CHAPTER 12. PARTIAL DIFFERENTIATION

Section 12.1 Functions of Several Variables (page 645)

1. $f(x, y) = \dfrac{x + y}{x - y}$.
The domain consists of all points in the xy-plane not on the line $x = y$.

2. $f(x, y) = \sqrt{xy}$.
Domain is the set of points (x, y) for which $xy \geq 0$, that is, points on the coordinate axes and in the first and third quadrants.

3. $f(x, y) = \dfrac{x}{x^2 + y^2}$.
The domain is the set of all points in the xy-plane except the origin.

4. $f(x, y) = \dfrac{xy}{x^2 - y^2}$.
The domain consists of all points not on the lines $x = \pm y$.

5. $f(x, y) = \sqrt{4x^2 + 9y^2 - 36}$.
The domain consists of all points (x, y) lying on or outside the ellipse $4x^2 + 9y^2 = 36$.

6. $f(x, y) = 1/\sqrt{x^2 - y^2}$.
The domain consists of all points in the part of the plane where $|x| > |y|$.

7. $f(x, y) = \ln(1 + xy)$.
The domain consists of all points satisfying $xy > -1$, that is, points lying between the two branches of the hyperbola $xy = -1$.

8. $f(x, y) = \sin^{-1}(x + y)$.
The domain consists of all points in the strip $-1 \leq x + y \leq 1$.

9. $f(x, y, z) = \dfrac{xyz}{x^2 + y^2 + z^2}$.
The domain consists of all points in 3-dimensional space except the origin.

10. $f(x, y, z) = \dfrac{e^{xyz}}{\sqrt{xyz}}$.
The domain consists of all points (x, y, z) where $xyz > 0$, that is, all points in the four octants $x > 0$, $y > 0$, $z > 0$; $x > 0$, $y < 0$, $z < 0$; $x < 0$, $y > 0$, $z < 0$; and $x < 0$, $y < 0$, $z > 0$.

11. $z = f(x, y) = x$

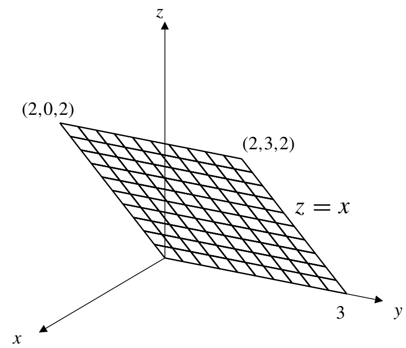


Fig. 12.1.11

12. $f(x, y) = \sin x, \ 0 \leq x \leq 2\pi, \ 0 \leq y \leq 1$

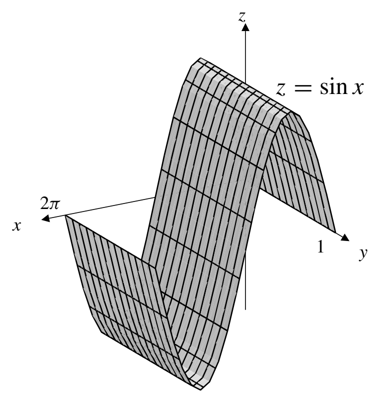


Fig. 12.1.12

13. $z = f(x, y) = y^2$

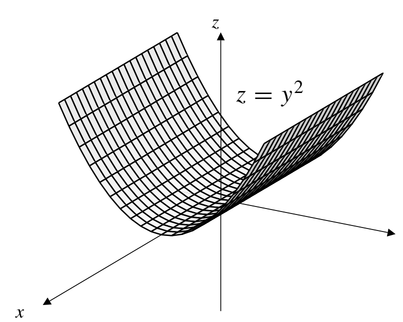


Fig. 12.1.13

14. $f(x, y) = 4 - x^2 - y^2, \ (x^2 + y^2 \leq 4, \ x \geq 0, \ y \geq 0)$

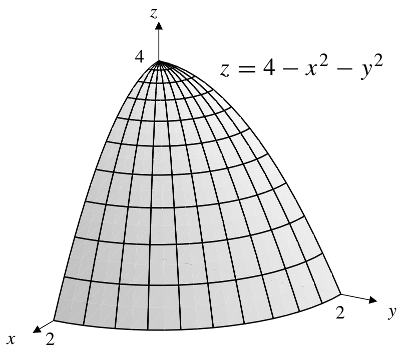


Fig. 12.1.14

15. $z = f(x, y) = \sqrt{x^2 + y^2}$

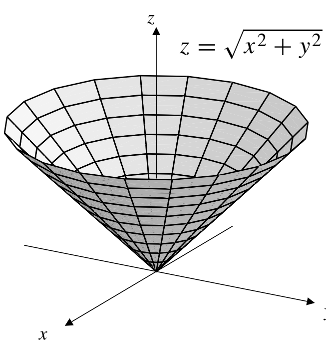


Fig. 12.1.15

16. $f(x, y) = 4 - x^2$

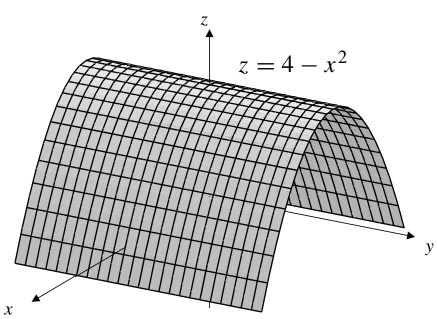


Fig. 12.1.16

17. $z = f(x, y) = |x| + |y|$

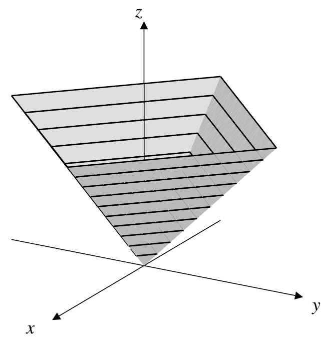


Fig. 12.1.17

18. $f(x, y) = 6 - x - 2y$

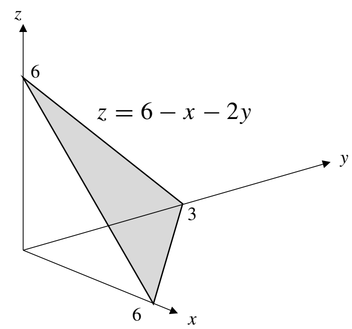


Fig. 12.1.18

19. $f(x, y) = x - y = C$, a family of straight lines of slope 1.

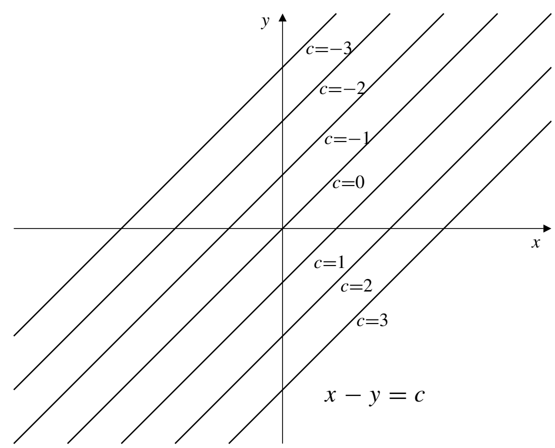


Fig. 12.1.19

20. $f(x, y) = x^2 + 2y^2 = C$, a family of similar ellipses centred at the origin.

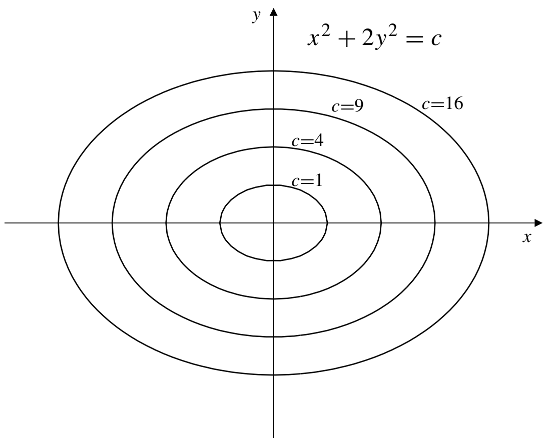


Fig. 12.1.20

21. $f(x, y) = xy = C$, a family of rectangular hyperbolas with the coordinate axes as asymptotes.

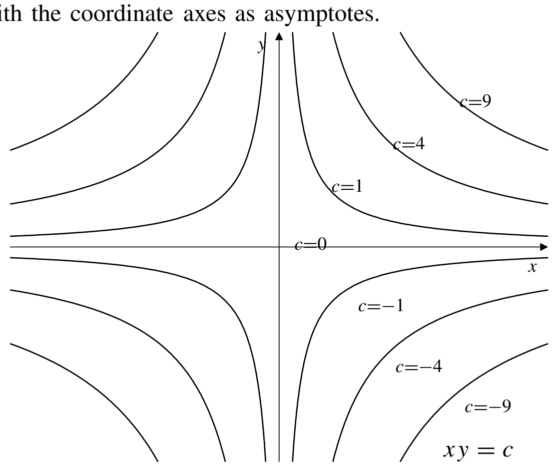


Fig. 12.1.21

22. $f(x, y) = \dfrac{x^2}{y} = C$, a family of parabolas, $y = x^2/C$, with vertices at the origin and vertical axes.

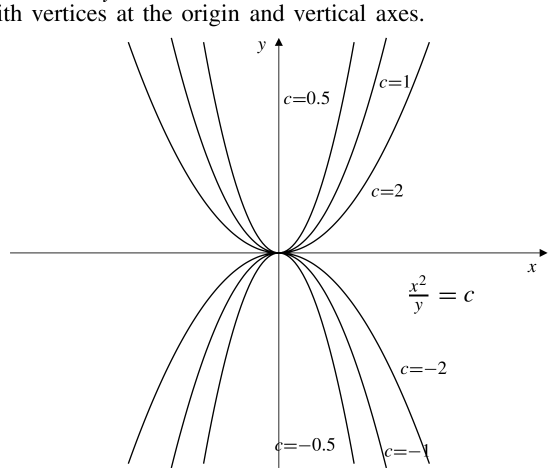


Fig. 12.1.22

23. $f(x, y) = \dfrac{x - y}{x + y} = C$, a family of straight lines through the origin, but not including the origin.

