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Classical geometries defined by exterior differential systems on higher frame bundles†

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Abstract. Exterior differential ideals are discussed, and sets of invariant generators presented, for Riemannian, conformal and projective geometries, and for specialisations such as Ricci-flat, self-dual and Einstein–Maxwell theories. The Cartan characteristic integers are explicitly calculated, and involutory basis forms found, for each of these (specialised to four dimensions), exposing their algebraic structure and showing how they generate well-posed sets of partial differential equations.

1. Introduction

Cartan's method of moving frames may best be understood as an application of another of his seminal techniques: the discovery of well-posed exterior differential systems to represent coupled sets of first-order partial differential equations [1]. In this paper we present Riemannian (and several other) geometries as sub-bundles of the first (or sometimes higher) frame bundle, i.e. as so-called G-structures, and in particular show that these sub-bundles are determined by well-posed invariant exterior differential systems [2] on the second (or sometimes higher) frame bundle. The integrability properties, transformations, invariances and other diagnostics of an exterior differential system can be studied by explicit calculation of its Cartan integer characters, by derivation of characteristic vectors and isovectors, and by discovery of conservation forms and prolongation forms involving auxiliary potential and pseudopotential fields [3]. The systematic application of this differential form analysis to moving frame systems may lead to new results for Riemannian geometries and gravitational field physics, and for invariant classification of equivalent solutions.

In § 2 the basic concept of a well-posed exterior system is briefly explained, with emphasis on the diagnostic test by calculations of Cartan integer characters and involutive bases. Because of the large dimensionality of higher frame bundles, we have developed computer programs to calculate these characters from the ranks of successive sets of the homogeneous linear equations involved. In § 3 we present exterior differential systems for a number of four-dimensional geometries. Calculation of the Cartan integer characters shows these systems indeed to be well-posed and involutory with respect to the usually chosen independent variables.

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Spinor notations, and specifically the Clifford algebra formalism introduced in § 4, are especially convenient for abstractly manipulating the 'pulled back' differential form structures for Riemannian and conformal geometries. In § 5 this is used to find a set of non-linear prolongation forms that expresses, and perhaps generalises, the coupled Einstein–Maxwell equations. The Cartan integer character analysis of this is given in four dimensions and shows that the system is indeed well-posed.

2. Cartan integer characters and well-posed systems, Cauchy characteristic vectors, and specialisation and prolongation

Consider a closed differential ideal I in an n-dimensional manifold, generated by sets of 1-forms α, 2-forms β, 3-forms γ, and so on, all of which are explicitly given in a local basis of 1-forms, say ω^i, $i=1,\ldots,n$. This frame need not be 'holonomic'—derived from a coordinate patch (in which case we would have $\omega^i = \mathrm{d}x^i$). If the ideal is *invariantly* given, the expansion of this set of generators of I, $\{\alpha, \beta, \gamma, \ldots\}$, on the basis ω^i has constant numerical coefficients.

The simplest integral submanifolds of I are one dimensional; denote a generic one of these by $\mathscr{V}_1$. They can each in principle be found by integration, beginning at an arbitrary initial point with a vector, say $\boldsymbol{V}_1$, chosen so that at successive points along the resultant trajectory $\mathscr{V}_1$

$$\boldsymbol{V}_1 \lrcorner \alpha = 0 \qquad \text{rank } s_0. \tag{1}$$

This is a set of linear homogeneous equations for the components of $\boldsymbol{V}_1$—in any independent basis—of rank, say s_0 (just the number of independent 1-forms α). So a choice of $\mathscr{L}_1 = n - s_0$ functions of an autonomous variable y^1 will thus have to be made.

We could express the result as an equivalent set of autonomous ordinary differential equations for the trajectory in terms of holonomic components of $\boldsymbol{V}_1$, i.e. if $V_1^i = \boldsymbol{V}_1 \lrcorner \mathrm{d}x^i$

$$\mathrm{d}x^i/\mathrm{d}y^1 = V_1^i \qquad \boldsymbol{V}_1 \lrcorner \mathrm{d}y^1 = 1. \tag{2}$$

The result is an explicit map of $\mathscr{V}_1 \to X$, the manifold of x^i. We could normalise y^1 as shown, and finally substitute back to find the induced map $\boldsymbol{V}_1 \to \partial/\partial y^1$.

Now use $\mathscr{V}_1$ as an initial boundary for a second set of integrations to find a one-parameter family of trajectories that will form a two-dimensional integral manifold $\mathscr{V}_2$. Start at each point of $\mathscr{V}_1$, labelled with y^1, and there, and at successive points along each resultant trajectory, choose a vector $\boldsymbol{V}_2$ such that

$$\left.\begin{aligned} &\boldsymbol{V}_2 \lrcorner \alpha = 0 \\ &\boldsymbol{V}_2 \lrcorner \boldsymbol{V}_1 \lrcorner \beta = 0 \end{aligned}\right\} \qquad \text{rank } s_0 + s_1. \tag{3}$$

Given $\boldsymbol{V}_1$, this is a set of linear homogeneous equations for the components of $\boldsymbol{V}_2$. The rank cannot be less than s_0, so with Cartan we denote it $s_0 + s_1$. Now one solution of (3) can be just $\boldsymbol{V}_1$, so for the construction we must have $\mathscr{L}_2 = n - s_0 - s_1 > 1$, and then that many functions of a second autonomous variable, y^2, can be chosen during the integrations along the trajectories of $\boldsymbol{V}_2$. This results in a two-dimensional integral manifold $\mathscr{V}_2$, as clearly from (1) and (3) all of I vanishes when restricted to $\mathscr{V}_2$.

