CHAPTER 11. VECTOR FUNCTIONS AND CURVES

Section 11.1 Vector Functions of One Variable (page 597)

1. Position: $\mathbf{r} = \mathbf{i} + t\mathbf{j}$
Velocity: $\mathbf{v} = \mathbf{j}$
Speed: $v = 1$
Acceleration : $\mathbf{a} = \mathbf{0}$
Path: the line $x = 1$ in the xy-plane.

2. Position: $\mathbf{r} = t^2\mathbf{i} + \mathbf{k}$
Velocity: $\mathbf{v} = 2t\mathbf{i}$
Speed: $v = 2|t|$
Acceleration : $\mathbf{a} = 2\mathbf{i}$
Path: the line $z = 1$, $y = 0$.

3. Position: $\mathbf{r} = t^2\mathbf{j} + t\mathbf{k}$
Velocity: $\mathbf{v} = 2t\mathbf{j} + \mathbf{k}$
Speed: $v = \sqrt{4t^2 + 1}$
Acceleration : $\mathbf{a} = 2\mathbf{j}$
Path: the parabola $y = z^2$ in the plane $x = 0$.

4. Position: $\mathbf{r} = \mathbf{i} + t\mathbf{j} + t\mathbf{k}$
Velocity: $\mathbf{v} = \mathbf{j} + \mathbf{k}$
Speed: $v = \sqrt{2}$
Acceleration : $\mathbf{a} = \mathbf{0}$
Path: the straight line $x = 1$, $y = z$.

5. Position: $\mathbf{r} = t^2\mathbf{i} - t^2\mathbf{j} + \mathbf{k}$
Velocity: $\mathbf{v} = 2t\mathbf{i} - 2t\mathbf{j}$
Speed: $v = 2\sqrt{2}t$
Acceleration: $\mathbf{a} = 2\mathbf{i} - 2\mathbf{j}$
Path: the half-line $x = -y \ge 0$, $z = 1$.

6. Position: $\mathbf{r} = t\mathbf{i} + t^2\mathbf{j} + t^2\mathbf{k}$
Velocity: $\mathbf{v} = \mathbf{i} + 2t\mathbf{j} + 2t\mathbf{k}$
Speed: $v = \sqrt{1 + 8t^2}$
Acceleration: $\mathbf{a} = 2\mathbf{j} + 2\mathbf{k}$
Path: the parabola $y = z = x^2$.

7. Position: $\mathbf{r} = a\cos t\mathbf{i} + a\sin t\mathbf{j} + ct\mathbf{k}$
Velocity: $\mathbf{v} = -a\sin t\mathbf{i} + a\cos t\mathbf{j} + c\mathbf{k}$
Speed: $v = \sqrt{a^2 + c^2}$
Acceleration: $\mathbf{a} = -a\cos t\mathbf{i} - a\sin t\mathbf{j}$
Path: a circular helix.

8. Position: $\mathbf{r} = a\cos\omega t\mathbf{i} + b\mathbf{j} + a\sin\omega t\mathbf{k}$
Velocity: $\mathbf{v} = -a\omega\sin\omega t\mathbf{i} + a\omega\cos\omega t\mathbf{k}$
Speed: $v = |a\omega|$
Acceleration: $\mathbf{a} = -a\omega^2\cos\omega t\mathbf{i} - a\omega^2\sin\omega t\mathbf{k}$
Path: the circle $x^2 + z^2 = a^2$, $y = b$.

9. Position: $\mathbf{r} = 3\cos t\mathbf{i} + 4\cos t\mathbf{j} + 5\sin t\mathbf{k}$
Velocity: $\mathbf{v} = -3\sin t\mathbf{i} - 4\sin t\mathbf{j} + 5\cos t\mathbf{k}$
Speed: $v = \sqrt{9\sin^2 t + 16\sin^2 t + 25\cos^2 t} = 5$
Acceleration : $\mathbf{a} = -3\cos t\mathbf{i} - 4\cos t\mathbf{j} - 5\sin t\mathbf{k} = -\mathbf{r}$
Path: the circle of intersection of the sphere $x^2 + y^2 + z^2 = 25$ and the plane $4x = 3y$.

10. Position: $\mathbf{r} = 3\cos t\mathbf{i} + 4\sin t\mathbf{j} + t\mathbf{k}$
Velocity: $\mathbf{v} = -3\sin t\mathbf{i} + 4\cos t\mathbf{j} + \mathbf{k}$
Speed: $v = \sqrt{9\sin^2 t + 16\cos^2 t + 1} = \sqrt{10 + 7\cos^2 t}$
Acceleration : $\mathbf{a} = -3\cos t\mathbf{i} - 4\sin t\mathbf{j} = t\mathbf{k} - \mathbf{r}$
Path: a helix (spiral) wound around the elliptic cylinder $(x^2/9) + (y^2/16) = 1$.

11. Position: $\mathbf{r} = ae^t\mathbf{i} + be^t\mathbf{j} + ce^t\mathbf{k}$
Velocity and acceleration: $\mathbf{v} = \mathbf{a} = \mathbf{r}$
Speed: $v = e^t\sqrt{a^2 + b^2 + c^2}$
Path: the half-line $\dfrac{x}{a} = \dfrac{y}{b} = \dfrac{z}{c} > 0$.

12. Position: $\mathbf{r} = at\cos\omega t\mathbf{i} + at\sin\omega t\mathbf{j} + b\ln t\mathbf{k}$
Velocity: $\mathbf{v} = a(\cos\omega t - \omega t\sin\omega t)\mathbf{i}$
$$+ a(\sin\omega t + \omega t\cos\omega t)\mathbf{j} + (b/t)\mathbf{k}$$
Speed: $v = \sqrt{a^2(1 + \omega^2 t^2) + (b^2/t^2)}$
Acceleration: $\mathbf{a} = -a\omega(2\sin\omega t + \omega\cos\omega t)\mathbf{i}$
$$+ a\omega(2\cos\omega t - \omega\sin\omega t)\mathbf{j} - (b/t^2)\mathbf{k}$$
Path: a spiral on the surface $x^2 + y^2 = a^2 e^{z/b}$.

13. Position: $\mathbf{r} = e^{-t}\cos(e^t)\mathbf{i} + e^{-t}\sin(e^t)\mathbf{j} - e^t\mathbf{k}$
Velocity: $\mathbf{v} = -\Big(e^{-t}\cos(e^t) + \sin(e^t)\Big)\mathbf{i}$
$$- \Big(e^{-t}\sin(e^t) - \cos(e^t)\Big)\mathbf{j} - e^t\mathbf{k}$$
Speed: $v = \sqrt{1 + e^{-2t} + e^{2t}}$
Acceleration: $\mathbf{a} = \Big((e^{-t} - e^t)\cos(e^t) + \sin(e^t)\Big)\mathbf{i}$
$$+ \Big((e^{-t} - e^t)\sin(e^t) - \cos(e^t)\Big)\mathbf{j} - e^t\mathbf{k}$$
Path: a spiral on the surface $z\sqrt{x^2 + y^2} = -1$.

14. Position: $\mathbf{r} = a\cos t\sin t\mathbf{i} + a\sin^2 t\mathbf{j} + a\cos t\mathbf{k}$
$$= \frac{a}{2}\sin 2t\mathbf{i} + \frac{a}{2}\Big(1 - \cos 2t\Big)\mathbf{j} + a\cos t\mathbf{k}$$
Velocity: $\mathbf{v} = a\cos 2t\mathbf{i} + a\sin 2t\mathbf{j} - a\sin t\mathbf{k}$
Speed: $v = a\sqrt{1 + \sin^2 t}$
Acceleration: $\mathbf{a} = -2a\sin 2t\mathbf{i} + 2a\cos 2t\mathbf{j} - a\cos t\mathbf{k}$
Path: the path lies on the sphere $x^2 + y^2 + z^2 = a^2$, on the surface defined in terms of spherical polar coordinates by $\phi = \theta$, on the circular cylinder $x^2 + y^2 = ay$, and on the parabolic cylinder $ay + z^2 = a^2$. Any two of these surfaces serve to pin down the shape of the path.

15. The position of the particle is given by
$$\mathbf{r} = 5\cos(\omega t)\mathbf{i} + 5\sin(\omega t)\mathbf{j},$$
where $\omega = \pi$ to ensure that $\mathbf{r}$ has period $2\pi/\omega = 2$ s. Thus
$$\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = -\omega^2\mathbf{r} = -\pi^2\mathbf{r}.$$
At $(3, 4)$, the acceleration is $-3\pi^2\mathbf{i} - 4\pi^2\mathbf{j}$.

16. When its x-coordinate is x, the particle is at position $\mathbf{r} = x\mathbf{i} + (3/x)\mathbf{j}$, and its velocity and speed are

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} - \frac{3}{x^2}\frac{dx}{dt}\mathbf{j}$$
$$v = \left|\frac{dx}{dt}\right|\sqrt{1 + \frac{9}{x^4}}.$$

We know that $dx/dt > 0$ since the particle is moving to the right. When $x = 2$, we have $10 = v = (dx/dt)\sqrt{1 + (9/16)} = (5/4)(dx/dt)$. Thus $dx/dt = 8$. The velocity at that time is $\mathbf{v} = 8\mathbf{i} - 6\mathbf{j}$.

17. The particle moves along the curve $z = x^2$, $x + y = 2$, in the direction of increasing y. Thus its position at time t is

$$\mathbf{r} = (2 - y)\mathbf{i} + y\mathbf{j} + (2 - y)^2\mathbf{k},$$

where y is an increasing function of time t. Thus

$$\mathbf{v} = \frac{dy}{dt}\Big[-\mathbf{i} + \mathbf{j} - 2(2 - y)\mathbf{k}\Big]$$
$$v = \frac{dy}{dt}\sqrt{1 + 1 + 4(2 - y)^2} = 3$$

since the speed is 3. When $y = 1$, we have $dy/dt = 3/\sqrt{6} = \sqrt{3/2}$. Thus

$$\mathbf{v} = \sqrt{\frac{3}{2}}(-\mathbf{i} + \mathbf{j} - 2\mathbf{k}).$$

18. The position of the object when its x-coordinate is x is

$$\mathbf{r} = x\mathbf{i} + x^2\mathbf{j} + x^3\mathbf{k},$$

so its velocity is $\mathbf{v} = \dfrac{dx}{dt}\Big[\mathbf{i} + 2x\mathbf{j} + 3x^2\mathbf{k}\Big]$. Since $dz/dt = 3x^2\,dx/dt = 3$, when $x = 2$ we have $12\,dx/dt = 3$, so $dx/dt = 1/4$. Thus

$$\mathbf{v} = \frac{1}{4}\mathbf{i} + \mathbf{j} + 3\mathbf{k}.$$

19. $\mathbf{r} = 3u\mathbf{i} + 3u^2\mathbf{j} + 2u^3\mathbf{k}$

$$\mathbf{v} = \frac{du}{dt}(3\mathbf{i} + 6u\mathbf{j} + 6u^2\mathbf{k})$$
$$\mathbf{a} = \frac{d^2u}{dt^2}(3\mathbf{i} + 6u\mathbf{j} + 6u^2\mathbf{k}) + \left(\frac{du}{dt}\right)^2(6\mathbf{j} + 12u\mathbf{k}).$$

Since u is increasing and the speed of the particle is 6,

$$6 = |\mathbf{v}| = 3\frac{du}{dt}\sqrt{1 + 4u^2 + 4u^4} = 3(1 + 2u^2)\frac{du}{dt}.$$

Thus $\dfrac{du}{dt} = \dfrac{2}{1 + 2u^2}$, and

$$\frac{d^2u}{dt^2} = \frac{-2}{(1 + 2u^2)^2}\,4u\,\frac{du}{dt} = \frac{-16u}{(1 + 2u^2)^3}.$$

The particle is at $(3, 3, 2)$ when $u = 1$. At this point $du/dt = 2/3$ and $d^2u/dt^2 = -16/27$, and so

$$\mathbf{v} = \frac{2}{3}(3\mathbf{i} + 6u\mathbf{j} + 6u^2\mathbf{k}) = 2\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}$$
$$\mathbf{a} = \frac{-16}{27}(3\mathbf{i} + 6\mathbf{j} + 6\mathbf{k}) + \left(\frac{2}{3}\right)^2(6\mathbf{j} + 12\mathbf{k})$$
$$= \frac{8}{9}(-2\mathbf{i} - \mathbf{j} + 2\mathbf{k}).$$

20. $\mathbf{r} = x\mathbf{i} - x^2\mathbf{j} + +x^2\mathbf{k}$

$$\mathbf{v} = \frac{dx}{dt}(\mathbf{i} - 2x\mathbf{j} + 2x\mathbf{k})$$
$$\mathbf{a} = \frac{d^2x}{dt^2}(\mathbf{i} - 2x\mathbf{j} + 2x\mathbf{k}) + \left(\frac{dx}{dt}\right)^2(-2\mathbf{j} + 2\mathbf{k}).$$

Thus $|\mathbf{v}| = \left|\dfrac{dx}{dt}\right|\sqrt{1 + 4x^4 + 4x^4} = \sqrt{1 + 8x^4}\dfrac{dx}{dt}$, since x is increasing. At $(1, -1, 1)$, $x = 1$ and $|\mathbf{v}| = 9$, so $dx/dt = 3$, and the velocity at that point is $\mathbf{v} = 3\mathbf{i} - 6\mathbf{j} + 6\mathbf{k}$. Now

$$\frac{d}{dt}|\mathbf{v}| = \sqrt{1 + 8x^4}\frac{d^2x}{dt^2} + \frac{16x^3}{\sqrt{1 + 8x^4}}\left(\frac{dx}{dt}\right)^2.$$

The left side is 3 when $x = 1$, so $3(d^2x/dt^2) + 48 = 3$, and $d^2x/dt^2 = -15$ at that point, and the acceleration there is

$$\mathbf{a} = -15(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}) + 9(-2\mathbf{j} + 2\mathbf{k}) = -15\mathbf{i} + 12\mathbf{j} - 12\mathbf{k}.$$

21. $\dfrac{d}{dt}|\mathbf{v}|^2 = \dfrac{d}{dt}\mathbf{v} \bullet \mathbf{v} = 2\mathbf{v} \bullet \mathbf{a}$.
If $\mathbf{v} \bullet \mathbf{a} > 0$ then the speed $v = |\mathbf{v}|$ is increasing.
If $\mathbf{v} \bullet \mathbf{a} < 0$ then the speed is decreasing.

22. If $\mathbf{u}(t) = u_1(t)\mathbf{i} + u_2(t)\mathbf{j} + u_3(t)\mathbf{k}$
$\mathbf{v}(t) = v_1(t)\mathbf{i} + v_2(t)\mathbf{j} + v_3(t)\mathbf{k}$
then $\mathbf{u} \bullet \mathbf{v} = u_1v_2 + u_2v_2 + u_3v_3$, so

$$\frac{d}{dt}\mathbf{u} \bullet \mathbf{v} = \frac{du_1}{dt}v_1 + u_1\frac{dv_1}{dt} + \frac{du_2}{dt}v_2 + u_2\frac{dv_2}{dt} + \frac{du_3}{dt}v_3 + u_3\frac{dv_3}{dt}$$
$$= \frac{d\mathbf{u}}{dt} \bullet \mathbf{v} + \mathbf{u} \bullet \frac{d\mathbf{v}}{dt}.$$

23. $$\frac{d}{dt}\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$
$$= \frac{d}{dt}\Big[a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}\Big]$$
$$\begin{aligned} &= a'_{11}a_{22}a_{33} + a_{11}a'_{22}a_{33} + a_{11}a_{22}a'_{33} \\ &+ a'_{12}a_{23}a_{31} + a_{12}a'_{23}a_{31} + a_{12}a_{23}a'_{31} \\ &+ a'_{13}a_{21}a_{32} + a_{13}a'_{21}a_{32} + a_{13}a_{21}a'_{32} \\ &- a'_{11}a_{23}a_{32} - a_{11}a'_{23}a_{32} - a_{11}a_{23}a'_{32} \\ &- a'_{12}a_{21}a_{33} - a_{12}a'_{21}a_{33} - a_{12}a_{21}a'_{33} \\ &- a'_{13}a_{22}a_{31} - a_{13}a'_{22}a_{31} - a_{13}a_{22}a'_{31} \end{aligned}$$
$$= \begin{vmatrix} a'_{11} & a'_{12} & a'_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a'_{21} & a'_{22} & a'_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a'_{31} & a'_{32} & a'_{33} \end{vmatrix}$$

24. $\frac{d}{dt}|\mathbf{r}|^2 = \frac{d}{dt}\mathbf{r} \bullet \mathbf{r} = 2\mathbf{r} \bullet \mathbf{v} = 0$ implies that $|\mathbf{r}|$ is constant. Thus $\mathbf{r}(t)$ lies on a sphere centred at the origin.

25. $$\frac{d}{dt}|\mathbf{r} - \mathbf{r}_0|^2 = \frac{d}{dt}(\mathbf{r} - \mathbf{r}_0) \bullet (\mathbf{r} - \mathbf{r}_0) = 2(\mathbf{r} - \mathbf{r}_0) \bullet \frac{d\mathbf{r}}{dt} = 0$$
implies that $|\mathbf{r} - \mathbf{r}_0|$ is constant. Thus $\mathbf{r}(t)$ lies on a sphere centred at the point P_0 with position vector $\mathbf{r}_0$.

26. If $\mathbf{r} \bullet \mathbf{v} > 0$ then $|\mathbf{r}|$ is increasing. (See Exercise 16 above.) Thus $\mathbf{r}$ is moving farther away from the origin. If $\mathbf{r} \bullet \mathbf{v} < 0$ then $\mathbf{r}$ is moving closer to the origin.

27. $$\frac{d}{dt}\left(\frac{d\mathbf{u}}{dt} \times \frac{d^2\mathbf{u}}{dt^2}\right) = \frac{d^2\mathbf{u}}{dt^2} \times \frac{d^2\mathbf{u}}{dt^2} + \frac{d\mathbf{u}}{dt} \times \frac{d^3\mathbf{u}}{dt^3} = \frac{d\mathbf{u}}{dt} \times \frac{d^3\mathbf{u}}{dt^3}.$$

28. $$\frac{d}{dt}\Big(\mathbf{u} \bullet (\mathbf{v} \times \mathbf{w})\Big) = \mathbf{u}' \bullet (\mathbf{v} \times \mathbf{w}) + \mathbf{u} \bullet (\mathbf{v}' \times \mathbf{w}) + \mathbf{u} \bullet (\mathbf{v} \times \mathbf{w}').$$

29. $$\frac{d}{dt}\Big(\mathbf{u} \times (\mathbf{v} \times \mathbf{w})\Big) = \mathbf{u}' \times (\mathbf{v} \times \mathbf{w}) + \mathbf{u} \times (\mathbf{v}' \times \mathbf{w}) + \mathbf{u} \times (\mathbf{v} \times \mathbf{w}').$$

30. $$\begin{aligned} &\frac{d}{dt}\left(\mathbf{u} \times \left(\frac{d\mathbf{u}}{dt} \times \frac{d^2\mathbf{u}}{dt^2}\right)\right) \\ &= \frac{d\mathbf{u}}{dt} \times \left(\frac{d\mathbf{u}}{dt} \times \frac{d^2\mathbf{u}}{dt^2}\right) + \mathbf{u} \times \left(\frac{d^2\mathbf{u}}{dt^2} \times \frac{d^2\mathbf{u}}{dt^2}\right) + \mathbf{u} \times \left(\frac{d\mathbf{u}}{dt} \times \frac{d^3\mathbf{u}}{dt^3}\right) \\ &= \frac{d\mathbf{u}}{dt} \times \left(\frac{d\mathbf{u}}{dt} \times \frac{d^2\mathbf{u}}{dt^2}\right) + \mathbf{u} \times \left(\frac{d\mathbf{u}}{dt} \times \frac{d^3\mathbf{u}}{dt^3}\right). \end{aligned}$$

31. $$\begin{aligned} &\frac{d}{dt}\Big[(\mathbf{u} + \mathbf{u}'') \bullet (\mathbf{u} \times \mathbf{u}')\Big] \\ &= (\mathbf{u}' + \mathbf{u}''') \bullet (\mathbf{u} \times \mathbf{u}') + (\mathbf{u} + \mathbf{u}'') \bullet (\mathbf{u}' \times \mathbf{u}') + (\mathbf{u} + \mathbf{u}'') \bullet (\mathbf{u} \times \mathbf{u}'') \\ &= \mathbf{u}''' \bullet (\mathbf{u} \times \mathbf{u}'). \end{aligned}$$

32. $$\begin{aligned} &\frac{d}{dt}\Big[(\mathbf{u} \times \mathbf{u}') \bullet (\mathbf{u}' \times \mathbf{u}'')\Big] \\ &= (\mathbf{u}' \times \mathbf{u}') \bullet (\mathbf{u}' \times \mathbf{u}'') + (\mathbf{u} \times \mathbf{u}'') \bullet (\mathbf{u}' \times \mathbf{u}'') + (\mathbf{u} \times \mathbf{u}') \bullet (\mathbf{u}'' \times \mathbf{u}'') + (\mathbf{u} \times \mathbf{u}') \bullet (\mathbf{u}' \times \mathbf{u}''') \\ &= (\mathbf{u} \times \mathbf{u}'') \bullet (\mathbf{u}' \times \mathbf{u}'') + (\mathbf{u} \times \mathbf{u}') \bullet (\mathbf{u}' \times \mathbf{u}'''). \end{aligned}$$

33. Since $\frac{d\mathbf{r}}{dt} = \mathbf{v}(t) = 2\mathbf{r}(t)$ and $\mathbf{r}(0) = \mathbf{r}_0$, we have
$$\begin{aligned} \mathbf{r}(t) &= \mathbf{r}(0)e^{2t} = \mathbf{r}_0 e^{2t}, \\ \mathbf{a}(t) &= \frac{d\mathbf{v}}{dt} = 2\frac{d\mathbf{r}}{dt} = 4\mathbf{r}_0 e^{2t}. \end{aligned}$$
The path is the half-line from the origin in the direction of $\mathbf{r}_0$.

