we shall do.

In order to define the tensor product of modules over a Lie algebra, we first recall the definition of the tensor product of two vector spaces. Given two vector spaces V_1 and V_2, we shall write an ordered pair having its first element v_1 in V_1 and its second v_2 in V_2 as $v_1 \otimes v_2$. The set of all such ordered pairs can be regarded as the basis of an infinite-dimensional vector space consisting of all their finite formal linear combinations. A typical vector in this space has the form $\sum_{i=1}^{n} \alpha_i(v_{1i} \otimes v_{2i})$.

Now consider the set of all vectors in this space of the form

$$(u_1 + v_1) \otimes v_2 - u_1 \otimes v_2 - v_1 \otimes v_2,$$

$$v_1 \otimes (u_2 + v_2) - v_1 \otimes u_2 - v_1 \otimes v_2,$$

$$\alpha(v_1 \otimes v_2) - (\alpha v_1) \otimes v_2$$

or

$$\alpha(v_1 \otimes v_2) - v_1 \otimes (\alpha v_2).$$

These vectors span a subspace. Taking the factor space, the cosets of the original space with respect to this subspace, we obtain a vector space which we call $V_1 \otimes V_2$. It is customary to write elements of the space $V_1 \otimes V_2$ in terms of coset representatives:

$$\sum_{i=1}^{n} v_{1i} \otimes v_{2i},$$

and to keep in mind that one must consider

$$\alpha v_1 \otimes v_2 = v_1 \otimes \alpha v_2$$

for any complex number α, and that

$$(u_1 + v_1) \otimes v_2 = u_1 \otimes v_2 + v_1 \otimes v_2$$

and

$$v_1 \otimes (u_2 + v_2) = v_1 \otimes u_2 + v_1 \otimes v_2 .$$

It is easily shown that the coset representative

$$\sum_{i=1}^{n} v_{1i} \otimes v_{2i}$$

can always be chosen so that the vectors v_{1i} are linearly independent in V_1 and the vectors v_{2i} are linearly independent in V_2. If the v_{1i} are not independent, then one of these vectors, say v_{11}, is a linear combination of the others:

$$v_{11} = \sum_{i=2}^{n} \alpha_i v_{1i} .$$

Then,

$$\sum_{i=1}^{n} v_{1i} \otimes v_{2i} = \sum_{i=2}^{n} v_{1i} \otimes (v_{2i} + \alpha_i v_{21}),$$

and we can proceed in this manner until we obtain a coset representative having its terms v_{1i} linearly independent. A similar argument can then be applied to the vectors v_{2i} without destroying the independence of the vectors v_{1i}. From this it is apparent that if $\{v_{1i}\}$ is a basis in V_1 and $\{v_{2j}\}$ a basis in V_2, then the cosets of $\{v_{1i} \otimes v_{2j}\}$ form a basis in $V_1 \otimes V_2$. Thus the dimension of the tensor product $V_1 \otimes V_2$ is the product of the dimensions of V_1 and V_2, and a coset representative can always be written as

$$\sum_{i,j} \alpha_{ij} v_{1i} \otimes v_{2j} .$$

The array of coefficients α_{ij} is uniquely determined by the element of $V_1 \otimes V_2$ being represented, and it changes appropriately when the bases in V_1 and V_2 are changed. Similarly, any array of coefficients which is made to change appropriately when the bases of V_1 and V_2 are varied determines a unique element of $V_1 \otimes V_2$. This accounts for the "folk" definition of a tensor as an array of quantities which varies appropriately when coordinate systems are changed. Frequently one has in mind the case $V_1 = V_2 = V$, and one calls an element of $V \otimes V$ a contravariant tensor of rank two.

The construction of the tensor product given above is not the only one possible.

Another rather clever way to construct the tensor product $V_1 \otimes V_2$ is to define it to be the dual space of the linear space of all bilinear forms on the Cartesian product $V_1 \times V_2$. This alternative definition can be shown to be equivalent to the one given above when the vector spaces involved are finite-dimensional, as we assume here.

There is a natural identification between the elements of $(V_1 \otimes V_2) \otimes V_3$ and those of $V_1 \otimes (V_2 \otimes V_3)$. Making this identification, we can define iteratively the tensor product of any number of vector spaces.

A useful property of the tensor product arises from the fact that any linear transformation on its factors V_1 and V_2 generates a linear mapping of the product $V_1 \otimes V_2$. In particular, if $f_1: V_1 \to V_1'$ and $f_2: V_2 \to V_2'$ are linear mappings, then $f_1 \otimes f_2: V_1 \otimes V_2 \to V_1' \otimes V_2'$ can be defined as the following linear transformation:

$$(f_1 \otimes f_2)\left\{\sum_{i=1}^{n} v_{1i} \otimes v_{2i})\right\} = \sum_{i=1}^{n} f_1(v_{1i}) \otimes f_2(v_{2i}).$$

Also, a bilinear mapping (function of two vector variables, linear in both) on V_1 and V_2 generates a linear mapping on the tensor product. A bilinear mapping $\beta: V_1 \times V_2 \to W$ generates a linear mapping $\beta^{\#}: V_1 \otimes V_2 \to W$ by

$$\beta^{\#}\left\{\sum_{i=1}^{n} v_{1i} \otimes v_{2i}\right\} = \sum_{i=1}^{n} \beta(v_{1i}, v_{2i}).$$

Notice that the operations on the tensor product $V_1 \otimes V_2$ are defined in terms of coset representatives, so that they must be shown to be well-defined, dependent only on the coset, not on the particular representation of the coset chosen. The selection of the generators determining the subspace with respect to which the cosets are defined, however, was made precisely with this end in view; the relations imposed on two representatives by being in the same coset are precisely those needed for the mappings to be well-defined.

Some special cases of such mappings should be mentioned. We use complex spaces for illustration. The real number field R could equally well appear in place of the complex number field C in the following paragraphs. If V_1 (or V_2) is just the complex number field and V_2 (or V_1) is a complex vector space, then

$$f(\alpha, v_2) = \alpha v_2$$

is a bilinear mapping and hence defines a linear mapping of $C \otimes V_2$ to V_2. This mapping is clearly an isomorphism and it is generally used to identify $C \otimes V_2$ with V_2. If V_1 is the dual space of V_2, the space of all complex-valued linear functions on V_2, then the function

$$f(v_1, v_2) = v_1(v_2)$$

is a bilinear complex-valued function on V_1 and V_2, and consequently defines a linear mapping of $V_1 \otimes V_2$ to C. In this latter case one generally refers to the elements of $V_1 \otimes V_2$ as mixed tensors of rank two, and one refers to the induced linear mapping from $V_1 \otimes V_2$ to C as contraction of the tensor. Similar remarks can be made, obviously, with the roles of V_1 and V_2 interchanged.

In general, if we have an iterated tensor product $V_1 \otimes \cdots \otimes V_p$, where each V_i is either a given vector space V, or else its dual V^*, one calls elements of the iterated tensor product tensors of rank p, contravariant as many times as V appears in the product and covariant as many times as its dual V^* appears. A contraction can be defined on any appearance of the vector space V and its dual V^* in $V_1 \otimes \cdots \otimes V_p$, and this is described as contracting on various upper and lower indices. The indices appear here because of the custom of choosing a basis $e_1, \cdots, e_n$ (say) in V and the corresponding dual basis $e^1, \cdots, e^n$ in the dual space V^*, where e^i is the linear function on V such that $e^i(e_j) = \delta_j^{\ i}$ (zero for $i \neq j$ and one for $i = j$). Every member of the tensor product space is then determined by an array of coefficients with p indices. When the basis in V is changed, say to $\bar{e}_1, \cdots, \bar{e}_n$, where $\bar{e}_i = \sum_j P_i^{\ j} e_j$, the corresponding dual basis changes automatically to $\bar{e}^1, \cdots, \bar{e}^n$, given by $\bar{e}^i = \sum_j Q_j^{\ i} e^j$, where the matrix Q is the inverse of the matrix P. The array of coefficients determining a fixed element of the tensor product space varies according to an obvious formula, that is, contravariantly on those indices corresponding to the space V and covariantly with respect to the indices corresponding to the dual space V^*. It is a custom in this case to write the covariant indices as subscripts and the contravariant indices as superscripts. A member of $V \otimes V \otimes V^*$, for instance, would be written

$$\sum_{i,j,k} \alpha_k^{ij} e_i \otimes e_j \otimes e^k.$$

If we substitute for the e-bases in terms of the new $\bar{e}$-bases, we obtain

$$\sum_{i,j,k} \bar{\alpha}_k^{\ ij} \bar{e}_i \times \bar{e}_j \times \bar{e}^k,$$

where

$$\bar{\alpha}_n^{\ lm} = \sum_{i,j,k} Q_i^{\ l} Q_j^{\ m} P_n^{\ k} \alpha_k^{\ ij}.$$