This second set of integrations along $\boldsymbol{V}_2$ may not appear to have been completely specified, however, as we only had previously determined $\boldsymbol{V}_1$ at a boundary. We also need the components of $\boldsymbol{V}_1$, as parameters, at each point of the integrations along the trajectories of $\boldsymbol{V}_2$. Cartan's deep insight was that, since we are working with a *closed* ideal I, it is consistent to require $[\boldsymbol{V}_1, \boldsymbol{V}_2] = 0$ as we go. This determines $\boldsymbol{V}_1$ as being 'dragged along' the V_2 congruence. Now since

$$\pounds_{V_2}(V_1 \lrcorner \alpha) = [V_2, V_1] \lrcorner \alpha + V_1 \lrcorner (\mathrm{d}(V_2 \lrcorner \alpha) + V_2 \lrcorner \mathrm{d}\alpha) \tag{4}$$

we see that all three right-hand terms vanish by our construction, and so the condition $V_1 \lrcorner \alpha = 0$, true initially, is itself dragged along V_2 and preserved. V_1 is thus simultaneously constructed throughout the two-dimensional manifold as everywhere belonging to one-dimensional integral manifolds, as indeed we initially took it. The result is that one has intersecting families of one-dimensional integral manifolds $\mathcal{V}_1$ from both V_1 and V_2, that these are 2-forming, and that the 2-manifolds $\mathcal{V}_2$ are also integral manifolds. Normalised autonomous variables y^1 and y^2 can be introduced by writing the holonomic components of the vectors as $V_1^i = \partial x^i/\partial y^1$, $V_2^i = \partial x^i/\partial y^2$ and the construction guarantees that these are consistent. They map V_1 and V_2 to $\partial/\partial y^1$ and $\partial/\partial y^2$.

The construction of three-dimensional integral manifolds proceeds entirely analogously. This time we begin with a bounding 2-manifold $\mathcal{V}_2$, everywhere containing V_1 and V_2, and search for a V_3 at each point of $\mathcal{V}_2$ such that

$$\begin{aligned} &V_3 \lrcorner \alpha = 0 \\ &V_3 \lrcorner V_1 \lrcorner \beta = 0 \\ &V_3 \lrcorner V_2 \lrcorner \beta = 0 \\ &V_3 \lrcorner V_2 \lrcorner V_1 \lrcorner \gamma = 0. \end{aligned} \tag{5}$$

The rank is denoted $s_0 + s_1 + s_2$, so s_2 must be $\geqslant 0$, and we can proceed if $\mathcal{L}_3 > 2$. As we integrate trajectories of V_3, we drag the integral 2-manifold $\mathcal{V}_2$ along by $[V_3, V_1] = 0$, $[V_3, V_2] = 0$, which preserves $[V_1, V_2] = 0$ and satisfies equations analogous to (4) for its propagation along V_3.

Integral manifolds constructed in this way, from nested integral manifolds of lower dimensionality, are called *regular.* Not all integral manifolds are regular—various 'singular' manifolds can also exist. But in any event, we recognise with Cartan that the positive integers s_0, s_1, s_2 are *numerical concomitants* of the closed ideal I and demonstrate in principle the integration of generic solutions.

Now at each integration we add more linear equations and further construction can only become more constrained, $\mathcal{L}_p \leqslant \mathcal{L}_{p-1}$. But if we have a $(p-1)$-dimensional integral manifold, we of course have $p-1$ trivial solutions of the linear homogeneous equations for V_p—thus we need $\mathcal{L}_p > p-1$ to proceed to construct $\mathcal{V}_p$. The process must terminate, so the regular integral manifolds of I must have a maximum dimension: this is Cartan's genus, g. If $\mathcal{L}_g > g-1$ but $\mathcal{L}_{g+1} \leqslant g$ we cannot proceed past g dimensions. This says in particular that if

$$\mathcal{L}_{g+1} = \mathcal{L}_g = n - (s_0 + \ldots s_{g-1}) = g \qquad \text{i.e. } s_g = 0 \tag{6}$$

there is no freedom left—no arbitrary function other than normalisation of V_g or the last autonomous variable y^g—in the final construction of maximum dimensional integral manifolds, $\mathcal{V}_g$.

Although there is no unique ideal I to represent a given set of partial differential equations, limiting the ideals considered to those that satisfy the criterion of (6) makes the choice of I as a practical matter quite limited. We denote such ideals as being 'well-posed'. The associated sets of partial differential equations will include all integrability conditions, and will be neither over nor under determined. All the geometric systems we deal with below will satisfy this criterion.

To summarise, determining 'well-posedness' and the Cartan characters requires calculation of the ranks of successive sets of equations such as those given in (1), (3) and (5). At each step these are linear homogeneous equations for the components of the new vector. However, the coefficients in the equations which are derived from forms of degree >1 depend on the components of all previous vectors, as in the β and γ equations of (5), and those components have themselves already been required to jointly annul all the forms in the ideal at previous steps. Consequently, they are subject to both linear and non-linear constraint equations, which must be taken into account in calculating the rank of the new equations. In general, p-degree forms will lead to pth-order polynomial constraints, so if one attempts to obtain a general solution, this nested algebraic problem is not in fact a linear one. Is this a serious problem?