34. $$\begin{aligned} \mathbf{r} &= \mathbf{r}_0 \cos\omega t + \left(\frac{\mathbf{v}_0}{\omega}\right)\sin\omega t \\ \frac{d\mathbf{r}}{dt} &= -\omega\mathbf{r}_0 \sin\omega t + \mathbf{v}_0 \cos\omega t \\ \frac{d^2\mathbf{r}}{dt^2} &= -\omega^2\mathbf{r}_0 \cos\omega t - \omega\mathbf{v}_0 \sin\omega t = -\omega^2\mathbf{r} \end{aligned}$$
$$\mathbf{r}(0) = \mathbf{r}_0, \qquad \left.\frac{d\mathbf{r}}{dt}\right|_{t=0} = \mathbf{v}_0.$$
Observe that $\mathbf{r} \bullet (\mathbf{r}_0 \times \mathbf{v}_0) = 0$ for all t. Therefore the path lies in a plane through the origin having normal $\mathbf{N} = \mathbf{r}_0 \times \mathbf{v}_0$.
Let us choose our coordinate system so that $\mathbf{r}_0 = a\mathbf{i}$ $(a > 0)$ and $\mathbf{v}_0 = \omega b\mathbf{i} + \omega c\mathbf{j}$ $(c > 0)$. Therefore, $\mathbf{N}$ is in the direction of $\mathbf{k}$. The path has parametric equations
$$\begin{aligned} x &= a\cos\omega t + b\sin\omega t \\ y &= c\sin\omega t. \end{aligned}$$
The curve is a conic section since it has a quadratic equation:
$$\frac{1}{a^2}\left(x - \frac{by}{c}\right)^2 + \frac{y^2}{c^2} = 1.$$
Since the path is bounded ($|\mathbf{r}(t)| \le |\mathbf{r}_0| + (|\mathbf{v}_0|/\omega)$), it must be an ellipse.

If $\mathbf{r}_0$ is perpendicular to $\mathbf{v}_0$, then $b = 0$ and the path is the ellipse $(x/a)^2 + (y/c)^2 = 1$ having semi-axes $a = |\mathbf{r}_0|$ and $c = |\mathbf{v}_0|/\omega$.

35. $$\frac{d^2\mathbf{r}}{dt^2} = -g\mathbf{k} - c\frac{d\mathbf{r}}{dt}$$
$$\mathbf{r}(0) = \mathbf{r}_0, \qquad \left.\frac{d\mathbf{r}}{dt}\right|_{t=0} = \mathbf{v}_0.$$

Let $\mathbf{w} = e^{ct}\dfrac{d\mathbf{r}}{dt}$. Then

$$\begin{aligned}\frac{d\mathbf{w}}{dt} &= ce^{ct}\frac{d\mathbf{r}}{dt} + e^{ct}\frac{d^2\mathbf{r}}{dt^2}\\ &= ce^{ct}\frac{d\mathbf{r}}{dt} - e^{ct}g\mathbf{k} - ce^{ct}\frac{d\mathbf{r}}{dt}\\ &= -e^{ct}g\mathbf{k}\\ \mathbf{w}(t) &= -\int e^{ct}g\mathbf{k}\,dt = -\frac{e^{ct}}{c}g\mathbf{k} + \mathbf{C}.\end{aligned}$$

Put $t = 0$ and get $\mathbf{v}_0 = -\dfrac{g}{c}\mathbf{k} + \mathbf{C}$, so

$$\begin{aligned}e^{ct}\frac{d\mathbf{r}}{dt} &= \mathbf{w} = \mathbf{v}_0 + \frac{g}{c}(1 - e^{ct})\mathbf{k}\\ \frac{d\mathbf{r}}{dt} &= e^{-ct}\mathbf{v}_0 - \frac{g}{c}(1 - e^{-ct})\mathbf{k}\\ \mathbf{r} &= -\frac{e^{-ct}}{c}\mathbf{v}_0 - \frac{g}{c}\left(t + \frac{e^{-ct}}{c}\right)\mathbf{k} + \mathbf{D}\\ \mathbf{r}_0 = \mathbf{r}(0) &= -\frac{1}{c}\mathbf{v}_0 - \frac{g}{c^2}\mathbf{k} + \mathbf{D}.\end{aligned}$$

Thus we have

$$\mathbf{r} = \mathbf{r}_0 + \frac{1 - e^{-ct}}{c}\mathbf{v}_0 - \frac{g}{c^2}(ct + e^{-ct} - 1)\mathbf{k}.$$

The limit of this solution, as $c \to 0$, is calculated via l'Hôpital's Rule:

$$\begin{aligned}\lim_{c\to 0}\mathbf{r}(t) &= \mathbf{r}_0 + \mathbf{v}_0\lim_{c\to 0}\frac{te^{-ct}}{1} - g\mathbf{k}\lim_{c\to 0}\frac{t - te^{-ct}}{2c}\\ &= \mathbf{r}_0 + \mathbf{v}_0 t - g\mathbf{k}\lim_{c\to 0}\frac{t^2e^{-ct}}{2}\\ &= \mathbf{r}_0 + \mathbf{v}_0 t - \frac{1}{2}gt^2\mathbf{k},\end{aligned}$$

which is the solution obtained in Example 4.

Section 11.2 Some Applications of Vector Differentiation (page 604)

1. It was shown in the text that

$$\mathbf{v}(T) - \mathbf{v}(0) = -\ln\left(\frac{m(0)}{m(T)}\right)\mathbf{v}_e.$$

If $\mathbf{v}(0) = \mathbf{0}$ and $\mathbf{v}(T) = -\mathbf{v}_e$ then $\ln(m(0)/m(T)) = 1$ and $m(T) = (1/e)m(0)$. The rocket must therefore burn fraction $\dfrac{e-1}{e}$ of its initial mass to accelerate to the speed of its exhaust gases.

Similarly, if $\mathbf{v}(T) = -2\mathbf{v}_e$, then $m(T) = (1/e^2)m(0)$, so the rocket must burn fraction $\dfrac{e^2-1}{e^2}$ of its initial mass to accelerate to twice the speed of its exhaust gases.

2. Let $v(t)$ be the speed of the tank car at time t seconds. The mass of the car at time t is $m(t) = M - kt$ kg. At full power, the force applied to the car is $F = Ma$ (since the motor can accelerate the full car at a m/s^2). By Newton's Law, this force is the rate of change of the momentum of the car. Thus

$$\begin{aligned}&\frac{d}{dt}\Big[(M - kt)v\Big] = Ma\\ &(M - kt)\frac{dv}{dt} - kv = Ma\\ &\frac{dv}{Ma + kv} = \frac{dt}{M - kt}\\ &\frac{1}{k}\ln(Ma + kv) = -\frac{1}{k}\ln(M - kt) + \frac{1}{k}\ln C\\ &Ma + kv = \frac{C}{M - kt}.\end{aligned}$$

At $t = 0$ we have $v = 0$, so $Ma = C/M$. Thus $C = M^2a$ and

$$kv = \frac{M^2a}{M - kt} - Ma = \frac{Makt}{M - kt}.$$

The speed of the tank car at time t (before it is empty) is

$$v(t) = \frac{Mat}{M - kt}\ \text{m/s}.$$

3. Given: $\dfrac{d\mathbf{r}}{dt} = \mathbf{k}\times\mathbf{r}$, $\quad\mathbf{r}(0) = \mathbf{i} + \mathbf{k}$.
Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. Then $x(0) = z(0) = 1$ and $y(0) = 0$.
Since $\mathbf{k}\bullet(d\mathbf{r}/dt) = \mathbf{k}\bullet(\mathbf{k}\times\mathbf{r}) = 0$, the velocity is always perpendicular to $\mathbf{k}$, so $z(t)$ is constant: $z(t) = z(0) = 1$ for all t. Thus

$$\frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} = \frac{d\mathbf{r}}{dt} = \mathbf{k}\times\mathbf{r} = x\mathbf{j} - y\mathbf{i}.$$

Separating this equation into components,

$$\frac{dx}{dt} = -y, \qquad \frac{dy}{dt} = x.$$

Therefore,

$$\frac{d^2x}{dt^2} = -\frac{dy}{dt} = -x,$$

and $x = A\cos t + B\sin t$. Since $x(0) = 1$ and $y(0) = 0$, we have $A = 1$ and $B = 0$. Thus $x(t) = \cos t$ and $y(t) = \sin t$. The path has equation

$$\mathbf{r} = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + \mathbf{k}.$$

Remark: This result also follows from comparing the given differential equation with that obtained for circular motion in the text. This shows that the motion is a rotation with angular velocity $\mathbf{k}$, that is, rotation about the z-axis with angular speed 1. The initial value given for $\mathbf{r}$ then forces

$$\mathbf{r} = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + \mathbf{k}.$$

4. First observe that

$$\frac{d}{dt}|\mathbf{r}-\mathbf{b}|^2 = 2(\mathbf{r}-\mathbf{b}) \bullet \frac{d\mathbf{r}}{dt} = 2(\mathbf{r}-\mathbf{b}) \bullet \Big(\mathbf{a} \times (\mathbf{r}-\mathbf{b})\Big) = 0,$$

so $|\mathbf{r} - \mathbf{b}|$ is constant; for all t the object lies on the sphere centred at the point with position vector $\mathbf{b}$ having radius $\mathbf{r}_0 - \mathbf{b}$.
Next, observe that

$$\frac{d}{dt}(\mathbf{r} - \mathbf{r}_0) \bullet \mathbf{a} = \Big(\mathbf{a} \times (\mathbf{r} - \mathbf{b})\Big) \bullet \mathbf{a} = 0,$$

so $\mathbf{r} - \mathbf{r}_0 \perp \mathbf{a}$; for all t the object lies on the plane through $\mathbf{r}_0$ having normal $\mathbf{a}$. Hence the path of the object lies on the circle in which this plane intersects the sphere described above. The angle between $\mathbf{r} - \mathbf{b}$ and $\mathbf{a}$ must therefore also be constant, and so the object's speed $|d\mathbf{r}/dt|$ is constant. Hence the path must be the whole circle.

5. Use a coordinate system with origin at the observer, $\mathbf{i}$ pointing east, and $\mathbf{j}$ pointing north. The angular velocity of the earth is $2\pi/24$ radians per hour northward:

$$\mathbf{\Omega} = \frac{\pi}{12}\mathbf{j}.$$

Because the earth is rotating west to east, the true north to south velocity of the satellite will appear to the observer to be shifted to the west by $\pi R/12$ km/h, where R is the radius of the earth in kilometres. Since the satellite circles the earth at a rate of π radians/h, its velocity, as observed at the moving origin, is

$$\mathbf{v}_R = -\pi R\mathbf{j} - \frac{\pi R}{12}\mathbf{i}.$$

$\mathbf{v}_R$ makes angle $\tan^{-1}\left(\dfrac{\pi R/12}{\pi R}\right) = \tan^{-1}(1/12) \approx 4.76°$ with the southward direction. Thus the satellite appears to the observer to be moving in a direction 4.76° west of south.

The apparent Coriolis force is

$$-2\mathbf{\Omega} \times \mathbf{v}_R = -\frac{2\pi}{12}\mathbf{j} \times \left(-\pi R\mathbf{j} - \frac{\pi R}{12}\mathbf{i}\right) = -\frac{\pi^2 R}{72}\mathbf{k},$$

which is pointing towards the ground.

6. We use the fixed and rotating frames as described in the text. Assume the satellite is in an orbit in the plane spanned by the fixed basis vectors $\mathbf{I}$ and $\mathbf{K}$. When the satellite passes overhead an observer at latitude 45°, its position is

$$\mathbf{R} = R\,\frac{\mathbf{I}+\mathbf{K}}{\sqrt{2}},$$

where R is the radius of the earth, and since it circles the earth in 2 hours, its velocity at that point is

$$\mathbf{V} = \pi R\,\frac{\mathbf{I}-\mathbf{K}}{\sqrt{2}}.$$

The angular velocity of the earth is $\mathbf{\Omega} = (\pi/12)\mathbf{K}$.

The rotating frame with origin at the observer's position has, at the instant in question, its basis vectors satisfying

$$\begin{aligned}
\mathbf{I} &= -\frac{1}{\sqrt{2}}\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k}\\
\mathbf{J} &= \mathbf{i}\\
\mathbf{K} &= \frac{1}{\sqrt{2}}\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k}.
\end{aligned}$$

As shown in the text, the velocity $\mathbf{v}$ of the satellite as it appears to the observer is given by $\mathbf{V} = \mathbf{v} + \mathbf{\Omega} \times \mathbf{R}$. Thus

$$\begin{aligned}
\mathbf{v} &= \mathbf{V} - \mathbf{\Omega} \times \mathbf{R}\\
&= \frac{\pi R}{\sqrt{2}}(\mathbf{I}-\mathbf{K}) - \frac{pi}{12}\mathbf{K} \times \frac{R}{\sqrt{2}}(\mathbf{I}+\mathbf{K})\\
&= \frac{\pi R}{\sqrt{2}}(\mathbf{I}-\mathbf{K}) - \frac{\pi R}{12\sqrt{2}}\mathbf{J}\\
&= -\pi R\mathbf{j} - \frac{\pi R}{12\sqrt{2}}\mathbf{i}.
\end{aligned}$$

$\mathbf{v}$ makes angle $\tan^{-1}\left(\dfrac{\pi R/12\sqrt{2}}{\pi R}\right) = \tan^{-1}(1/(12\sqrt{2}) \approx 3.37°$ with the southward direction. Thus the satellite appears to the observer to be moving in a direction 3.37° west of south.

The apparent Coriolis force is

$$\begin{aligned}
-2\mathbf{\Omega} \times \mathbf{v} &= -2\frac{\pi}{12}\mathbf{K} \times \left(\frac{\pi R}{\sqrt{2}}(\mathbf{I} - \mathbf{K} - \frac{\pi R}{12\sqrt{2}}\mathbf{J}\right)\\
&= -\frac{\pi^2 R}{6\sqrt{2}}\left(\mathbf{J} + \frac{1}{12}\mathbf{I}\right)\\
&= -\frac{\pi^2 R}{6\sqrt{2}}\left(\mathbf{i} + \frac{1}{12\sqrt{2}}(-\mathbf{j}+\mathbf{k})\right).
\end{aligned}$$

7. The angular velocity of the earth is $\mathbf{\Omega}$, pointing due north. For a particle moving with horizontal velocity $\mathbf{v}$, the tangential and normal components of the Coriolis force $\mathbf{C}$, and of $\mathbf{\Omega}$, are related by

$$\mathbf{C}_T = -2\mathbf{\Omega}_N \times \mathbf{v}, \qquad \mathbf{C}_N = -2\mathbf{\Omega}_T \times \mathbf{v}.$$

At the north or south pole, $\mathbf{\Omega}_T = \mathbf{0}$ and $\mathbf{\Omega}_N = \mathbf{\Omega}$. Thus $\mathbf{C}_N = \mathbf{0}$ and $\mathbf{C}_T = -2\mathbf{\Omega} \times \mathbf{v}$. The Coriolis force is horizontal. It is 90° east of $\mathbf{v}$ at the north pole and 90° west of $\mathbf{v}$ at the south pole.

At the equator, $\mathbf{\Omega}_N = \mathbf{0}$ and $\mathbf{\Omega}_T = \mathbf{\Omega}$. Thus $\mathbf{C}_T = \mathbf{0}$ and $\mathbf{C}_N = -2\mathbf{\Omega} \times \mathbf{v}$. The Coriolis force is vertical.

8. We continue with the same notation as in Example 4. Since $\mathbf{j}$ points northward at the observer's position, the angle μ between the direction vector of the sun, $\mathbf{S} = \cos\sigma\mathbf{I} + \sin\sigma\mathbf{J}$ and north satisfies

$$\cos\mu = \mathbf{S} \bullet \mathbf{j} = -\cos\sigma\cos\phi\cos\theta + \sin\sigma\sin\phi.$$

For the sun, $\theta = 0$ and at sunrise and sunset we have, by Example 4, $\cos\theta = -\tan\sigma/\tan\phi$, so that

$$\begin{aligned}\cos\mu &= \cos\sigma\cos\phi\frac{\tan\sigma}{\tan\phi} + \sin\sigma\sin\phi \\ &= \sin\sigma\frac{\cos^2\phi}{\sin\phi} + \sin\sigma\sin\phi \\ &= \frac{\sin\sigma}{\sin\phi}.\end{aligned}$$

9. At Vancouver, $\phi = 90° - 49.2° = 40.8°$. On June 21st, $\sigma = 23.3°$. Ignoring the mountains and the rain, by Example 4 there will be

$$\frac{24}{\pi}\cos^{-1}\left(-\frac{\tan 23.3°}{\tan 40.8°}\right) \approx 16$$

hours between sunrise and sunset. By Exercise 8, the sun will rise and set at an angle

$$\cos^{-1}\left(\frac{\sin 23.3°}{\sin 40.8°}\right) \approx 52.7°$$

to the east and west of north.

10. At Umeå, $\phi = 90° - 63.5° = 26.5°$. On June 21st, $\sigma = 23.3°$. By Example 4 there will be

$$\frac{24}{\pi}\cos^{-1}\left(-\frac{\tan 23.3°}{\tan 26.5°}\right) \approx 20$$

hours between sunrise and sunset. By Exercise 8, the sun will rise and set at an angle

$$\cos^{-1}\left(\frac{\sin 23.3°}{\sin 26.5°}\right) \approx 27.6°$$

to the east and west of north.

Section 11.3 Curves and Parametrizations (page 611)

1. On the first quadrant part of the circle $x^2 + y^2 = a^2$ we have $x = \sqrt{a^2 - y^2}$, $0 \le y \le a$. The required parametrization is

$$\mathbf{r} = \mathbf{r}(y) = \sqrt{a^2 - y^2}\mathbf{i} + y\mathbf{j}, \quad (0 \le y \le a).$$

2. On the first quadrant part of the circle $x^2 + y^2 = a^2$ we have $y = \sqrt{a^2 - x^2}$, $0 \le x \le a$. The required parametrization is

$$\mathbf{r} = \mathbf{r}(x) = x\mathbf{i} + \sqrt{a^2 - x^2}\mathbf{j}, \quad (0 \le x \le a).$$

3. From the figure we see that

$$\begin{aligned}\phi &= \theta + \frac{\pi}{2}, \qquad 0 \le \theta \le \frac{\pi}{2} \\ x &= a\cos\theta = a\cos\left(\phi - \frac{\pi}{2}\right) = a\sin\phi \\ y &= a\sin\theta = a\sin\left(\phi - \frac{\pi}{2}\right) = -a\cos\phi.\end{aligned}$$

The required parametrization is

$$\mathbf{r} = a\sin\phi\mathbf{i} - a\cos\phi\mathbf{j}, \qquad \left(\frac{\pi}{2} \le \phi \le \pi\right).$$

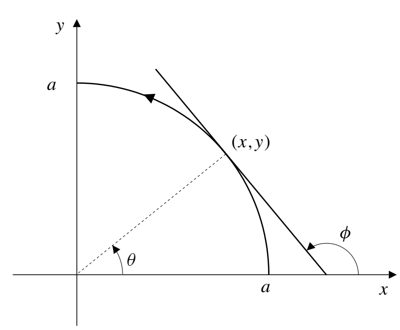

Fig. 11.3.3

4. $x = a\sin\frac{s}{a}, \quad y = a\cos\frac{s}{a}, \qquad 0 \le \frac{s}{a} \le \frac{\pi}{2}$

$$\mathbf{r} = a\sin\frac{s}{a}\mathbf{i} + a\cos\frac{s}{a}\mathbf{j}, \qquad \left(0 \le s \le \frac{a\pi}{2}\right).$$

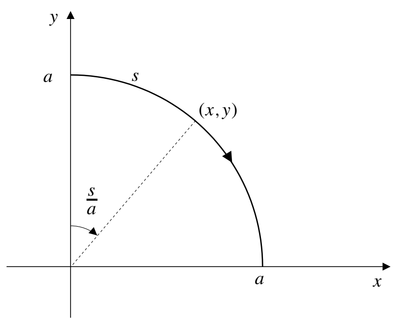

Fig. 11.3.4

5. $z = x^2$, $z = 4y^2$. If $t = y$, then $z = 4t^2$, so $x = \pm 2t$. The curve passes through $(2, -1, 4)$ when $t = -1$, so $x = -2t$. The parametrization is $\mathbf{r} = -2t\mathbf{i} + t\mathbf{j} + 4t^2\mathbf{k}$.

6. $z = x^2$, $x + y + z = 1$. If $t = x$, then $z = t^2$ and $y = 1 - t - t^2$. The parametrization is $\mathbf{r} = t\mathbf{i} + (1 - t - t^2)\mathbf{j} + t^2\mathbf{k}$.

7. $z = x + y$, $x^2 + y^2 = 9$. One possible parametrization is $\mathbf{r} = 3\cos t\mathbf{i} + 3\sin t\mathbf{j} + 3(\cos t + \sin t)\mathbf{k}$.