In this sense the upper indices require transformations contrary to the basis e_i (hence, the term "contravariant"), while the lower indices require transformations akin to that of the basis e_i (hence, "covariant"). For this same tensor, contraction on the first and third indices, and using the standard identification of $C \otimes V$ with V, yields

$$\sum_{j=1}^{n} \left[\sum_{i=1}^{n} \alpha_i^{\ ij} \right] e_j$$

in V. Thus contraction in this notation corresponds to setting an upper index of the coefficient array equal to a lower index and summing on it.

While on the subject of tensor products, we should mention that two elements of different iterated tensor products can always be tensored together to get a member of a larger iterated tensor product by defining, for instance,

$$(\sum_i u_i \otimes v_i) \otimes (\sum_j w_j \otimes x_j) = \sum_{i,j} u_i \otimes v_i \otimes w_j \otimes x_j.$$

This leads to the idea of the (contravariant) tensor algebra on a vector space V, which is simply the infinite weak external direct sum

$$T(V) = C \oplus V \oplus V^2 \oplus V^3 \oplus \cdots,$$

where

$$V^n = V \otimes V \otimes \cdots \otimes V \quad (n \quad \text{copies}).$$

Again, C denotes the complex numbers. The infinite direct sum is weak in the sense that we consider only finite linear combinations of the elements of the spaces $C, V, V^2, \cdots$. No actual infinite summations are involved, so no convergence questions arise. As a direct sum of vector spaces, the contravariant tensor algebra is obviously a vector space. With the tensoring together operation $\otimes$ as multiplication, the tensor algebra $T(V)$ is an associative algebra.

Having discussed at some length the tensor product for vector spaces, we are now ready to discuss tensor products for modules. Let M_1 and M_2 be modules over a Lie group G. We can make the tensor product space $M_1 \otimes M_2$ a module over the group G by defining

$$g\,[\sum_i m_{1i} \otimes m_{2i}] = \sum_i (gm_{1i}) \otimes (gm_{2i}).$$

We can pass from modules over Lie groups to modules over Lie algebras by differentiation, or vice versa by exponentiation. Choosing the latter method, if x is an element of a Lie algebra L, then $g = e^{tx} = 1 + tx + \cdots$ is an element of a Lie group. For sufficiently small real numbers t, we could ignore higher order terms. Then, inserting this expansion for g in the above definition of its action on the tensor product space, and comparing powers of t, we arrive at the formula

$$x\,[\sum_i m_{1i} \otimes m_{2i}] = \sum_i [xm_{1i} \otimes m_{2i} + m_{1i} \otimes xm_{2i}].$$

If M_1 and M_2 are modules over a Lie algebra L, this formula is used to define a multiplication which makes the tensor product $M_1 \otimes M_2$ a module over L. It can be verified directly that the axioms for a module are satisfied without reference to Lie groups. We only started with Lie groups for the purpose of motivating the definition.

Consider the tensor product of two modules over a Lie algebra. If the Lie algebra is semisimple, then the original two modules are completely reducible, and so can be written as a direct sum of irreducible modules. The tensor product is then just the direct sum of all the tensor products of these irreducible modules. To reduce the tensor product of arbitrary modules over a semisimple Lie algebra, therefore, it is sufficient to study the reductions of tensor products of irreducible modules as direct sums of irreducible modules. This expansion of a tensor product of irreducible modules as a direct sum of irreducible submodules is frequently called a Clebsch-Gordan series (although this name applies more specifically to the case of the rotation group) and it is quite important in applications.

Given any module M over a Lie algebra L, the dual space M^*, consisting of all linear complex-valued functions on M, can be made into a module over L by defining the action of any x in L on any f in M^* to be given by $(xf)(m) = -f(xm)$ for all m in M. Straightforward computation verifies that M^* satisfies the axioms of a module. If a basis $\{e_i\}$ is chosen in M, and the corresponding dual basis $\{e^i\}$ is chosen in M^*, then the representation matrices in the dual module are the negative transposes of the representation matrices in the original module. The action of x on the basis elements e_i can be written as $xe_i = \sum_j D_i^{\,j}(x)e_j$. Then, using $xe^i(e_j) = -e^i(xe_j)$, a simple computation shows that $xe^i = \sum_j \{-D_j^{\,i}(x)\}e^j$. Thus here $D_i^{\,j}(x)$ is the matrix representative associated with the module M, and the negative transpose $-D_j^{\,i}(x)$ gives the corresponding matrix representation associated with the dual module M^*.

For Lie algebras of type B_l, C_l, and also for the Lie algebras A_1 and G_2, F_4, E_7, E_8, each module is isomorphic with its own dual. Most modules over D_l-type algebras also are self-dual [26]. The dual operation thus plays no particularly important role for these Lie algebras. In fact, the main application for the concept of dual module in the theory of semisimple Lie algebras occurs for the Lie algebras of type A_l with $l \geqq 2$.

Yet another way of building new modules from given ones can be obtained by antisymmetrizing the tensor product. Starting with the tensor algebra $T(M)$ of a module M, which is an associative algebra under the $\otimes$ operation, consider the two-sided ideal A generated by all elements of the form $m \otimes m$. The quotient algebra $T(M)/A$ forms a new associative algebra $E(M)$ called the Grassmann algebra or the exterior algebra of M. It has become the custom to denote the multiplication operation induced by $\otimes$ in $E(M)$ by the symbol $\wedge$. Thus $m_1 \wedge m_2$ means the coset $(m_1 \otimes m_2) + A$. It is clear that $m \wedge m = 0$ (that is, the coset $0 + A$) since $m \otimes m$ is in A by definition. Since $(m_1 + m_2) \wedge (m_1 + m_2) = 0$, writing it out and using $m_1 \wedge m_1 = m_2 \wedge m_2 = 0$ yields

$$m_1 \wedge m_2 = -m_2 \wedge m_1 .$$

We see that the multiplication in the exterior algebra is antisymmetric as in a Lie algebra. Unlike a Lie algebra, however, the multiplication is also associative. The exterior algebra can be broken up into pieces each having a given degree by considering the direct summands of the tensor algebra $T(M)$, the tensor products of M with itself. Let $T^r(M)$ denote the tensor product of M with itself r times, the tensors of rank r. Since $T(M)$ is the direct sum of the various $T^r(M)$, we can look individually at the portion A^r of the ideal A lying in each $T^r(M)$ and form the vector space $\wedge^r(M) = T^r(M)/A^r$. The exterior algebra is then the direct sum of these spaces $\wedge^r(M)$. Now A^r is simply the subspace generated by those members of $T^r(M)$ having a repeated factor $m \otimes m$ appearing somewhere in their composition, being of the form $a \otimes m \otimes m \otimes b$, where a, b denote tensors whose ranks sum to $r - 2$. It is clear that $T^r(M)$ is a module over the Lie algebra L. A computation of the effect of multiplying an element such as $a \otimes m \otimes m \otimes b$ by an element x of the Lie algebra shows that A^r is invariant under L; hence it is a submodule of $T^r(M)$. The quotient space $\wedge^r(M) = T^r(M)/A^r$ is therefore also a module. The construction of these "antisymmetric tensor modules" $\wedge^r(M)$ from a basic module M is a standard method for generating modules and will be dealt with later in connection with our discussion of elementary representations. The action of an element x in L on the cosets in $\bigwedge^r(M)$ is induced by the action of x in $T^r(M)$, namely,

$$\begin{aligned} x\cdot(m_1 \wedge m_2 \wedge \cdots \wedge m_r) = {} & (xm_1) \wedge m_2 \wedge \cdots \wedge m_r \\ & + m_1 \wedge (xm_2) \wedge \cdots \wedge m_r \\ & + \cdots \\ & + m_1 \wedge m_2 \wedge \cdots \wedge (xm_r). \end{aligned}$$