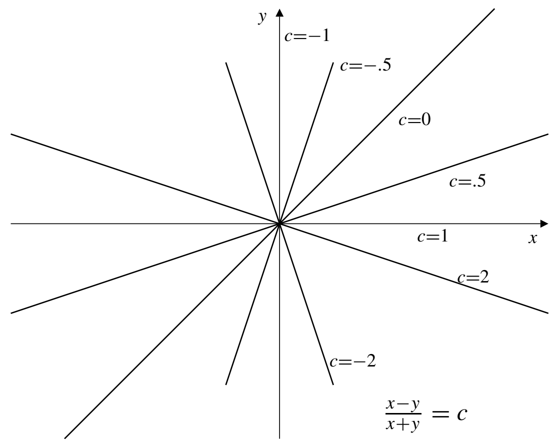


Fig. 12.1.23

24. $f(x, y) = \dfrac{y}{x^2 + y^2} = C$.

This is the family $x^2 + \left(y - \frac{1}{2C}\right)^2 = \frac{1}{4C^2}$ of circles passing through the origin and having centres on the y-axis. The origin itself is, however, not on any of the level curves.

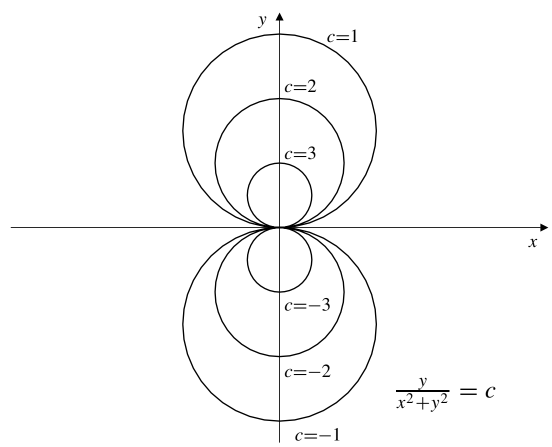


Fig. 12.1.24

25. $f(x, y) = xe^{-y} = C$.

This is the family of curves $y = \ln \dfrac{x}{C}$.

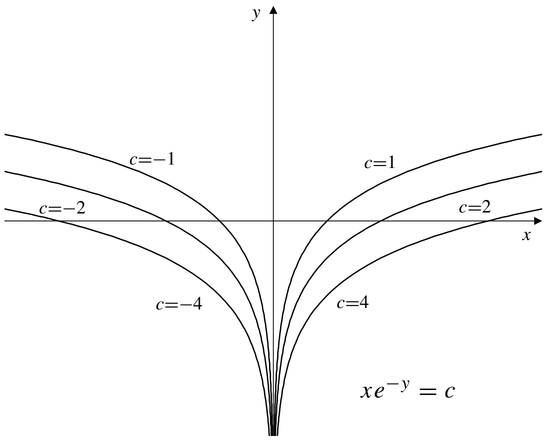


Fig. 12.1.25

26. $f(x, y) = \sqrt{\frac{1}{y} - x^2} = C \Rightarrow y = \frac{1}{x^2 + C^2}$.

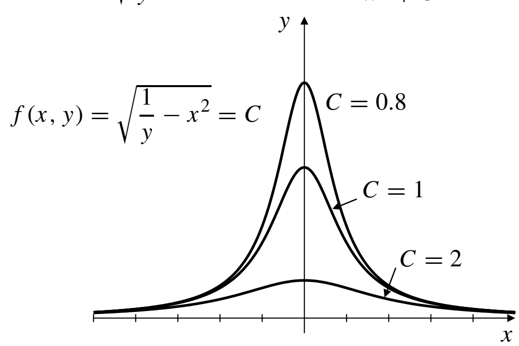


Fig. 12.1.26

27. The landscape is steepest at B where the level curves are closest together.

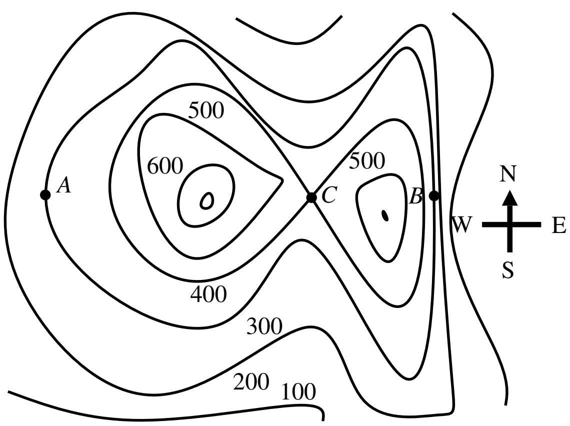


Fig. 12.1.27

28. C is a "pass" between two peaks to the east and west. The land is level at C and rises as you move to the east or west, but falls as you move to the north or south.

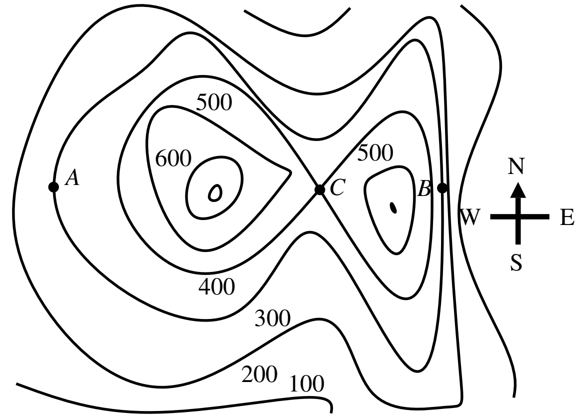


Fig. 12.1.28

29. The graph of the function whose level curves are as shown in part (a) of Figure 12.1.29 is a plane containing the y-axis and sloping uphill to the right. It is consistent with, say, a function of the form $f(x, y) = y$.

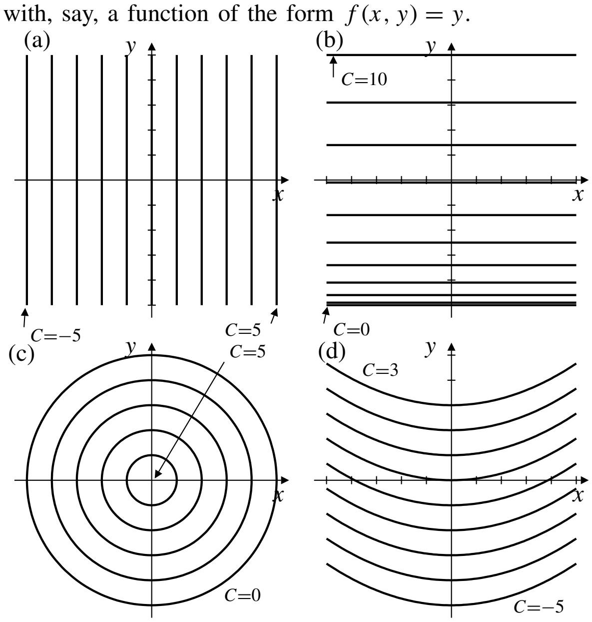


Fig. 12.1.29

30. The graph of the function whose level curves are as shown in part (b) of Figure 12.1.29 is a cylinder parallel to the x-axis, rising from height zero first steeply and then more and more slowly as y increases. It is consistent with, say, a function of the form $f(x, y) = \sqrt{y + 5}$.

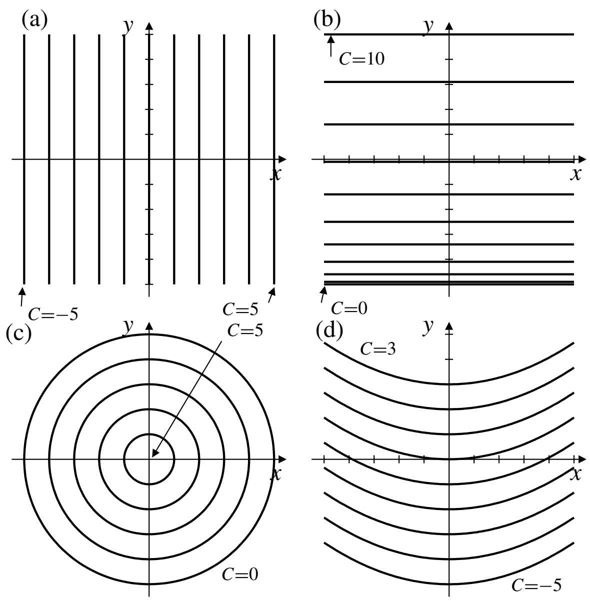


Fig. 12.1.30

31. The graph of the function whose level curves are as shown in part (c) of Figure 12.1.29 is an upside down circular cone with vertex at height 5 on the z-axis and base circle in the xy-plane. It is consistent with, say, a function of the form $f(x, y) = 5 - \sqrt{x^2 + y^2}$.

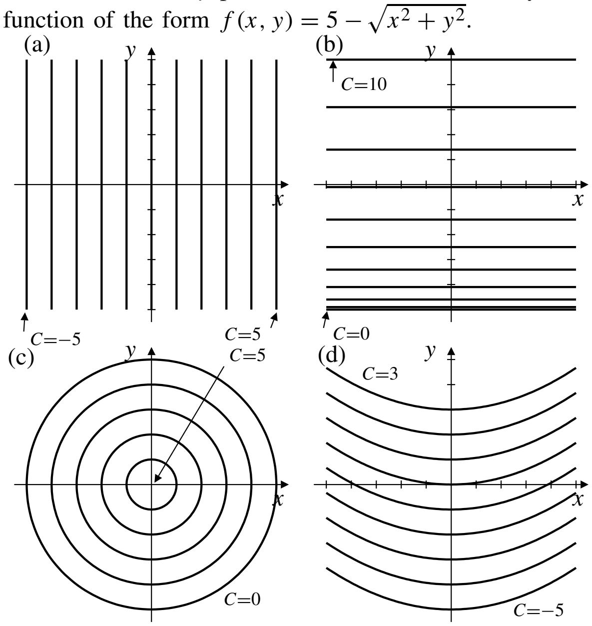


Fig. 12.1.31

32. The graph of the function whose level curves are as shown in part (d) of Figure 12.1.29 is a cylinder (possibly parabolic) with axis in the yz-plane, sloping upwards in the direction of increasing y. It is consistent with, say, a function of the form $f(x, y) = y - x^2$.