8. $x + y = 1$, $z = \sqrt{1 - x^2 - y^2}$. If $x = t$, then $y = 1 - t$ and $z = \sqrt{1 - t^2 - (1-t)^2} = \sqrt{2(t - t^2)}$. One possible parametrization is

$$\mathbf{r} = t\mathbf{i} + (1 - t)\mathbf{j} + \sqrt{2(t - t^2)}\mathbf{k}.$$

9. $z = x^2 + y^2$, $2x - 4y - z - 1 = 0$. These surfaces intersect on the vertical cylinder

$$x^2 + y^2 = 2x - 4y - 1, \quad \text{that is}$$
$$(x - 1)^2 + (y + 2)^2 = 4.$$

One possible parametrization is

$$x = 1 + 2\cos t$$
$$y = -2 + 2\sin t$$
$$z = -1 + 2(1 + 2\cos t) - 4(-2 + 2\sin t) = 9 + 4\cos t - 8\sin t$$
$$\mathbf{r} = (1 + 2\cos t)\mathbf{i} - 2(1 - \sin t)\mathbf{j} + (9 + 4\cos t - 8\sin t)\mathbf{k}.$$

10. $yz + x = 1$, $xz - x = 1$. One possible parametrization is $x = t$, $z = (1 + t)/t$, and $y = (1 - t)/z = (1 - t)t/(1 + t)$, that is,

$$\mathbf{r} = t\mathbf{i} + \frac{t - t^2}{1 + t}\mathbf{j} + \frac{1 + t}{t}\mathbf{k}.$$

11. $z^2 = x^2 + y^2$, $z = 1 + x$.

a) If $t = x$, then $z = 1 + t$, so $1 + 2t + t^2 = t^2 + y^2$, and $y = \pm\sqrt{1 + 2t}$. Two parametrizations are needed to get the whole parabola, one for $y \le 0$ and one for $y \ge 0$.

b) If $t = y$, then $x^2 + t^2 = z^2 = 1 + 2x + x^2$, so $2x + 1 = t^2$, and $x = (t^2 - 1)/2$. Thus $z = 1 + x = (t^2 + 1)/2$. The whole parabola is parametrized by

$$\mathbf{r} = \frac{t^2 - 1}{2}\mathbf{i} + t\mathbf{j} + \frac{t^2 + 1}{2}\mathbf{k}.$$

c) If $t = z$, then $x = t - 1$ and $t^2 = t^2 - 2t + 1 + y^2$, so $y = \pm\sqrt{2t - 1}$. Again two parametrizations are needed to get the whole parabola.

12. By symmetry, the centre of the circle $\mathcal{C}$ of intersection of the plane $x + y + z = 1$ and the sphere $x^2 + y^2 + z^2 = 1$ must lie on the plane and must have its three coordinates equal. Thus the centre has position vector

$$\mathbf{r}_0 = \frac{1}{3}(\mathbf{i} + \mathbf{j} + \mathbf{k}).$$

Since $\mathcal{C}$ passes through the point $(0, 0, 1)$, its radius is

$$\sqrt{\left(0 - \frac{1}{3}\right)^2 + \left(0 - \frac{1}{3}\right)^2 + \left(1 - \frac{1}{3}\right)^2} = \sqrt{\frac{2}{3}}.$$

Any vector $\mathbf{v}$ that satisfies $\mathbf{v} \bullet (\mathbf{i} + \mathbf{j} + \mathbf{k}) = 0$ is parallel to the plane $x + y + z = 1$ containing $\mathcal{C}$. One such vector is $\mathbf{v}_1 = \mathbf{i} - \mathbf{j}$. A second one, perpendicular to $\mathbf{v}_1$, is

$$\mathbf{v}_2 = (\mathbf{i} + \mathbf{j} + \mathbf{k}) \times (\mathbf{i} - \mathbf{j}) = \mathbf{i} + \mathbf{j} - 2\mathbf{k}.$$

Two perpendicular unit vectors that are parallel to the plane of $\mathcal{C}$ are

$$\hat{\mathbf{v}}_1 = \frac{\mathbf{i} - \mathbf{j}}{\sqrt{2}}, \quad \hat{\mathbf{v}}_2 = \frac{\mathbf{i} + \mathbf{j} - 2\mathbf{k}}{\sqrt{6}}.$$

Thus one possible parametrization of $\mathcal{C}$ is

$$\mathbf{r} = \mathbf{r}_0 + \sqrt{\frac{2}{3}}(\cos t\hat{\mathbf{v}}_1 + \sin t\hat{\mathbf{v}}_2)$$
$$= \frac{\mathbf{i} + \mathbf{j} + \mathbf{k}}{3} + \frac{\cos t}{\sqrt{3}}(\mathbf{i} - \mathbf{j}) + \frac{\sin t}{3}(\mathbf{i} + \mathbf{j} - 2\mathbf{k}).$$

13.
$$\mathbf{r} = t^2\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}, \quad (0 \le t \le 1)$$
$$v = \sqrt{(2t)^2 + (2t)^2 + (3t^2)^2} = t\sqrt{8 + 9t^2}$$
$$\text{Length} = \int_0^1 t\sqrt{8 + 9t^2}\,dt \quad \text{Let } u = 8 + 9t^2, \quad du = 18t\,dt$$
$$= \frac{1}{18}\,\frac{2}{3}u^{3/2}\Big|_8^{17} = \frac{17\sqrt{17} - 16\sqrt{2}}{27} \text{ units.}$$

14. $\mathbf{r} = t\mathbf{i} + \lambda t^2\mathbf{j} + t^3\mathbf{k}, \quad (0 \le t \le T)$
$v = \sqrt{1 + (2\lambda t)^2 + 9t^4} = \sqrt{(1+3t^2)^2}$
if $4\lambda^2 = 6$, that is, if $\lambda = \pm\sqrt{3/2}$. In this case, the length of the curve is

$$s(T) = \int_0^T (1+3t^2)\,dt = T + T^3.$$

15.
$$\text{Length} = \int_1^T \left|\frac{d\mathbf{r}}{dt}\right| dt = \int_1^T \sqrt{4a^2t^2 + b^2 + \frac{c^2}{t^2}}\,dt \text{ units.}$$

If $b^2 = 4ac$ then

$$\text{Length} = \int_1^T \sqrt{\left(2at + \frac{c}{t}\right)^2}\,dt = \int_1^T \left(2at + \frac{c}{t}\right) dt = a(T^2 - 1) + c\ln T \text{ units.}$$

16. $x = a\cos t\sin t = \dfrac{a}{2}\sin 2t,$
$y = a\sin^2 t = \dfrac{a}{2}(1 - \cos 2t),$
$z = bt.$
The curve is a circular helix lying on the cylinder

$$x^2 + \left(y - \frac{a}{2}\right)^2 = \frac{a^2}{4}.$$

Its length, from $t = 0$ to $t = T$, is

$$L = \int_0^T \sqrt{a^2\cos^2 2t + a^2\sin^2 2t + b^2}\,dt = T\sqrt{a^2 + b^2} \text{ units.}$$

17. $\mathbf{r} = t\cos t\mathbf{i} + t\sin t\mathbf{j} + t\mathbf{k}, \qquad 0 \le t \le 2\pi$
$\mathbf{v} = (\cos t - t\sin t)\mathbf{i} + (\sin t + t\cos t)\mathbf{j} + \mathbf{k}$
$v = |\mathbf{v}| = \sqrt{(1+t^2)+1} = \sqrt{2+t^2}.$
The length of the curve is

$$L = \int_0^{2\pi} \sqrt{2+t^2}\,dt \quad \begin{array}{l}\text{Let } t = \sqrt{2}\tan\theta \\ dt = \sqrt{2}\sec^2\theta\,d\theta\end{array}$$
$$= 2\int_{t=0}^{t=2\pi} \sec^3\theta\,d\theta$$
$$= \left(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right)\Big|_{t=0}^{t=2\pi}$$
$$= \frac{t\sqrt{2+t^2}}{2} + \ln\left(\frac{\sqrt{2+t^2}}{\sqrt{2}} + \frac{t}{\sqrt{2}}\right)\Bigg|_0^{2\pi}$$
$$= \pi\sqrt{2+4\pi^2} + \ln\left(\sqrt{1+2\pi^2} + \sqrt{2}\pi\right) \text{ units.}$$

The curve is called a conical helix because it is a spiral lying on the cone $x^2 + y^2 = z^2$.

18. One-eighth of the curve $\mathcal{C}$ lies in the first octant. That part can be parametrized

$$x = \cos t, \quad z = \frac{1}{\sqrt{2}}\sin t, \quad (0 \le t \le \pi/2)$$
$$y = \sqrt{1 - \cos^2 t - \frac{1}{2}\sin^2 t} = \frac{1}{\sqrt{2}}\sin t.$$

Since the first octant part of $\mathcal{C}$ lies in the plane $y = z$, it must be a quarter of a circle of radius 1. Thus the length of all of $\mathcal{C}$ is $8 \times (\pi/2) = 4\pi$ units.
If you wish to use an integral, the length is

$$8\int_0^{\pi/2} \sqrt{\sin^2 t + \frac{1}{2}\cos^2 t + \frac{1}{2}\cos^2 t}\,dt = 8\int_0^{\pi/2} dt = 4\pi \text{ units.}$$

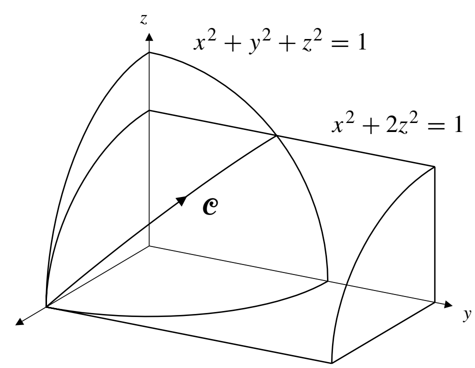

Fig. 11.3.18

19. If $\mathcal{C}$ is the curve

$$x = e^t\cos t, \quad y = e^t\sin t, \quad z = t, \qquad (0 \le t \le 2\pi),$$

then the length of $\mathcal{C}$ is

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$
$$= \int_0^{2\pi} \sqrt{e^{2t}(\cos t - \sin t)^2 + e^{2t}(\sin t + \cos t)^2 + 1}\,dt$$
$$= \int_0^{2\pi} \sqrt{2e^{2t}+1}\,dt \quad \begin{array}{l}\text{Let } 2e^{2t} + 1 = v^2 \\ 2e^{2t}\,dt = v\,dv\end{array}$$
$$= \int_{t=0}^{t=2\pi} \frac{v^2\,dv}{v^2 - 1} = \int_{t=0}^{t=2\pi} \left(1 + \frac{1}{v^2-1}\right) dv$$
$$= \left(v + \frac{1}{2}\ln\left|\frac{v-1}{v+1}\right|\right)\Bigg|_{t=0}^{t=2\pi}$$

$$= \sqrt{2e^{4\pi}+1} - \sqrt{3} + \frac{1}{2}\ln \frac{\sqrt{2e^{2t}+1}-1}{\sqrt{2e^{2t}+1}+1}\Bigg|_0^{2\pi}$$
$$= \sqrt{2e^{4\pi}+1} - \sqrt{3} + \ln \frac{\sqrt{2e^{2t}+1}-1}{\sqrt{2}e^t}\Bigg|_0^{2\pi}$$
$$= \sqrt{2e^{4\pi}+1} - \sqrt{3} + \ln\left(\sqrt{2e^{4\pi}+1}-1\right)$$
$$- 2\pi - \ln(\sqrt{3}-1) \text{ units.}$$

Remark: This answer appears somewhat different from that given in the answers section of the text. The two are, however, equal. Somewhat different simplifications were used in the two.

20. $\mathbf{r} = t^3\mathbf{i} + t^2\mathbf{j}$
$\mathbf{v} = 3t^2\mathbf{i} + 2t\mathbf{j}$
$v = |\mathbf{v}| = \sqrt{9t^4+4t^2} = |t|\sqrt{9t^2+4}$
The length L between $t=-1$ and $t=2$ is

$$L = \int_{-1}^0 (-t)\sqrt{9t^2+4}\,dt + \int_0^2 t\sqrt{9t^2+4}\,dt.$$

Making the substitution $u = 9t^2+4$ in each integral, we obtain

$$L = \frac{1}{18}\left[\int_4^{13} u^{1/2}\,du + \int_4^{40} u^{1/2}\,du\right]$$
$$= \frac{1}{27}\left(13^{3/2}+40^{3/2}-16\right) \quad \text{units.}$$

21. $\mathbf{r}_1 = t\mathbf{i} + t\mathbf{j}$, $(0 \le t \le 1)$ represents the straight line segment from the origin to $(1,1)$ in the xy-plane.

$\mathbf{r}_2 = (1-t)\mathbf{i} + (1+t)\mathbf{j}$, $(0 \le t \le 1)$ represents the straight line segment from $(1,1)$ to $(0,2)$.

Thus $\mathcal{C} = \mathcal{C}_1 + \mathcal{C}_2$ is the 2-segment polygonal line from the origin to $(1,1)$ and then to $(0,2)$.

22. (Solution due to Roland Urbanek, a student at Okanagan College.) Suppose the spool is vertical and the cable windings make angle θ with the horizontal at each point.

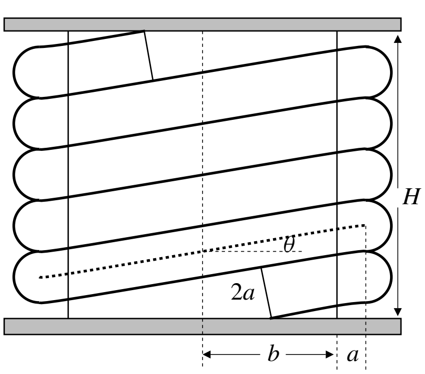

Fig. 11.3.22

The centreline of the cable is wound around a cylinder of radius $a+b$ and must rise a vertical distance $\dfrac{2a}{\cos\theta}$ in one revolution. The figure below shows the cable unwound from the spool and inclined at angle θ. The total length of spool required is the total height H of the cable as shown in that figure.

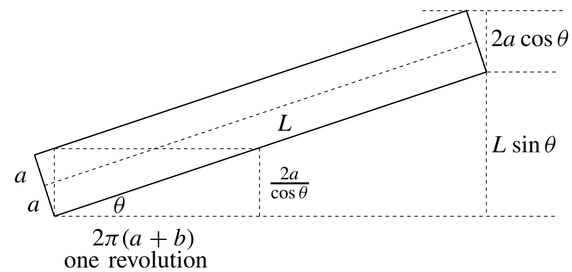

Fig. 11.3.22

Observe that $\tan\theta = \dfrac{2a}{\cos\theta} \times \dfrac{1}{2\pi(a+b)}$. Therefore

$$\sin\theta = \frac{a}{\pi(a+b)}$$
$$\cos\theta = \sqrt{1 - \frac{a^2}{\pi^2(a+b)^2}} = \frac{\sqrt{\pi^2(a+b)^2-a^2}}{\pi(a+b)}.$$

The total length of spool required is

$$H = L\sin\theta + 2a\cos\theta$$
$$= \frac{a}{\pi(a+b)}\left(L + 2\sqrt{\pi^2(a+b)^2-a^2}\right) \text{ units.}$$

23. $\mathbf{r} = At\mathbf{i} + Bt\mathbf{j} + Ct\mathbf{k}$.
The arc length from the point where $t=0$ to the point corresponding to arbitrary t is

$$s = s(t) = \int_0^t \sqrt{A^2+B^2+C^2}\,du = \sqrt{A^2+B^2+C^2}\,t.$$

Thus $t = s/\sqrt{A^2+B^2+C^2}$. The required parametrization is

$$\mathbf{r} = \frac{As\mathbf{i} + Bs\mathbf{j} + Cs\mathbf{k}}{\sqrt{A^2+B^2+C^2}}.$$

24. $\mathbf{r} = e^t\mathbf{i} + \sqrt{2}t\mathbf{j} - e^{-t}\mathbf{k}$
$\mathbf{v} = e^t\mathbf{i} + \sqrt{2}\mathbf{j} + e^{-t}\mathbf{k}$
$v = |\mathbf{v}| = \sqrt{e^{2t}+2+e^{-2t}} = e^t + e^{-t}$.
The arc length from the point where $t=0$ to the point corresponding to arbitrary t is

$$s = s(t) = \int_0^t (e^u + e^{-u})\,du = e^t - e^{-t} = 2\sinh t.$$

Thus $t = \sinh^{-1}(s/2) = \ln\left(\dfrac{s+\sqrt{s^2+4}}{2}\right)$,

and $e^t = \dfrac{s+\sqrt{s^2+4}}{2}$. The required parametrization is

$$\mathbf{r} = \frac{s+\sqrt{s^2+4}}{2}\mathbf{i} + \sqrt{2}\ln\left(\frac{s+\sqrt{s^2+4}}{2}\right)\mathbf{j} - \frac{2\mathbf{k}}{s+\sqrt{s^2+4}}.$$

25. $\mathbf{r} = a\cos^3 t\mathbf{i} + a\sin^3 t\mathbf{j} + b\cos 2t\mathbf{k}, \quad 0 \le t \le \dfrac{\pi}{2}$

$\mathbf{v} = -3a\cos^2 t\sin t\mathbf{i} + 3a\sin^2 t\cos t\mathbf{j} - 4b\sin t\cos t\mathbf{k}$

$v = \sqrt{9a^2+16b^2}\sin t\cos t$

$s = \displaystyle\int_0^t \sqrt{9a^2+16b^2}\sin u\cos u\,du$

$= \dfrac{1}{2}\sqrt{9a^2+16b^2}\sin^2 t = K\sin^2 t$

where $K = \dfrac{1}{2}\sqrt{9a^2+16b^2}$

Therefore $\sin t = \sqrt{\dfrac{s}{K}}$, $\cos t = \sqrt{1-\dfrac{s}{K}}$,

$\cos 2t = 1 - 2\sin^2 t = 1 - \dfrac{2s}{K}$.

The required parametrization is

$$\mathbf{r} = a\left(1-\frac{s}{K}\right)^{3/2}\mathbf{i} + a\left(\frac{s}{K}\right)^{3/2} + b\left(1-\frac{2s}{K}\right)\mathbf{k}$$

for $0 \le s \le K$, where $K = \dfrac{1}{2}\sqrt{9a^2+16b^2}$.

26. $\mathbf{r} = 3t\cos t\mathbf{i} + 3t\sin t\mathbf{j} + 2\sqrt{2}t^{3/2}\mathbf{k}, \ (t \ge 0)$

$\mathbf{v} = 3(\cos t - t\sin t)\mathbf{i} + 3(\sin t + t\cos t)\mathbf{j} + 3\sqrt{2}\sqrt{t}\mathbf{k}$

$v = |\mathbf{v}| = 3\sqrt{1+t^2+2t} = 3(1+t)$

$s = \displaystyle\int_0^t 3(1+u)\,du = 3\left(t+\frac{t^2}{2}\right)$

Thus $t^2+2t = \dfrac{2s}{3}$, so $t = -1+\sqrt{1+\dfrac{2s}{3}}$ since $t \ge 0$. The required parametrization is the given one with t replaced by $-1+\sqrt{1+(2s)/3}$.

27. As claimed in the statement of the problem, $\mathbf{r}_1(t) = \mathbf{r}_2\big(u(t)\big)$, where u is a function from $[a,b]$ to $[c,d]$, having $u(a) = c$ and $u(b) = d$. We assume u is differentiable. Since u is one-to-one and orientation-preserving, $du/dt \ge 0$ on $[a,b]$. By the Chain Rule:

$$\frac{d}{dt}\mathbf{r}_1(t) = \frac{d}{du}\mathbf{r}_2(u)\,\frac{du}{dt},$$

and so

$$\int_a^b \left|\frac{d}{dt}\mathbf{r}_1(t)\right| dt = \int_a^b \left|\frac{d}{du}\mathbf{r}_2\big(u(t)\big)\right| \frac{du}{dt}\,dt = \int_c^d \left|\frac{d}{du}\mathbf{r}_2(u)\right| du.$$

28. If $\mathbf{r} = \mathbf{r}(t)$ has nonvanishing velocity $\mathbf{v} = d\mathbf{r}/dt$ on $[a,b]$, then for any t_0 in $[a,b]$, the function

$$s = g(t) = \int_{t_0}^t |\mathbf{v}(u)|\,du,$$

which gives the (signed) arc length s measured from $\mathbf{r}(t_0)$ along the curve, is an increasing function:

$$\frac{ds}{dt} = g'(t) = |\mathbf{v}(t)| > 0$$

on $[a,b]$, by the Fundamental Theorem of Calculus. Hence g is invertible, and defines t as a function of arc length s:

$$t = g^{-1}(s) \Leftrightarrow s = g(t).$$

Then

$$\mathbf{r} = \mathbf{r}_2(s) = \mathbf{r}\Big(g^{-1}(s)\Big)$$

is a parametrization of the curve $\mathbf{r} = \mathbf{r}(t)$ in terms of arc length.

Section 11.4 Curvature, Torsion, and the Frenet Frame (page 619)

1. $\mathbf{r} = t\mathbf{i} - 2t^2\mathbf{j} + 3t^3\mathbf{k}$

$\mathbf{v} = \mathbf{i} - 4t\mathbf{j} + 9t^2\mathbf{k}$

$v = \sqrt{1+16t^2+81t^4}$

$\hat{\mathbf{T}} = \dfrac{\mathbf{v}}{v} = \dfrac{\mathbf{i} - 4t\mathbf{j} + 9t^2\mathbf{k}}{\sqrt{1+16t^2+81t^4}}$.

2. $\mathbf{r} = a\sin\omega t\mathbf{i} + a\cos\omega t\mathbf{k}$

$\mathbf{v} = a\omega\cos\omega t\mathbf{i} - a\omega\sin\omega t\mathbf{k}, \quad v = |a\omega|$

$\hat{\mathbf{T}} = \mathrm{sgn}(a\omega)\Big[\cos\omega t\mathbf{i} - \sin\omega t\mathbf{k}\Big]$.