It is of interest also to note that for modules M of dimension n, the tensor algebra $T(M)$ is infinite-dimensional, each $T^r(M)$ having dimension n^r, but the exterior algebra $E(M)$ is of finite dimension equal to 2^n, each $\bigwedge^r(M)$ having dimension $\binom{n}{r} = n!/r!(n-r)!$ for $0 \leqq r \leqq n$ and zero for $r > n$. This is readily seen by choosing a basis $e_1, e_2, \cdots, e_n$ in M. The products $e_{j1} \wedge e_{j2} \wedge \cdots \wedge e_{j_r}$ with strictly increasing indices $j_1 < j_2 < \cdots < j_r$ then form a basis for $\bigwedge^r(M)$, since a change in ordering of a product changes at most its sign.

2.3. The universal enveloping algebra. Intuitively, the universal enveloping associative algebra $U(L)$ of a Lie algebra L is obtained by forming all possible formal products and sums of elements of the Lie algebra and an identity element, where we identify the commutator $[x, y]$ for x and y in L with the expression $xy - yx$.

More formally, we can construct the universal enveloping algebra U as follows. Since the Lie algebra L is a vector space, it is possible to construct its (contravariant) tensor algebra $T(L) = C \oplus L \oplus L^2 \oplus \cdots$, where $L^n = L \otimes L \otimes \cdots \otimes L$ (n factors), as in the preceding section. In this associative algebra $T(L)$, we consider the two-sided ideal K generated by the set of elements of the form

$$[x, y] - (x \otimes y - y \otimes x),$$

where x, y are elements of L. The ideal K thus contains the differences between Lie algebra products and the corresponding commutators in the associative tensor algebra. If we consider the associative quotient algebra $U(L) = T(L)/K$, then Lie algebra products will not be distinguishable from commutators since they belong to the same coset. The associative algebra $U(L) = T(L)/K$ is called the universal enveloping algebra of L. As with any associative algebra, we can also make $U(L)$ a Lie algebra using the commutator operation as the Lie product. If we do this, we can consider L to be injected homomorphically into $U(L)$, considered as a Lie algebra. (The injection is induced by regarding L as a direct summand of $T(L)$, for the ideal K in $T(L)$ has been constructed so that for x, y in L we have $[x, y] + K = x \otimes y - y \otimes x + K$, so the coset of $[x, y]$ is identical with that of the commutator Lie product in $T(L)$.)

The associative algebra $U(L)$ plays a unique role with respect to this property. Suppose that A is an arbitrary associative algebra, and A is also given the commutator Lie algebra structure. Any homomorphism of L into A, considered as a

Lie algebra, has a unique extension to an associative algebra homomorphism of $U(L)$ into A, i.e., if ϕ is the canonical homomorphism of L into $U(L)$ and θ is the homomorphism of L into A, then there is a unique associative algebra homomorphism ψ such that $\psi\phi(l) = \theta(l)$ for all l in L.

Now a representation of a Lie algebra is a homomorphism into the associative algebra of linear transformations on the module with Lie multiplication of the linear transformations being the commutator operator. Thus every representation of a Lie algebra L can be extended to a representation of its associative universal enveloping algebra $U(L)$. We define a module over an associative algebra A in much the same way we defined a module over a Lie algebra except that the requirement that $[x, y]m = x(ym) - y(xm)$ for all x and y in the Lie algebra and m in the module is replaced by $(xy)m = x(ym)$ for all x, y in the associative algebra and m in the module. Thus we see that every module over L can also be regarded as a module over its enveloping algebra $U(L)$. The associative algebra $U(L)$ acts on the module M by letting

$$(x_1 \otimes x_2 \otimes \cdots \otimes x_n)m = x_1x_2 \cdots x_nm$$

for all n, x_i in L, and m in M, and extending the definition to all of $U(L)$. It is easily checked that this definition is independent of the choice of a representative from a K-coset of $T(L)$ and so actually defines an action of $U(L) = T(L)/K$ on M.

The enveloping algebra $U(L)$ has a structure which may be described as follows. Let $\{x_i\}$ be a basis for L. The "monomials" of the form

$$x_{i_1} \otimes x_{i_2} \otimes \cdots \otimes x_{i_n}, \qquad n = 0, 1, 2, 3, \cdots,$$

where we take the "trivial monomial" 1 for the case $n = 0$, yield a basis for $T(L)$ and hence their cosets span $U(L)$. Results of Poincaré, Birkhoff and Witt show that if we only take monomials having their indices i_j in ascending order (allowing repetition), then the cosets of these monomials (again, including 1) form a basis for $U(L)$. The universal enveloping algebra bears much the same sort of "functorial" relationship to the Lie algebra that the Lie algebra itself bears to a Lie group, i.e., homomorphisms of one Lie algebra into another induce a corresponding homomorphism of their universal enveloping algebras, and moreover, the ideal generated in the universal enveloping algebra by the kernel of the Lie algebra homomorphism is the kernel of the induced homomorphism on the universal enveloping algebras. More generally, Lie subalgebras and Lie algebra ideals yield corresponding associative subalgebras and ideals in the universal enveloping algebra.

It can be shown that $U(L)$ has no nonzero zero divisors and that, for finite-dimensional L, there cannot be an infinite sequence of ideals (or one-sided ideals) in $U(L)$ each properly contained in the next. Every ideal of U is finitely generated. From this, one could conclude that U has right and left quotient division rings (Goldie-Ore theorem, cf. [24, p. 165]).

The universal enveloping algebra plays an important role in constructing irreducible modules over semisimple Lie algebras. For the case of a semisimple Lie algebra we can make some further remarks about the structure of the enveloping algebra based on the decomposition of the Lie algebra in terms of its root spaces. Let L be a semisimple Lie algebra, and let H be a Cartan subalgebra of L. A choice of an arbitrary ordered basis for the real space H_R^*, generated by the roots, induces an ordering on that space. The ordering is produced by calling a vector in H_R^* positive when its first nonzero component with respect to the basis is positive and saying one vector is greater than another when their difference is positive (see [4, p. 26]). A simple root is a positive root which is not the sum of two others, and these simple roots form a basis for the dual of the Cartan subalgebra. The root vectors of the simple roots are called simple raising operators. The negative of a simple root is always a root, and a convenient multiple of the root vector of the negative of a simple root is called a lowering operator (see [4, p. 27]). We shall denote by L^+ the subspace of the Lie algebra L spanned by the root vectors of positive roots and by L^- that spanned by the root vectors of negative roots. A simple computation using Jacobi's identity shows that if e_α and e_β are root vectors corresponding to roots α and β, then $[e_\alpha, e_\beta]$ is a root vector corresponding to $\alpha + \beta$ if $\alpha + \beta$ is a root, and $[e_\alpha, e_\beta]$ is zero otherwise. Since L is the direct sum of H and the one-dimensional subalgebras spanned by the root vectors, the total number of nonzero roots is the dimension of L less that of H. L^+ and L^- are nilpotent subalgebras and we can write

$$L = H \oplus L^+ \oplus L^-.$$

Each positive root is a linear combination with nonnegative integer coefficients of the simple roots and each negative root is such a combination with nonpositive integer coefficients. Thus L^+ is generated by the simple raising operators and L^- is generated by the simple lowering operators.