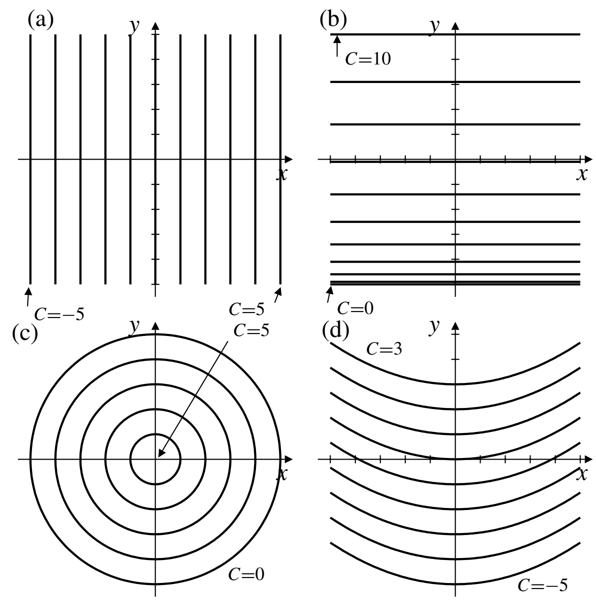
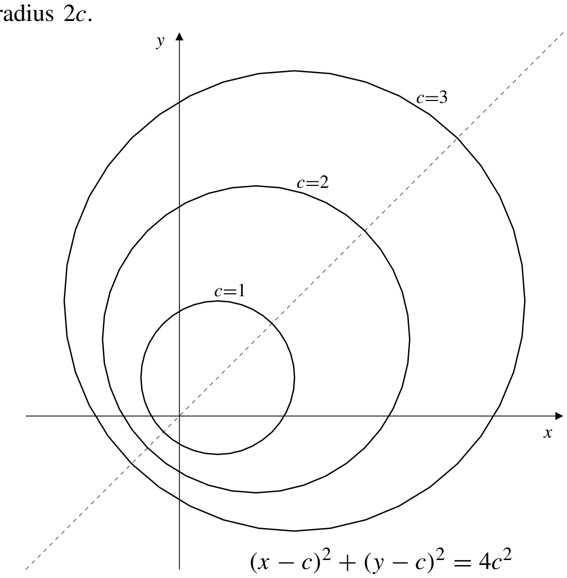


Fig. 12.1.32

33. The curves $y = (x - C)^2$ are all horizontally shifted versions of the parabola $y = x^2$, and they all lie in the half-plane $y \geq 0$. Since each of these curves intersects all of the others, they cannot be level curves of a function $f(x, y)$ defined in $y \geq 0$. To be a family of level curves of a function $f(x, y)$ in a region, the various curves in the family cannot intersect one another in that region.

34. $4z^2 = (x - z)^2 + (y - z)^2$.
If $z = c > 0$, we have $(x - c)^2 + (y - c)^2 = 4c^2$, which is a circle in the plane $z = c$, with centre (c, c, c) and radius $2c$.



$(x - c)^2 + (y - c)^2 = 4c^2$

Fig. 12.1.34

The graph of the function $z = z(x, y) \geq 0$ defined by the given equation is (the upper half of) an elliptic cone with axis along the line $x = y = z$, and circular cross-sections in horizontal planes.

35. a) $f(x, y) = C$ is $x^2 + y^2 = C^2$ implies that $f(x, y) = \sqrt{x^2 + y^2}$.

b) $f(x, y) = C$ is $x^2 + y^2 = C^4$ implies that $f(x, y) = (x^2 + y^2)^{1/4}$.

c) $f(x, y) = C$ is $x^2 + y^2 = C$ implies that $f(x, y) = x^2 + y^2$.

d) $f(x, y) = C$ is $x^2 + y^2 = (\ln C)^2$ implies that $f(x, y) = e^{\sqrt{x^2+y^2}}$.

36. If the level surface $f(x, y, z) = C$ is the plane

$$\frac{x}{C^3} + \frac{y}{2C^3} + \frac{z}{3C^3} = 1,$$

that is, $x + \dfrac{y}{2} + \dfrac{z}{3} = C^3$, then

$$f(x, y, z) = \left(x + \frac{y}{2} + \frac{z}{3}\right)^{1/3}.$$

37. $f(x, y, z) = x^2 + y^2 + z^2$.
The level surface $f(x, y, z) = c > 0$ is a sphere of radius $\sqrt{c}$ centred at the origin.

38. $f(x, y, z) = x + 2y + 3z$.
The level surfaces are parallel planes having common normal vector $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

39. $f(x, y, z) = x^2 + y^2$.
The level surface $f(x, y, z) = c > 0$ is a circular cylinder of radius $\sqrt{c}$ with axis along the z-axis.

40. $f(x, y, z) = \dfrac{x^2 + y^2}{z^2}$.
The equation $f(x, y, z) = c$ can be rewritten $x^2 + y^2 = C^2 z^2$. The level surfaces are circular cones with vertices at the origin and axes along the z-axis.

41. $f(x, y, z) = |x| + |y| + |z|$.
The level surface $f(x, y, z) = c > 0$ is the surface of the octahedron with vertices $(\pm c, 0, 0)$, $(0, \pm c, 0)$, and $(0, 0, \pm c)$. (An octahedron is a solid with eight planar faces.)

42. $f(x, y, z, t) = x^2 + y^2 + z^2 + t^2$.
The "level hypersurface" $f(x, y, z, t) = c > 0$ is the "4-sphere" of radius $\sqrt{c}$ centred at the origin in $\mathbb{R}^4$. That is, it consists of all points in $\mathbb{R}^4$ at distance $\sqrt{c}$ from the origin.

43.

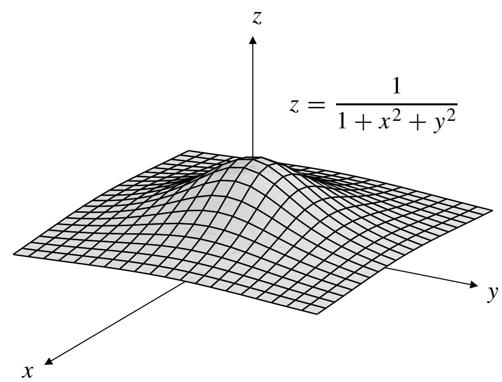


Fig. 12.1.43

44.

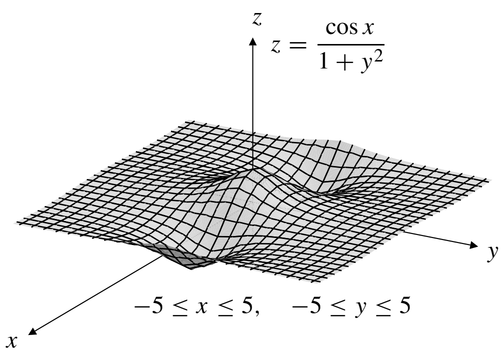


Fig. 12.1.44

45.

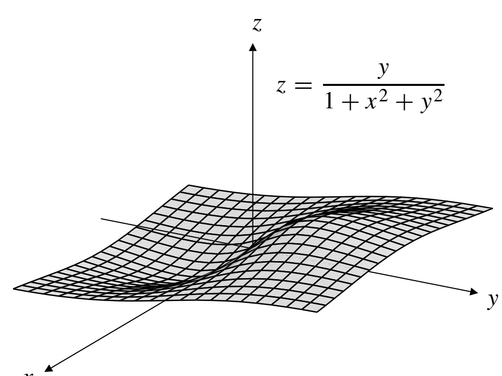


Fig. 12.1.45

46.

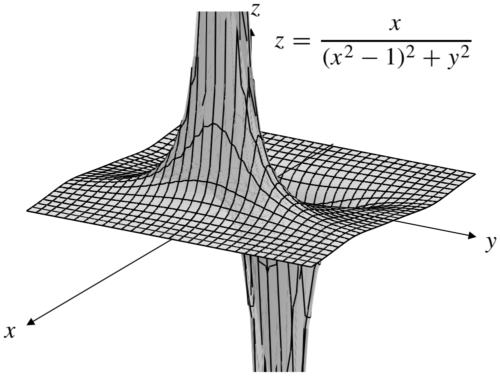


Fig. 12.1.46

47.

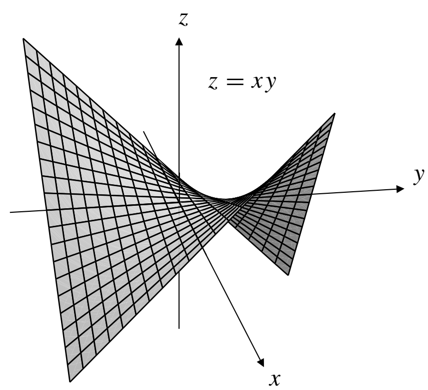


Fig. 12.1.47

48. The graph is asymptotic to the coordinate planes.

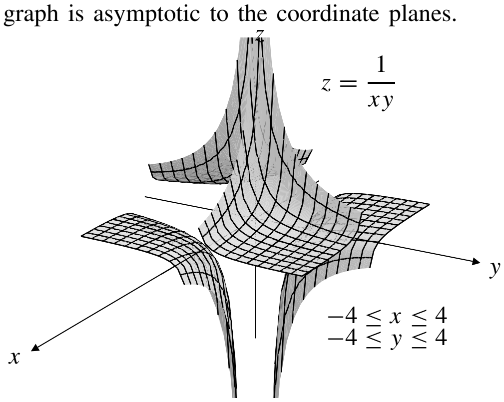


Fig. 12.1.48

Section 12.2 Limits and Continuity (page 650)

1. $\lim_{(x,y)\to(2,-1)} xy + x^2 = 2(-1) + 2^2 = 2$

2. $\lim_{(x,y)\to(0,0)} \sqrt{x^2+y^2} = 0$

3. $\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{y}$ does not exist.

If $(x, y) \to (0, 0)$ along $x = 0$, then $\frac{x^2+y^2}{y} = y \to 0$.
If $(x, y) \to (0, 0)$ along $y = x^2$, then $\frac{x^2+y^2}{y} = 1 + x^2 \to 1$.

4. Let $f(x, y) = \frac{x}{x^2+y^2}$.
Then $|f(x, 0)| = |1/x| \to \infty$ as $x \to 0$.
But $|f(0, y)| = 0 \to 0$ as $y \to 0$.
Thus $\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist.