As an example from what follows, consider the ideal of (12), which is set on the manifold of the second frame bundle over a base space of dimension m; its dimension n is $\frac{1}{2}m(m+1)(m+2)$, so if $m=4$, the vectors V_p will each have 60 components. Calculating the ranks of equations solving the ideal, when we come to consider the equations for V_6 we would have 100 quadratic constraints and 40 cubic constraints involving the 300 components of the previous five vectors $V_1, \ldots, V_5$! In special situations, of course, it may well be that symmetry properties can be used to obtain the answer easily. But in a general case it is unlikely that even a highly sophisticated algebraic manipulation system would be capable of correctly evaluating the rank of the equations for V_6 in the presence of such constraints.

Since the rank evaluation need only be done at a single *generic* point of the manifold, we can finesse the algebraic problem by calculating particular solutions using fixed numerical components $V_p \lrcorner \omega^j$. At each step these components are determined to satisfy the appropriate linear equations at that step, and any components left undetermined by the equations are assigned *random* numerical values. This purely numerical problem is now linear throughout and standard techniques to determine the ranks of linear equations can be used. We also have elected to assign integer values to the vector components and to do the numerical calculations using integer arithmetic. While not really essential, this does avoid the possible problem of having fortuitously small values become zero due to numerical round-off errors.

Clearly, this technique suffers from the difficulty of any Monte Carlo approach in that it may not give the generic answer in any single calculation. A particular set of assigned random numerical components may here give a lower rank than is true in general. This does in fact happen, but quite infrequently, since the large number of random components provides a very large ensemble of random vectors. Furthermore, even on a personal computer the calculation only requires from a few seconds to a few minutes, depending on size, so it can easily be repeated many times over to check for accidental degeneracies.

It is also important to explicitly calculate which basis 1-forms remain independent when restricted to the maximal dimension integral submanifolds $\mathcal{V}_g$. A set of g of these—neither they nor any linear combinations of them vanishing—are then a suitable 1-form basis to span $\mathcal{V}_g$. Cartan denotes these as *involutory*, and when holonomic they belong to a possible set of independent variables for a set of partial differential equations locally equivalent to I. The involutory property of a particular set of g bases can also be obtained from the Monte Carlo calculation. This can be done simply by modifying the program to require that the first g vectors span the g-dimensional subspace of those bases. When g such non-zero vectors result, involution of the corresponding bases is demonstrated.

Cauchy characteristic vectors are vector fields V such that $V \lrcorner I \subset I$. This is a set of linear homogeneous equations for the n components of V. The set of all independent

Cauchy characteristic vectors V_a, $a = 1, \ldots, q$, generates a q-dimensional fibration and Cartan has shown that, in fact, a set of generators for I can be found involving q fewer basis 1-forms (and, for a non-invariantly expressed ideal, q fewer explicit coordinate dependences). If inspection shows some of the basis forms not to be explicitly present in the set of generators of an invariant (closed) ideal, for each of these one immediately has a Cauchy characteristic vector. The consequence of importance here, however, is that by their defining property these Cauchy characteristic vectors satisfy all the successive equations (1), (3), (5), etc, of the construction of regular integral manifolds, could have been used at any of the steps, *and must all be present in the maximal manifolds* $\mathcal{V}_g$. At least locally, each $\mathcal{V}_g$ has the structure of a fibre bundle: q-dimensional fibres over a $(g-q)$-dimensional base. Although the Cauchy characteristic vectors solve homogeneous equations and so are undetermined in length, it is, as we will see later, sometimes possible to explicitly 'calibrate' them to give $\mathcal{V}_g$ the further structure of a principal fibre bundle.

We have previously introduced a kind of prolongation [3] for sets of partial differential equations with $g = 2$ (two independent variables), in which auxiliary 1-forms, say α', are adjoined to the ideal I, to form an ideal $I' = \{I, \alpha'\}$ that is still closed and well-posed. These also involve new variables, which become essentially potential fields (or, most generally, pseudopotentials or 'Miura' transforms) in the integral manifolds of I'. This prolongation can now be formulated in the general case. We seek to adjoin additional forms, say α', β', $\gamma, \ldots$, of degrees 1, 2, 3, etc, involving n' additional coordinates $x' = \{x^{n+1}, \ldots, x^{n+n'}\}$ and basis 1-forms, to construct an ideal $I' = \{I, \alpha', \beta', \ldots\}$ with the following properties: (i) $\mathrm{d}I' \subset I'$, (ii) $\mathcal{L}'_1 \geq \mathcal{L}_1$, $\mathcal{L}'_2 \geq \mathcal{L}_2$, $\mathcal{L}'_3 \geq \mathcal{L}_3$, etc, (iii) $g' = g$ and (iv) I' and I involutory with respect to the same g bases. In a solution manifold, the adjoined prolongation variables are seen as auxiliary fields, functions of the same independent variables. We again calculate the numerology with the Monte Carlo programs.