Corresponding to this break-up of the Lie algebra L into H, L^+ and L^-, we can also break up the universal enveloping algebra $U(L)$ into three parts U^0, U^+, and U^-. Here U^0 is the Abelian subalgebra generated by the identity element and the Cartan subalgebra H, and U^+ and U^- are generated by the identity element together with L^+ and L^- respectively. We may call U^+ and U^- the raising and lowering algebras of the Lie algebra L. The whole enveloping algebra $U(L)$ may be written as the product of the three subalgebras U^0, U^+, and U^-, taken in any order. The algebra U^+ is generated by the identity and the simple raising operators, while U^- is generated by the identity and the simple lowering operators. The whole algebra $U(L)$ is generated by 1 and the simple raising and lowering operators.

Let e_i, $i = 1, \cdots, l$, denote the simple raising operators in a semisimple Lie algebra L. If a nonzero vector x in a module M is annihilated by all the simple raising operators, it is called an extreme vector:

$$e_i x = 0, \qquad i = 1, \cdots, l.$$

The extreme vectors together with the zero vector form a subspace, the extreme subspace, of the module. If a module is irreducible its extreme subspace is one-dimensional. In any module, $U^- x$ is an irreducible submodule if x is an extreme vector. Every module is the direct sum of the irreducible submodules $U^- x_i$, where $\{x_i\}$ is any basis for the extreme subspace. Thus a knowledge of the extreme subspace yields a reduction of the module to the sum of irreducible submodules. This gives a useful technique for obtaining Clebsch-Gordan coefficients.

We saw previously that there was a one-to-one correspondence between representations of a Lie algebra L and representations of its universal enveloping algebra $U(L)$. Using the fact that every irreducible module over a semisimple Lie algebra is generated by the action of U on an extreme vector, it becomes possible to construct the modules themselves from $U(L)$. The space $U(L)$ can, of course, be regarded as a module over itself. A submodule in $U(L)$ is then simply a left ideal. If A is a left ideal of $U(L)$, the cosets $U(L)/A$ form a module over $U(L)$. Conversely, if x is an extreme vector in any irreducible module M, the members of $U(L)$ which annihilate x form a left ideal of $U(L)$, which we shall again call A. Any vector y in the module M is given by $y = ux$ for some u in $U(L)$, by the fact that the entire irreducible module is generated by x under the action of U. Taking the coset $u + A$, we have $y = (u + A)x$, since $Ax = 0$ by definition of A. We then have a homomorphism of $U(L)/A$ onto the module M, the homomorphism being obtained by making the coset $u + A$ correspond to the module vector ux. This is in fact an isomorphism, since if $u_1 x = u_2 x$, then $(u_1 - u_2)x = 0$ and u_1 and u_2 belong to the same A-coset. Thus, every irreducible module is isomorphic to $U(L)/A$ for some left ideal A in $U(L)$. It is easy to see that $U(L)/A$ is irreducible if and only if A is a maximal left ideal of $U(L)$. Thus the problem of finding all the irreducible representations of $U(L)$ (and consequently of L) is precisely equivalent to that of finding all the maximal left ideals of $U(L)$.

The center $C(L)$ of the universal enveloping algebra $U(L)$ plays an important role in the computational aspects of representation theory. It is an Abelian subalgebra which consists of those elements z in $U(L)$ which commute with every element of the Lie algebra L. The second order Casimir operator is a simple example of an element of $C(L)$, and we may think of $C(L)$ as consisting of the Casimir operator and its various higher order generalizations. For any module M, we can define a linear form on $C(L)$ by means of the trace:

$$\lambda(z) = \mathrm{Tr}_M\, z.$$

The linear form $\lambda \in C(L)^*$ characterizes the modules in the sense that if two modules M and M' have linear forms λ and λ' which are proportional, then the modules are isomorphic [19].

For irreducible representations each element of $C(L)$ is represented by a multiple of the identity. Recently much work has been done in order to find the exact relation between these "higher order Casimir operators" and the more conventional classification of modules by means of their highest weight.

2.4. Theory of weights and the Weyl group. A useful tool for the classification of modules is the concept of weight. Weights play the same role in the classification of modules over semisimple Lie algebras that the roots played in the classification of the semisimple Lie algebras themselves. The concept of root is in fact just the special case of the concept of weight obtained when we specialize to the adjoint representation of the algebra. The study of weights also has a practical value for obtaining a qualitative interpretation of what a given module contains. The set of all the weights of a module is called the weight diagram. The symmetry properties of the weight diagrams are described by the Weyl group.

Although the concept of weight is most useful in connection with the semisimple Lie algebras, it can be introduced in a more general setting, and we therefore consider a module M over an arbitrary Lie algebra L. A linear form $\mu \in L^*$ is called a weight of M if there exists a nonzero vector v in M such that $xv = \mu(x)v$. The vector v is thus an eigenvector for all elements of L simultaneously, with $\mu(x)$ being the eigenvalue corresponding to the operator x acting in M. The weight μ may thus be considered as this collection of eigenvalues. To obtain direct sum decompositions, it is useful to generalize slightly the concept of a simultaneous eigenvector. A nonzero vector v in M is called a weight vector, or generalized simultaneous eigenvector, if there exists an integer p such that $(x - \mu(x)I)^p v = 0$ for all x in L. The set of all weight vectors, together with zero, is a submodule $M_L{}^\mu$ of the module M called the weight module corresponding to the weight μ. In the weight submodule $M_L{}^\mu$ each element x in L is represented by an operator which differs from a multiple of the unit matrix by a nilpotent operator. Sophus Lie proved two theorems about the existence and completeness of weight vectors. Lie's first theorem says that any module over a solvable (and a fortiori any nilpotent or Abelian) Lie algebra has a weight μ. Lie's second theorem (also called the Lie-Zassenhaus theorem) asserts that for a nilpotent Lie algebra, the weight vectors are also complete, that is, we can find a basis for any module consisting of weight vectors. In other words, a module over a nilpotent Lie algebra is the direct sum of its weight submodules. For a semisimple Lie algebra L, we consider its Cartan subalgebra H. Since the Cartan subalgebra is Abelian, both of Lie's theorems will apply. Any module M over a semisimple Lie algebra L can then be written as the direct sum of its weight submodules with respect to H:

$$M = \bigoplus_{\mu} M_H{}^\mu.$$

We may regard L itself as a module over H via the adjoint representation, obtaining the decomposition of L as a direct sum of its root spaces $L_H{}^\alpha$. The Cartan subalgebra H itself is just the root space corresponding to $\alpha = 0$.

A useful property of weights and roots is their additivity with respect to tensor products. If $M_H{}^\mu$ and $N_H{}^\nu$ are weight modules over a nilpotent Lie algebra H, then the tensor product $M_H{}^\mu \otimes N_H{}^\nu$ is a weight module with weight $\mu + \nu$. To make this result plausible, we momentarily disregard the complications arising from the generalized eigenvalue problem, considering just the case $p = 1$. There exist vectors v and w in $M_H{}^\mu$ and $N_H{}^\nu$ respectively, such that $hv = \mu(h)v$ and $hw = \nu(h)w$ for all h in H. Then

$$\begin{aligned} h(v \otimes w) &= (hv) \otimes w + v \otimes (hw) \\ &= (\mu(h) + \nu(h))(v \otimes w). \end{aligned}$$

This shows that $\mu + \nu$ is at least one of the weights of the tensor product, but it does not prove that it is the only one. However, the proof that it is the only weight does not require any new ideas, but just more tedious algebra, and we omit it.

In general, if we have an arbitrary module M, and another one N, then the collection of weights for the tensor product $M \otimes N$ can be obtained as the collection of sums of a weight of M and a weight of N. That is, the weight diagram of a tensor product of two modules can be obtained from the weight diagrams of the two individual modules by a process of vector addition.