5. $\lim_{(x,y)\to(1,\pi)} \frac{\cos(xy)}{1-x-\cos y} = \frac{\cos\pi}{1-1-\cos\pi} = -1$

6. $\lim_{(x,y)\to(0,1)} \frac{x^2(y-1)^2}{x^2+(y-1)^2} = 0$, because

$$0 \le \left|\frac{x^2(y-1)^2}{x^2+(y-1)^2}\right| \le x^2$$

and $x^2 \to 0$ as $(x, y) \to (0, 1)$.

7. $\left|\frac{y^3}{x^2+y^2}\right| \le \frac{y^2}{x^2+y^2}|y| \le |y| \to 0$
as $(x, y) \to (0, 0)$. Thus $\lim_{(x,y)\to(0,0)} \frac{y^3}{x^2+y^2} = 0$.

8. $\lim_{(x,y)\to(0,0)} \frac{\sin(x-y)}{\cos(x+y)} = \frac{\sin 0}{\cos 0} = 0$.

9. Let $f(x, y) = \frac{\sin(xy)}{x^2+y^2}$.
Now $f(0, y) = 0/x^2 = 0 \to 0$ as $x \to 0$.
However, $f(x, x) = \frac{\sin x^2}{2x^2} \to \frac{1}{2}$ as $x \to 0$.
Therefore $\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist.

10. The fraction is not defined at points of the line $y = 2x$ and so cannot have a limit at $(1, 2)$ by Definition 4. However, if we use the extended Definition 6, then, cancelling the common factor $2x - y$, we get

$$\lim_{(x,y)\to(1,2)} \frac{2x^2-xy}{4x^2-y^2} = \lim_{(x,y)\to(1,2)} \frac{x}{2x+y} = \frac{1}{4}.$$

11. $x^2 \le x^2 + y^4$. Thus $\frac{x^2y^2}{x^2+y^4} \le y^2 \to 0$ as $y \to 0$. Thus $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^4} = 0$.

12. If $x = 0$ and $y \neq 0$, then $\dfrac{x^2y^2}{2x^4 + y^4} = 0$.

If $x = y \neq 0$, then $\dfrac{x^2y^2}{2x^4 + y^4} = \dfrac{x^4}{2x^4 + x^4} = \dfrac{1}{3}$.

Therefore $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{2x^4 + y^4}$ does not exist.

13. $f(x, y) = \dfrac{x^2 + y^2 - x^3y^3}{x^2 + y^2} = 1 - \dfrac{x^3y^3}{x^2 + y^2}$. But

$$\left|\frac{x^3y^3}{x^2 + y^2}\right| = \left|\frac{x^2}{x^2 + y^2}\right| |xy^3| \leq |xy^3| \to 0$$

as $(x, y) \to (0, 0)$. Thus $\displaystyle\lim_{(x,y)\to(0,0)} f(x, y) = 1 - 0 = 1$. Define $f(0, 0) = 1$.

14. For $x \neq y$, we have

$$f(x, y) = \frac{x^3 - y^3}{x - y} = x^2 + xy + y^2.$$

The latter expression has the value $3x^2$ at points of the line $x = y$. Therefore, if we extend the definition of $f(x, y)$ so that $f(x, x) = 3x^2$, then the resulting function will be equal to $x^2 + xy + y^2$ everywhere, and so continuous everywhere.

15. $f(x, y) = \dfrac{x - y}{x^2 - y^2} = \dfrac{x - y}{(x - y)(x + y)}$.
Since $f(x, y) = 1/(x + y)$ at all points off the line $x = y$ and so is defined at some points in any neighbourhood of $(1, 1)$, it approaches $1/(1 + 1) = 1/2$ as $(x, y) \to (1, 1)$; If we define $f(1, 1) = 1/2$, then f becomes continuous at $(1, 1)$. Similarly, $f(x, y)$ can be defined to be $1/(2x)$ at any point on the line $x = y$ except the origin, and becomes continuous at such points.

However there is no way to define $f(x, -x)$ so that f becomes continuous on $y = -x$, since $|f(x, y)| = 1/|x + y| \to \infty$ as $y \to -x$.

16. Let f be the function of Example 3 of Section 3.2:

$$f(x, y) = \begin{cases} \dfrac{2xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

Let $a = b = 0$. If $g(x) = f(x, 0)$ and $h(y) = f(0, y)$, then $g(x) = 0$ for all x, and $h(y) = 0$ for all y, so g and h are continuous at 0. But, as shown in Example 3 of Section 3.2, f is not continuous at $(0, 0)$.

If $f(x, y)$ is continuous at (a, b), then $g(x) = f(x, b)$ is continuous at $x = a$ because

$$\lim_{x\to a} g(x) = \lim_{\substack{x\to a \\ y=b}} f(x, y) = f(a, b).$$

Similarly, $h(y) = f(a, y)$ is continuous at $y = b$.

17. $f_{\mathbf{u}}(t) = f(a + tu, b + tv)$, where $\mathbf{u} = u\mathbf{i} + v\mathbf{j}$ is a unit vector.

$f(x, y)$ may not be continuous at (a, b) even if $f_{\mathbf{u}}(t)$ is continuous at $t = 0$ for every unit vector $\mathbf{u}$. A counterexample is the function f of Example 4 in this section. Here $a = b = 0$. The condition that each $f_{\mathbf{u}}$ should be continuous is the condition that f should be continuous on each straight line through $(0, 0)$, which it is if we extend the domain of f to include $(0, 0)$ by defining $f(0, 0) = 0$. (We showed that $f(x, y) \to 0$ as $(x, y) \to (0, 0)$ along every straight line.) However, we also showed that $\lim_{(x,y)\to(0,0)} f(x, y)$ did not exist.

On the other hand, if $f(x, y)$ is continuous at (a, b), then $f(x, y) \to f(a, b)$ if (x, y) approaches (a, b) in any way, in particular, along the line through (a, b) parallel to $\mathbf{u}$. Thus all such functions $f_{\mathbf{u}}(t)$ must be continuous at $t = 0$.

18. Since $|x| \leq \sqrt{x^2 + y^2}$ and $|y| \leq \sqrt{x^2 + y^2}$, we have

$$\left|\frac{x^m y^n}{(x^2 + y^2)^p}\right| \leq \frac{(x^2 + y^2)^{(m+n)/2}}{(x^2 + y^2)^p} = (x^2 + y^2)^{-p+(m+n)/2}.$$

The expression on the right $\to 0$ as $(x, y) \to (0, 0)$, provided $m + n > 2p$. In this case

$$\lim_{(x,y)\to(0,0)} \frac{x^m y^n}{(x^2 + y^2)^p} = 0.$$

19. Suppose $(x, y) \to (0, 0)$ along the ray $y = kx$. Then

$$f(x, y) = \frac{xy}{ax^2 + bxy + cy^2} = \frac{k}{a + bk + ck^2}.$$

Thus $f(x, y)$ has different constant values along different rays from the origin unless $a = c = 0$ and $b \neq 0$. If this condition is not satisfied, $\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist. If the condition is satisfied, then $\lim_{(x,y)\to(0,0)} f(x, y) = 1/b$ does exist.

20. $f(x, y) = \dfrac{\sin x \sin^3 y}{1 - \cos(x^2 + y^2)}$ cannot be defined at $(0, 0)$ so as to become continuous there, because $f(x, y)$ has no limit as $(x, y) \to (0, 0)$. To see this, observe that $f(x, 0) = 0$, so the limit must be 0 if it exists at all. However,

$$f(x, x) = \frac{\sin^4 x}{1 - \cos(2x^2)} = \frac{\sin^4 x}{2\sin^2(x^2)}$$

which approaches 1/2 as $x \to 0$ by l'Hôpital's Rule or by using Maclaurin series.

21.

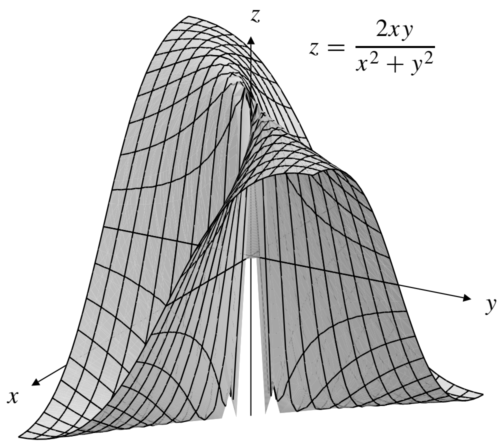


Fig. 12.2.21

The graphing software is unable to deal effectively with the discontinuity at $(x, y) = (0, 0)$ so it leaves some gaps and rough edges near the z-axis. The surface lies between a ridge of height 1 along $y = x$ and a ridge of height -1 along $y = -x$. It appears to be creased along the z-axis. The level curves are straight lines through the origin.

22. The graphing software is unable to deal effectively with the discontinuity at $(x, y) = (0, 0)$ so it leaves some gaps and rough edges near the z-axis. The surface lies between a ridge along $y = x^2$, $z = 1$, and a ridge along $y = -x^2$, $z = -1$. It appears to be creased along the z-axis. The level curves are parabolas $y = kx^2$ through the origin. One of the families of rulings on the surface is the family of contours corresponding to level curves.

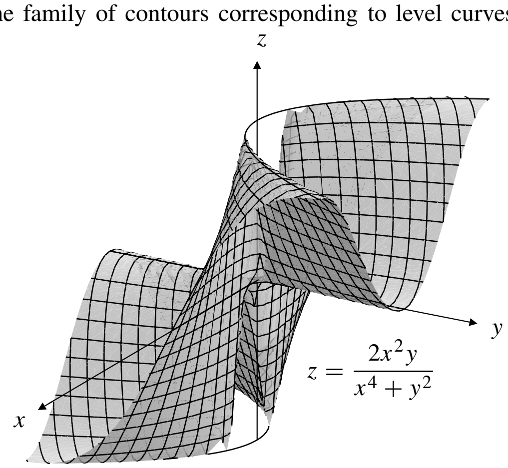


Fig. 12.2.22

23. The graph of a function $f(x, y)$ that is continuous on region R in the xy-plane is a surface with no breaks or tears in it and that intersects each line parallel to the z-axis through a point (x, y) of R at exactly one point.

Section 12.3 Partial Derivatives (page 656)

1. $f(x, y) = x - y + 2$,
$f_1(x, y) = 1 = f_1(3, 2)$, $f_2(x, y) = -1 = f_2(3, 2)$.