3. $\mathbf{r} = \cos t\sin t\mathbf{i} + \sin^2 t + \cos t\mathbf{k}$

$= \dfrac{1}{2}\sin 2t\mathbf{i} + \dfrac{1}{2}(1-\cos 2t)\mathbf{j} + \cos t\mathbf{k}$

$\mathbf{v} = \cos 2t\mathbf{i} + \sin 2t\mathbf{j} - \sin t\mathbf{k}$

$v = |\mathbf{v}| = \sqrt{1+\sin^2 t}$

$\hat{\mathbf{T}} = \dfrac{1}{\sqrt{1+\sin^2 t}}\Big(\cos 2t\mathbf{i} + \sin 2t\mathbf{j} - \sin t\mathbf{k}\Big)$.

4. $\mathbf{r} = a\cos t\mathbf{i} + b\sin t\mathbf{j} + t\mathbf{k}$

$\mathbf{v} = -a\sin t\mathbf{i} + b\cos t\mathbf{j} + \mathbf{k}$

$v = \sqrt{a^2\sin^2 t + b^2\cos^2 t + 1}$

$\hat{\mathbf{T}} = \dfrac{\mathbf{v}}{v} = \dfrac{-a\sin t\mathbf{i} + b\cos t\mathbf{j} + \mathbf{k}}{\sqrt{a^2\sin^2 t + b^2\cos^2 t + 1}}$.

5. If $\kappa(s) = 0$ for all s, then $\frac{d\hat{\mathbf{T}}}{ds} = \kappa\hat{\mathbf{N}} = 0$, so $\hat{\mathbf{T}}(s) = \hat{\mathbf{T}}(0)$ is constant. This says that $\frac{d\mathbf{r}}{ds} = \hat{\mathbf{T}}(0)$, so $\mathbf{r} = \hat{\mathbf{T}}(0)s + \mathbf{r}(0)$, which is the vector parametric equation of a straight line.

6. If $\tau(s) = 0$ for all s, then $\frac{d\hat{\mathbf{B}}}{ds} = -\tau\hat{\mathbf{N}} = 0$, so $\hat{\mathbf{B}}(s) = \hat{\mathbf{B}}(0)$ is constant. Therefore,

$$\frac{d}{ds}\Big(\mathbf{r}(s) - \mathbf{r}(0)\Big) \bullet \hat{\mathbf{B}}(s) = \frac{d\mathbf{r}}{ds} \bullet \hat{\mathbf{B}}(s) = \hat{\mathbf{T}}(s) \bullet \hat{\mathbf{B}}(s) = 0.$$

It follows that

$$\Big(\mathbf{r}(s) - \mathbf{r}(0)\Big) \bullet \hat{\mathbf{B}}(0) = \Big(\mathbf{r}(s) - \mathbf{r}(0)\Big) \bullet \hat{\mathbf{B}}(s) = 0$$

for all s. This says that $\mathbf{r}(s)$ lies in the plane through $\mathbf{r}(0)$ having normal $\hat{\mathbf{B}}(0)$.

7. The circle $\mathcal{C}_1$ given by

$$\mathbf{r} = \frac{1}{C}\cos Cs\,\mathbf{i} + \frac{1}{C}\sin Cs\,\mathbf{j}$$

is parametrized in terms of arc length, and has curvature C and torsion 0. (See Examples 2 and 3.)
If curve $\mathcal{C}$ has constant curvature $\kappa(s) = C$ and constant torsion $\tau(s) = 0$, then $\mathcal{C}$ is congruent to $\mathcal{C}_1$ by Theorem 3. Thus $\mathcal{C}$ must itself be a circle (with radius $1/C$).

8. The circular helix $\mathcal{C}_1$ given by

$$\mathbf{r} = a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j} + bt\mathbf{k}$$

has curvature and torsion given by

$$\kappa(s) = \frac{a}{a^2+b^2}, \qquad \tau(s) = \frac{b}{a^2+b^2},$$

by Example 3.
if a curve $\mathcal{C}$ has constant curvature $\kappa(s) = C > 0$, and constant torsion $\tau(s) = T \neq 0$, then we can choose a and b so that

$$\frac{a}{a^2+b^2} = C, \qquad \frac{b}{a^2+b^2} = T.$$

(Specifically, $a = \frac{C}{C^2+T^2}$, and $b = \frac{T}{C^2+T^2}$.) By Theorem 3, $\mathcal{C}$ is itself a circular helix, congruent to $\mathcal{C}_1$.

Section 11.5 Curvature and Torsion for General Parametrizations (page 625)

1. For $y = x^2$ we have

$$\kappa(x) = \frac{|d^2y/dx^2|}{(1+(dy/dx)^2)^{3/2}} = \frac{2}{(1+4x^2)^{3/2}}.$$

Hence $\kappa(0) = 2$ and $\kappa(\sqrt{2}) = 2/27$. The radii of curvature at $x = 0$ and $x = \sqrt{2}$ are $1/2$ and $27/2$, respectively.

2. For $y = \cos$ we have

$$\kappa(x) = \frac{|d^2y/dx^2|}{(1+(dy/dx)^2)^{3/2}} = \frac{|\cos x|}{(1+\sin^2 x)^{3/2}}.$$

Hence $\kappa(0) = 1$ and $\kappa(\pi/2) = 0$. The radius of curvature at $x = 0$ is 1. The radius of curvature at $x = \pi/2$ is infinite.

3.
$$\begin{aligned}
\mathbf{r} &= 2t\mathbf{i} + (1/t)\mathbf{j} - 2t\mathbf{k}\\
\mathbf{v} &= 2\mathbf{i} - (1/t^2)\mathbf{j} - 2\mathbf{k}\\
\mathbf{a} &= (2/t^3)\mathbf{j}\\
\mathbf{v}\times\mathbf{a} &= (4/t^3)\mathbf{i} + (4/t^3)\mathbf{k}
\end{aligned}$$
At $(2, 1, -2)$, that is, at $t = 1$, we have

$$\kappa = \kappa(1) = \frac{|\mathbf{v}\times\mathbf{a}|}{v^3} = \frac{4\sqrt{2}}{27}.$$

Thus the radius of curvature is $27/(4\sqrt{2})$.

4.
$$\begin{aligned}
\mathbf{r} &= t^3\mathbf{i} + t^2\mathbf{j} + t\mathbf{k}\\
\mathbf{v} &= 3t^2\mathbf{i} + 2t\mathbf{j} + \mathbf{k}\\
\mathbf{a} &= 6t\mathbf{i} + 2\mathbf{j}\\
\mathbf{v}(1) &= 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}, \quad \mathbf{a}(1) = 6\mathbf{i} + 2\mathbf{j}\\
\mathbf{v}(1)\times\mathbf{a}(1) &= -2\mathbf{i} + 6\mathbf{j} - 6\mathbf{k}\\
\kappa(1) &= \frac{\sqrt{4+36+36}}{(9+4+1)^{3/2}} = \frac{2\sqrt{19}}{14^{3/2}}
\end{aligned}$$
At $t = 1$ the radius of curvature is $14^{3/2}/(2\sqrt{19})$.

5.
$$\begin{aligned}
\mathbf{r} &= t\mathbf{i} + t^2\mathbf{j} + 2\mathbf{k}\\
\mathbf{v} &= \mathbf{i} + 2t\mathbf{j}\\
\mathbf{a} &= 2\mathbf{j}\\
\mathbf{v}\times\mathbf{a} &= 2\mathbf{k}
\end{aligned}$$
At $(1, 1, 2)$, where $t = 1$, we have
$$\begin{aligned}
\hat{\mathbf{T}} &= \mathbf{v}/|\mathbf{v}| = (\mathbf{i} + 2\mathbf{j})/\sqrt{5}\\
\hat{\mathbf{B}} &= (\mathbf{v}\times\mathbf{a})/|\mathbf{v}\times\mathbf{a}| = \mathbf{k}\\
\hat{\mathbf{N}} &= \hat{\mathbf{B}}\times\hat{\mathbf{T}} = (-2\mathbf{i} + \mathbf{j})/\sqrt{5}.
\end{aligned}$$

6.
$$\begin{aligned}\mathbf{r} &= t\mathbf{i}+t^2\mathbf{j}+t\mathbf{k}\\ \mathbf{v} &= \mathbf{i}+2t\mathbf{j}+\mathbf{k}\\ \mathbf{a} &= 2\mathbf{j}\\ \mathbf{v}\times\mathbf{a} &= -2\mathbf{i}+2\mathbf{k}\end{aligned}$$

At $(1,1,1)$, where $t=1$, we have
$$\begin{aligned}\hat{\mathbf{T}} &= \mathbf{v}/|\mathbf{v}| = (\mathbf{i}+2\mathbf{j}+\mathbf{k})/\sqrt{6}\\ \hat{\mathbf{B}} &= (\mathbf{v}\times\mathbf{a})/|\mathbf{v}\times\mathbf{a}| = -(\mathbf{i}-\mathbf{k})/\sqrt{2}\\ \hat{\mathbf{N}} &= \hat{\mathbf{B}}\times\hat{\mathbf{T}} = -(\mathbf{i}-\mathbf{j}+\mathbf{k})/\sqrt{3}.\end{aligned}$$

7.
$$\begin{aligned}\mathbf{r} &= t\mathbf{i}+\frac{t^2}{2}\mathbf{j}+\frac{t^3}{3}\mathbf{k}\\ \mathbf{v} &= \mathbf{i}+t\mathbf{j}+t^2\mathbf{k}\\ \mathbf{a} &= \mathbf{j}+2t\mathbf{k}, \qquad \frac{d\mathbf{a}}{dt}=2\mathbf{k}\\ \mathbf{v}\times\mathbf{a} &= t^2\mathbf{i}-2t\mathbf{j}+\mathbf{k}\\ v &= |\mathbf{v}| = \sqrt{1+t^2+t^4}, \qquad |\mathbf{v}\times\mathbf{a}| = \sqrt{1+4t^2+t^4}\\ (\mathbf{v}\times\mathbf{a})\bullet\frac{d\mathbf{a}}{dt} &= 2\end{aligned}$$

$$\begin{aligned}\hat{\mathbf{T}} &= \frac{\mathbf{v}}{v} = \frac{\mathbf{i}+t\mathbf{j}+t^2\mathbf{k}}{\sqrt{1+t^2+t^4}}\\ \hat{\mathbf{B}} &= \frac{\mathbf{v}\times\mathbf{a}}{|\mathbf{v}\times\mathbf{a}|} = \frac{t^2\mathbf{i}-2t\mathbf{j}+\mathbf{k}}{\sqrt{1+4t^2+t^4}}\\ \hat{\mathbf{N}} &= \hat{\mathbf{B}}\times\hat{\mathbf{T}} = \frac{-(2t^3+t)\mathbf{i}+(1-t^4)\mathbf{j}+(t^3+2t)\mathbf{k}}{\sqrt{(1+t^2+t^4)(1+4t^2+t^4)}}\\ \kappa &= \frac{|\mathbf{v}\times\mathbf{a}|}{v^3} = \frac{\sqrt{1+4t^2+t^4}}{(1+t^2+t^4)^{3/2}}\\ \tau &= \frac{(\mathbf{v}\times\mathbf{a})\bullet\frac{d\mathbf{a}}{dt}}{|\mathbf{v}\times\mathbf{a}|^2} = \frac{2}{1+4t^2+t^4}.\end{aligned}$$

8.
$$\begin{aligned}\mathbf{r} &= e^t\cos t\mathbf{i}+e^t\sin t\mathbf{j}+e^t\mathbf{k}\\ \mathbf{v} &= e^t(\cos t-\sin t)\mathbf{i}+e^t(\sin t+\cos t)\mathbf{j}+e^t\mathbf{k}\\ \mathbf{a} &= -2e^t\sin t\mathbf{i}+2e^t\cos t\mathbf{j}+e^t\mathbf{k}\\ \frac{d\mathbf{a}}{dt} &= -2e^t(\cos t+\sin t)\mathbf{i}+2e^t(\cos t-\sin t)\mathbf{j}+e^t\mathbf{k}\\ \mathbf{v}\times\mathbf{a} &= e^{2t}(\sin t-\cos t)\mathbf{i}-e^{2t}(\cos t+\sin t)\mathbf{j}+2e^{2t}\mathbf{k}\\ v &= |\mathbf{v}| = \sqrt{3}e^t, \qquad |\mathbf{v}\times\mathbf{a}| = \sqrt{6}e^{2t}\\ (\mathbf{v}\times\mathbf{a})\bullet\frac{d\mathbf{a}}{dt} &= 2e^{3t}\\ \hat{\mathbf{T}} &= \frac{\mathbf{v}}{v} = \frac{(\cos t-\sin t)\mathbf{i}+(\cos t+\sin t)\mathbf{j}+\mathbf{k}}{\sqrt{3}}\\ \hat{\mathbf{B}} &= \frac{\mathbf{v}\times\mathbf{a}}{|\mathbf{v}\times\mathbf{a}|} = \frac{(\sin t-\cos t)\mathbf{i}-(\cos t+\sin t)\mathbf{j}+2\mathbf{k}}{\sqrt{6}}\\ \hat{\mathbf{N}} &= \hat{\mathbf{B}}\times\hat{\mathbf{T}} = -\frac{(\cos t+\sin t)\mathbf{i}-(\cos t-\sin t)\mathbf{j}}{\sqrt{2}}\\ \kappa &= \frac{|\mathbf{v}\times\mathbf{a}|}{v^3} = \frac{\sqrt{2}}{3e^t}\\ \tau &= \frac{(\mathbf{v}\times\mathbf{a})\bullet\frac{d\mathbf{a}}{dt}}{|\mathbf{v}\times\mathbf{a}|^2} = \frac{1}{3e^t}.\end{aligned}$$

9.
$$\begin{aligned}\mathbf{r} &= (2+\sqrt{2}\cos t)\mathbf{i}+(1-\sin t)\mathbf{j}+(3+\sin t)\mathbf{k}\\ \mathbf{v} &= -\sqrt{2}\sin t\mathbf{i}-\cos t\mathbf{j}+\cos t\mathbf{k}\\ v &= \sqrt{2\sin^2 t+\cos^2 t+\cos^2 t} = \sqrt{2}\\ \mathbf{a} &= -\sqrt{2}\cos t\mathbf{i}+\sin t\mathbf{j}-\sin t\mathbf{k}\\ \frac{d\mathbf{a}}{dt} &= \sqrt{2}\sin t\mathbf{i}+\cos t\mathbf{j}-\cos t\mathbf{k}\\ \mathbf{v}\times\mathbf{a} &= -\sqrt{2}\mathbf{j}-\sqrt{2}\mathbf{k}\\ \kappa &= \frac{|\mathbf{v}\times\mathbf{a}|}{v^3} = \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}}\\ (\mathbf{v}\times\mathbf{a})\bullet\frac{d\mathbf{a}}{dt} &= -\sqrt{2}\cos t+\sqrt{2}\cos t = 0\\ \tau &= 0.\end{aligned}$$

Since $\kappa = 1/\sqrt{2}$ is constant, and $\tau = 0$, the curve is a circle. Its centre is $(2,1,3)$ and its radius is $\sqrt{2}$. It lies in a plane with normal $\mathbf{j}+\mathbf{k}(=-\sqrt{2}\hat{\mathbf{B}})$.

10.
$$\begin{aligned}\mathbf{r} &= x\mathbf{i}+\sin x\mathbf{j}\\ \mathbf{v} &= \frac{dx}{dt}\mathbf{i}+\cos x\frac{dx}{dt}\mathbf{j} = k(\mathbf{i}+\cos x\mathbf{j})\\ v &= k\sqrt{1+\cos^2 x}\\ \mathbf{a} &= -k\sin x\frac{dx}{dt}\mathbf{j} = -k^2\sin x\mathbf{j}\\ \mathbf{v}\times\mathbf{a} &= -k^3\sin x\mathbf{k}\\ \kappa &= \frac{|\mathbf{v}\times\mathbf{a}|}{v^3} = \frac{|\sin x|}{(1+\cos^2 x)^{3/2}}.\end{aligned}$$

The tangential and normal components of acceleration are

$$\frac{dv}{dt} = \frac{k}{2\sqrt{1+\cos^2 x}} 2\cos x)(-\sin x)\frac{dx}{dt} = -\frac{k^2\cos x \sin x}{\sqrt{1+\cos^2 x}}$$
$$v^2\kappa = \frac{k^2|\sin x|}{\sqrt{1+\cos^2 x}}.$$

11. $$\mathbf{r} = \sin t\cos t\mathbf{i} + \sin^2 t\mathbf{j} + \cos t\mathbf{k}$$
$$\mathbf{v} = \cos 2t\mathbf{i} + \sin 2t\mathbf{j} - \sin t\mathbf{k}$$
$$\mathbf{a} = -2\sin 2t\mathbf{i} + 2\cos 2t\mathbf{j} - \cos t\mathbf{k}$$
$$\frac{d\mathbf{a}}{dt} = -4\cos 2t\mathbf{i} - 4\sin 2t\mathbf{j} + \sin t\mathbf{k}.$$

At $t = 0$ we have $\mathbf{v} = \mathbf{i}$, $\mathbf{a} = 2\mathbf{j} - \mathbf{k}$, $\dfrac{d\mathbf{a}}{dt} = -4\mathbf{i}$,
$\mathbf{v}\times\mathbf{a} = \mathbf{j} + 2\mathbf{k}$, $(\mathbf{v}\times\mathbf{a}) \bullet \dfrac{d\mathbf{a}}{dt} = 0$.
Thus $\hat{\mathbf{T}} = \mathbf{i}$, $\hat{\mathbf{B}} = (\mathbf{j}+2\mathbf{k})/\sqrt{5}$, $\hat{\mathbf{N}} = (2\mathbf{j}-\mathbf{k})/\sqrt{5}$,
$\kappa = \sqrt{5}$, and $\tau = 0$.

At $t = \pi/4$ we have $\mathbf{v} = \mathbf{j} - \dfrac{1}{\sqrt{2}}\mathbf{k}$, $\mathbf{a} = -2\mathbf{i} - \dfrac{1}{\sqrt{2}}\mathbf{k}$,
$\dfrac{d\mathbf{a}}{dt} = -4\mathbf{j} + \dfrac{1}{\sqrt{2}}\mathbf{k}$, $\mathbf{v}\times\mathbf{a} = -\dfrac{1}{\sqrt{2}}\mathbf{i} + \sqrt{2}\mathbf{j} + 2\mathbf{k}$,
$(\mathbf{v}\times\mathbf{a}) \bullet \dfrac{d\mathbf{a}}{dt} = -3\sqrt{2}$.
Thus

$$\hat{\mathbf{T}} = \frac{1}{\sqrt{3}}(\sqrt{2}\mathbf{j} - \mathbf{k})$$
$$\hat{\mathbf{B}} = \frac{1}{\sqrt{13}}(-\mathbf{i} + 2\mathbf{j} + 2\sqrt{2}\mathbf{k})$$
$$\hat{\mathbf{N}} = -\frac{1}{\sqrt{39}}(6\mathbf{i} + \mathbf{j} + \sqrt{2}\mathbf{k})$$
$$\kappa = \frac{2\sqrt{39}}{9}, \qquad \tau = -\frac{6\sqrt{2}}{13}.$$

12. $$\mathbf{r} = a\cos t\mathbf{i} + b\sin t\mathbf{j}$$
$$\mathbf{v} = -a\sin t\mathbf{i} + b\cos t\mathbf{j}$$
$$\mathbf{a} = -a\cos t\mathbf{i} - b\sin t\mathbf{j}$$
$$\mathbf{v}\times\mathbf{a} = ab\mathbf{k}$$
$$v = \sqrt{a^2\sin^2 t + b^2\cos^2 t}.$$

The tangential component of acceleration is

$$\frac{dv}{dt} = \frac{(a^2-b^2)\sin t\cos t}{\sqrt{a^2\sin^2 t + b^2\cos^2 t}},$$

which is zero if t is an integer multiple of $\pi/2$, that is, at the ends of the major and minor axes of the ellipse.
The normal component of acceleration is

$$v^2\kappa = v^2\frac{|\mathbf{v}\times\mathbf{a}|}{v^3} = \frac{ab}{\sqrt{a^2\sin^2 t + b^2\cos^2 t}}.$$

13. The ellipse is the same one considered in Exercise 16, so its curvature is

$$\kappa = \frac{ab}{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}} = \frac{ab}{\left((a^2-b^2)\sin^2 t + b^2\right)^{3/2}}.$$

If $a > b > 0$, then the maximum curvature occurs when $\sin t = 0$, and is a/b^2. The minimum curvature occurs when $\sin t = \pm 1$, and is b/a^2.

14. By Example 2, the curvature of $y = x^2$ at (1, 1) is

$$\kappa = \left.\frac{2}{(1+4x^2)^{3/2}}\right|_{x=1} = \frac{2}{5\sqrt{5}}.$$

Thus the magnitude of the normal acceleration of the bead at that point is $v^2\kappa = 2v^2/(5\sqrt{5})$.

The rate of change of the speed, dv/dt, is the tangential component of the acceleration, and is due entirely to the tangential component of the gravitational force since there is no friction:

$$\frac{dv}{dt} = g\cos\theta = g(-\mathbf{j}) \bullet \hat{\mathbf{T}},$$

where θ is the angle between $\hat{\mathbf{T}}$ and $-\mathbf{j}$. (See the figure.) Since the slope of $y = x^2$ at (1, 1) is 2, we have $\hat{\mathbf{T}} = -(\mathbf{i}+2\mathbf{j})/\sqrt{5}$, and therefore $dv/dt = 2g/\sqrt{5}$.