Suppose now that we have a semisimple Lie algebra L, with its Cartan subalgebra H. Let M be a module over L, and hence also a module over H. The action of L on M can be formally identified with the tensor product $L \otimes M$. In particular, the action of a root space $L_H{}^\alpha$ on a weight submodule $M_H{}^\mu$ is given by $L_H{}^\alpha M_H{}^\mu \subset M_H{}^{\mu+\alpha}$. This "shifting rule" says that if a root vector e_α acts on a weight vector v with weight μ, then $e_\alpha v$ is a weight vector corresponding to the shifted weight $\mu + \alpha$. Acting on a weight submodule $M_H{}^\mu$ repeatedly with e_α and $e_{-\alpha}$ we get weight submodules corresponding to a whole ladder of weights $\mu + z\alpha$, where $z = 0, \pm 1, \pm 2, \cdots$. The direct sum of the corresponding weight modules, which we may call a weight-ladder module, can be regarded as a module over the Lie algebra $H + L_H{}^\alpha + L_H{}^{-\alpha}$. Since we are dealing with finite-dimensional modules, all but finitely many of the weight modules corresponding to the infinite weight-ladder are trivial, and we really only need to consider a finite set $\mu - p\alpha, \cdots, \mu + q\alpha$.

Both weights and roots are vectors in H^*, the dual space of the Cartan subalgebra, and in fact it can be shown that the weights are all real linear combinations of the roots, and thus lie in $H_R{}^*$, the real vector space of such linear combinations. A positive definite inner product is induced on $H_R{}^*$ by the Killing form (cf. [4, pp. 26–27]), thereby making it a Euclidean space, the inner product of two vectors α and β being denoted by (α, β). The weight diagram of any module, as well as the root diagram of the Lie algebra, may thus be regarded as a set of points in a Euclidean space of l dimensions, l being the rank of the Lie algebra. Since the Euclidean space $H_R{}^*$ was given an order induced by ordering some arbitrary basis, the weights are also ordered, and since any given finite-dimensional module has only finitely many weights, there must be some weight which is higher than all the others. This weight is called the highest weight of the module. If v is a weight vector for the highest weight λ, then for any raising operator e_α with $\alpha > 0$, $e_\alpha v$ would be a weight vector for the higher weight $\lambda + \alpha$ if it were not zero. Thus it follows that a weight vector for a highest weight is an extreme vector. For an irreducible module, there is only one linearly independent weight vector for the highest weight. In general, the multiplicity of a weight μ is the dimension of the corresponding weight submodule: $n_\mu = \dim M_H{}^\mu$. Thus the highest weight of an irreducible module has multiplicity one. Two irreducible modules over a semisimple Lie algebra are isomorphic if and only if their highest weights are equal. Thus the problem of classifying the irreducible modules is that of finding all possible highest weights.

To study the structure of a given irreducible module with highest weight λ, we must find some way to calculate its weight diagram. The weight diagram has certain symmetry properties which we now study. If μ is any weight, and α any root, then the ladder of weights $\mu + z\alpha$ for $-p \leqq z \leqq +q$ also belongs to the weight diagram. It can be shown that $2(\mu, \alpha)/(\alpha, \alpha)$ is an integer and

$$\mu - \frac{2(\mu, \alpha)}{(\alpha, \alpha)} \alpha$$

is a weight in the ladder. This transformation may be interpreted geometrically as a reflection w_α in the plane perpendicular to the root α in H_R^*. The weight μ and the reflected weight $w_\alpha \mu$ have the same multiplicity. These reflections are called Weyl reflections, and the group generated by them is called the Weyl group. Note that $w_\alpha^2 = 1$, and $w_\alpha = w_{-\alpha}$ (cf. [4, p. 28]). The weight diagrams are invariant under the Weyl group, the multiplicities of weights are invariant, and the Killing form is invariant. If w is any element of the Weyl group W, then $(w\alpha, w\beta) = (\alpha, \beta)$. The root diagram is not only invariant under the Weyl group, but also under inversion $\alpha \to -\alpha$, and thus in general may have a higher degree of symmetry than some of the other weight diagrams.

Let $\alpha_1, \cdots, \alpha_l$ denote the simple roots of a semisimple Lie algebra L, that is, those positive roots which are not the sum of two positive roots. The corresponding Weyl reflections $w_i = w_{\alpha_i}$ are called simple Weyl reflections. The simple Weyl reflections are a set of generators for the whole Weyl group W.

We now study the set of vectors in H_R^* which can be highest weights for irreducible modules. If M_1 and M_2 are irreducible modules with extreme vectors x_1 and x_2 corresponding to the highest weights λ_1 and λ_2, respectively, then $x_1 \otimes x_2$ is an extreme vector in $M_1 \otimes M_2$ and $U^-(x_1 \otimes x_2)$ is an irreducible module having $\lambda_1 + \lambda_2$ as its highest weight. Thus the set of highest weights is closed under the addition. The submodule $U^-(x_1 \otimes x_2)$, sometimes denoted as $\overline{M_1 \otimes M_2}$, is called the Cartan composition of the modules M_1 and M_2. It is the irreducible submodule of the tensor product which has the highest weight.

We call a highest weight basic if it is not a sum of two other highest weights. For a semisimple Lie algebra of rank l, there are exactly l basic weights. The basic weights $\lambda_1, \cdots, \lambda_l$ can be indexed in such a way that they correspond to the l simple roots, the relation between them being

$$\frac{2(\lambda_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta_{ij},$$

where δ_{ij} is the Kronecker delta, zero if $i \neq j$, one if $i = j$. Note that under simple Weyl reflections we have $w_i\lambda_j = \lambda_j$ if $i \neq j$, while $w_i\lambda_i = \lambda_i - \alpha_i$. Thus each basic weight is invariant under all but one of the simple Weyl reflections.

Every highest weight λ is a linear combination of basic weights with nonnegative integral coefficients,

$$\lambda = n_1\lambda_1 + \cdots + n_l\lambda_l,$$

where the coefficients $n_i = 2(\lambda, \alpha_i)/(\alpha_i, \alpha_i)$ are called the components of the highest weight. Of course, any weight whatever, not just a highest weight, can be written in a similar way if we allow the components to be any integers, positive, zero or negative. In particular, the components of the roots α_j are given by the Cartan matrix $A_{ij} = 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i)$ and we may write $\alpha_j = \sum \lambda_i A_{ij}$ (cf. [4, p. 27]).

We can conversely use the Cartan composition process to construct an irreducible module corresponding to the highest weight $\lambda = \sum n_i \lambda_i$, given a set of basic modules $M_1, \cdots, M_l$. Let x_i in M_i be a weight vector corresponding to the basic highest weight λ_i. Take the tensor product

$$x = x_1 \otimes \cdots \otimes x_1 \otimes x_2 \otimes \cdots \otimes x_2 \otimes \cdots \otimes x_l,$$

where x_i appears n_i times. The Cartan composition $U^- x$ will then be an irreducible module having λ as its highest weight. Thus the process of Cartan composition reduces the problem of constructing the irreducible representations of a semisimple Lie algebra to that of constructing l basic irreducible representations.

Dynkin introduced a very convenient way of denoting any particular irreducible module over a semisimple Lie algebra [13], [32]. One simply writes over the ith vertex of the Dynkin diagram, which corresponds to the ith simple root, the nonnegative integer n_i giving the ith component of the highest weight with respect to a basis consisting of the basic weights. In Fig. 1 this is illustrated for a particular representation of the Lie algebra B_3, the complexification of the Lie algebra of $SO(7, R)$. This particular module is 8-dimensional, and can be used to define the exceptional Lie algebra G_2 (see [3]).

There are many different ways of denoting modules. In addition to the Dynkin diagram method, another common notation for modules is to write $\{N\}$, where N is the dimension of the module. In some cases, there are several nonisomorphic irreducible modules with the same dimension. In this case these may be distinguished by adding primes, asterisks, etc. The dual of the module $\{N\}$ would be denoted by $\{N^*\}$ in general. The main advantage of this notation is that it is concise, while the Dynkin diagram notation has the advantage of supplying more pertinent information which is useful for computations. We shall use both notations in the sequel.