If the second of these conditions is not fulfilled, but the others are, the enlarged exterior differential system I' is itself still well-posed and of interest in its own right. Since the freedom of the Cartan construction is at some point limited, however, solutions of such an I' must be *specialised*, i.e. must be a consistent and well formulated *subset* of solutions of the ideal I, while at the same time an interesting set of auxiliary or pseudopotential variables is now also solved for. Examples of this, as we will see, are the Ricci-flat and Einstein–Maxwell solutions, particular specialised *subsets* of general Riemannian geometries. The ideal I' for the latter of these involves six auxiliary variables, $n' = 6$, and two additional 3-forms γ, and explicit calculation has shown it to be well-posed.

Several examples of true prolongation, when none of the $\mathcal{L}_p$ decreases, are discussed below: (i) extension of the ideal for conformal geometry to that for so-called *normal* conformal geometry, (ii) adjoining a 3-form to the ideal for general three-dimensional Riemannian geometry, and (iii) adjoining forms for *test* electromagnetic fields to the I for general four-dimensional Riemannian geometry.

3. Exterior differential systems for various well-posed four-dimensional geometries

Frame bundles come partially equipped with canonical or 'solder' basis 1-forms and structure identities [2]. Given a differentiable manifold M, dim m, the second frame bundle over M, denoted $F_2(M)$, is a principal fibre bundle that contains independent intrinsic (or canonical) sets of 1-forms denoted ω^i, ω^i_j ($i, j, k = 1, \ldots, m$), numbering m and m^2 respectively. A complete basis also requires a third set ω^i_{jk} (symmetric on the pair of lower indices) that is in fact only given canonically on the third frame bundle, a cross section of which gives a suitable set on the second frame bundle. These basis forms satisfy the structure 2-form identities:

$$\mathrm{d}\omega^i + \omega^i_s \wedge \omega^s = 0 \tag{7}$$

$$\mathrm{d}\omega^i_j + \omega^i_s \wedge \omega^s_j + \omega^i_{js} \wedge \omega^s = 0 \tag{8}$$

plus the exterior derivative of (8), a 3-form identity. The essence of the method of moving frames is to set exterior differential systems using these canonical (but anholonomic) basis forms, and avoiding—or deferring—use of specific (holonomic) coordinates in $F_2(M)$, or any submanifold, as much as possible. Using the Cartan character analysis, such a system must be shown to be well-posed; then the regular integral manifolds which it determines, when properly fibred, can be G-structures, principal fibre sub-bundles of $F_1(M)$. Generalised G-structures extend this construction to even higher frame bundles.

Cartan's exterior differential system for a general Riemannian geometry is a closed ideal, I, generated by $\frac{1}{2}m(m+1)$ 1-forms and their exterior derivatives, the latter calculable using (8):

$$\omega^i_j + \omega^j_i \tag{9}$$

$$\omega^i_{js} \wedge \omega^s + \omega^j_{is} \wedge \omega^s. \tag{10}$$

For I we calculate [4] that $s_0 = \frac{1}{2}m(m+1) = s_1 = s_2 = \ldots = s_m$, $s_{m+1} = 0 = s_{m+2} = \ldots$, and it follows that $g = \frac{1}{2}m(m+1)$. ω^i and $\omega^i_j - \omega^j_i$ are an involutory set. The $\frac{1}{2}m(m-1)$ basis forms $\omega^i_j - \omega^j_i$ do not appear in (3) and (4), and consequently there are $q = \frac{1}{2}m(m-1)$ Cauchy characteristic vectors determined by I, flows that will lie in the maximal integral submanifolds, as fibres. When $m = 4$, $n = 60$, $s_0 = 10$, $s_1 = 10$, $s_2 = 10$, $s_3 = 10$, $s_4 = 10$, $g = 10$, the maximal integral submanifolds of I are ten dimensional with six-dimensional fibres. This is the first well-posed system summarised in table 1.

The $q = \frac{1}{2}m(m-1)$ Cauchy characteristic vectors contracted on the $g - q$ forms ω^i must all give zero, i.e. the ω^i further pulled back into the fibres then vanish. The canonical structure 2-form identities from (8) that survive in the fibres involve only

Table 1. Closed exterior differential systems on $F_2(M)$ and $F_3(M)$, dim $M = 4$.

	n	s_0	s_1	s_2	s_3	s_4	g
General Riemannian geometry	60	10	10	10	10	10	10
Ricci-flat Riemannian geometry	60	10	10	14	12	4	10
Self-dual Riemannian geometry	60	10	13	13	10	4	10
Flat spacetime	60	10	16	12	8	4	10
Einstein-Maxwell theory	66	10	10	16	16	4	10
Riemannian geometry with test Maxwell field	66	10	10	12	14	10	10
Conformal geometry	60	9	9	9	9	9	15
Normal conformal geometry	70	9	9	13	15	9	15
Flat conformal geometry	70	9	19	14	9	4	15
Projective geometry	70	0	0	0	10	36	24
Flat projective geometry	70	0	19	14	9	4	24
Flat affine geometry	140	40	40	24	12	4	20

the $\omega^i_j - \omega^j_i$ and are Cartan–Maurer equations. This gives the fibres the structure of group manifolds, in this case O(m). The Cauchy characteristic vectors finally can be 'calibrated' so as to be algebraically dual to the ω^i_j $(=-\omega^j_i)$, and then their action will explicitly represent the Lie algebra o(m):

$$\begin{aligned} &V^i_j \lrcorner \omega^a = 0 \\ &V^i_j \lrcorner \omega^a_b = \delta^i_b \delta^a_j - \delta^i_a \delta^b_j \\ &V^i_j \lrcorner \omega^a_{bc} = 0. \end{aligned} \tag{11}$$