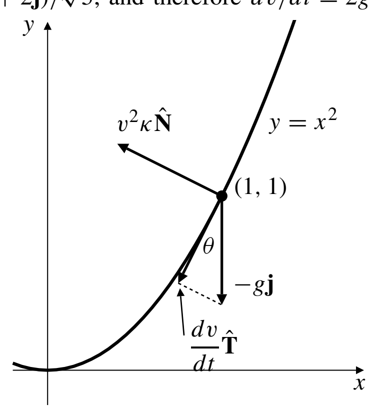

Fig. 11.5.14

15. Curve: $\mathbf{r} = x\mathbf{i} + e^x\mathbf{j}$.
Velocity: $\mathbf{v} = \mathbf{i} + e^x\mathbf{j}$. Speed: $v = \sqrt{1+e^{2x}}$.
Acceleration: $\mathbf{a} = e^x\mathbf{j}$. We have

$$\mathbf{v}\times\mathbf{a} = e^x\mathbf{k}, \qquad |\mathbf{v}\times\mathbf{a}| = e^x.$$

The curvature is $\kappa = \dfrac{e^x}{(1+e^{2x})^{3/2}}$. Therefore, the radius of curvature is $\rho = \dfrac{(1+e^{2x})^{3/2}}{e^x}$.

The unit normal is

$$\hat{\mathbf{N}} = \hat{\mathbf{B}} \times \hat{\mathbf{T}} = \frac{(\mathbf{v}\times\mathbf{a})\times\mathbf{v}}{|(\mathbf{v}\times\mathbf{a})\times\mathbf{v}|} = \frac{-e^x\mathbf{i}+\mathbf{j}}{\sqrt{1+e^{2x}}}.$$

The centre of curvature is

$$\begin{aligned}\mathbf{r}_c &= \mathbf{r}+\rho\hat{\mathbf{N}}\\ &= x\mathbf{i}+e^x\mathbf{j}+(1+e^{2x})\left(-\mathbf{i}+\frac{1}{e^x}\mathbf{j}\right)\\ &= (x-1-e^{2x})\mathbf{i}+(2e^x+e^{-x})\mathbf{j}.\end{aligned}$$

This is the equation of the evolute.

16. The curve with polar equation $r = f(\theta)$ is given parametrically by

$$\mathbf{r} = f(\theta)\cos\theta\,\mathbf{i} + f(\theta)\sin\theta\,\mathbf{j}.$$

Thus we have

$$\begin{aligned}\mathbf{v} &= \Big(f'(\theta)\cos\theta - f(\theta)\sin\theta\Big)\mathbf{i}\\ &\quad + \Big(f'(\theta)\sin\theta + f(\theta)\cos\theta\Big)\mathbf{j}\\ \mathbf{a} &= \Big(f''(\theta)\cos\theta - 2f'(\theta)\sin\theta - f(\theta)\cos\theta\Big)\mathbf{i}\\ &\quad + \Big(f''(\theta)\sin\theta + 2f'(\theta)\cos\theta - f(\theta)\sin\theta\Big)\mathbf{j}\\ v &= |\mathbf{v}| = \sqrt{\Big(f'(\theta)\Big)^2 + \Big(f(\theta)\Big)^2}\\ \mathbf{v}\times\mathbf{a} &= \left[2\Big(f'(\theta)\Big)^2 + \Big(f(\theta)\Big)^2 - f(\theta)f''(\theta)\right]\mathbf{k}.\end{aligned}$$

The curvature is, therefore,

$$\frac{\left|2\Big(f'(\theta)\Big)^2 + \Big(f(\theta)\Big)^2 - f(\theta)f''(\theta)\right|}{\left[\Big(f'(\theta)\Big)^2 + \Big(f(\theta)\Big)^2\right]^{3/2}}.$$

17. If $r = a(1-\cos\theta)$, then $r' = a\sin\theta$, and $r'' = a\cos\theta$. By the result of Exercise 20, the curvature of this cardioid is

$$\begin{aligned}\kappa &= \frac{1}{\Big(a^2\sin^2\theta + a^2(1-\cos\theta)^2\Big)^{3/2}} \times \Big|2a^2\sin^2\theta\\ &\qquad + a^2(1-\cos\theta)^2 - a^2(\cos\theta - \cos^2\theta)\Big|\\ &= \frac{3a^2(1-\cos\theta)}{\Big(2a^2(1-\cos\theta)\Big)^{3/2}} = \frac{3}{2\sqrt{2ar}}.\end{aligned}$$

18. By Exercise 8 of Section 2.4, the required curve must be a circular helix with parameters $a = 1/2$ (radius), and $b = 1/2$. Its equation will be

$$\mathbf{r} = \frac{1}{2}\cos t\,\mathbf{i}_1 + \frac{1}{2}\sin t\,\mathbf{j}_1 + \frac{1}{2}t\mathbf{k}_1 + \mathbf{r}_0$$

for some right-handed basis $\{\mathbf{i}_1, \mathbf{j}_1, \mathbf{k}_1\}$, and some constant vector $\mathbf{r}_0$. Example 3 of Section 2.4 provides values for $\hat{\mathbf{T}}(0)$, $\hat{\mathbf{N}}(0)$, and $\hat{\mathbf{B}}(0)$, which we can equate to the given values of these vectors:

$$\begin{aligned}\mathbf{i} &= \hat{\mathbf{T}}(0) = \frac{1}{\sqrt{2}}\mathbf{j}_1 + \frac{1}{\sqrt{2}}\mathbf{k}_1\\ \mathbf{j} &= \hat{\mathbf{N}}(0) = -\mathbf{i}_1\\ \mathbf{k} &= \hat{\mathbf{B}}(0) = -\frac{1}{\sqrt{2}}\mathbf{j}_1 + \frac{1}{\sqrt{2}}\mathbf{k}_1.\end{aligned}$$

Solving these equations for $\mathbf{i}_1$, $\mathbf{j}_1$, and $\mathbf{k}_1$ in terms of the given basis vectors, we obtain

$$\begin{aligned}\mathbf{i}_1 &= -\mathbf{j}\\ \mathbf{j}_1 &= \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{k}\\ \mathbf{k}_1 &= \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{k}.\end{aligned}$$

Therefore

$$\mathbf{r}(t) = \frac{t+\sin t}{2\sqrt{2}}\mathbf{i} - \frac{\cos t}{2}\mathbf{j} + \frac{t-\sin t}{2\sqrt{2}}\mathbf{k} + \mathbf{r}_0.$$

We also require that $\mathbf{r}(0) = \mathbf{i}$, so $\mathbf{r}_0 = \mathbf{i} + \frac{1}{2}\mathbf{j}$. The required equation is, therefore,

$$\mathbf{r}(t) = \left(\frac{t+\sin t}{2\sqrt{2}} + 1\right)\mathbf{i} + \frac{1-\cos t}{2}\mathbf{j} + \frac{t-\sin t}{2\sqrt{2}}\mathbf{k}.$$

19. Given that $\dfrac{d\mathbf{r}}{dt} = \mathbf{c}\times\mathbf{r}(t)$, we have

$$\begin{aligned}&\frac{d}{dt}|\mathbf{r}|^2 = \frac{d}{dt}\mathbf{r}\bullet\mathbf{r} = 2\mathbf{r}\bullet(\mathbf{c}\times\mathbf{r}) = 0\\ &\frac{d}{dt}\Big(\mathbf{r}(t)-\mathbf{r}(0)\Big)\bullet\mathbf{c} = \frac{d\mathbf{r}}{dt}\bullet\mathbf{c} = (\mathbf{c}\times\mathbf{r})\bullet\mathbf{c} = 0.\end{aligned}$$

Thus $|\mathbf{r}(t)| = |\mathbf{r}(0)|$ is constant, and $\Big(\mathbf{r}(t)-\mathbf{r}(0)\Big)\bullet\mathbf{c} = 0$ is constant. Thus $\mathbf{r}(t)$ lies on the sphere centred at the origin with radius $|\mathbf{r}(0)|$, and also on the plane through $\mathbf{r}(0)$ with normal $\mathbf{c}$. The curve is the circle of intersection of this sphere and this plane.

20. For $\mathbf{r} = a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j} + bt\mathbf{k}$, we have, by Example 3 of Section 2.4,

$$\hat{\mathbf{N}} = -\cos t\,\mathbf{i} - \sin t\,\mathbf{j}, \qquad \kappa = \frac{a}{a^2+b^2}.$$

The centre of curvature $\mathbf{r}_c$ is given by

$$\mathbf{r}_c = \mathbf{r} + \rho\hat{\mathbf{N}} = \mathbf{r} + \frac{1}{\kappa}\hat{\mathbf{N}}.$$

Thus the evolute has equation

$$\begin{aligned}\mathbf{r} &= a\cos t\mathbf{i} + a\sin t\mathbf{j} + bt\mathbf{k}\\ &\quad - \frac{a^2+b^2}{a}(\cos t\mathbf{i} + \sin t\mathbf{j})\\ &= -\frac{b^2}{a}\cos t\mathbf{i} - \frac{b^2}{a}\sin t\mathbf{j} + bt\mathbf{k}.\end{aligned}$$

The evolute is also a circular helix.

21. The parabola $y = x^2$ has curvature

$$\kappa = \frac{2}{(1+4x^2)^{3/2}},$$

by Exercise 18. The normal at (x, x^2) is perpendicular to the tangent, so has slope $-1/(2x)$. Since the unit normal points upward (the concave side of the parabola), we have

$$\hat{\mathbf{N}} = \frac{-2x\mathbf{i} + \mathbf{j}}{\sqrt{1+4x^2}}.$$

Thus the evolute of the parabola has equation

$$\begin{aligned}\mathbf{r} &= x\mathbf{i} + x^2\mathbf{j} + \frac{(1+4x^2)^{3/2}}{2}\left(\frac{-2x\mathbf{i}+\mathbf{j}}{\sqrt{1+4x^2}}\right)\\ &= x\mathbf{i} + x^2\mathbf{j} - (1+4x^2)x\mathbf{i} + \frac{1+4x^2}{2}\mathbf{j}\\ &= -4x^3\mathbf{i} + \left(3x^2 + \frac{1}{2}\right)\mathbf{j}.\end{aligned}$$

22. For the ellipse $\mathbf{r} = 2\cos t\mathbf{i} + \sin t\mathbf{j}$, we have

$$\begin{aligned}\mathbf{v} &= -2\sin t\mathbf{i} + \cos t\mathbf{j}\\ \mathbf{a} &= -2\cos t\mathbf{i} - \sin t\mathbf{j}\\ \mathbf{v}\times\mathbf{a} &= 2\mathbf{k}\\ v &= \sqrt{4\sin^2 t + \cos^2 t} = \sqrt{3\sin^2 t + 1}.\end{aligned}$$

The curvature is $\kappa = \dfrac{2}{(3\sin^2 t + 1)^{3/2}}$, so the radius of curvature is $\rho = \dfrac{(3\sin^2 t + 1)^{3/2}}{2}$. We have

$$\hat{\mathbf{T}} = \frac{-2\sin t\mathbf{i} + \cos t\mathbf{j}}{\sqrt{3\sin^2 t + 1}}, \qquad \hat{\mathbf{B}} = \mathbf{k}$$
$$\hat{\mathbf{N}} = -\frac{\cos t\mathbf{i} + 2\sin t\mathbf{j}}{\sqrt{3\sin^2 t + 1}}.$$

Therefore the evolute has equation

$$\begin{aligned}\mathbf{r} &= 2\cos t\mathbf{i} + \sin t\mathbf{j} - \frac{3\sin^2 t + 1}{2}(\cos t\mathbf{i} + 2\sin t\mathbf{j})\\ &= \frac{3}{2}\cos^3\mathbf{i} - 3\sin^3 t\mathbf{j}.\end{aligned}$$

23. We require that

$$\begin{aligned}f(1) &= 1, & f'(1) &= 0, & f''(1) &= 0,\\ f(-1) &= -1, & f'(-1) &= 0, & f''(-1) &= 0.\end{aligned}$$

As in Example 5, we try a polynomial of degree 5. However, here it is clear that an odd function will do, and we need only impose the conditions at $x = 1$. Thus we try

$$\begin{aligned}f(x) &= Ax + Bx^3 + Cx^5\\ f'(x) &= A + 3Bx^2 + 5Cx^4\\ f''(x) &= 6Bx + 20Cx^3.\end{aligned}$$

The conditions at $x = 1$ become

$$\begin{aligned}A + B + C &= 1\\ A + 3B + 5C &= 0\\ 6B + 20C &= 0.\end{aligned}$$

This system has solution $A = 15/8$, $B = -5/4$, and $C = 3/8$. Thus

$$f(x) = \frac{15}{8}x - \frac{5}{4}x^3 + \frac{3}{8}x^5$$

is one possible solution.

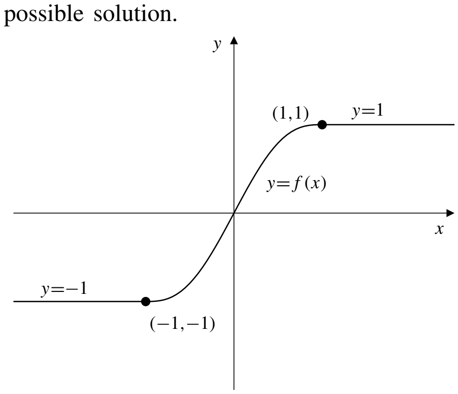

Fig. 11.5.23

24. We require

$$\begin{aligned}f(0) &= 1, & f'(0) &= 0, & f''(0) &= -1,\\ f(-1) &= 1, & f'(-1) &= 0, & f''(-1) &= 0.\end{aligned}$$

The condition $f''(0) = -1$ follows from the fact that

$$\left.\frac{d^2}{dx^2}\sqrt{1-x^2}\right|_{x=0} = -1.$$

As in Example 5, we try

$$\begin{aligned} f(x) &= A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 \\ f'(x) &= B + 2Cx + 3Dx^2 + 4Ex^3 + 5Fx^4 \\ f'' &= 2C + 6Dx + 12Ex^2 + 20Fx^3. \end{aligned}$$

The required conditions force the coefficients to satisfy the system of equations

$$\begin{aligned} A - B + C - D + E - F &= 1 \\ B - 2C + 3D - 4E + 5F &= 0 \\ 2C - 6D + 12E - 20F &= 0 \\ A &= 1 \\ B &= 0 \\ 2C &= -1 \end{aligned}$$

which has solution $A = 1$, $B = 0$, $C = -1/2$, $D = -3/2$, $E = -3/2$, $F = -1/2$. Thus we can use a track section in the shape of the graph of

$$f(x) = 1 - \frac{1}{2}x^2 - \frac{3}{2}x^3 - \frac{3}{2}x^4 - \frac{1}{2}x^5 = 1 - \frac{1}{2}x^2(1+x)^3.$$

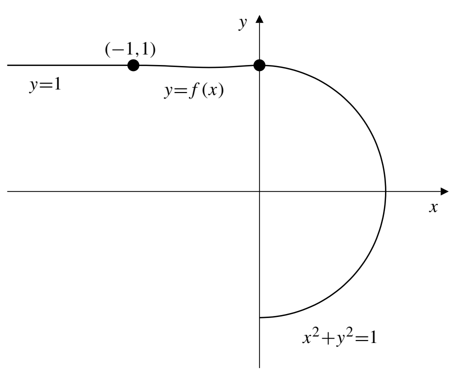

Fig. 11.5.24

25. Given: $\mathbf{a}(t) = \lambda(t)\mathbf{r}(t) + \mu(t)\mathbf{v}(t)$, $\mathbf{v} \times \mathbf{a} \neq \mathbf{0}$. We have

$$\begin{aligned} \mathbf{v} \times \mathbf{a} &= \lambda \mathbf{v} \times \mathbf{r} + \mu \mathbf{v} \times \mathbf{v} = \lambda \mathbf{v} \times \mathbf{r} \\ \frac{d\mathbf{a}}{dt} &= \lambda' \mathbf{r} + \lambda \mathbf{v} + \mu' \mathbf{v} + \mu \mathbf{a} \\ &= \lambda' \mathbf{r} + (\lambda + \mu')\mathbf{v} + \mu(\lambda \mathbf{r} + \mu \mathbf{v}) \\ &= (\lambda' + \mu\lambda)\mathbf{r} + (\lambda + \mu' + \mu^2)\mathbf{v}. \end{aligned}$$

Since $\mathbf{v} \times \mathbf{r}$ is perpendicular to both $\mathbf{v}$ and $\mathbf{r}$, we have

$$(\mathbf{v} \times \mathbf{a}) \bullet \frac{d\mathbf{a}}{dt} = 0.$$

Thus the torsion $\tau(t)$ of the curve is identically zero. It remains zero when expressed in terms of arc length: $\tau(s) = 0$. By Exercise 6 of Section 2.4, $\mathbf{r}(t)$ must be a plane curve.

26. After loading the LinearAlgebra and VectorCalculus packages, issue the following commands:

```
> R := t -> <cos(t), 2*sin(t),
cos(t)>;
> assume(t::real):
> interface(showassumed=0):
> V := t -> diff(R(t),t):
> A := t -> diff(V(t),t):
> v := t -> Norm(V(t),2):
> VxA := t -> V(t) &x A(t):
> vxa := t -> Norm(VxA(t),2):
> Ap := t -> diff(A(t),t):
> Curv := t ->
> simplify(vxa(t)/(v(t))^3):
> Tors := t -> simplify(
> (VxA(t).Ap(t))/(vxa(t))^2):
> Curv(t); Tors(t);
```

This leads to the values

$$\frac{\sqrt{2}}{(\cos(t)^2 + 1)\sqrt{2\cos(t)^2 + 2}} \quad \text{and} \quad 0$$

for the curvature and torsion, respectively. Maple doesn't seem to recognize that the curvature simplifies to $1/(\cos^2 t + 1)^{3/2}$. The torsion is zero because the curve is lies in the plane $z = x$. It is the ellipse in which this plane intersects the ellipsoid $2x^2 + y^2 + 2z^2 = 4$. The maximum and minimum values of the curvature are 1 and $1/2^{3/2}$, respectively, at the ends of the major and minor axes of the ellipse.

27. After loading the LinearAlgebra and VectorCalculus packages, issue the following commands:

```
> R := t -> <t-sin(t), 1-cos(t), t>;
> assume(t::real):
> interface(showassumed=0):
> V := t -> diff(R(t),t):
> A := t -> diff(V(t),t):
> v := t -> Norm(V(t),2):
> VxA := t -> V(t) &x A(t):
> vxa := t -> Norm(VxA(t),2):
> Ap := t -> diff(A(t),t):
> Curv := t ->
> simplify(vxa(t)/(v(t))^3):
> Tors := t -> simplify(
> (VxA(t).Ap(t))/(vxa(t))^2):
> Curv(t); Tors(t);
```

This leads to the values

$$\frac{\sqrt{\cos(t)^2 + 2 - 2\cos(t)}}{(3 - 2\cos(t))^{3/2}}$$

and

$$-\frac{1}{2\cos(t)^2+\sin(t)^2-2\cos(t)+1}$$

for the curvature and torsion, respectively. Each of these formulas can be simplified somewhat:

$$\text{Curv}(t)=\frac{\sqrt{2-2\cos t+\cos^2 t}}{(3-2\cos t)^{3/2}}$$
$$\text{Tors}(t)=\frac{-1}{2-2\cos t+\cos^2 t}.$$

Since $3-2\cos t>0$ and $2-2\cos t+\cos^2 t=1+(1-\cos t)^2>0$ for all t, the curvature and torsion are both continuous for all t. The curve appears to be some sort of helix (but not a circular one) with central axis along the line $x=z$, $y=1$.

28. After loading the LinearAlgebra and VectorCalculus packages, issue the following commands:

```
> R := t -> <cos(t)*cos(2*t),
cos(t)*sin(2*t), sin(t)>;
> assume(t::real):
> interface(showassumed=0):
> V := t -> diff(R(t),t):
> A := t -> diff(V(t),t):
> v := t -> Norm(V(t),2):
> VxA := t -> V(t) &x A(t):
> vxa := t -> Norm(VxA(t),2):
> Ap := t -> diff(A(t),t):
> Curv := t ->
> simplify(vxa(t)/(v(t))^3):
> Tors := t -> simplify(
> (VxA(t).Ap(t))/(vxa(t))^2):
> Curv(t); Tors(t);
> simplify(%,trig);
```

The last line simplifies the rather complicated expression that `Tors(t)` returns by applying some trigonometric identities. The values for the curvature and torsion are

$$\text{Curv}(t)=\frac{\sqrt{17+60\cos(t)^2+48\cos(t)^4}}{\left(4\cos(t)^2+1\right)^{3/2}}$$
$$\text{Tors}(t)=\frac{12\cos t\,(2\cos(t)^2+3)}{17+60\cos(t)^2+48\cos(t)^4}.$$

Plotting the curvature as a function of t, (`plot(Curv(t),t=-2*Pi..2*Pi)`), shows that the minimum curvature occurs at $t=0$ (and any integer multiple of π). The minimum curvature is $\sqrt{125}/5^{3/2}=1$.

The command `simplify(Norm(R(t),2));` gives output 1, indicating that the curve lies on the sphere $x^2+y^2+z^2=1$.

29. After loading the LinearAlgebra and VectorCalculus packages, issue the following commands:

```
> R := t -> <t+cos(t), t+sin(t), 1+t-
cos(t)>;
> assume(t::real):
> interface(showassumed=0):
> V := t -> diff(R(t),t):
> A := t -> diff(V(t),t):
> v := t -> Norm(V(t),2):
> VxA := t -> V(t) &x A(t):
> vxa := t -> Norm(VxA(t),2):
> Ap := t -> diff(A(t),t):
> Curv := t ->
> simplify(vxa(t)/(v(t))^3):
> Tors := t -> simplify(
> (VxA(t).Ap(t))/(vxa(t))^2):
> Curv(t); Tors(t);
```

This leads to the values

$$\text{Curv}(t)=\frac{2\sqrt{\cos(t)^2+\cos t+1}}{\left(5-\cos(t)^2+2\cos t\right)^{3/2}}$$
$$\text{Tors}(t)=\frac{1}{2(\cos(t)^2)+\cos t+1}$$

This appears to be an elliptical helix with central axis along the line $x=y=z-1$.