The Dynkin diagram allows one to compute the angles between the simple roots and their relative lengths (cf. [4, p. 28]). The absolute lengths of the roots may be obtained from an observation of Brown [7] which says that the sum of the squares of the lengths of all of the roots of a semisimple Lie algebra is equal to the rank l. From this information one can then draw the root diagram, which is a set of points in a Euclidean space of l dimensions. From the formulas given above, one can construct geometrically or algebraically the basic weights. The only further

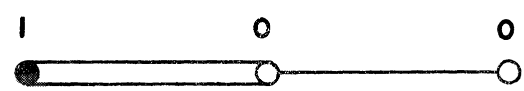

FIG. 1. *Dynkin diagram for the* 8-*dimensional spinor module of* B_3, *which can be used to define the exceptional Lie algebra* G_2

information we need for practical calculation then is an algorithm for computing the complete weight diagram of a module, given its highest weight.

2.5. Characters. The characters of group representations are useful as a general tool for such basic computations as finding the Clebsch-Gordan series. If we have a representation f of a group G by linear operators acting on a finite-dimensional vector space (module) M, then the character $\chi: G \rightarrow C$ of the representation is a complex-valued function defined on the group by taking traces:

$$\chi(g) = \mathrm{Tr}_M f(g) = \sum_{i=1}^{\dim M} f_{ii}(g)$$

for all g in G. Since the trace of the linear transformation $f(g)$, i.e., the sum of its diagonal matrix elements $f_{ii}(g)$, is equal to the sum of its eigenvalues, counting multiple roots of the characteristic equation, the trace does not depend on the choice of basis in M with respect to which the matrix elements are computed. The character χ of a representation thus is invariant under changes of basis in the module M. It is clear that isomorphic modules (equivalent representations) have the same character, since the trace is invariant under similarity transformations.

If the Lie group corresponding to a semisimple Lie algebra is compact and connected, then every element of the group is conjugate to an element of the "maximal torus" subgroup corresponding to a given Cartan subalgebra of the Lie algebra [15]. Elements g_1 and g_2 are said to be conjugate, we recall, if there exists some element s in the group for which $g_2 = sg_1s^{-1}$. It is clear that the character takes the same value at conjugate elements of the group. Thus the character is determined by its values on such a maximal torus subgroup. Conversely, given a semisimple Lie algebra, we can start with its compact real form, and by exponentiation arrive at a compact, connected Lie group. Accordingly, for a semisimple Lie algebra, we may define a character χ for a module M to be the function defined on a Cartan subalgebra by taking traces of exponentials:

$$\chi(h) = \mathrm{Tr}_M (\exp h).$$

Here Tr_M denotes the trace as an operator on the particular module M being considered. (The exponential function can be defined by its Taylor series, which is convergent here because we can regard h as being a linear transformation in a finite-dimensional vector space M.)

We note here some of the elementary properties of characters. If we set $h = 0$, we get the trace of the unit operator in the module M, which is equal to the dimension of the module

$$\chi(0) = \dim M.$$

The character χ of the direct sum $M = M_1 \oplus M_2$ of two modules M_1 and M_2 is the sum of their respective characters χ_1 and χ_2 :

$$\chi(h) = \chi_1(h) + \chi_2(h).$$

This may be proved by choosing a basis in the direct sum which is the union of bases in the summands. The character χ of the tensor product $M = M_1 \otimes M_2$ of two modules is similarly seen to be the product of the characters of the two modules:

$$\chi(h) = \chi_1(h) \cdot \chi_2(h).$$

One way to prove this is by direct computation, substituting

$$h^n(x \otimes y) = \sum_{r=0}^{n} \frac{n!}{r!(n-r)!} (h^r x) \otimes (h^{n-r} y)$$

into

$$\exp h = \sum_{n=0}^{\infty} \frac{h^n}{n!},$$

and interchanging orders of summation. Another way is to note that if $hx = \lambda x$ and $hy = \mu y$, then $h(x \otimes y) = (\lambda + \mu)(x \otimes y)$, and hence $e^h(x \otimes y) = e^{\lambda+\mu}(x \otimes y)$, while $e^h x = e^\lambda x$ and $e^h y = e^\mu y$. One then uses the fact that the trace is equal to the sum of the eigenvalues.

The representation matrices in the dual of a given module are obtained from those in the original module by taking the negative transpose. Since the trace is unaffected by the operation of taking tranposes, we see that

$$\chi_{M^*}(h) = \chi_M(-h).$$

Finally, we note that since any module M is the direct sum of its weight submodules M_H^μ, the trace is given by

$$\chi(h) = \mathrm{Tr}_M \exp h = \sum_\mu \mathrm{Tr}_{M_H^\mu} \exp h.$$

Since $hx = \mu(h)x$ for all x in M_H^μ and all h in H, we have $e^h = e^{\mu(h)}1$ in M_H^μ, and since the trace of the unit operator gives the dimension $n_\mu = \dim M_H^\mu$, we have

$$\chi(h) = \sum_{\mu \in \Lambda} n_\mu e^{\mu(h)},$$

where Λ is the weight diagram of the module M. This formula shows that a knowledge of the character of a module is equivalent to knowing its weight diagram and the multiplicity of each weight. This formula may be used in two ways. If we know how to compute the multiplicities of the weights by means of some algorithm, then we can use this formula to calculate the characters, and from this we can obtain the Clebsch-Gordan series. Alternatively, if we have some formula for computing the characters, then we can use the above formula to calculate the weight diagrams and multiplicities of the irreducible modules. A further discussion of such computational methods will be given in the third paper, where we discuss Freudenthal's algorithm and Weyl's character formula. For rank two simple Lie algebras some of these calculations have been carried out by Behrends, Dreitlein, Fronsdal and Lee [3].

2.6. Application to irreducible representations of A_1. We now apply some of the preceding ideas to compute the irreducible representations of A_1, the complexification of the real Lie algebra of the ordinary rotation group $SO(3, R)$. Similar formulas also apply to representations of $A_1 \oplus A_1$ which is the complexification of the real Lie algebra of the usual homogeneous Lorentz group $SO(3, 1; R)$.

The Lie algebra A_1 has the structure discussed above in §1.4, with basis $h = j_3$, $e_{\pm\alpha} = j_1 \pm ij_2 = j_\pm$, satisfying

$$[h, e_\alpha] = e_\alpha, \quad [h, e_{-\alpha}] = -e_{-\alpha}, \quad [e_\alpha, e_{-\alpha}] = 2h.$$

A Cartan subalgebra H has basis h. The roots are 0, $\pm\alpha$, where $\alpha(h) = 1$.

Since the sum of the squares of the lengths of the roots is equal to the rank, which is one, we have $(\alpha, \alpha) = \frac{1}{2}$. There is one basic weight λ in the one-dimensional space H_R^*. It must be some multiple of α satisfying $2(\lambda, \alpha)/(\alpha, \alpha) = 1$. Consequently, $\lambda = \alpha/2$.

By means of a Weyl reflection, the basic weight λ goes over into its negative $-\lambda$. Thus the basic module has a weight diagram consisting of at least these two weights. Let x be a weight vector corresponding to λ, so that

$$hx = \tfrac{1}{2}x.$$

The raising and lowering algebras U^+ and U^- consist, respectively, of all polynomials in e_α and $e_{-\alpha}$. Since x is an extreme vector, we have

$$e_\alpha x = 0.$$

Since $e_\alpha e_{-\alpha} x = [e_\alpha, e_{-\alpha}]x + e_{-\alpha}e_\alpha x = 2hx + 0 = x$, it follows that

$$y = e_{-\alpha}x$$

is nonzero. It is clearly a weight vector corresponding to the shifted weight $-\lambda = \lambda - \alpha$. We also have

$$e_{-\alpha}y = 0,$$

because if it were not zero, then we would obtain a weight lower than $-\lambda$, and hence by Weyl reflection a weight higher than λ. Hence $U^- x$ is spanned just by the two vectors x and y. The basic module over A_1 is therefore a two-dimensional module spanned by x and y with

$$hx = \tfrac{1}{2}x, \quad hy = -\tfrac{1}{2}y,$$

$$e_\alpha x = e_{-\alpha}y = 0,$$

$$e_{-\alpha}x = y, \quad e_\alpha y = x.$$

It is called the spin half representation [29].