2. $f(x, y) = xy + x^2$,
$f_1(x, y) = y + 2x$, $f_2(x, y) = x$,
$f_1(2, 0) = 4$, $f_2(2, 0) = 2$.

3. $f(x, y, z) = x^3y^4z^5$,
$f_1(x, y, z) = 3x^2y^4z^5$, $f_1(0, -1, -1) = 0$,
$f_2(x, y, z) = 4x^3y^3z^5$, $f_2(0, -1, -1) = 0$,
$f_3(x, y, z) = 5x^3y^4z^4$, $f_3(0, -1, -1) = 0$.

4. $g(x, y, z) = \dfrac{xz}{y+z}$,
$g_1(x, y, z) = \dfrac{z}{y+z}$, $g_1(1, 1, 1) = \dfrac{1}{2}$,
$g_2(x, y, z) = \dfrac{-xz}{(y+z)^2}$, $g_2(1, 1, 1) = -\dfrac{1}{4}$,
$g_3(x, y, z) = \dfrac{xy}{(y+z)^2}$, $g_3(1, 1, 1) = \dfrac{1}{4}$.

5. $z = \tan^{-1}\left(\dfrac{y}{x}\right)$

$$\frac{\partial z}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2}$$

$$\frac{\partial z}{\partial y} = \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2}$$

$$\left.\frac{\partial z}{\partial x}\right|_{(-1,1)} = -\frac{1}{2}, \quad \left.\frac{\partial z}{\partial y}\right|_{(-1,1)} = -\frac{1}{2}.$$

6. $w = \ln(1 + e^{xyz})$, $\dfrac{\partial w}{\partial x} = \dfrac{yze^{xyz}}{1+e^{xyz}}$,
$\dfrac{\partial w}{\partial y} = \dfrac{xze^{xyz}}{1+e^{xyz}}$, $\dfrac{\partial w}{\partial z} = \dfrac{xye^{xyz}}{1+e^{xyz}}$,
At $(2, 0, -1)$: $\dfrac{\partial w}{\partial x} = 0$, $\dfrac{\partial w}{\partial y} = -1$, $\dfrac{\partial w}{\partial z} = 0$.

7. $f(x, y) = \sin(x\sqrt{y})$,
$f_1(x, y) = \sqrt{y}\cos(x\sqrt{y})$, $f_1\left(\dfrac{\pi}{3}, 4\right) = -1$,
$f_2(x, y) = \dfrac{x}{2\sqrt{y}}\cos(x\sqrt{y})$, $f_2\left(\dfrac{\pi}{3}, 4\right) = -\dfrac{\pi}{24}$.

8. $f(x,y) = \dfrac{1}{\sqrt{x^2+y^2}}$,
$f_1(x,y) = -\dfrac{1}{2}(x^2+y^2)^{-3/2}(2x) = -\dfrac{x}{(x^2+y^2)^{3/2}}$,
By symmetry, $f_2(x,y) = -\dfrac{y}{(x^2+y^2)^{3/2}}$,
$f_1(-3,4) = \dfrac{3}{125}$, $f_2(-3,4) = -\dfrac{4}{125}$.

9. $w = x^{y \ln z}$,
$\dfrac{\partial w}{\partial x} = y \ln z \, x^{y \ln z - 1}$, $\left.\dfrac{\partial w}{\partial x}\right|_{(e,2,e)} = 2e$,
$\dfrac{\partial w}{\partial y} = \ln x \ln z \, x^{y \ln z}$, $\left.\dfrac{\partial w}{\partial y}\right|_{(e,2,e)} = e^2$,
$\dfrac{\partial w}{\partial z} = \dfrac{y}{z} \ln x \, x^{y \ln z}$, $\left.\dfrac{\partial w}{\partial z}\right|_{(e,2,e)} = 2e$.

10. If $g(x_1,x_2,x_3,x_4) = \dfrac{x_1 - x_2^2}{x_3 + x_4^2}$, then

$$g_1(x_1,x_2,x_3,x_4) = \frac{1}{x_3+x_4^2} \qquad g_1(3,1,-1,-2) = \frac{1}{3}$$
$$g_2(x_1,x_2,x_3,x_4) = \frac{-2x_2}{x_3+x_4^2} \qquad g_2(3,1,-1,-2) = -\frac{2}{3}$$
$$g_3(x_1,x_2,x_3,x_4) = \frac{x_2^2 - x_1}{(x_3+x_4^2)^2} \qquad g_3(3,1,-1,-2) = -\frac{2}{9}$$
$$g_4(x_1,x_2,x_3,x_4) = \frac{(x_2^2 - x_1)2x_4}{(x_3+x_4^2)^2} \qquad g_4(3,1,-1,-2) = \frac{8}{9}.$$

11. $$f(x,y) = \begin{cases} \dfrac{2x^3 - y^3}{x^2+3y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$
$$f_1(0,0) = \lim_{h\to 0} \frac{2h^3 - 0}{h(h^2+0)} = 2$$
$$f_2(0,0) = \lim_{k\to 0} \frac{-k^3 - 0}{k(0+3k^2)} = -\frac{1}{3}.$$

12. $$f(x,y) = \begin{cases} \dfrac{x^2 - 2y^2}{x-y} & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$
$$f_1(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{h-0}{h} = 1,$$
$$f_2(0,0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k} = \lim_{k\to 0} \frac{2k}{k} = 2.$$

13. $f(x,y) = x^2 - y^2$ $\quad f(-2,1) = 3$
$f_1(x,y) = 2x$ $\quad f_1(-2,1) = -4$
$f_2(x,y) = -2y$ $\quad f_2(-2,1) = -2$
Tangent plane: $z = 3 - 4(x+2) - 2(y-1)$, or $4x + 2y + z = -3$.
Normal line: $\dfrac{x+2}{-4} = \dfrac{y-1}{-2} = \dfrac{z-3}{-1}$.

14. $f(x,y) = \dfrac{x-y}{x+y}$, $f(1,1) = 0$,
$f_1(x,y) = \dfrac{(x+y)-(x-y)}{(x+y)^2}$, $\quad f_1(1,1) = \dfrac{1}{2}$
$f_2(x,y) = \dfrac{(x+y)(-1)-(x-y)}{(x+y)^2}$, $\quad f_2(1,1) = -\dfrac{1}{2}$.
Tangent plane to $z = f(x,y)$ at (1,1) has equation $z = \dfrac{x-1}{2} - \dfrac{y-1}{2}$, or $2z = x - y$.
Normal line: $2(x-1) = -2(y-1) = -z$.

15. $f(x,y) = \cos\dfrac{x}{y}$ $\qquad f(\pi,4) = \dfrac{1}{\sqrt{2}}$
$f_1(x,y) = -\dfrac{1}{y}\sin\dfrac{x}{y}$ $\qquad f_1(\pi,4) = -\dfrac{1}{4\sqrt{2}}$
$f_2(x,y) = \dfrac{x}{y^2}\sin\dfrac{x}{y}$ $\qquad f_2(\pi,4) = \dfrac{\pi}{16\sqrt{2}}$
The tangent plane at $x = \pi$, $y = 4$ is
$$z = \frac{1}{\sqrt{2}}\left(1 - \frac{1}{4}(x-\pi) + \frac{\pi}{16}(y-4)\right),$$
or $4x - \pi y + 16\sqrt{2}z = 16$.
Normal line:
$-4\sqrt{2}(x-\pi) = \dfrac{16\sqrt{2}}{\pi}(y-4) = -\left(z - (1/\sqrt{2})\right)$.

16. $f(x,y) = e^{xy}$, $f_1(x,y) = ye^{xy}$, $f_2(x,y) = xe^{xy}$,
$f(2,0) = 1$, $f_1(2,0) = 0$, $f_2(2,0) = 2$.
Tangent plane to $z = e^{xy}$ at (2,0) has equation $z = 1 + 2y$.
Normal line: $x = 2$, $y = 2 - 2z$.

17. $f(x,y) = \dfrac{x}{x^2+y^2}$
$f_1(x,y) = \dfrac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2} = \dfrac{y^2 - x^2}{(x^2+y^2)^2}$
$f_2(x,y) = -\dfrac{2xy}{(x^2+y^2)^2}$
$f(1,2) = \dfrac{1}{5}$, $\quad f_1(1,2) = \dfrac{3}{25}$, $\quad f_2(1,2) = -\dfrac{4}{25}$.
The tangent plane at $x = 1$, $y = 2$ is
$$z = \frac{1}{5} + \frac{3}{25}(x-1) - \frac{4}{25}(y-2),$$
or $3x - 4y - 25z = -10$.
Normal line: $\dfrac{x-1}{3} = \dfrac{y-2}{-4} = \dfrac{5z-1}{-125}$.

18. $f(x,y) = ye^{-x^2}$, $f_1 = -2xye^{-x^2}$, $f_2 = e^{-x^2}$,
$f(0,1) = 1$, $f_1(0,1) = 0$, $f_2(0,1) = 1$.
Tangent plane to $z = f(x,y)$ at $(0,1)$ has equation $z = 1 + 1(y-1)$, or $z = y$.
Normal line: $x = 0$, $y + z = 2$.

19. $f(x, y) = \ln(x^2 + y^2) \qquad f(1, -2) = \ln 5$
$f_1(x, y) = \dfrac{2x}{x^2 + y^2} \qquad f_1(1, -2) = \dfrac{2}{5}$
$f_2(x, y) = \dfrac{2y}{x^2 + y^2} \qquad f_2(1, -2) = -\dfrac{4}{5}$
The tangent plane at $(1, -2, \ln 5)$ is

$$z = \ln 5 + \frac{2}{5}(x - 1) - \frac{4}{5}(y + 2),$$

or $2x - 4y - 5z = 10 - 5\ln 5$.
Normal line: $\dfrac{x - 1}{2/5} = \dfrac{y + 2}{-4/5} = \dfrac{z - \ln 5}{-1}$.

20. $f(x, y) = \dfrac{2xy}{x^2 + y^2}, \qquad f(0, 2) = 0$
$f_1(x, y) = \dfrac{(x^2 + y^2)2y - 2xy(2x)}{(x^2 + y^2)^2} = \dfrac{2y(y^2 - x^2)}{(x^2 + y^2)^2}$
$f_2(x, y) = \dfrac{2x(x^2 - y^2)}{(x^2 + y^2)^2}$ (by symmetry)
$f_1(0, 2) = 1, \qquad f_2(0, 2) = 0.$
Tangent plane at $(0, 2)$: $z = x$.
Normal line: $z + x = 0$, $y = 2$.