30.

```
evolute := R -> (t ->
     R(t)+TNBFrame(R)[2](t)
      *(1/Curvature(R)(t)));
```

31.

```
tanline := R ->
     ((t,u)    ->
R(t)+TNBFrame(R)[1](t)*u);
```

Section 11.6 Kepler's Laws of Planetary Motion (page 634)

1. $r=\dfrac{\ell}{1+\epsilon\cos\theta}\quad\Longrightarrow\quad r+\epsilon x=\ell$

$r=\ell-\epsilon x$

$x^2+y^2=r^2=\ell^2-2\ell\epsilon x+\epsilon^2x^2$

$(1-\epsilon^2)x^2+2\ell\epsilon x+y^2=\ell^2$

$$(1-\epsilon^2)\left(x+\frac{\ell\epsilon}{1-\epsilon^2}\right)^2+y^2=\ell^2+\frac{\ell^2\epsilon^2}{1-\epsilon^2}=\frac{\ell^2}{1-\epsilon^2}$$

$$\frac{\left(x+\dfrac{\ell\epsilon}{1-\epsilon^2}\right)^2}{\left(\dfrac{\ell}{1-\epsilon^2}\right)^2}+\frac{y^2}{\left(\dfrac{\ell}{\sqrt{1-\epsilon^2}}\right)^2}=1.$$

2. Position: $\mathbf{r} = r\hat{\mathbf{r}} = k\hat{\mathbf{r}}$.
Velocity: $\mathbf{v} = k\dot{\hat{\mathbf{r}}} = k\dot{\theta}\hat{\boldsymbol{\theta}}$; speed: $v = k\dot{\theta}$.
Acceleration: $k\ddot{\theta}\hat{\boldsymbol{\theta}} + k\dot{\theta}\dot{\hat{\boldsymbol{\theta}}} = -k\dot{\theta}^2\hat{\mathbf{r}} + k\ddot{\theta}\hat{\boldsymbol{\theta}}$.
Radial component of acceleration: $-k\dot{\theta}^2$.
Transverse component of acceleration: $k\ddot{\theta} = \dot{v}$ (the rate of change of the speed).

3. Position: on the curve $r = e^{\theta}$.
Radial velocity: $\dot{r} = e^{\theta}\dot{\theta}$.
Transverse velocity: $r\dot{\theta} = e^{\theta}\dot{\theta}$.
Speed $v = \sqrt{2}e^{\theta}\dot{\theta} = 1 \quad\Longrightarrow\quad \dot{\theta} = (1/\sqrt{2})e^{-\theta}$.
Thus $\ddot{\theta} = -(1/\sqrt{2})e^{-\theta}\dot{\theta} = -e^{-2\theta}/2$.
Radial velocity = transverse velocity = $1/\sqrt{2}$.
Radial acceleration:
$\ddot{r} - r\dot{\theta}^2 = e^{\theta}\dot{\theta}^2 + e^{\theta}\ddot{\theta} - e^{\theta}\dot{\theta}^2 = e^{\theta}\ddot{\theta} = -e^{-\theta}/2$.
Transverse acceleration:
$r\ddot{\theta} + 2\dot{r}\dot{\theta} = -(e^{-\theta})/2 + e^{-\theta} = e^{-\theta}/2$.

4. Path: $r = \theta$. Thus $\dot{r} = \dot{\theta}$, $\ddot{r} = \ddot{\theta}$.
Speed: $v = \sqrt{(\dot{r})^2 + (r\dot{\theta})^2} = \dot{\theta}\sqrt{1 + r^2}$.
Transverse acceleration = 0 (central force). Thus
$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0$, or $\ddot{\theta} = -2\dot{\theta}^2/r$.
Radial acceleration:

$$\begin{aligned}\ddot{r} - r\dot{\theta}^2 &= \ddot{\theta} - r\dot{\theta}^2\\ &= -\left(\frac{2}{r} + r\right)\dot{\theta}^2 = -\frac{(2 + r^2)v^2}{r(1 + r^2)}.\end{aligned}$$

The magnitude of the acceleration is, therefore,
$\dfrac{(2 + r^2)v^2}{r(1 + r^2)}$.

5. For a central force, $r^2\dot{\theta} = h$ (constant), and the acceleration is wholly radial, so

$$|\mathbf{a}| = |\ddot{r} - r\dot{\theta}^2|.$$

For $r = \theta^{-2}$, we have

$$\dot{r} = -2\theta^{-3}\dot{\theta} = -2\theta^{-3}\frac{h}{r^2} = -2h\theta.$$

Thus $\ddot{r} = -2h\dot{\theta} = -2h^2/r^2$. The speed v is given by

$$v^2 = \dot{r}^2 + r^2\dot{\theta}^2 = 4h^2\theta^2 + (h^2/r^2).$$

Since the speed is v_0 when $\theta = 1$ (and so $r = 1$), we have $v_0^2 = 5h^2$, and $h = v_0/\sqrt{5}$. Hence the magnitude of the acceleration at any point on the path is

$$|\mathbf{a}| = \left|-2\frac{h^2}{r^2} - r\frac{h^2}{r^4}\right| = \frac{v_0^2}{5}\left(\frac{2}{r^2} + \frac{1}{r^3}\right).$$

6. Let the period and the semi-major axis of the orbit of Halley's comet be $T_H = 76$ years and a_H km respectively. Similar parameters for the earth's orbit are $T_E = 1$ year and $a_E = 150 \times 10^6$ km. By Kepler's third law

$$\frac{T_H^2}{a_H^3} = \frac{T_E^2}{a_E^3}.$$

Thus

$$a_H = 150 \times 10^6 \times 76^{2/3} \approx 2.69 \times 10^9.$$

The major axis of Halley's comet's orbit is
$2a_H \approx 5.38 \times 10^9$ km.

7. The period and semi-major axis of the moon's orbit around the earth are

$$T_M \approx 27 \text{ days}, \qquad a_M \approx 385,000 \text{ km}.$$

The satellite has a circular orbit of radius a_S and period $T_S = 1$ day. (If the orbit is in the plane of the equator, the satellite will remain above the same point on the earth.) By Kepler's third law,

$$\frac{T_S^2}{a_S^3} = \frac{T_M^2}{a_M^3}.$$

Thus $a_S = 385,000 \times (1/27)^{2/3} \approx 42,788$. The satellite's orbit should have radius about 42,788 km, and should lie in the equatorial plane.

8. The period T (in years) and radius R (in km) of the asteroid's orbit satisfies

$$\frac{T^2}{R^3} = \frac{T_{\text{earth}}^2}{R_{\text{earth}}^3} = \frac{1^2}{(150 \times 10^6)^3}.$$

Thus the radius of the asteroid's orbit is
$R \approx 150 \times 10^6 T^{2/3}$ km.

9. If R is the radius and T is the period of the asteroid's circular orbit, then *almost* stopping the asteroid causes it to drop into a very eccentric elliptical orbit with major axis approximately R. (Thus, $a = R/2$.) The period T_e of the new elliptical orbit satisfies

$$\frac{T_e^2}{T^2} = \frac{(R/2)^3}{R^3} = \frac{1}{8}.$$

Thus $T_e = T/(2\sqrt{2})$. The time the asteroid will take to fall into the sun is half of T_e. Thus it is $T/(4\sqrt{2})$.

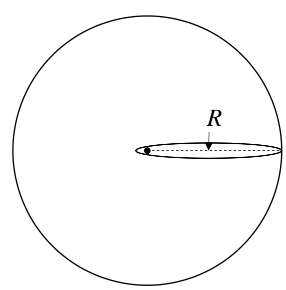

Fig. 11.6.9

10. At perihelion, $r = a - c = (1-\epsilon)a$.
At aphelion $r = a + c = (1+\epsilon)a$.
Since $\dot{r} = 0$ at perihelion and aphelion, the speed is $v = r\dot{\theta}$ at each point. Since $r^2\dot{\theta} = h$ is constant over the orbit, $v = h/r$. Therefore

$$v_{\text{perihelion}} = \frac{h}{a(1-\epsilon)}, \qquad v_{\text{aphelion}} = \frac{h}{a(1+\epsilon)}.$$

If $v_{\text{perihelion}} = 2v_{\text{aphelion}}$ then

$$\frac{h}{a(1-\epsilon)} = \frac{2h}{a(1+\epsilon)}.$$

Hence $1+\epsilon = 2(1-\epsilon)$, and $\epsilon = 1/3$. The eccentricity of the orbit is 1/3.

11. The orbital speed v of a planet satisfies (by conservation of energy)

$$\frac{v^2}{2} - \frac{k}{r} = K \quad \text{(total energy)}.$$

If v is constant so must be r, and the orbit will therefore be circular.

12. Since $r^2\dot{\theta} = h =$ constant for the planet's orbit, and since the speed is $v = r\dot{\theta}$ at perihelion and at aphelion (the radial velocity is zero at these points), we have

$$r_p v_p = r_a v_a,$$

where the subscripts p and a refer to perihelion and aphelion, respectively. Since $r_p/r_a = 8/10$, we must have $v_p/v_a = 10/8 = 1.25$. Also,

$$r_p = \frac{\ell}{1+\epsilon\cos 0} = \frac{\ell}{1+\epsilon}, \quad r_a = \frac{\ell}{1+\epsilon\cos\pi} = \frac{\ell}{1-\epsilon}.$$

Thus $\ell/(1+\epsilon) = (8/10)\ell/(1-\epsilon)$, and so $10 - 10\epsilon = 8 + 8\epsilon$. Hence $2 = 18\epsilon$. The eccentricity of the orbit is $\epsilon = 1/9$.

13. Let the radius of the circular orbit be R, and let the parameters of the new elliptical orbit be a and c, as shown in the figure. Then $R = a + c$. At the moment of the collision, r does not change ($r = R$), but the speed $r\dot{\theta}$ is cut in half. Therefore $\dot{\theta}$ is cut in half, and so $h = r^2\dot{\theta}$ is cut in half. Let H be the value of $r^2\dot{\theta}$ for the circular orbit, and let h be the value for the new elliptical orbit. Thus $h = H/2$. We have

$$R = \frac{H^2}{k}, \qquad a = \frac{h^2}{k(1-\epsilon^2)} = \frac{H^2}{4k(1-\epsilon^2)} = \frac{R}{4(1-\epsilon^2)}.$$

Similarly, $c = \epsilon a = \dfrac{\epsilon R}{4(1-\epsilon^2)}$, so

$$R = c + a = \frac{(1+\epsilon)R}{4(1-\epsilon^2)} = \frac{R}{4(1-\epsilon)}.$$

It follows that $1 = 4 - 4\epsilon$, so $\epsilon = 3/4$. The new elliptical orbit has eccentricity $\epsilon = 3/4$.

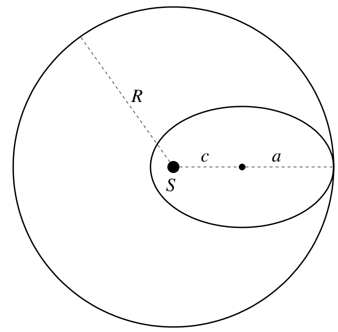

Fig. 11.6.13

14. As in Exercise 12, $r_P v_P = r_A v_A$, where $r_A = \ell/(1-\epsilon)$ and $r_P = \ell/(1+\epsilon)$, ϵ being the eccentricity of the orbit. Thus

$$\frac{v_P}{v_A} = \frac{r_A}{r_P} = \frac{1+\epsilon}{1-\epsilon}.$$

Solving this equation for ϵ in terms of v_P and v_A, we get

$$\epsilon = \frac{v_P - v_A}{v_P + v_A}.$$

By conservation of energy the speed v at the ends of the minor axis of the orbit (where $r = a$) satisfies

$$\frac{v^2}{2} - \frac{k}{a} = \frac{v_P^2}{2} - \frac{k}{r_P} = \frac{v_A^2}{2} - \frac{k}{r_A}.$$

The latter equality shows that

$$v_P^2 - v_A^2 = 2k\left(\frac{1}{r_P} - \frac{1}{r_A}\right) = \frac{4k\epsilon}{\ell}.$$

Using this result and the parameters of the orbit given in the text, we obtain

$$\begin{aligned} v^2 &= v_P^2 + 2k\left(\frac{1}{a} - \frac{1}{r_P}\right) \\ &= v_P^2 + \frac{2k}{\ell}\Big(1 - \epsilon^2 - (1+\epsilon)\Big) \\ &= v_P^2 - \frac{2k\epsilon}{\ell}(1+\epsilon) \\ &= v_P^2 - \frac{v_P^2 - v_A^2}{2}\left(1 + \frac{v_P - v_A}{v_P + v_A}\right) \\ &= v_P^2 - \frac{v_P - v_A}{2}(2v_P) = v_P v_A. \end{aligned}$$

Thus $v = \sqrt{v_P v_A}$.

15. Since the radial line from the sun to the planet sweeps out equal areas in equal times, the fraction of the planet's period spend on the same side of the minor axis as the sun is equal to the shaded area in the figure to the total area of the ellipse, that is,

$$\frac{\frac{1}{2}\pi ab - \frac{1}{2}(2bc)}{\pi ab} = \frac{\frac{1}{2}\pi ab - \epsilon ab}{\pi ab} = \frac{1}{2} - \frac{\epsilon}{\pi},$$

where $\epsilon = c/a$ is the eccentricity of the orbit.

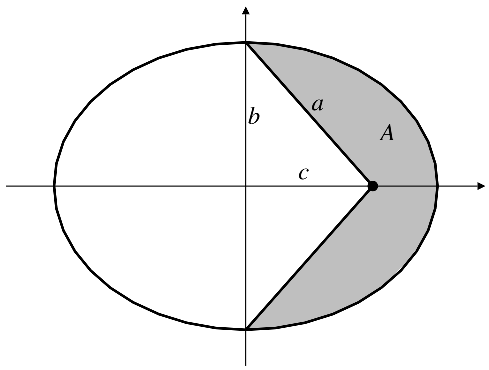

Fig. 11.6.15

16. By conservation of energy, we have

$$\frac{k}{r} - \frac{1}{2}\left(\dot{r}^2 + \frac{h^2}{r^2}\right) = -K$$

where K is a constant for the orbit (the total energy). The term in the parentheses is v^2, the square of the speed. Thus

$$\frac{k}{r} - \frac{1}{2}v^2 = -K = \frac{k}{r_0} - \frac{1}{2}v_0^2,$$

where r_0 and v_0 are the given distance and speed. We evaluate $-K$ at perihelion.
The parameters of the orbit are

$$\ell = \frac{h^2}{k}, \quad a = \frac{h^2}{k(1-\epsilon^2)}, \quad b = \frac{h^2}{k\sqrt{1-\epsilon^2}}, \quad c = \epsilon a.$$

At perihelion P we have

$$r = a - c = (1-\epsilon)a = \frac{h^2}{k(1+\epsilon)}.$$

Since $\dot{r} = 0$ at perihelion, the speed there is $v = r\dot{\theta}$. By Kepler's second law, $r^2\dot{\theta} = h$, so $v = h/r = k(1+\epsilon)/h$. Thus

$$\begin{aligned} -K &= \frac{k}{r} - \frac{v^2}{2} \\ &= \frac{k^2}{h^2}(1+\epsilon) - \frac{1}{2}\frac{k^2}{h^2}(1+\epsilon)^2 \\ &= \frac{k^2}{2h^2}(1+\epsilon)\Big[2 - (1+\epsilon)\Big] \\ &= \frac{k^2}{2h^2}(1-\epsilon^2) = \frac{k}{2a}. \end{aligned}$$

Thus $a = \dfrac{k}{-2K}$. By Kepler's third law,

$$T^2 = \frac{4\pi^2}{k}a^3 = \frac{4\pi^2}{k}\left(\frac{k}{-2K}\right)^3.$$

Thus $T = \dfrac{2\pi}{\sqrt{k}}\left(\dfrac{2}{r_0} - \dfrac{v_0^2}{k}\right)^{-3/2}$.

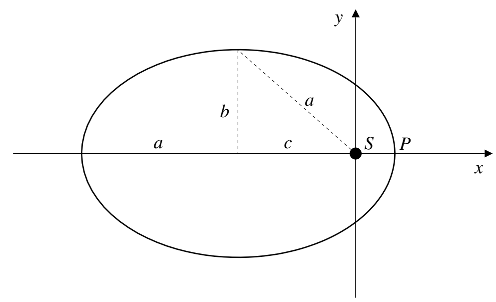

Fig. 11.6.16

17. Let $r_1(s)$ and $r_2(s)$ be the distances from the point $P = \mathbf{r}(s)$ on the ellipse $\mathcal{E}$ to the two foci. (Here s denotes arc length on $\mathcal{E}$, measured from any convenient point.) By symmetry

$$\int_{\mathcal{E}} r_1(s)\,ds = \int_{\mathcal{E}} r_2(s)\,ds.$$

But $r_1(s) + r_2(s) = 2a$ for any s. Therefore,

$$\int_{\mathcal{E}} r_1(s)\,ds + \int_{\mathcal{E}} r_2(s)\,ds = \int_{\mathcal{E}} 2a\,ds = 2ac(\mathcal{E}).$$

Hence $\int_{\mathcal{E}} r_1(s)\,ds = ac(\mathcal{E})$, and

$$\frac{1}{c(\mathcal{E})}\int_{\mathcal{E}} r_1(s)\,ds = a.$$

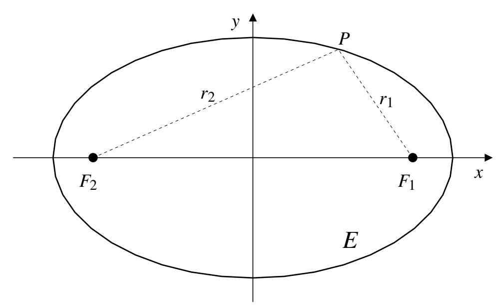

Fig. 11.6.17

18. Start with
$$\ddot{r} - \frac{h^2}{r^3} = -\frac{k}{r^2}.$$

Let $r(t) = \dfrac{1}{u(\theta)}$, where $\theta = \theta(t)$. Since $r^2\dot{\theta} = h$ (constant), we have

$$\dot{r} = -\frac{1}{u^2}\frac{du}{d\theta}\dot{\theta} = -r^2\frac{du}{d\theta}\frac{h}{r^2} = -h\frac{du}{d\theta}$$
$$\ddot{r} = -h\frac{d^2u}{d\theta^2}\dot{\theta} = -\frac{h^2}{r^2}\frac{d^2u}{d\theta^2} = -h^2u^2\frac{d^2u}{d\theta^2}.$$

Thus $-h^2u^2\dfrac{d^2u}{d\theta^2} - h^2u^3 = -ku^2$, or

$$\frac{d^2u}{d\theta^2} + u = \frac{k}{h^2}.$$

This is the DE for simple harmonic motion with a constant forcing term (nonhomogeneous term) on the right-hand side. It is easily verified that

$$u = \frac{k}{h^2}\Big(1 + \epsilon\cos(\theta - \theta_0)\Big)$$

is a solution for any choice of the constants ϵ and θ_0. Expressing the solution in terms of r, we have

$$r = \frac{h^2/k}{1 + \epsilon\cos(\theta - \theta_0)},$$

which is an ellipse if $|\epsilon| < 1$.

19. For inverse cube attraction, the equation of motion is

$$\ddot{r} - \frac{h^2}{r^3} = -\frac{k}{r^3},$$

where $r^2\dot{\theta} = h$ is constant, since the force is central. Making the same change of variables used in Exercise 18, we obtain

$$-h^2u^2\frac{d^2u}{d\theta^2} - h^2u^3 = -ku^3,$$

or
$$\frac{d^2u}{d\theta^2} - \frac{k - h^2}{h^2}u = 0.$$

There are three cases to consider.

CASE I. If $k < h^2$ the DE is $\dfrac{d^2u}{d\theta^2} + \omega^2 u = 0$, where $\omega^2 = (h^2 - k)/h^2$. This has solution $u = A\cos\omega(\theta - \theta_0)$. Thus
$$r = \frac{1}{A\cos\omega(\theta - \theta_0)}.$$

Note that $r \to \infty$ as $\theta \to \theta_0 + \dfrac{\pi}{2\omega}$. There are no bounded orbits in this case.

CASE II. If $k > h^2$ the DE is $\dfrac{d^2u}{d\theta^2} - \omega^2 u = 0$, where $\omega^2 = (k - h^2)/h^2$. This has solution $u = Ae^{\omega\theta} + Be^{-\omega\theta}$. Since $u \to 0$ or ∞ as $\theta \to \infty$, the corresponding solution $r = 1/u$ cannot be both bounded and bounded away from zero. (Note that $\dot{\theta} = h/r^2 \ge K > 0$ for any orbit which is bounded away from zero, so we can be sure $\theta \to \infty$ on such an orbit.)

CASE III. If $k = h^2$ the DE is $\dfrac{d^2u}{d\theta^2} = 0$, which has solutions $u = A\theta + B$, corresponding to

$$r = \frac{1}{A\theta + B}.$$

Such orbits are bounded away from zero and infinity only if $A = 0$, in which case they are circular.

Thus, the only possible orbits which are bounded away from zero and infinity (i.e., which do not escape to infinity or plunge into the sun) in a universe with an inverse cube gravitational attraction are some circular orbits for which $h^2 = k$. Such orbits cannot be considered "stable" since even slight loss of energy would result in decreased h and the condition $h^2 = k$ would no longer be satisfied. Now aren't you glad you live in an inverse square universe?

20. Since $\dfrac{k}{r} = \dfrac{1}{2}v^2 - K$ by conservation of energy, if $K < 0$, then
$$\frac{k}{r} \ge -K > 0,$$

so $r \le -\dfrac{k}{K}$. The orbit is, therefore, bounded.