Every irreducible module over A_1 is generated by U^- (i.e., repeated applications of $e_{-\alpha}$) acting on a tensor product $x \otimes x \otimes \cdots \otimes x$. Each irreducible module is therefore characterized by a single integer, n, the rank of the tensor $x \otimes x \otimes \cdots \otimes x$ (i.e., the number of repetitions of x). The tensor $x \otimes x$

$\otimes \cdots \otimes x$ is the highest weight of the module and since

$$\begin{aligned} h(x \otimes x \otimes \cdots \otimes x) &= (hx) \otimes x \otimes \cdots \otimes x + x \otimes (hx) \otimes x \otimes \cdots \otimes x \\ &\qquad + \cdots + x \otimes x \otimes \cdots \otimes hx \\ &= \frac{n}{2}(x \otimes x \otimes \cdots \otimes x), \end{aligned}$$

the highest weight for the module is $n/2 = j$. The number $j = n/2$ is called the spin in the quantum mechanical theory of angular momentum. The entire module structure can readily be calculated. For example, in the case $n = 2$ (spin one), we easily compute

$$\begin{gathered} h(x \otimes x) = x \otimes x, \quad e_\alpha(x \otimes x) = 0, \\ e_{-\alpha}(x \otimes x) = x \otimes y + y \otimes x, \\ e_{-\alpha}(x \otimes y + y \otimes x) = 2y \otimes y, \\ h(x \otimes y + y \otimes x) = 0, \\ e_\alpha(x \otimes y + y \otimes x) = 2x \otimes x, \\ e_\alpha(y \otimes y) = x \otimes y + y \otimes x, \quad h(y \otimes y) = -y \otimes y, \\ e_{-\alpha}(y \otimes y) = 0. \end{gathered}$$

For symmetry, it is convenient to take as basis for the module

$$u_1 = x \otimes x, \quad u_0 = \frac{1}{\sqrt{2}}(x \otimes y + y \otimes x), \quad u_{-1} = y \otimes y.$$

More generally, for the module M_j of spin j (highest weight $j = n/2$), we can compute a basis of $2j + 1$ vectors u_m, $m = -j, -j + 1, \cdots, +j$, with

$$\begin{aligned} u_j &= x \otimes x \otimes \cdots \otimes x \quad (n = 2j \text{ factors}), \\ u_{-j} &= y \otimes y \otimes \cdots \otimes y \end{aligned}$$

and

$$\begin{aligned} hu_m &= mu_m, \\ e_\alpha u_m &= \sqrt{(j - m)(j + m + 1)}\,u_{m+1}, \\ e_{-\alpha} u_m &= \sqrt{(j + m)(j - m + 1)}\,u_{m-1}. \end{aligned}$$

These formulas play a basic role in the theories of the representations of the ordinary rotation group and of the homogeneous Lorentz group (see [14], [28]).

2.7. Application to irreducible representations of A_2. The simple Lie algebra A_2 is the complexification of the real Lie algebra of the special unitary group $SU(3)$. This particular group has recently been studied extensively in applications dealing with the strongly interacting elementary particles. As a consequence there now exist excellent review articles devoted just to the study of the representations of this algebra alone [10]. We recall from our earlier study (cf. [4, p. 32]) that the root diagram of the Lie algebra A_2 has six nonzero roots of equal length, forming the six vertices of a regular hexagon, together with the zero root located at the center of the hexagon, having multiplicity two (cf. Fig. 2). Since the sum of the squares of the lengths of all the roots is equal to the rank, which is two, it is clear that each nonzero root has length $1/\sqrt{3}$. The inner product on the Cartan subalgebra, i.e., the Killing form, can now be computed in terms of the simple roots α_1 and α_2, since we know that the angle between them is 120°. We thus find

$$(\alpha_1, \alpha_1) = (\alpha_2, \alpha_2) = \tfrac{1}{3},$$

$$(\alpha_1, \alpha_2) = -\tfrac{1}{6}.$$

The equations defining the basic weights λ_1 and λ_2 can be solved either algebraically or geometrically. For example, the equations say that λ_1 is perpendicular to α_2, and its projection onto α_1 is half the length of α_1 (cf. Fig. 2).

Since every representation can be obtained from the two basic representations, we naturally study these. The representation which corresponds to the basic weight λ_2 is just the dual of the one corresponding to λ_1. In the applied literature dealing with the elementary particles, these are known as the quark and antiquark representations. (The word "quark" was coined by M. Gell-

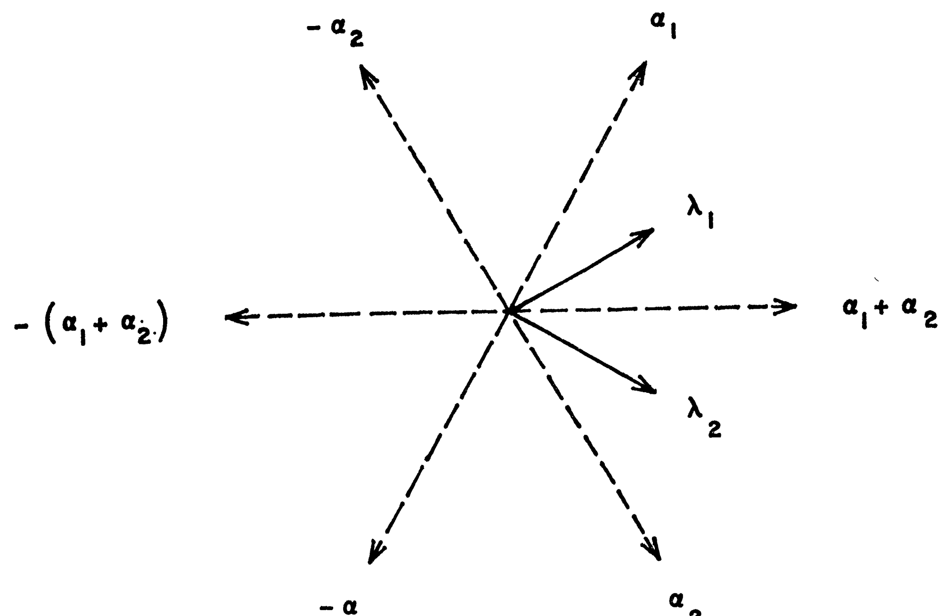

FIG. 2. *Basic weights of the simple Lie algebra A_2 shown superimposed on the root diagram*

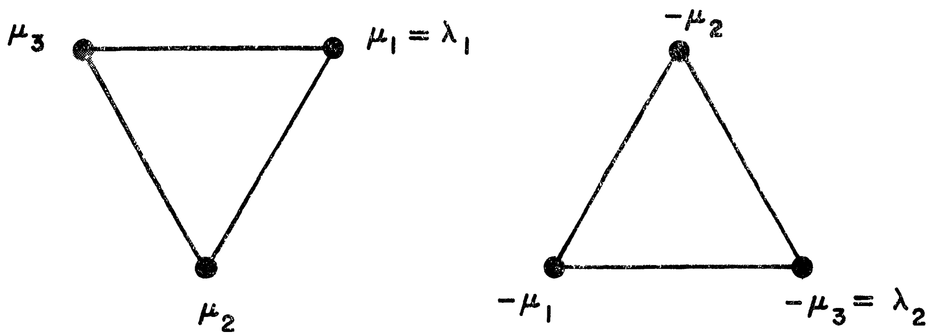

FIG. 3. *Weight diagrams of the two basic modules of* A_2

Mann, who took the word from a poem appearing in James Joyce's famous work *Finnegan's Wake* [18].)