Each integral submanifold thus has the structure of an O(m) principal bundle over an m-dimensional base. This is the G-structure. Since there is a natural map $F_2(M) \to F_1(M)$, it could also be seen as a sub-bundle of $F_1(M)$. The set $\omega^i_j = -\omega^j_i$ canonically given in each such integral submanifold is usually now called the (torsion-free) connection. Holonomic coordinates can be introduced such that an arbitrary m-dimensional cross section is a realisation of an orthogonal frame field on an m-dimensional metric space. This is illustrated in figure 1. Evidently different $[\frac{1}{2}m(m+1)]$-dimensional integral submanifolds of I can belong to intrinsically different Riemannian geometries, and indeed Cartan's approach to the equivalence problem for such geometries was to formulate them in this language.

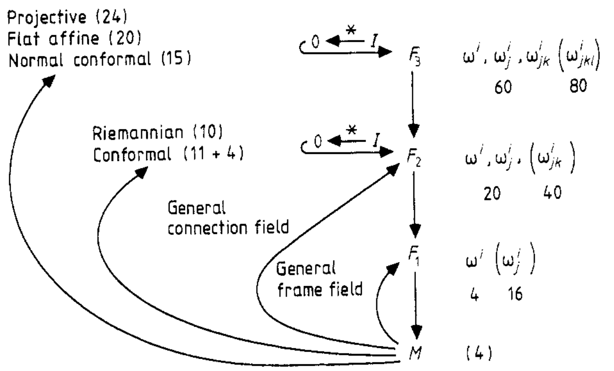

Figure 1. Exterior differential ideals, I, determine various geometries as sub-bundles of frame bundles over a base manifold M. Invariant basis forms are indicated, and dimensions given for the case dim $M = 4$.

Ricci-flat four-dimensional Riemannian geometries [4] are sub-bundles of $F_2(4)$ determined by an exterior system I' containing four additional 3-forms adjoined to the I of general Riemannian geometry:

$$\begin{aligned} &\omega^i_j + \omega^j_i \\ &\omega^i_{js} \wedge \omega^s + \omega^j_{is} \wedge \omega^s \\ &\omega^i_{js} \wedge \omega^s \wedge \omega^k + \omega^j_{ks} \wedge \omega^s \wedge \omega^i + \omega^k_{is} \wedge \omega^s \wedge \omega^j \\ &i, j, k = \{1, 2, 3, 4\}. \end{aligned} \tag{12}$$

Explicit calculation gives $s_0 = 10$, $s_1 = 10$, $s_2 = 14$, $s_3 = 12$, $s_4 = 4$, $g = 10$ and the ω^i, ω^i_j are in involution. There are evidently the same six Cauchy characteristic vectors; the integral manifolds are (a subset of all possible) orthogonal frame bundles. That this is a specialisation is of course also shown by calculating the $\mathscr{L}_p$, two of which decrease. It is noteworthy how the Ricci-flat formulation varies with dimension, for example in five dimensions ($n=105$) one must form I' by adjoining to I the 4-forms $\varepsilon^{ijklm}\omega^i_{js}\wedge\omega^s\wedge\omega^k\wedge\omega^l$. In six dimensions one must adjoin 5-forms, and so on. We have calculated the Cartan characters for several other low-dimensional Ricci-flat geometries to verify that these ideals are all indeed well-posed. This result is of course gratifying, though not to us obvious; the 'naturalness' of the adjoined higher rank forms is perhaps clearer in Clifford algebra formalism (§ 4).

In four dimensions a well known further specialisation is to Ricci-flat self-dual geometries. Now I' is generated by

$$\begin{aligned}&\omega^i_j+\omega^j_i\\&\omega^i_{js}\wedge\omega^s+\omega^j_{is}\wedge\omega^s\\&\omega^1_{2s}\wedge\omega^s+\omega^3_{4s}\wedge\omega^s\\&\omega^1_{3s}\wedge\omega^s+\omega^4_{2s}\wedge\omega^s\\&\omega^1_{4s}\wedge\omega^s+\omega^2_{3s}\wedge\omega^s.\end{aligned}\tag{13}$$

We calculate $s_0=10$, $s_1=13$, $s_2=13$, $s_3=10$, $s_4=4$, well-posed with $g=10$. The final specialisation in this line is to adjoin *all* the 2-forms $\omega^i_{js}\wedge\omega^s$. One then obtains $s_0=10$, $s_1=16$, $s_2=12$, $s_3=8$, $s_4=4$, well-posed with $g=10$ and, in fact, an orthogonal bundle over flat 4-space.