21. $f(x, y) = \tan^{-1}\left(\dfrac{y}{x}\right), \ f(1, -1) = -\dfrac{\pi}{4},$
$f_1(x, y) = \dfrac{1}{1 + \dfrac{y^2}{x^2}}\left(-\dfrac{y}{x^2}\right) = -\dfrac{y}{x^2 + y^2},$
$f_2(x, y) = \dfrac{1}{1 + \dfrac{y^2}{x^2}}\left(\dfrac{1}{x}\right) = \dfrac{x}{x^2 + y^2},$
$f_1(1, -1) = f_2(1, -1) = \dfrac{1}{2}$. The tangent plane is
$z = -\dfrac{\pi}{4} + \dfrac{1}{2}(x - 1) + \dfrac{1}{2}(y + 1)$, or $z = -\dfrac{\pi}{4} + \dfrac{1}{2}(x + y)$.
Normal line: $2(x - 1) = 2(y + 1) = -z - \dfrac{\pi}{4}$.

22. $f(x, y) = \sqrt{1 + x^3y^2} \qquad f(2, 1) = 3$
$f_1(x, y) = \dfrac{3x^2y^2}{2\sqrt{1 + x^3y^2}} \qquad f_1(2, 1) = 2$
$f_2(x, y) = \dfrac{2x^3y}{2\sqrt{1 + x^3y^2}} \qquad f_2(2, 1) = \dfrac{8}{3}$
Tangent plane: $z = 3 + 2(x - 2) + \frac{8}{3}(y - 1)$, or
$6x + 8y - 3z = 11$.
Normal line: $\dfrac{x - 2}{2} = \dfrac{y - 1}{8/3} = \dfrac{z - 3}{-1}$.

23. $z = x^4 - 4xy^3 + 6y^2 - 2$
$\dfrac{\partial z}{\partial x} = 4x^3 - 4y^3 = 4(x - y)(x^2 + xy + y^2)$
$\dfrac{\partial z}{\partial y} = -12xy^2 + 12y = 12y(1 - xy).$
The tangent plane will be horizontal at points where both first partials are zero. Thus we require $x = y$ and either $y = 0$ or $xy = 1$.
If $x = y$ and $y = 0$, then $x = 0$.
If $x = y$ and $xy = 1$, then $x^2 = 1$, so $x = y = \pm 1$.
The tangent plane is horizontal at the points $(0, 0)$, $(1, 1)$, and $(-1, -1)$.

24. $z = xye^{-(x^2+y^2)/2}$
$\dfrac{\partial z}{\partial x} = ye^{-(x^2+y^2)/2} - x^2ye^{-(x^2+y^2)/2} = y(1 - x^2)e^{-(x^2+y^2)/2}$
$\dfrac{\partial z}{\partial y} = x(1 - y^2)e^{-(x^2+y^2)/2}$ (by symmetry)
The tangent planes are horizontal at points where both of these first partials are zero, that is, points satisfying

$$y(1 - x^2) = 0 \qquad \text{and} \qquad x(1 - y^2) = 0.$$

These points are $(0, 0)$, $(1, 1)$, $(-1, -1)$, $(1, -1)$ and $(-1, 1)$.
At (0,0) the tangent plane is $z = 0$.
At $(1, 1)$ and $(-1, -1)$ the tangent plane is $z = 1/e$.
At $(1, -1)$ and $(-1, 1)$ the tangent plane is $z = -1/e$.

25. If $z = xe^y$, then $\dfrac{\partial z}{\partial x} = e^y$ and $\dfrac{\partial z}{\partial y} = xe^y$.
Thus $x\dfrac{\partial z}{\partial x} = xe^y = \dfrac{\partial z}{\partial y}$.

26. $z = \dfrac{x + y}{x - y}$,
$\dfrac{\partial z}{\partial x} = \dfrac{(x - y)(1) - (x + y)(1)}{(x - y)^2} = \dfrac{-2y}{(x - y)^2},$
$\dfrac{\partial z}{\partial y} = \dfrac{(x - y)(1) - (x + y)(-1)}{(x - y)^2} = \dfrac{2x}{(x - y)^2}.$
Therefore
$x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = -\dfrac{2xy}{(x - y)^2} + \dfrac{2xy}{(x - y)^2} = 0.$

27. If $z = \sqrt{x^2 + y^2}$, then $\dfrac{\partial z}{\partial x} = \dfrac{x}{\sqrt{x^2 + y^2}}$, and
$\dfrac{\partial z}{\partial y} = \dfrac{y}{\sqrt{x^2 + y^2}}$. Thus

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{x^2 + y^2}{\sqrt{x^2 + y^2}} = z.$$

28. $w = x^2 + yz, \quad \frac{\partial w}{\partial x} = 2x, \quad \frac{\partial w}{\partial y} == z, \quad \frac{\partial w}{\partial z} = y.$
Therefore

$$\begin{aligned} x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} + z\frac{\partial w}{\partial z} & \\ &= 2x^2 + yz + yz \\ &= 2(x^2 + yz) = 2w. \end{aligned}$$

29. If $w = \frac{1}{x^2 + y^2 + z^2}$, then $\frac{\partial w}{\partial x} = -\frac{2x}{(x^2 + y^2 + z^2)^2}$, $\frac{\partial w}{\partial y} = -\frac{2y}{(x^2 + y^2 + z^2)^2}$, and $\frac{\partial w}{\partial z} = -\frac{2z}{(x^2 + y^2 + z^2)^2}$.
Thus

$$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} + z\frac{\partial w}{\partial z} = -2\frac{x^2 + y^2 + z^2}{(x^2 + y^2 + z^2)^2} = -2w.$$

30. $z = f(x^2 + y^2)$,
$\frac{\partial z}{\partial x} = f'(x^2 + y^2)(2x), \qquad \frac{\partial z}{\partial y} = f'(x^2 + y^2)(2y).$
Thus $y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = 2xyf'(x^2 + y^2) - 2xyf'(x^2 + y^2) = 0.$

31. $z = f(x^2 - y^2)$,
$\frac{\partial z}{\partial x} = f'(x^2 - y^2)(2x), \qquad \frac{\partial z}{\partial y} = f'(x^2 - y^2)(-2y).$
Thus $y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = (2xy - 2xy)f'(x^2 - y^2) = 0.$

32.
$$\begin{aligned} f_1(x, y, z) &= \lim_{h\to 0}\frac{f(x+h, y, z) - f(x, y, z)}{h} \\ f_2(x, y, z) &= \lim_{k\to 0}\frac{f(x, y+k, z) - f(x, y, z)}{k} \\ f_3(x, y, z) &= \lim_{\ell\to 0}\frac{f(x, y, z+\ell) - f(x, y, z)}{\ell} \end{aligned}$$

33. At $\big(a, b, c, f(a, b, c)\big)$ the graph of $w = f(x, y, z)$ has tangent hyperplane

$$\begin{aligned} w = f(a, b, c) + f_1(a, b, c)(x - a) + f_2(a, b, c)(y - b) \\ + f_3(a, b, c)(z - c). \end{aligned}$$

34. If $Q = (X, Y, Z)$ is the point on the surface $z = x^2 + y^2$ that is closest to $P = (1, 1, 0)$, then

$$\overrightarrow{PQ} = (X - 1)\mathbf{i} + (Y - 1)\mathbf{j} + Z\mathbf{k}$$

must be normal to the surface at Q, and hence must be parallel to $\mathbf{n} = 2X\mathbf{i} + 2Y\mathbf{j} - \mathbf{k}$. Hence $\overrightarrow{PQ} = t\mathbf{n}$ for some real number t, so

$$X - 1 = 2tX, \qquad Y - 1 = 2tY, \qquad Z = -t.$$

Thus $X = Y = \frac{1}{1 - 2t}$, and, since $Z = X^2 + Y^2$, we must have

$$-t = \frac{2}{(1 - 2t)^2}.$$

Evidently this equation is satisfied by $t = -\frac{1}{2}$. Since the left and right sides of the equation have graphs similar to those in Figure 12.18(b) (in the text), the equation has only this one real solution. Hence $X = Y = \frac{1}{2}$, and so $Z = \frac{1}{2}$.
The distance from $(1, 1, 0)$ to $z = x^2$ is the distance from $(1, 1, 0)$ to $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$, which is $\sqrt{3}/2$ units.

35. If $Q = (X, Y, Z)$ is the point on the surface $z = x^2 + 2y^2$ that is closest to $P = (0, 0, 1)$, then

$$\overrightarrow{PQ} = X\mathbf{i} + Y\mathbf{j} + (Z - 1)\mathbf{k}$$

must be normal to the surface at Q, and hence must be parallel to $\mathbf{n} = 2X\mathbf{i} + 4Y\mathbf{j} - \mathbf{k}$. Hence $\overrightarrow{PQ} = t\mathbf{n}$ for some real number t, so

$$X = 2tX, \qquad Y = 4tY, \qquad Z - 1 = -t.$$

If $X \neq 0$, then $t = 1/2$, so $Y = 0$, $Z = 1/2$, and $X = \sqrt{Z} = 1/\sqrt{2}$. The distance from $(1/\sqrt{2}, 0, 1/2)$ to $(0, 0, 1)$ is $\sqrt{3}/2$ units.
If $Y \neq 0$, then $t = 1/4$, so $X = 0$, $Z = 3/4$, and $Y = \sqrt{Z/2} = \sqrt{3/8}$. The distance from $(0, \sqrt{3/8}, 3/4)$ to $(0, 0, 1)$ is $\sqrt{7}/4$ units.
If $X = Y = 0$, then $Z = 0$ (and $t = 1$). The distance from $(0, 0, 0)$ to $(0, 0, 1)$ is 1 unit.
Since

$$\frac{\sqrt{7}}{4} < \frac{\sqrt{3}}{2} < 1,$$

the closest point to $(0, 0, 1)$ on $z = x^2 + 2y^2$ is $(0, \sqrt{3/8}, 3/4)$, and the distance from $(0, 0, 1)$ to that surface is $\sqrt{7}/4$ units.