Let us consider now the basic module with highest weight λ_1. By means of the Weyl group it is easy to show that the complete weight diagram of this module is an equilateral triangle, consisting of three weights μ_1, μ_2 and μ_3, as shown in Fig. 3. Let us use the canonical basis for A_2 (given in [4, p. 27]), consisting of the simple raising operators $e_1 = e_{\alpha_1}$, $e_2 = e_{\alpha_2}$, the simple lowering operators f_1 and f_2 proportional to $e_{-\alpha_1}$ and $e_{-\alpha_2}$, respectively, and their commutators $e_{12} = [e_1, e_2]$ and $f_{12} = [f_1, f_2]$, and elements h_1 and h_2 proportional to h_{α_1} and h_{α_2}, respectively. For the basic module M_1 we choose a basis x_1, x_2, x_3 consisting of weight vectors corresponding to the weights μ_1, μ_2, μ_3, respectively. Having chosen the extreme vector x_1, it is clearly possible to define x_2 and x_3 by the equations

$$x_2 = f_1 x_1,$$

$$x_3 = f_2 x_2.$$

This serves to fix the relative normalization of the three basis vectors for the module. It is now possible to use the commutation relations of the Lie algebra A_2 (cf. [4, p. 31]) to work out explicitly the matrices for the representation. We also need to note here that $\lambda_i(h_j) = \delta_{ij}$ and $h_i x_j = \mu_j(h_i)x_j$. Note that $\mu_1 = \lambda_1$, $\mu_2 = \lambda_2 - \lambda_1$ and $\mu_3 = -\lambda_2$. As a sample calculation, we note that

$$\begin{aligned} e_1 x_2 &= e_1 f_1 x_1 = [e_1, f_1]x_1 + f_1 e_1 x_1 \\ &= h_1 x_1 + 0 = \lambda_1(h_1)x_1 = x_1. \end{aligned}$$

Since the shifting rule $L_H^{\alpha} M_H^{\mu} \subset M_H^{\mu+\alpha}$ allows us to conclude from inspection of the weight diagram that $e_1 x_1 = e_1 x_3 = 0$, it follows that the matrix corresponding to e_1 is a matrix whose only nonzero entry is its entry in the first row and second column. A complete calculation, which we leave to the reader to verify, leads to the following matrix representation in the basic module M_1 over A_2 :

$$e_1 \rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad e_2 \rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad e_{12} \rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$f_1 \rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad f_2 \rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad f_{12} \rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix},$$

$$h_1 \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad h_2 \rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

For the dual module $M_2 \simeq M_1^*$, the weight diagram is the triangle obtained from the weight diagram of the original module M_1 by inversion in the origin. Thus if μ_1, μ_2 and μ_3 are the weights for the quark module, then $-\mu_1$, $-\mu_2$ and $-\mu_3$ are the weights for the antiquark module. The representation matrices in the dual module can be obtained as the negative transposes of the representation matrices calculated above, provided we use the corresponding dual basis x_1^*, x_2^*, x_3^* in the dual module M_1^*.

The representation matrices in an arbitrary irreducible representation of A_2 can be obtained by a similar calculation to the one performed above. A general formula can also be derived, making use of the second and third order Casimir operators for A_2 (see [1]).

REFERENCES

[1] G. E. BAIRD AND L. C. BIEDENHARN, JR., *On the representations of semisimple Lie groups. II*, J. Mathematical Phys., 4 (1963), pp. 1449–1466.

[2] V. BARGMANN, *Irreducible unitary representations of the Lorentz group*, Ann. Math., 48 (1947), pp. 568–642.

[3] R. E. BEHRENDS, J. DREITLEIN, C. FRONSDAL AND B. W. LEE, *Simple groups and strong interaction symmetries*, Rev. Modern Phys., 34 (1962), pp. 1–40.

[4] J. G. BELINFANTE, B. KOLMAN AND H. A. SMITH, *An introduction to Lie groups and Lie algebras, with applications*, this Review, 8 (1966), pp. 11–46.

[5] L. C. BIEDENHARN, JR. AND W. J. HOLMAN, III, *Complex angular momenta and the groups SU*(1, 1) *and SU*(2), Ann Physics, 39 (1966), pp. 1–42.

[6] R. BIVINS, N. METROPOLIS, M. ROTENBERG AND J. K. WOOTEN, JR., *The 3-j and 6-j Symbols*, The M.I.T. Press, Cambridge, Massachusetts, 1959.

[7] G. BROWN, *A remark on semisimple Lie algebras*, Proc. Amer. Math. Soc., 15 (1964), p. 518.

[8] C. CHEVALLEY, *Theory of Lie Groups*, vol. I, Princeton University Press, Princeton, 1946.

[9] F. CHILTON AND P. MCNAMEE, *Tables of Clebsch-Gordan coefficients of SU*(3), Rev. Modern Phys., 36 (1964), pp. 1005–1024.

[10] J. J. DESWART, *The octet model and its Clebsch-Gordan coefficients*, Ibid., 35 (1963), pp. 916–939. (Erratum, Ibid., 37 (1965), p. 326.)

[11] N. DUNFORD AND J. T. SCHWARTZ, *Linear Operators*, Interscience, New York, 1963.

[12] E. B. DYNKIN, *Semisimple subalgebras of semisimple Lie algebras*, Amer. Math. Soc. Transl. (2), 6 (1957), pp. 111–244.

[13] ———, *The maximal subgroups of the classical groups. Supplement: Survey of the basic*

concepts and facts in the theory of linear representations of semisimple Lie algebras, Ibid., 6 (1957), pp. 245–378.

[14] A. R. EDMONDS, *Angular Momentum in Quantum Mechanics*, Princeton University Press, Princeton, New Jersey, 1957.

[15] H. FREUDENTHAL, *Lie Groups*, Lecture Notes, Yale University, New Haven, 1961.

[16] F. GANTMACHER, *Canonical representations of automorphisms of a complex semisimple Lie group*, Mat. Sb., 5 (1939), pp. 101–146.

[17] ——— , *On the classification of real simple Lie groups*, Ibid., 5 (1939), pp. 217–249.

[18] M. GELL-MANN, *A schematic model of baryons and mesons*, Phys. Lett., 8 (1964), pp. 214–215.

[19] HARISH-CHANDRA, *On representations of Lie algebras*, Ann. Math., 50 (1949), pp. 900–915.

[20] S. HELGASON, *Differential Geometry and Symmetric Spaces*, Academic Press, New York, 1962.

[21] R. HERMANN, *Some properties of representations of noncompact groups*, High-Energy Physics and Elementary Particles, Lectures Presented at Trieste, 1965, International Atomic Energy Agency, Vienna, 1965, pp. 625–639.

[22] ——— , *Lie Groups for Physicists*, Benjamin, New York, 1966.

[23] G. HOCHSCHILD, *The Structure of Lie Groups*, Holden-Day, San Francisco, 1965.

[24] N. JACOBSON, *Lie Algebras*, Interscience Tracts in Pure and Applied Mathematics, no. 10, Interscience, New York, 1962.

[25] G. W. MACKEY, *Infinite-dimensional group representations*, Bull. Amer. Math. Soc., 69 (1963), pp. 628–686.

[26] A. I. MAL'CEV, *On semisimple subgroups of Lie groups*, Amer. Math. Soc. Transl. (1), 9 (1962), pp. 172–213.

[27] M. A. NAIMARK, *Normed Rings*, Noordhoff, Groningen, The Netherlands, 1959.

[28] ———, *Linear Representations of the Lorentz Group*, Pergamon Press, London, 1964.

[29] W. PAULI, JR., *Continuous groups in quantum mechanics*, Springer Tracts in Modern Physics, 37 (1965), pp. 85–104.

[30] L. PONTRYAGIN, *Topological Groups*, Gordon and Breach, New York, 1966.

[31] L. PUKÁNSZKY, *On the Kronecker products of irreducible representations of the 2 × 2 real unimodular group I*, Trans. Amer. Math. Soc., 100 (1961), pp. 116–152.

[32] P. A. ROWLATT, *Group Theory and Elementary Particle Physics*, American Elsevier, New York, 1966.

[33] T. SATAKE, *On representations and compactifications of symmetric spaces*, Ann. Math., 71 (1960), pp. 77–110.

[34] J. A. SCHOUTEN, *Tensor Analysis for Physicists*, Clarendon Press, Oxford, 1954.

[35] G. F. SIMMONS, *Introduction to Topology and Modern Analysis*, McGraw-Hill, New York, 1963.