Other classical geometries can also be treated with this approach. Conformal geometry follows from an ideal on $F_2(M)$ generated by a set of traceless 1-forms and their closure 2-forms:

$$\begin{aligned}&\omega^i_j+\omega^j_i-2\delta^i_j\tau\\&\omega^i_{js}\wedge\omega^s+\omega^j_{is}\wedge\omega^s-2\delta^i_j\tau_t\wedge\omega^t\end{aligned}\tag{14}$$

where $\tau=(1/m)\omega^s_s$ and $\tau_i=(1/m)\omega^s_{si}$. In four dimensions one obtains $s_0=9$, $s_1=9$, $s_2=9$, $s_3=9$, $s_4=9$, well-posed with $g=15$. The solutions have $(q=11)$-dimensional fibres, only seven of which can be calibrated by pulling back the remaining basis 1-forms, $\omega^i_j-\omega^j_i$ and τ. Thus the solutions cannot strictly be seen as sub-bundles of F_1. The forms τ_i are not canonically given on F_2, and so we do not have explicit 2-form relations for $\mathrm{d}\tau_i$ that would pull back to give complete dual Cartan–Maurer structure to the fibres, as subgroups of the GL(m) fibres of F_1. Thus conformal geometry is better set as a higher-order G-structure [5].

On the third frame bundle $F_3(M)$ the forms ω^i_{jk} are canonical and satisfy in general the structure equations

$$\mathrm{d}\omega^i_{jk}+\omega^i_s\wedge\omega^s_{jk}-\omega^s_j\wedge\omega^i_{sk}-\omega^s_k\wedge\omega^i_{js}+\omega^i_{jkl}\wedge\omega^l=0\tag{15}$$

together with their closure. For $m=4$, some 80 new forms ω^i_{jkl} are introduced to complete a basis; the dimensionality is 140. Consider first however the specialised case (which uses the trace of (15)) when just $\frac{1}{2}m(m+1)$ new basis forms $\tau_{ij}\equiv(1/m)\omega^s_{sij}$ are introduced into invariant structure equations for the τ_i:

$$\mathrm{d}\tau_i-\omega^s_i\wedge\tau_s+\tau_{is}\wedge\omega^s=0.\tag{16}$$

For $m=4$, now $n=70$ and, returning to the discussion of conformal geometry, the extra freedom allows us to supplement the ideal of (14) with the additional 3-forms and their closure 4-forms:

$$\begin{aligned} &\varepsilon_{ijkl}\Omega^i_j \wedge \omega^k \\ &\varepsilon_{ijkl}\Omega_i \wedge \omega^j \wedge \omega^k \end{aligned} \tag{17}$$

where the 'curvature' forms are respectively defined as

$$\begin{aligned} &\Omega^i_j \equiv \omega^i_{js} \wedge \omega^s + \tau_i \wedge \omega^j - \tau_j \wedge \omega^i - \delta^i_j \tau_s \wedge \omega^s \\ &\Omega_i \equiv \tau_{it} \wedge \omega^t. \end{aligned} \tag{18}$$

They identically satisfy $\Omega^i_j \wedge \omega^j = 0$ and $\Omega_i \wedge \omega^i = 0$. The additional forms (17) express the condition for a 'normal Cartan connection' that, in any final cross section, where $\Omega^i_j = K^i_{jkl}\omega^k \wedge \omega^l$, the components K^i_{jkl} will satisfy $K^i_{jil} = 0$. The Cartan characters for the augmented ideal (14) plus (17) now are computed to be $s_0 = 9$, $s_1 = 9$, $s_2 = 13$, $s_3 = 15$, $s_4 = 9$; $g = 15$, still with eleven Cauchy characteristics. There are now eleven structure relations, pulled back, to calibrate solutions as principal fibre bundles. This extension and completion of the ideal for conformal geometry to the third frame bundle is clearly no restriction on the generality of the solutions, since none of the $\mathscr{L}_p$ decreases. The well-posedness of this completed ideal is, of course, related to mathematical theorems on the uniqueness of normal Cartan connections [5].

In the case of *flat* conformal geometry, one adjoins to (14) the 2-forms Ω^i_j and Ω_i. We then calculate $s_0 = 9$, $s_1 = 19$, $s_2 = 14$, $s_3 = 9$, $s_4 = 4$, well-posed with $n = 70$ and $g = 15$.

Projective geometries are submanifolds of $F_3(M)$ determined by ideals that do not restrict the ω^i_j. Consider, when $m = 4$, a set of 4-forms and their closure 5-forms:

$$\begin{aligned} &\varepsilon_{ijkl}\Omega^i_s \wedge \omega^j \wedge \omega^k \\ &\varepsilon_{abcd}\sigma_{ij} \wedge \omega^a \wedge \omega^b \wedge \omega^c \wedge \omega^d. \end{aligned} \tag{19}$$

These express again the normal Cartan condition $K^i_{jik} = 0$ for curvature forms now identified as [5]

$$\Omega^i_j \equiv \omega^i_{js} \wedge \omega^s - \sigma_j \wedge \omega^i - \delta^i_j \sigma_s \wedge \omega^s \tag{20}$$

where $\sigma_i \equiv [1/(m+1)]\omega^t_{ti}$ and $\sigma_{ij} = [1/(m+1)]\omega^t_{tij}$. The ideal generated by (19) yields the numerology $n = 70$, $s_0 = s_1 = s_2 = 0$, $s_3 = 10$, $s_4 = 36$ and $g = 24$. The problem is well-posed: there are twenty-dimensional fibres which can be calibrated by the involutory bases ω^i_j, σ_i. Flat projective geometry is generated by the 2-forms Ω^i_j and Ω_i (the forms of (19) are then in the ideal). Since $\Omega^s_s = 0$ there are just nineteen of these and the result is $n = 70$, $s_0 = 0$, $s_1 = 19$, $s_2 = 14$, $s_3 = 9$, $s_4 = 4$ and $g = 24$.