21. $r = \dfrac{\ell}{1 + \epsilon\cos\theta}$, $(\epsilon > 1)$.

See the following figure.
Vertices: At V_1, $\theta = 0$ and $r = \ell/(1 + \epsilon)$.
At V_2, $\theta = \pi$ and $r = \ell/(1 - \epsilon) = -\ell/(\epsilon - 1)$.
Semi-focal separation:

$$c = \frac{1}{2}\left(\frac{\ell}{1 + \epsilon} + \frac{\ell}{1 - \epsilon}\right) = \frac{\ell\epsilon}{\epsilon^2 - 1}.$$

The centre is $(c, 0)$.
Semi-transverse axis:

$$a = \frac{\ell\epsilon}{\epsilon^2 - 1} - \frac{\ell}{\epsilon + 1} = \frac{\ell}{\epsilon^2 - 1}.$$

Semi-conjugate axis:

$$b = \sqrt{c^2 - a^2} = \frac{\ell}{\sqrt{\epsilon^2 - 1}}.$$

Direction of asymptotes (see figure):

$$\theta = \tan^{-1}\frac{b}{a} = \cos^{-1}\frac{a}{c} = \cos^{-1}\frac{1}{\epsilon}.$$

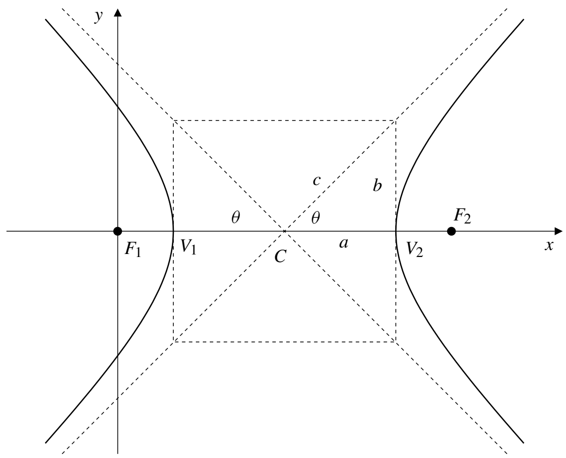

Fig. 11.6.21

22. By Exercise 17, the asymptotes make angle $\theta = \cos^{-1}(1/\epsilon)$ with the transverse axis, as shown in the figure. The angle of deviation δ satisfies $2\theta + \delta = \pi$, so $\theta = \dfrac{\pi}{2} - \dfrac{\delta}{2}$, and

$$\cos\theta = \sin\frac{\delta}{2}, \qquad \sin\theta = \cos\frac{\delta}{2}.$$

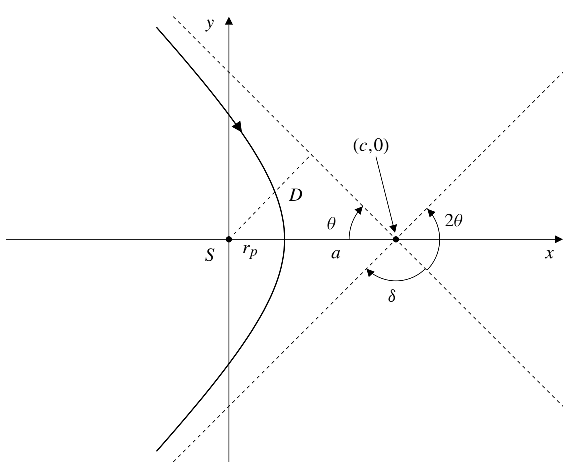

Fig. 11.6.22

By conservation of energy,

$$\frac{v^2}{2} - \frac{k}{r} = \text{constant} = \frac{v_\infty^2}{2}$$

for all points on the orbit. At perihelion,

$$r = r_p = c - a = (\epsilon - 1)a = \frac{\ell}{\epsilon + 1},$$
$$v = v_p = r_p\dot\theta = \frac{h}{r_p} = \frac{h(\epsilon+1)}{\ell}.$$

Since $h^2 = k\ell$, we have

$$\begin{aligned} v_\infty^2 &= v_p^2 - \frac{2k}{r_p} \\ &= \frac{h^2}{\ell^2}(\epsilon+1)^2 - \frac{2k}{\ell}(\epsilon+1) \\ &= \frac{k}{\ell}\Big[(\epsilon+1)^2 - 2(\epsilon+1)\Big] \\ &= \frac{k}{\ell}(\epsilon^2 - 1) = \frac{k}{a}. \end{aligned}$$

Thus $av_\infty^2 = k$.

If D is the perpendicular distance from the sun S to an asymptote of the orbit (see the figure) then

$$\begin{aligned} D &= c\sin\theta = \epsilon a \sin\theta = a\frac{\sin\theta}{\cos\theta} \\ &= a\frac{\cos(\delta/2)}{\sin(\delta/2)} = a\cot\frac{\delta}{2}. \end{aligned}$$

Therefore

$$\frac{Dv_\infty^2}{k} = \frac{v_\infty^2 a}{k}\cot\frac{\delta}{2} = \cot\frac{\delta}{2}.$$

Review Exercises 11 (page 636)

1. Given that $\mathbf{a} \bullet \mathbf{r} = 0$ and $\mathbf{a} \bullet \mathbf{v} = 0$, we have

$$\begin{aligned} &\frac{d}{dt}|\mathbf{r}(t) - t\mathbf{v}(t)|^2 \\ &= 2\Big(\mathbf{r}(t) - t\mathbf{v}(t)\Big) \bullet \Big(\mathbf{v}(t) - \mathbf{v}(t) - t\mathbf{a}(t)\Big) \\ &= 2\Big(\mathbf{r}(t) - t\mathbf{v}(t)\Big) \bullet \mathbf{a}(t) = 0 - 0 = 0. \end{aligned}$$

2. $\mathbf{r} = t\cos t\,\mathbf{i} + t\sin t\,\mathbf{j} + (2\pi - t)\mathbf{k}$, $(0 \le t \le 2\pi)$ is a conical helix wound around the cone $z = 2\pi - \sqrt{x^2+y^2}$ starting at the vertex $(0, 0, 2\pi)$, and completing one revolution to end up at $(2\pi, 0, 0)$. Since

$$\mathbf{v} = (\cos t - t\sin t)\mathbf{i} + (\sin t + t\cos t)\mathbf{j} - \mathbf{k},$$

the length of the curve is

$$L = \int_0^{2\pi}\sqrt{2+t^2}\,dt = \pi\sqrt{2+4\pi^2} + \ln\left(\frac{2\pi + \sqrt{2+4\pi^2}}{\sqrt{2}}\right)$$

units.

3. The position of the particle at time t is

$$\mathbf{r} = x\mathbf{i} + x^2\mathbf{j} + \tfrac{2}{3}x^3\mathbf{k},$$

where x is an increasing function of t. Thie velocity is

$$\mathbf{v} = \frac{dx}{dt}\Big(\mathbf{i} + 2x\mathbf{j} + 2x^2\mathbf{k}\Big).$$

Since the speed is 6, we have

$$6 = \frac{dx}{dt}\sqrt{1 + 4x^2 + 4x^4} = (2x^2 + 1)\frac{dx}{dt},$$

so that $dx/dt = 6/(2x^2 + 1)$. The particle is at $(1, 1, \frac{2}{3})$ when $x = 1$. At this time its velocity is

$$\mathbf{v}(1) = 2(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}).$$

Also

$$\begin{aligned}\frac{d^2x}{dt^2} &= -\frac{6}{(2x^2+1)^2}(4x)\frac{dx}{dt} = -\frac{144x}{(2x^2+1)^3}\\ \mathbf{a} &= \frac{d^2x}{dt^2}(\mathbf{i} + 2x\mathbf{j} + 2x^2\mathbf{k})\\ &\quad + \frac{dx}{dt}\left(2\frac{dx}{dt}\mathbf{j} + 4x\frac{dx}{dt}\mathbf{k}\right).\end{aligned}$$

At $x = 1$, we have

$$\begin{aligned}\mathbf{a}(1) &= -\frac{16}{3}(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}) + 2(4\mathbf{j} + 8\mathbf{k})\\ &= \frac{8}{3}(-2\mathbf{i} - \mathbf{j} + 2\mathbf{k}).\end{aligned}$$

4. The position, velocity, speed, and acceleration of the particle are given by

$$\begin{aligned}\mathbf{r} &= x\mathbf{i} + x^2\mathbf{j}\\ \mathbf{v} &= \frac{dx}{dt}(\mathbf{i} + 2x\mathbf{j}), \quad v = \left|\frac{dx}{dt}\right|\sqrt{1 + 4x^2}\\ \mathbf{a} &= \frac{d^2x}{dt^2}(\mathbf{i} + 2x\mathbf{j}) + 2\left(\frac{dx}{dt}\right)^2\mathbf{j}.\end{aligned}$$

Let us assume that the particle is moving to the right, so that $dx/dt > 0$. Since the speed is t, we have

$$\begin{aligned}\frac{dx}{dt} &= \frac{t}{\sqrt{1+4x^2}}\\ \frac{d^2x}{dt^2} &= \frac{\sqrt{1+4x^2} - \dfrac{4tx}{\sqrt{1+4x^2}}\dfrac{dx}{dt}}{1 + 4x^2}.\end{aligned}$$

If the particle is at $(\sqrt{2}, 2)$ at $t = 3$, then $dx/dt = 1$ at that time, and

$$\frac{d^2x}{dt^2} = \frac{3 - 4\sqrt{2}}{9}.$$

Hence the acceleration is

$$\mathbf{a} = \frac{3 - 4\sqrt{2}}{9}(\mathbf{i} + 2\sqrt{2}\mathbf{j}) + 2\mathbf{j}.$$

If the particle is moving to the left, so that $dx/dt < 0$, a similar calculation shows that at $t = 3$ its acceleration is

$$\mathbf{a} = -\frac{3 + 4\sqrt{2}}{9}(\mathbf{i} + 2\sqrt{2}\mathbf{j}) + 2\mathbf{j}.$$

5.

$$\begin{aligned}\mathbf{r} &= e^t\mathbf{i} + \sqrt{2}t\mathbf{j} + e^{-t}\mathbf{k}\\ \mathbf{v} &= e^t\mathbf{i} + \sqrt{2}\mathbf{j} - e^{-t}\mathbf{k}\\ \mathbf{a} &= e^t\mathbf{i} + e^{-t}\mathbf{k}\\ \frac{d\mathbf{a}}{dt} &= e^t\mathbf{i} - e^{-t}\mathbf{k}\\ \mathbf{v}\times\mathbf{a} &= \sqrt{2}e^{-t}\mathbf{i} - 2\mathbf{j} - \sqrt{2}e^t\mathbf{k}\\ v &= \sqrt{e^{2t} + 2 + e^{-2t}} = e^t + e^{-t}\\ |\mathbf{v}\times\mathbf{a}| &= \sqrt{2}(e^t + e^{-t})\\ \kappa &= \frac{|\mathbf{v}\times\mathbf{a}|}{v^3} = \frac{\sqrt{2}}{(e^t + e^{-t})^2}\\ \tau &= \frac{(\mathbf{v}\times\mathbf{a})\bullet\dfrac{d\mathbf{a}}{dt}}{|\mathbf{v}\times\mathbf{a}|^2} = \frac{\sqrt{2}}{(e^t + e^{-t})^2} = \kappa.\end{aligned}$$

6. Tangential acceleration: $dv/dt = e^t - e^{-t}$.
Normal acceleration: $v^2\kappa = \sqrt{2}$.
Since $v = 2\cosh t$, the minimum speed is 2 at time $t = 0$.

7. For $x(s) = \displaystyle\int_0^s \cos\frac{kt^2}{2}\,dt$, $y(s) = \displaystyle\int_0^s \sin\frac{kt^2}{2}\,dt$, we have

$$\frac{dx}{ds} = \cos\frac{ks^2}{2}, \quad \frac{dy}{ds} = \sin\frac{ks^2}{2},$$

so that the speed is unity:

$$v = \sqrt{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2} = 1.$$

Since $x(0) = y(0) = 0$, the arc length along the curve, measured from the origin, is s. Also,

$$\begin{aligned}\mathbf{v} &= \cos\frac{ks^2}{2}\mathbf{i} + \sin\frac{ks^2}{2}\mathbf{j}\\ \mathbf{a} &= -ks\sin\frac{ks^2}{2}\mathbf{i} + ks\cos\frac{ks^2}{2}\mathbf{j}\\ \mathbf{v}\times\mathbf{a} &= ks\mathbf{k}.\end{aligned}$$

Therefore the curvature at position s is
$\kappa = |\mathbf{v}\times\mathbf{a}|/v^3 = ks$.

8. If $r = e^{-\theta}$, and $\dot\theta = k$, then $\dot r = -e^{-\theta}\dot\theta = -kr$, and $\ddot r = k^2 r$. Since $\mathbf{r} = r\hat{\mathbf{r}}$, we have

$$\begin{aligned}\mathbf{v} &= \dot r\hat{\mathbf{r}} + r\dot\theta\hat{\boldsymbol\theta} = -kr\hat{\mathbf{r}} + kr\hat{\boldsymbol\theta}\\ \mathbf{a} &= (\ddot r - r\dot\theta^2)\hat{\mathbf{r}} + (r\ddot\theta + 2\dot r\dot\theta)\hat{\boldsymbol\theta}\\ &= (k^2r - k^2r)\hat{\mathbf{r}} + (0 - 2k^2r)\hat{\boldsymbol\theta} = -2k^2r\hat{\boldsymbol\theta}.\end{aligned}$$

9.
$$\begin{aligned}\mathbf{r} &= a(t - \sin t)\mathbf{i} + a(1 - \cos t)\mathbf{j}\\ \mathbf{v} &= a(1 - \cos t)\mathbf{i} + a\sin t\mathbf{j}\\ v &= a\sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t}\\ &= a\sqrt{2}\sqrt{1-\cos t} = 2a\sin\frac{t}{2} \quad \text{if } 0 \le t \le 2\pi.\end{aligned}$$

The length of the cycloid from $t = 0$ to $t = T \le 2\pi$ is

$$s(T) = \int_0^T 2a\sin\frac{t}{2}\,dt = 4a\left(1 - \cos\frac{T}{2}\right) \text{ units.}$$

10. $s = 4a\left(1 - \cos\dfrac{t}{2}\right) \Rightarrow t = 2\cos^{-1}\left(1 - \dfrac{s}{4a}\right) = t(s)$.

The required arc length parametrization of the cycloid is

$$\mathbf{r} = a\Big(t(s) - \sin t(s)\Big)\mathbf{i} + a\Big(1 - \cos t(s)\Big)\mathbf{j}.$$

11. From Exercise 9 we have

$$\begin{aligned}\hat{\mathbf{T}}(t) &= \frac{\mathbf{v}}{v} = \frac{(1-\cos t)\mathbf{i} + \sin t\mathbf{j}}{2\sin(t/2)}\\ &= \sin\frac{t}{2}\mathbf{i} + \cos\frac{t}{2}\mathbf{j}\\ \frac{d\hat{\mathbf{T}}}{ds} &= \frac{1}{v}\frac{d\hat{\mathbf{T}}}{dt} = \frac{\frac12\cos\frac{t}{2}\mathbf{i} - \frac12\sin\frac{t}{2}\mathbf{j}}{2a\sin\frac{t}{2}}\\ &= \frac{1}{4a}\left(\cot\frac{t}{2}\mathbf{i} - \mathbf{j}\right)\\ \kappa(t) &= \left|\frac{d\hat{\mathbf{T}}}{ds}\right| = \frac{1}{4a\sin(t/2)}\end{aligned}$$

$$\begin{aligned}\mathbf{r}_C(t) &= \mathbf{r}(t) + \rho(t)\hat{\mathbf{N}}(t) = \mathbf{r}(t) + \frac{1}{(\kappa(t))^2}\frac{d\hat{\mathbf{T}}}{ds}\\ &= \mathbf{r}(t) + \frac{16a^2\sin^2(t/2)}{4a}\left(\cot\frac{t}{2}\mathbf{i} - \mathbf{j}\right)\\ &= \mathbf{r}(t) + 4a\cos\frac{t}{2}\sin\frac{t}{2}\mathbf{i} - 4a\sin^2\frac{t}{2}\mathbf{j}\\ &= a(t - \sin t)\mathbf{i} + a(1-\cos t)\mathbf{j}\\ &\qquad + 2a\sin t\mathbf{i} - 2a(1-\cos t)\mathbf{j}\\ &= a(t + \sin t)\mathbf{i} - a(1 - \cos t)\mathbf{j} \quad (\text{let } t = u - \pi)\\ &= a(u - \sin u - \pi)\mathbf{i} + a(1 - \cos u - 2)\mathbf{j}.\end{aligned}$$

This is the same cycloid as given by $\mathbf{r}(t)$ but translated πa units to the right and $2a$ units downward.

12. Let P be the point with position vector $\mathbf{r}(t)$ on the cycloid. By Exercise 9, the arc OP has length $4a - 4a\cos(t/2)$, and so PQ has length $4a$ - arc $OP = 4a\cos(t/2)$ units. Thus

$$\begin{aligned}\overrightarrow{PQ} &= 4a\cos\frac{t}{2}\hat{\mathbf{T}}(t)\\ &= 4a\cos\frac{t}{2}\left(\sin\frac{t}{2}\mathbf{i} + \cos\frac{t}{2}\mathbf{j}\right)\\ &= 2a\sin t\mathbf{i} + 2a(1 + \cos t)\mathbf{j}.\end{aligned}$$

It follows that Q has position vector

$$\begin{aligned}\mathbf{r}_Q &= \mathbf{r} + \overrightarrow{PQ}\\ &= a(t - \sin t)\mathbf{i} + a(1-\cos t)\mathbf{j} + 2a\sin t\mathbf{i} + 2a(1+\cos t)\mathbf{j}\\ &= a(t + \sin t)\mathbf{i} + a(1 + \cos t + 2)\mathbf{j} \quad (\text{let } t = u + \pi)\\ &= a(u - \sin u + \pi)\mathbf{i} + a(1 - \cos u + 2)\mathbf{j}.\end{aligned}$$

Thus $\mathbf{r}_Q(t)$ represents the same cycloid as $\mathbf{r}(t)$, but translated πa units to the left and $2a$ units upward. From Exercise 11, the given cycloid is the evolute of its involute.

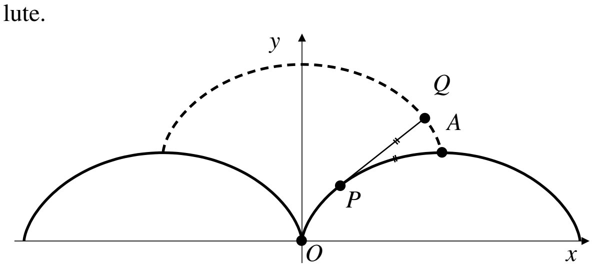

Fig. R-11.12

13. The position vector of P is given by

$$\mathbf{r} = \rho\sin\phi\cos\theta\mathbf{i} + \rho\sin\phi\sin\theta\mathbf{j} + \rho\cos\phi\mathbf{k}.$$

Mutually perpendicular unit vectors in the directions of increasing ρ, ϕ and θ can be found by differentiating $\mathbf{r}$ with respect to each of these coordinates and dividing the resulting vectors by their lengths. They are

$$\begin{aligned}\hat{\boldsymbol\rho} &= \frac{d\mathbf{r}}{d\rho} = \sin\phi\cos\theta\mathbf{i} + \sin\phi\sin\theta\mathbf{j} + \cos\phi\mathbf{k}\\ \hat{\boldsymbol\phi} &= \frac{1}{\rho}\frac{d\mathbf{r}}{d\phi} = \cos\phi\cos\theta\mathbf{i} + \cos\phi\sin\theta\mathbf{j} - \sin\phi\mathbf{k}\\ \hat{\boldsymbol\theta} &= \frac{1}{\rho\sin\phi}\frac{d\mathbf{r}}{d\theta} = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}.\end{aligned}$$

The triad$\{\hat{\boldsymbol\rho}, \hat{\boldsymbol\phi}, \hat{\boldsymbol\theta}\}$ is right-handed. This is the reason for ordering the spherical polar coordinates (ρ, ϕ, θ) rather than (ρ, θ, ϕ).

14. By Kepler's Second Law the position vector $\mathbf{r}$ from the origin (the sun) to the planet sweeps out area at a constant rate, say $h/2$:

$$\frac{dA}{dt} = \frac{h}{2}.$$

As observed in the text, $dA/dt = r^2\dot{\theta}/2$, so $r^2\dot{\theta} = h$, and

$$\mathbf{r} \times \mathbf{v} = (r\hat{\mathbf{r}}) \times (\dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}) = r^2\dot{\theta}\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} = h\mathbf{k} = \mathbf{h}$$

is a constant vector.

15. By Exercise 14, $\mathbf{r} \times \dot{\mathbf{r}} = \mathbf{r} \times \mathbf{v} = \mathbf{h}$ is constant, so, by Newton's second law of motion,

$$\mathbf{r} \times \mathbf{F}(\mathbf{r}) = m\mathbf{r} \times \ddot{\mathbf{r}} = m\frac{d}{dt}(\mathbf{r} \times \dot{\mathbf{r}}) = \mathbf{0}.$$

Thus $\mathbf{F}(\mathbf{r})$ is parallel to $\mathbf{r}$, and therefore has zero transverse component:

$$\mathbf{F}(\mathbf{r}) = -f(\mathbf{r})\hat{\mathbf{r}}$$

for some scalar function $f(\mathbf{r})$.