36. $f(x, y) = \frac{2xy}{x^2 + y^2}$ if $(x, y) \neq (0, 0), \qquad f(0, 0) = 0$

$$\begin{aligned} f_1(0, 0) &= \lim_{h\to 0}\frac{f(h, 0) - f(0, 0)}{h} = \lim_{h\to 0}\frac{0 - 0}{h} = 0 \\ f_2(0, 0) &= \lim_{k\to 0}\frac{f(0, k) - f(0, 0)}{h} = \lim_{k\to 0}\frac{0 - 0}{k} = 0 \end{aligned}$$

Thus $f_1(0, 0)$ and $f_2(0, 0)$ both exist even though f is not continuous at $(0, 0)$ (as shown in Example 2 of Section 3.2).

37. $f(x, y) = \begin{cases} (x^3 + y)\sin \dfrac{1}{x^2+y^2} & \text{if } (x, y) \neq (0,0) \\ 0 & \text{if } (x, y) = (0,0) \end{cases}$

$$\begin{aligned} f_1(0,0) &= \lim_{h\to 0} \frac{1}{h}\left(h^3 \sin \frac{1}{h^2}\right) \\ &= \lim_{h\to 0} h^2 \sin \frac{1}{h^2} = 0 \\ f_2(0,0) &= \lim_{k\to 0} \frac{1}{k}\left(k \sin \frac{1}{k^2}\right) \\ &= \lim_{k\to 0} \sin \frac{1}{k^2} \quad \text{does not exist.} \end{aligned}$$

38. If $(x, y) \neq (0, 0)$, then

$$f_1(x, y) = 3x^2 \sin \frac{1}{x^2+y^2} - \frac{(x^3+y)2x}{(x^2+y^2)^2} \cos \frac{1}{x^2+y^2}.$$

The first term on the right $\to 0$ as $(x, y) \to (0, 0)$, but the second term has no limit at $(0, 0)$. (It is 0 along $x = 0$, but along $x = y$ it is

$$-\frac{2x^4+2x^2}{4x^4} \cos \frac{1}{2x^2} = -\frac{1}{2}\left(1 + \frac{1}{x^2}\right) \cos \frac{1}{2x^2},$$

which has no limit as $x \to 0$.) Thus $f_1(x, y)$ has no limit at $(0, 0)$ and is not continuous there.

39. $f(x, y) = \begin{cases} \dfrac{x^3 - y^3}{x^2+y^2} & \text{if } (x, y) \neq (0,0) \\ 0 & \text{if } (x, y) = (0,0). \end{cases}$

If $(x, y) \neq (0, 0)$, then

$$\begin{aligned} f_1(x, y) &= \frac{(x^2+y^2)3x^2 - (x^3-y^3)2x}{(x^2+y^2)^2} \\ &= \frac{x^4 + 3x^2y^2 + 2xy^3}{(x^2+y^2)^2} \\ f_2(x, y) &= \frac{(x^2+y^2)(-3y^2) - (x^3-y^3)2y}{(x^2+y^2)^2} \\ &= -\frac{y^4 + 3x^2y^2 + 2x^3y}{(x^2+y^2)^2}. \end{aligned}$$

Also, at $(0, 0)$,

$$f_1(0,0) = \lim_{h\to 0} \frac{h^3}{h \cdot h^2} = 1, \quad f_2(0,0) = \lim_{k\to 0} \frac{-k^3}{k \cdot k^2} = -1.$$

Neither f_1 nor f_2 has a limit at $(0, 0)$ (the limits along $x = 0$ and $y = 0$ are different in each case), so neither function is continuous at $(0, 0)$. However, f is continuous at $(0, 0)$ because

$$|f(x, y)| \leq \left|\frac{x^3}{x^2+y^2}\right| + \left|\frac{y^3}{x^2+y^2}\right| \leq |x| + |y|,$$

which $\to 0$ as $(x, y) \to (0, 0)$.

40. $f(x, y, z) = \begin{cases} \dfrac{xy^2z}{x^4+y^4+z^4} & \text{if } (x, y, z) \neq (0,0,0) \\ 0 & \text{if } (x, y, z) = (0,0,0). \end{cases}$

By symmetry we have

$$f_3(0,0,0) = f_1(0,0,0) = \lim_{h\to 0} \frac{0}{h^5} = 0.$$

Also,

$$f_2(0,0,0) = \lim_{k\to 0} \frac{0}{k^5} = 0.$$

f is not continuous at $(0, 0, 0)$; it has different limits as $(x, y, z) \to (0, 0, 0)$ along $x = 0$ and along $x = y = z$. None of f_1, f_2, and f_3 is continuous at $(0, 0, 0)$ either. For example,

$$f_1(x, y, z) = \frac{(y^4 + z^4 - 3x^4)y^2z}{(x^4+y^4+z^4)^2},$$

which has no limit as $(x, y, z) \to (0, 0, 0)$ along the line $x = y = z$.

Section 12.4 Higher-Order Derivatives (page 662)

1. $z = x^2(1 + y^2)$

$$\frac{\partial z}{\partial x} = 2x(1+y^2), \quad \frac{\partial z}{\partial y} = 2x^2y,$$
$$\frac{\partial^2 z}{\partial x^2} = 2(1+y^2), \quad \frac{\partial^2 z}{\partial y^2} = 2x^2,$$
$$\frac{\partial^2 z}{\partial y \partial x} = 4xy = \frac{\partial^2 z}{\partial x \partial y}.$$

2. $f(x, y) = x^2 + y^2$, $\quad f_1(x, y) = 2x$, $\quad f_2(x, y) = 2y$, $f_{11}(x, y) = f_{22}(x, y) = 2$, $\quad f_{12}(x, y) = f_{21}(x, y) = 0$.

3. $w = x^3y^3z^3$,

$$\frac{\partial w}{\partial x} = 3x^2y^3z^3, \quad \frac{\partial w}{\partial y} = 3x^3y^2z^3, \quad \frac{\partial w}{\partial z} = 3x^3y^3z^2,$$
$$\frac{\partial^2 w}{\partial x^2} = 6xy^3z^3, \quad \frac{\partial^2 w}{\partial y^2} = 6x^3yz^3, \quad \frac{\partial^2 w}{\partial z^2} = 6x^3y^3z,$$
$$\frac{\partial^2 w}{\partial x \partial y} = 9x^2y^2z^3 = \frac{\partial^2 w}{\partial y \partial x},$$
$$\frac{\partial^2 w}{\partial x \partial z} = 9x^2y^3z^2 = \frac{\partial^2 w}{\partial z \partial x},$$
$$\frac{\partial^2 w}{\partial y \partial z} = 9x^3y^2z^2 = \frac{\partial^2 w}{\partial z \partial y}.$$

4. $z = \sqrt{3x^2 + y^2}$,

$$\frac{\partial z}{\partial x} = \frac{3x}{\sqrt{3x^2+y^2}}, \quad \frac{\partial z}{\partial y} = \frac{y}{\sqrt{3x^2+y^2}},$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\sqrt{3x^2+y^2}(3) - 3x\dfrac{3x}{\sqrt{3x^2+y^2}}}{3x^2+y^2} = \frac{3y^2}{(3x^2+y^2)^{3/2}},$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\sqrt{3x^2+y^2} - y\dfrac{y}{\sqrt{3x^2+y^2}}}{3x^2+y^2} = \frac{3x^2}{(3x^2+y^2)^{3/2}},$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} = -\frac{3xy}{(3x^2+y^2)^{3/2}}.$$

5. $z = xe^y - ye^x$,

$$\frac{\partial z}{\partial x} = e^y - ye^x, \quad \frac{\partial z}{\partial y} = xe^y - e^x,$$

$$\frac{\partial^2 z}{\partial x^2} = -ye^x, \quad \frac{\partial^2 z}{\partial y^2} = xe^y,$$

$$\frac{\partial^2 z}{\partial y \partial x} = e^y - e^x = \frac{\partial^2 z}{\partial x \partial y}.$$

6. $f(x, y) = \ln(1 + \sin(xy))$

$$f_1(x, y) = \frac{y\cos(xy)}{1+\sin(xy)}, \quad f_2(x, y) = \frac{x\cos(xy)}{1+\sin(xy)}$$

$f_{11}(x, y)$

$$= \frac{(1+\sin(xy))(-y^2\sin(xy)) - (y\cos(xy))(y\cos(xy))}{(1+\sin(xy))^2}$$

$$= -\frac{y^2}{1+\sin(xy)}$$

$$f_{22}(x, y) = -\frac{x^2}{1+\sin(xy)} \quad \text{(by symmetry)}$$

$f_{12}(x, y) =$

$$\frac{(1+\sin(xy))(\cos(xy) - xy\sin(xy)) - (y\cos(xy))(x\cos(xy))}{(1+\sin(xy))^2}$$

$$= \frac{\cos(xy) - xy}{1+\sin(xy)} = f_{21}(x, y).$$

7. A function $f(x, y, z)$ of three variables can have $3^3 = 27$ partial derivatives of order 3. Of these, ten can have different values, namely f_{111}, f_{222}, f_{333}, f_{112}, f_{122}, f_{223}, f_{233}, f_{113}, f_{133}, and f_{123}.
For $f(x, y, z) = xe^{xy}\cos(xz)$, we have

$$f_{133} = f_{313} = f_{331} = \frac{\partial}{\partial x}\Big(-x^3e^{xy}\cos(xz)\Big)$$
$$= -(3x^2 + x^3y)e^{xy}\cos(xz) + x^3ze^{xy}\sin(xz).$$

8. $f(x, y) = A(x^2 - y^2) + Bxy, \quad f_1 = 2Ax + By,$

$f_2 = -2Ay + Bx,$

$f_{11} = 2A, \quad f_{22} = -2A,$

Thus $f_{11} + f_{22} = 0$, and f is harmonic.

9. $f(x, y) = 3x^2y - y^3$,

$f_1(x, y) = 6xy, \quad f_{11}(x, y) = 6y,$

$f_2(x, y) = 3x^2 - 3y^2, \quad f_{22}(x, y) = -6y.$

Thus $f_{11} + f_{22} = 0$ and f is harmonic.

Also $g(x, y) = x^3 - 3xy^2$ is harmonic.