Flat affine geometry is set on $F_3(M)$ in analogy to flat metric geometry on $F_2(M)$. That is, the ideal is generated by all of the ω^i_{jk} and their closure $\omega^i_{jks} \wedge \omega^s$. $n = 140$, $s_0 = s_1 = 40$, $s_2 = 24$, $s_3 = 12$, $s_4 = 4$, $g = 20$, and the fibres can use the ω^i_j to have the structure of gl(4).

The Cartan numerologies calculated for all these geometries are summarised in table 1.

4. Clifford-algebra-valued forms [6]

The large set of structure equations that remain when (7) and (8) are restricted to any integral submanifold of the ideal I generated by (9) and (10) can be written in various concise spinor notations that allow convenient manipulation. The sets of forms entering

(7) and (8), when restricted with (9) and (10), are antisymmetric on pairs of indices, and the contractions on indices are those of matrix algebra. The straightforward formalism of m-dimensional Clifford algebra is, we believe, consequently heuristic for the derivation of further results or of consistent further restrictions to special cases, and since no complex conjugating is involved it is perhaps easier to use than any of its spinor isomorphs. At the same time, any such notation can conceal essential details, in particular all the information contained in the Cartan characters themselves, or results peculiar to a particular value of m, so one must be able at any point to return to the explicit canonical frame basis.

We introduce symbols I, γ^i, γ^{ij}, γ^{ijk}, etc, each totally antisymmetric on its indices, as a basis set for the Clifford algebra in m dimensions. The entire algebra is generated by the anticommutator relation $\gamma^i\gamma^j+\gamma^j\gamma^i=2\delta^{ij}I$ together with the definitions $\gamma^{ij}=\gamma^{[i}\gamma^{j]}$, $\gamma^{ijk}=\gamma^{[i}\gamma^j\gamma^{k]}$, and so on. One can think of the γ^i, etc, as square matrices, and indeed in the case $m=4$, there is a 4×4 (invertible) matrix representation of this algebra (Eddington's E numbers), but we will not in fact need a matrix representation, nor be concerned with its rank. We will treat the Clifford algebra as generated by 2^m independent basis elements $I,\ldots,\gamma^{ij\ldots k}$, where there are precisely m indices in the last of these. Explicit equations in terms of the basis set of differential forms can be recovered from a Clifford-algebra-valued expression by expansion on this basis. In the Clifford algebra there is a grading such that I is of grade zero, the γ^i are of grade 1, γ^{ij} of grade 2, etc. After expansion of a vanishing expression into this basis the sets of forms of each grade can be independently set to zero.

The antisymmetry of the sets of forms ω^i, ω^i_j and $\omega^i_{js}\wedge\omega^s$ in an integral submanifold allows us to define Clifford-algebra-valued forms there:

$$\theta=\omega^i\gamma^i \tag{21}$$

$$\Gamma=\tfrac{1}{4}\omega^i_j\gamma^{ij} \tag{22}$$

$$R=\tfrac{1}{8}R^i_j\gamma^{ij}=-\tfrac{1}{4}\omega^i_{js}\wedge\omega^s\gamma^{ij} \tag{23}$$

(also defining the antisymmetric set of Riemann 2-forms $R^i_j=-R^j_i$).

Entirely equivalent to (7) and (8), restricted with (9) and (10), are the relations

$$\mathrm{d}\theta+\Gamma\theta+\theta\Gamma=0 \tag{24}$$

$$\mathrm{d}\Gamma+\Gamma\Gamma-R=0 \tag{25}$$

$$R\theta-\theta R=0 \tag{26}$$

and their identical consequence

$$\mathrm{d}R+\Gamma R-R\Gamma=0. \tag{27}$$

We have omitted the wedge symbol, $\wedge$, for simplicity.

As an example of expansion of a Clifford-algebra-valued expression on the set of completely antisymmetric independent basis elements, consider (26). We write this first as

$$R\theta-\theta R=R^a_b\omega^i(\gamma^{ab}\gamma^i-\gamma^i\gamma^{ab}) \tag{28}$$

and use the basic anticommutator relation to expand the products on basis elements to find

$$\begin{aligned}&=R^a_b\omega^i(2\delta^{ib}\gamma^a-2\delta^{ai}\gamma^b)\\&=4R^a_b\omega^b\gamma^a.\end{aligned} \tag{29}$$

So, taking the γ^a as independent, we recover

$$R^a_b \wedge \omega^b = 0 \tag{30}$$

which also is obvious from the definition (23).

The Ricci-flat specialisation of Riemannian geometries becomes in this formalism the adjoining to I of the Clifford-algebra-valued forms $R\theta$ (when $m=4$), or $R\theta\theta$ (when $m=5$), etc.