16. By Exercise 15, $\mathbf{F}(\mathbf{r}) = m(\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} = -f(\mathbf{r})\hat{\mathbf{r}}$. We are given that $r = \ell/(1 + \epsilon\cos\theta)$. Thus

$$\begin{aligned}
\dot{r} &= -\frac{\ell}{(1+\epsilon\cos\theta)^2}(-\epsilon\sin\theta)\dot{\theta}\\
&= \frac{\epsilon\ell\sin\theta}{(1+\epsilon\cos\theta)^2}\dot{\theta}\\
&= \frac{\epsilon\sin\theta}{\ell}r^2\dot{\theta} = \frac{h\epsilon}{\ell}\sin\theta\\
\ddot{r} &= \frac{h\epsilon}{\ell}(\cos\theta)\dot{\theta} = \frac{h^2\epsilon\cos\theta}{\ell r^2}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\ddot{r} - r\dot{\theta}^2 &= \frac{h^2\epsilon\cos\theta}{\ell r^2} - \frac{h^2}{r^3}\\
&= \frac{h^2}{\ell r^2}\left(\epsilon\cos\theta - \frac{\ell}{r}\right) = -\frac{h^2}{\ell r^2},
\end{aligned}$$

(because $(\ell/r) = 1 + \epsilon\cos\theta$). Hence

$$f(\mathbf{r}) = \frac{mh^2}{\ell r^2}.$$

This says that the magnitude of the force on the planet is inversely proportional to the square of its distance from the sun. Thus Newton's law of gravitation follows from Kepler's laws and the second law of motion.

Challenging Problems 11 (page 636)

1. a) The angular velocity $\boldsymbol{\Omega}$ of the earth points northward in the direction of the earth's axis; in terms of the basis vectors defined at a point P at 45° north latitude, it points in the direction of $\mathbf{j} + \mathbf{k}$:

$$\boldsymbol{\Omega} = \Omega\frac{\mathbf{j}+\mathbf{k}}{\sqrt{2}}, \quad \Omega = \frac{2\pi}{24 \times 3{,}600} \text{ rad/s.}$$

b) If $\mathbf{v} = -v\mathbf{k}$, then

$$\mathbf{a}_C = 2\boldsymbol{\Omega} \times \mathbf{v} = -\frac{2\Omega v}{\sqrt{2}}(\mathbf{j}+\mathbf{k}) \times \mathbf{k} = -\sqrt{2}\Omega v\mathbf{i}.$$

c) If $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ is the position of the falling object at time t, then $\mathbf{r}(t)$ satisfies the DE

$$\frac{d^2\mathbf{r}}{dt^2} = -g\mathbf{k} + 2\boldsymbol{\Omega} \times \frac{d\mathbf{r}}{dt}$$

and the initial conditions $\mathbf{r}(0) = 100\mathbf{k}$, $\mathbf{r}'(0) = \mathbf{0}$. If we use the approximation

$$\frac{d\mathbf{r}}{dt} \approx \frac{dz}{dt}\mathbf{k},$$

which is appropriate since Ω is much smaller than g, then

$$2\boldsymbol{\Omega} \times \frac{d\mathbf{r}}{dt} \approx \sqrt{2}\Omega\frac{dz}{dt}\mathbf{i}.$$

Breaking the DE into its components, we get

$$\frac{d^2x}{dt^2} = \sqrt{2}\Omega\frac{dz}{dt}, \quad \frac{d^2y}{dt^2} = 0, \quad \frac{d^2z}{dt^2} = -g.$$

Solving these equations (beginning with the last one), using the initial conditions, we get

$$z(t) = 100 - \frac{gt^2}{2}, \quad y(t) = 0, \quad x(t) = -\frac{\Omega g t^3}{3\sqrt{2}}.$$

Since $g \approx 9.8$ m/s^2, the time of fall is

$$t = \sqrt{\frac{200}{g}} \approx 4.52,$$

at which time we have

$$x \approx -\frac{2\pi}{24 \times 3{,}600}\,\frac{9.8}{3\sqrt{2}}\,(4.52)^3 \approx -0.0155 \text{ m.}$$

The object strikes the ground about 15.5 cm west of P.

2. $\begin{cases} \dfrac{d\mathbf{v}}{dt} = \mathbf{k}\times\mathbf{v} - 32\mathbf{k} \\ \mathbf{v}(0) = 70\mathbf{i} \end{cases}$

a) If $\mathbf{v} = v_1\mathbf{i}+v_2\mathbf{j}+v_3\mathbf{k}$, then $\mathbf{k}\times\mathbf{v} = v_1\mathbf{j}-v_2\mathbf{i}$. Thus the initial-value problem breaks down into component equations as

$$\begin{cases} \dfrac{dv_1}{dt} = -v_2 \\ v_1(0) = 70 \end{cases} \quad \begin{cases} \dfrac{dv_2}{dt} = v_1 \\ v_2(0) = 0 \end{cases} \quad \begin{cases} \dfrac{dv_3}{dt} = -32 \\ v_3(0) = 0. \end{cases}$$

b) If $\mathbf{r} = x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ denotes the position of the baseball t s after it is thrown, then $x(0) = y(0) = z(0) = 0$ and we have

$$\frac{dz}{dt} = v_3 = -32t \ \Rightarrow\ z = -16t^2.$$

Also, $\dfrac{d^2v_1}{dt^2} = -\dfrac{dv_2}{dt} = -v_1$ (the equation of simple harmonic motion), so

$$v_1(t) = A\cos t + B\sin t, \quad v_2(t) = A\sin t - B\cos t.$$

Since $v_1(0) = 70$, $v_2(0) = 0$, $x(0) = 0$, and $y(0) = 0$, we have

$$\begin{aligned} \frac{dx}{dt} &= v_1 = 70\cos t & \frac{dy}{dt} &= v_2 = 70\sin t \\ x(t) &= 70\sin t & y(t) &= 70(1-\cos t). \end{aligned}$$

At time t seconds after it is thrown, the ball is at position

$$\mathbf{r} = 70\sin t\,\mathbf{i} + 70(1-\cos t)\mathbf{j} - 16t^2\mathbf{k}.$$

c) At $t = 1/5$ s, the ball is at about $(13.9, 1.40, -0.64)$. If it had been thrown without the vertical spin, its position at time t would have been

$$\mathbf{r} = 70t\mathbf{i} - 16t^2\mathbf{k},$$

so its position at $t = 1/5$ s would have been $(14, 0, -0.64)$. Thus the spin has deflected the ball approximately 1.4 ft to the left (as seen from above) of what would have been its parabolic path had it not been given the spin.

3. $\begin{cases} \dfrac{d\mathbf{v}}{dt} = \omega\mathbf{v}\times\mathbf{k}, \quad \omega = \dfrac{qB}{m} \\ \mathbf{v}(0) = \mathbf{v}_0 \end{cases}$

a) $\dfrac{d}{dt}(\mathbf{v}\bullet\mathbf{k}) = \dfrac{d\mathbf{v}}{dt}\bullet\mathbf{k} = \omega(\mathbf{v}\times\mathbf{k})\bullet\mathbf{k} = 0.$
Thus $\mathbf{v}\bullet\mathbf{k} = \text{constant} = \mathbf{v}_0\bullet\mathbf{k}$.
Also, $\dfrac{d}{dt}|v|^2 = 2\dfrac{d\mathbf{v}}{dt}\bullet\mathbf{v} = 2\omega(\mathbf{v}\times\mathbf{k})\bullet\mathbf{v} = 0,$
so $|\mathbf{v}| = \text{constant} = |\mathbf{v}_0|$ for all t.

b) If $\mathbf{w}(t) = \mathbf{v}(t) - (\mathbf{v}_0\bullet\mathbf{k})\mathbf{k}$, then $\mathbf{w}\bullet\mathbf{k} = 0$ by part (a). Also, using the result of Exercise 23 of Section 1.3, we have

$$\begin{aligned} \frac{d^2\mathbf{w}}{dt^2} = \frac{d^2\mathbf{v}}{dt^2} &= \omega\frac{d\mathbf{v}}{dt}\times\mathbf{k} = \omega^2(\mathbf{v}\times\mathbf{k})\times\mathbf{k} \\ &= -\omega^2\Big[(\mathbf{k}\bullet\mathbf{k})\mathbf{v} - (\mathbf{k}\bullet\mathbf{v})\mathbf{k}\Big] \\ &= -\omega^2\Big[\mathbf{v} - (\mathbf{v}_0\bullet\mathbf{k})\mathbf{k}\Big] = -\omega^2\mathbf{w}, \end{aligned}$$

the equation of simple harmonic motion. Also,

$$\begin{aligned} \mathbf{w}(0) &= \mathbf{v}_0 - (\mathbf{v}_0\bullet\mathbf{k})\mathbf{k} \\ \mathbf{w}'(0) &= \omega\mathbf{v}_0\times\mathbf{k}. \end{aligned}$$

c) Solving the above initial-value problem for $\mathbf{w}$, we get

$$\begin{aligned} \mathbf{w} &= \mathbf{A}\cos(\omega t) + \mathbf{B}\sin(\omega t), \quad \text{where} \\ \mathbf{A} &= \mathbf{w}(0) = \mathbf{v}_0 - (\mathbf{v}_0\bullet\mathbf{k})\mathbf{k}, \quad \text{and} \\ \omega\mathbf{B} &= \mathbf{w}'(0) = \omega\times\mathbf{k}. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbf{v}(t) &= \mathbf{w}(t) + (\mathbf{v}_0\bullet\mathbf{k})\mathbf{k} \\ &= \Big[\mathbf{v}_0 - (\mathbf{v}_0\bullet\mathbf{k})\mathbf{k}\Big]\cos(\omega t) + (\mathbf{v}_0\times\mathbf{k})\sin(\omega t) \\ &\quad + (\mathbf{v}_0\bullet\mathbf{k})\mathbf{k}. \end{aligned}$$

d) If $d\mathbf{r}/dt = \mathbf{v}$ and $\mathbf{r}(0) = \mathbf{0}$, then

$$\begin{aligned} \mathbf{r}(t) &= \frac{\mathbf{v}_0 - (\mathbf{v}_0\bullet\mathbf{k})\mathbf{k}}{\omega}\sin(\omega t) \\ &\quad + \frac{\mathbf{v}_0\times\mathbf{k}}{\omega}\Big(1-\cos(\omega t)\Big) + (\mathbf{v}_0\bullet\mathbf{k})t\mathbf{k}. \end{aligned}$$

Since the three constant vectors

$$\frac{\mathbf{v}_0 - (\mathbf{v}_0\bullet\mathbf{k})\mathbf{k}}{\omega}, \quad \frac{\mathbf{v}_0\times\mathbf{k}}{\omega}, \text{ and } (\mathbf{v}_0\bullet\mathbf{k})\mathbf{k}$$

are mutually perpendicular, and the first two have the same length because

$$|\mathbf{v}_0 - (\mathbf{v}_0\bullet\mathbf{k})\mathbf{k}| = |\mathbf{v}_0|\sin\theta = |\mathbf{v}_0\times\mathbf{k}|,$$

where θ is the angle between $\mathbf{v}_0$ and $\mathbf{k}$, the curve $\mathbf{r}(t)$ is generally a circular helix with axis in the z direction. However, it will be a circle if $\mathbf{v}_0\bullet\mathbf{k} = 0$, that is, if $\mathbf{v}_0$ is horizontal, and it will be a straight line if $\mathbf{v}_0\times\mathbf{k} = \mathbf{0}$, that is, if $\mathbf{v}_0$ is vertical.

4. The arc length element on $x = a(\theta - \sin\theta)$, $y = a(\cos\theta - 1)$ is (for $\theta \le \pi$)

$$\begin{aligned} ds &= a\sqrt{(1-\cos\theta)^2 + \sin^2\theta}\, d\theta \\ &= a\sqrt{2(1-\cos\theta)}\, d\theta = 2a\sin(\theta/2)\, d\theta. \end{aligned}$$

If the bead slides downward from rest at height $y(\theta_0)$ to height $y(\theta)$, its gravitational potential energy has decreased by

$$mg\Big[y(\theta_0) - y(\theta)\Big] = mga(\cos\theta_0 - \cos\theta).$$

Since there is no friction, all this potential energy is converted to kinetic energy, so its speed v at height $y(\theta)$ is given by

$$\frac{1}{2}mv^2 = mga(\cos\theta_0 - \cos\theta),$$

and so $v = \sqrt{2ga(\cos\theta_0 - \cos\theta)}$. The time required for the bead to travel distance ds at speed v is $dt = ds/v$, so the time T required for the bead to slide from its starting position at $\theta = \theta_0$ to the lowest point on the wire, $\theta = \pi$, is

$$\begin{aligned} T &= \int_{\theta=\theta_0}^{\theta=\pi} \frac{ds}{v} = \int_{\theta_0}^{\pi} \frac{1}{v}\frac{ds}{d\theta}\, d\theta \\ &= \sqrt{\frac{2a}{g}} \int_{\theta_0}^{\pi} \frac{\sin(\theta/2)}{\sqrt{\cos\theta_0 - \cos\theta}}\, d\theta \\ &= \sqrt{\frac{2a}{g}} \int_{\theta_0}^{\pi} \frac{\sin(\theta/2)}{\sqrt{2\cos^2(\theta_0/2) - 2\cos^2(\theta/2)}}\, d\theta \\ &\qquad \text{Let } u = \cos(\theta/2) \\ &\qquad du = -\tfrac{1}{2}\sin(\theta/2)\, d\theta \\ &= 2\sqrt{\frac{a}{g}} \int_0^{\cos(\theta_0/2)} \frac{du}{\sqrt{\cos^2(\theta_0/2) - u^2}} \\ &= 2\sqrt{\frac{a}{g}} \sin^{-1}\left(\frac{u}{\cos(\theta_0/2)}\right)\Bigg|_0^{\cos(\theta_0/2)} \\ &= \pi\sqrt{ag} \end{aligned}$$

which is independent of θ_0.

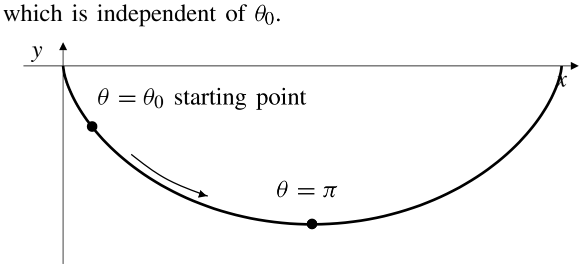

Fig. C-11.4

5. a) The curve BCD is the graph of an even function; a fourth degree polynomial with terms of even degree only will enable us to match the height, slope, and curvature at D, and therefore also at C. We have

$$\begin{aligned} f(x) &= ax^4 + bx^2 + c \\ f'(x) &= 4ax^3 + 2bx \\ f''(x) &= 12ax^2 + 2b. \end{aligned}$$

At D we have $x = 2$, so we need

$$\begin{aligned} 2 &= f(2) = 16a + 4b + c \\ 1 &= f'(2) = 32a + 4b \\ 0 &= f''(2) = 48a + 2b. \end{aligned}$$

These equations yield $a = -1/64$, $b = 3/8$, $c = 3/4$, so the curved track BCD is the graph of $y = f(x) = \dfrac{1}{64}(-x^4 + 24x^2 + 48)$.

b) Since we are ignoring friction, the speed v of the car during its drop is given by $v = \sqrt{2gs}$, where s is the vertical distance dropped. (See the previous solution.) At B the car has dropped about 7.2 m, so its speed there is $v \approx \sqrt{2(9.8)(7.2)} \approx 11.9$ m/s. At C the car has dropped $10 - (c/\sqrt{2}) \approx 9.47$ m, so its speed there is $v = 13.6$ m/s. At D the car has dropped 10 m, so its speed is $v = 14.0$ m/s.

c) At C we have $x = 0$, $f'(0) = 0$, and $f''(0) = 2b = 3/4$. Thus the curvature of the track at C is

$$\kappa = \frac{|f''(0)|}{(1 + (f'(0))^2)^{3/2}} = \frac{3}{4}.$$

The normal acceleration is $v^2\kappa \approx 138.7$ m/s^2 (or about $14g$). Since $v = \sqrt{2gs}$, we have

$$\frac{dv}{dt} = \frac{\sqrt{2g}}{2\sqrt{s}}\frac{ds}{dt} = \frac{\sqrt{2g}}{2\sqrt{s}}\, v \approx \frac{\sqrt{19.6}}{2\sqrt{9.47}}(13.6) \approx 9.78 \text{ m/s}^2,$$

so the total acceleration has magnitude approximately

$$\sqrt{(138.7)^2 + (9.78)^2} \approx 139 \text{ m/s}^2,$$

which is again about $14g$.

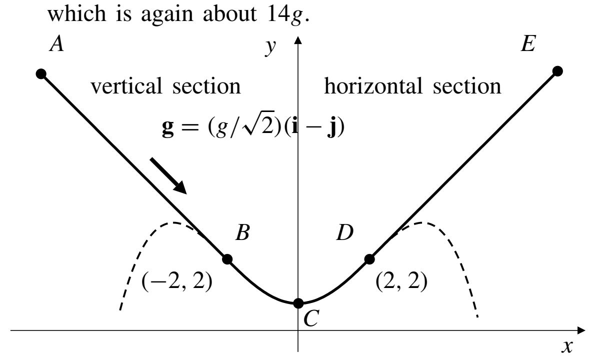

Fig. C-11.5

6. a) At time t, the hare is at $P = (0, vt)$ and the fox is at $Q = \big(x(t), y(t)\big)$, where x and y are such that the slope dy/dx of the fox's path is the slope of the line PQ:

$$\frac{dy}{dx} = \frac{y - vt}{x}.$$

b) Since $\dfrac{d}{dt}\dfrac{dy}{dx} = \dfrac{d^2y}{dx^2}\dfrac{dx}{dt}$, we have

$$\begin{aligned}
\frac{dx}{dt}\frac{d^2y}{dx^2} &= \frac{d}{dt}\left(\frac{y - vt}{x}\right)\\
&= \frac{x\left(\dfrac{dy}{dt} - v\right) - (y - vt)\dfrac{dx}{dt}}{x^2}\\
&= \frac{1}{x}\left(\frac{dy}{dx}\frac{dx}{dt} - v\right) - \frac{1}{x^2}(y - vt)\frac{dx}{dt}\\
&= \frac{1}{x^2}(y - vt)\frac{dx}{dt} - \frac{v}{x} - \frac{1}{x^2}(y - vt)\frac{dx}{dt}\\
&= -\frac{v}{x}.
\end{aligned}$$

Thus $x\dfrac{d^2y}{dx^2} = -\dfrac{v}{dx/dt}$.
Since the fox's speed is also v, we have

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = v^2.$$

Also, the fox is always running to the left (towards the y-axis from points where $x > 0$), so $dx/dt < 0$. Hence

$$\frac{v}{-\left(\dfrac{dx}{dt}\right)} = \sqrt{1 + \frac{(dy/dt)^2}{(dx/dt)^2}} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2},$$

and so the fox's path $y = y(x)$ satisfies the DE

$$x\frac{d^2y}{dx^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

c) If $u = dy/dx$, then $u = 0$ and $y = 0$ when $x = a$, and

$$\begin{aligned}
x\frac{du}{dx} &= \sqrt{1 + u^2}\\
\int \frac{du}{\sqrt{1 + u^2}} &= \int \frac{dx}{x} \qquad \text{Let } u = \tan\theta,\ du = \sec^2\theta\, d\theta\\
\int \sec\theta\, d\theta &= \ln x + \ln C\\
\ln(\tan\theta + \sec\theta) &= \ln(Cx)\\
u + \sqrt{1 + u^2} &= Cx.
\end{aligned}$$

Since $u = 0$ when $x = a$, we have $C = 1/a$.

$$\begin{aligned}
\sqrt{1 + u^2} &= \frac{x}{a} - u\\
1 + u^2 &= \frac{x^2}{a^2} - \frac{2xu}{a} + u^2
\end{aligned}$$

Since $y = 0$ when $x = a$, we have $C_1 = -\dfrac{a}{4} + \dfrac{a}{2}\ln a$, so

$$y = \frac{x^2 - a^2}{4} - \frac{a}{2}\ln\frac{x}{a}$$

is the path of the fox.

7. a) Since you are always travelling northeast at speed v, you are always moving north at rate $v/\sqrt{2}$. Therefore you will reach the north pole in finite time

$$T = \frac{\pi a/2}{v/\sqrt{2}} = \frac{\pi a}{\sqrt{2}v}.$$

b) Since your velocity at any point has a northward component $v/\sqrt{2}$, and progress northward is measured along a circle of radius a (a meridian), your colatitude $\phi(t)$ satisfies

$$a\frac{d\phi}{dt} = -\frac{v}{\sqrt{2}}.$$

Since $\phi(0) = \pi/2$, it follows that

$$\phi(t) = \frac{\pi}{2} - \frac{vt}{a\sqrt{2}}.$$

Since your velocity also has an eastward component $v/\sqrt{2}$ measured along a parallel of latitude that is a circle of radius $a\sin\phi$, your longitude coordinate θ satisfies

$$\begin{aligned}
(a\sin\phi)\frac{d\theta}{dt} &= \frac{v}{\sqrt{2}}\\
\left(\cos\frac{vt}{a\sqrt{2}}\right)\frac{d\theta}{dt} &= \frac{v}{a\sqrt{2}}\\
\theta &= \frac{v}{a\sqrt{2}}\int \sec\left(\frac{vt}{a\sqrt{2}}\right) dt\\
&= \ln\left(\sec\frac{vt}{a\sqrt{2}} + \tan\frac{vt}{a\sqrt{2}}\right) + C.
\end{aligned}$$

As $\theta = 0$ at $t = 0$, we have $C = 0$, and so

$$\theta(t) = \ln\left(\sec\frac{vt}{a\sqrt{2}} + \tan\frac{vt}{a\sqrt{2}}\right).$$

c) As $t \to T = \pi a/(\sqrt{2}v)$, the expression for $\theta(t) \to \infty$, so your path spirals around the north pole, crossing any meridian infinitely often.