5. Einstein–Maxwell ideals

We have found a quadratically non-linear set of 3-forms, in the abstract Clifford notation, which can apparently sometimes lead to interesting auxiliary or prolongation fields for Riemannian geometries. Introducing a Clifford-algebra-valued scalar, ϕ, we can construct the 3-form

$$\Xi \equiv \mathrm{d}(\theta\phi\theta) + \Gamma\theta\phi\theta - \theta\phi\theta\Gamma + \theta\phi\theta\phi\theta + R\theta \tag{31}$$

which may quickly be shown to be closed, modulo itself and I (that is, using (24)-(27))! Thus the prolonged ideal $I' \equiv \{I, \Xi\}$ may be a self-consistent way to introduce auxiliary fields, at least in those cases which the Cartan character analysis shows to be well set.

The first case we have worked out is when $m=3$, and moreover we take ϕ to be purely grade 1: $\phi = \phi_i\gamma^i$. Recall that the ideal I for Riemannian geometry has $s_0 = 6 = s_1 = s_2 = s_3$; dim $F_2(3) = 30$, $g = 6$. The prolongation Ξ is in this case a single 3-form:

$$\begin{aligned}\mathrm{d}\phi_i \wedge \omega^k \wedge \omega^j + \phi_s\omega^s_i \wedge \omega^j \wedge \omega^k - \phi_s\phi_s\omega^i \wedge \omega^j \wedge \omega^k \\ + 2\phi_j\phi_s\omega^i \wedge \omega^s \wedge \omega^k - \tfrac{1}{4}\omega^i_{js} \wedge \omega^s \wedge \omega^k + \text{antisym on } i, j, k.\end{aligned} \tag{32}$$

With the ϕ, the dimension is 33. The Cartan characters become $s_0 = 6$, $s_1 = 6$, $s_2 = 7$, $s_3 = 8$, $s_4 = s_5 = 0$ (none of which is a decrease, so none of the $\mathscr{L}_p$ decreases), the genus is unaltered, there are still three Cauchy congruences and the ω^i remain in involution, independent in the solution manifolds. We have apparently found a sort of non-linear vector potential for the Ricci scalar R^i_i of any Riemannian 3-manifold.

When $m=4$, and when we specialise ϕ to the second grade, we recover the Rainich-Misner-Wheeler 'already unified' theory of electromagnetism in general relativity. Whenever m is even, setting $\phi = F_{ij}\gamma^{ij}$ in (31) gives two separate equations:

$$\mathrm{d}(\theta\phi\theta) + \Gamma\theta\phi\theta - \theta\phi\theta\Gamma = 0 \tag{33}$$

$$\theta\phi\theta\phi\theta + R\theta = 0. \tag{34}$$

The first of these, when $m=4$, is of grades 0 and 4, giving two 3-forms for Maxwell's field equations:

$$\begin{aligned}&\mathrm{d}F_{ij} \wedge \omega^k \wedge \omega^l - 2F_{is}\omega^s_j \wedge \omega^k \wedge \omega^l + \text{antisym on } i, j, k, l = 0 \\ &\mathrm{d}F_{ij} \wedge \omega^i \wedge \omega^j - 2F_{ij}\omega^i_s \wedge \omega^s \wedge \omega^j = 0\end{aligned} \tag{35}$$

while the second, (34), incorporates all the non-linearity, is pure grade three and is precisely the Einstein-Maxwell field equation coupling gravitation to electromagnetic energy-momentum. The total number of dimensions is now 66, the Cartan characters become $s_0 = 10$, $s_1 = 10$, $s_2 = 16$, $s_3 = 16$, $s_4 = 4$, $s_5 = 0 = s_6 =$ etc; the genus is 10. The ideal I' is in involution with respect to ω^i, ω^i_j, and the coupled Einstein-Maxwell equations are thus shown to be well-posed. Note however that $\mathscr{L}_4$ has decreased! As is well known from Rainich-Misner-Wheeler theory, this is a *specialisation*: it formulates a consistent *subset* of Riemannian 4-geometries (in which the Ricci tensor has a particular canonical form).

We have performed many computer calculations of ideals including generalised Einstein-Maxwell forms (31), i.e. when ϕ is not purely of grade two. Surprisingly, none of these passed the requirements of being well-posed, with the ω^i remaining independent in the solution manifolds. It remains for future research to ascertain if the generalisation remains empty for $m > 4$.

As a final remark in this particular case $m = 4$, the Maxwell 3-forms given by (35) are in fact algebraically equivalent to a pair of exact 3-forms [4]

$$\mathrm{d}(F_{ij}\omega^k \wedge \omega^l \varepsilon^{ijkl}) = 0 \qquad \text{and} \qquad \mathrm{d}(F_{ij}\omega^i \wedge \omega^j) = 0 \tag{36}$$

so the grade-3 part of (31) is not needed to ensure closure. Thus it can be dropped and one thus finds a linear prolongation; namely just the imposition of *test* electromagnetic fields on an arbitrary 4-geometry. (Otherwise said, that the integrability condition $R\theta\phi\theta - \theta\phi\theta R = 0$ is identically satisfied for pure grade-2 ϕ, when $m = 4$, for any R.) The Cartan characters of I, augmented with the two 3-forms in (36), are $s_0 = 10$, $s_1 = 10$, $s_2 = 12$, $s_3 = 14$, $s_4 = 10$, $s_5 = 0$, etc $g = 10$ and this is truly a prolongation and not a specialisation.

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