10. $f(x, y) = \dfrac{x}{x^2+y^2}$

$$f_1(x, y) = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$f_2(x, y) = -\frac{2xy}{(x^2+y^2)^2}$$

$$f_{11}(x, y) = \frac{(x^2+y^2)^2(-2x) - (y^2-x^2)2(x^2+y^2)(2x)}{(x^2+y^2)^4}$$

$$= \frac{2x^3 - 6xy^2}{(x^2+y^2)^3}$$

$$f_{22}(x, y) = -\frac{(x^2+y^2)^2(2x) - 2xy2(x^2+y^2)(2y)}{(x^2+y^2)^4}$$

$$= \frac{-2x^3 + 6xy^2}{(x^2+y^2)^3}.$$

Evidently $f_{11}(x, y) + f_{22}(x, y) = 0$ for $(x, y) \neq (0, 0)$. Hence f is harmonic except at the origin.

11. $f(x, y) = \ln(x^2 + y^2), \quad f_1 = \dfrac{2x}{x^2+y^2}, \quad f_2 = \dfrac{2y}{x^2+y^2}$

$$f_{11} = \frac{(x^2+y^2)(2) - 2x(2x)}{(x^2+y^2)^2} = \frac{2(y^2-x^2)}{(x^2+y^2)^2}$$

$$f_{22} = \frac{2(x^2-y^2)}{(x^2+y^2)^2} \quad \text{(by symmetry)}$$

Thus $f_{11} + f_{22} = 0$ (everywhere except at the origin), and f is harmonic.

12. $f(x, y) = \tan^{-1}\left(\dfrac{y}{x}\right), \quad (x \neq 0).$

$$f_1(x, y) = \frac{1}{1+\dfrac{y^2}{x^2}}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2},$$

$$f_2(x, y) = \frac{1}{1+\dfrac{y^2}{x^2}}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2},$$

$$f_{11} = \frac{2xy}{(x^2+y^2)^2}, \qquad f_{22} = -\frac{2xy}{(x^2+y^2)^2}.$$

Thus $f_{11} + f_{22} = 0$ and f is harmonic.

13. $w = e^{3x+4y}\sin(5z)$,

$w_1 = 3w, \quad w_2 = 4w, \quad w_{11} = 9w, \quad w_{22} = 16w,$

$w_3 = 5e^{3x+4y}\cos(5z), \quad w_{33} = -25w.$

Thus $w_{11} + w_{22} + w_{33} = (9 + 16 - 25)w = 0$, and w is harmonic in 3-space.

14. Let $g(x, y, z) = zf(x, y)$. Then

$$g_1(x, y, z) = zf_1(x, y), \qquad g_{11}(x, y, z) = zf_{11}(x, y)$$
$$g_2(x, y, z) = zf_2(x, y), \qquad g_{22}(x, y, z) = zf_{22}(x, y)$$
$$g_3(x, y, z) = f(x, y), \qquad g_{33}(x, y, z) = 0.$$

Thus $g_{11} + g_{22} + g_{33} = z(f_{11} + f_{22}) = 0$ and g is harmonic because f is harmonic. This proves (a). The proofs of (b) and (c) are similar.

If $h(x, y, z) = f(ax + by, cz)$, then $h_{11} = a^2 f_{11}$, $h_{22} = b^2 f_{11}$ and $h_{33} = c^2 f_{22}$. If $a^2 + b^2 = c^2$ and f is harmonic then

$$h_{11} + h_{22} + h_{33} = c^2(f_{11} + f_{22}) = 0,$$

so h is harmonic.

15. Since $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$, $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$, and the second partials of u are continuous, we have

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\frac{\partial v}{\partial x} = -\frac{\partial^2 u}{\partial y^2}.$$

Thus $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$, and u is harmonic. The proof that v is harmonic is similar.

16. Let

$$f(x, y) = \begin{cases} \dfrac{2xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

For $(x, y) \neq (0, 0)$, we have

$$\begin{aligned} f_1(x, y) &= \frac{(x^2 + y^2)2y - 2xy(2x)}{(x^2 + y^2)^2} = \frac{2y(y^2 - x^2)}{(x^2 + y^2)^2} \\ f_2(x, y) &= \frac{2x(x^2 - y^2)}{(x^2 + y^2)^2} \qquad \text{(by symmetry)}. \end{aligned}$$

Let $F(x, y) = (x^2 - y^2) f(x, y)$. Then we calculate

$$\begin{aligned} F_1(x, y) &= 2xf(x, y) + (x^2 - y^2) f_1(x, y) \\ &= 2xf(x, y) - \frac{2y(y^2 - x^2)^2}{(x^2 + y^2)^2} \\ F_2(x, y) &= -2yf(x, y) + (x^2 - y^2) f_2(x, y) \\ &= -2yf(x, y) + \frac{2x(x^2 - y^2)^2}{(x^2 + y^2)^2} \\ F_{12}(x, y) &= \frac{2(x^6 + 9x^4y^2 - 9x^2y^4 - y^6)}{(x^2 + y^2)^3} = F_{21}(x, y). \end{aligned}$$

For the values at $(0, 0)$ we revert to the definition of derivative to calculate the partials:

$$\begin{aligned} F_1(0, 0) &= \lim_{h\to 0} \frac{F(h, 0) - F(0, 0)}{h} = 0 = F_2(0, 0) \\ F_{12}(0, 0) &= \lim_{k\to 0} \frac{F_1(0, k) - F_1(0, 0)}{k} = \lim_{k\to 0} \frac{-2k(k^4)}{k(k^4)} = -2 \\ F_{21}(0, 0) &= \lim_{h\to 0} \frac{F_2(h, 0) - F_2(0, 0)}{h} = \lim_{h\to 0} \frac{2h(h^4)}{h(h^4)} = 2 \end{aligned}$$

This does not contradict Theorem 1 since the partials F_{12} and F_{21} are not continuous at $(0, 0)$. (Observe, for instance, that $F_{12}(x, x) = 0$, while $F_{12}(x, 0) = 2$ for $x \neq 0$.)

17. $u(x, t) = t^{-1/2} e^{-x^2/4t}$

$$\begin{aligned} \frac{\partial u}{\partial t} &= \left(-\frac{1}{2}t^{-3/2} + \frac{1}{4}t^{-5/2}x^2\right) e^{-x^2/4t} \\ \frac{\partial u}{\partial x} &= -\frac{1}{2}xt^{-3/2}e^{-x^2/4t} \\ \frac{\partial^2 u}{\partial x^2} &= \left(-\frac{1}{2}t^{-3/2} + \frac{1}{4}t^{-5/2}x^2\right) e^{-x^2/4t} \\ &= \frac{\partial u}{\partial t}. \end{aligned}$$

18. $u(x, y, t) = t^{-1} e^{-(x^2+y^2)/4t}$

$$\begin{aligned} \frac{\partial u}{\partial t} &= -\frac{1}{t^2}e^{-(x^2+y^2)/4t} + \frac{x^2 + y^2}{4t^3}e^{-(x^2+y^2)/4t} \\ \frac{\partial u}{\partial x} &= -\frac{x}{2t^2}e^{-(x^2+y^2)/4t} \\ \frac{\partial^2 u}{\partial x^2} &= -\frac{1}{2t^2}e^{-(x^2+y^2)/4t} + \frac{x^2}{4t^3}e^{-(x^2+y^2)/4t} \\ \frac{\partial^2 u}{\partial y^2} &= -\frac{1}{2t^2}e^{-(x^2+y^2)/4t} + \frac{y^2}{4t^3}e^{-(x^2+y^2)/4t} \end{aligned}$$

Thus $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}$.

19. For $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}$ the solution is

$$u(x, y, z, t) = t^{-3/2} e^{-(x^2+y^2+z^2)/4t},$$

which is verified similarly to the previous Exercise.

20. $u(x, y)$ is biharmonic $\Leftrightarrow \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}$ is harmonic

$$\begin{aligned} &\Leftrightarrow \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0 \\ &\Leftrightarrow \frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} = 0 \end{aligned}$$

by the equality of mixed partials.

21. If $u(x, y) = x^4 - 3x^2y^2$, then

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}(4x^3 - 6xy^2) = 12x^2 - 6y^2\\
\frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial y}(-6x^2y) = -6x^2\\
\frac{\partial^4 u}{\partial x^4} &= \frac{\partial}{\partial x}(24x) = 24\\
\frac{\partial^4 u}{\partial x^2 \partial y^2} &= \frac{\partial}{\partial x}(-12x) = -12\\
\frac{\partial^4 u}{\partial y^4} &= 0\\
\frac{\partial^4 u}{\partial x^4} &+ 2\frac{\partial^4 u}{\partial x^2\partial y^2} + \frac{\partial^4 u}{\partial y^4} = 24 - 24 = 0.
\end{aligned}$$

Thus u is biharmonic.

22. If u is harmonic, then $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$. If $v(x, y) = xu(x, y)$, then

$$\begin{aligned}
\frac{\partial^2 v}{\partial x^2} &= \frac{\partial}{\partial x}\left(u + x\frac{\partial u}{\partial x}\right) = 2\frac{\partial u}{\partial x} + x\frac{\partial^2 u}{\partial x^2}\\
\frac{\partial^2 v}{\partial y^2} &= \frac{\partial}{\partial y}\left(x\frac{\partial u}{\partial y}\right) = x\frac{\partial^2 u}{\partial y^2}\\
\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} &= 2\frac{\partial u}{\partial x} + x\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 2\frac{\partial u}{\partial x}.
\end{aligned}$$

Since u is harmonic, so is $\partial u/dx$:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \frac{\partial}{\partial x}(0) = 0.$$

Thus $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}$ is harmonic, and so v is biharmonic. The proof that $w(x, y) = yu(x, y)$ is biharmonic is similar.

23. By Example 3, $e^x \sin y$ is harmonic. Therefore $xe^x \sin y$ is biharmonic by Exercise 22.

24. By Exercise 11, $\ln(x^2 + y^2)$ is harmonic (except at the origin). Therefore $y \ln(x^2 + y^2)$ is biharmonic by Exercise 22.

25. By Exercise 10, $\frac{x}{x^2 + y^2}$ is harmonic (except at the origin). Therefore $\frac{xy}{x^2 + y^2}$ is biharmonic by Exercise 22.

26. $u(x, y, z)$ is biharmonic $\Leftrightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$ is harmonic

$$\begin{aligned}
&\Leftrightarrow \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = 0\\
&\Leftrightarrow \frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4} + \frac{\partial^4 u}{\partial z^4} + 2\left(\frac{\partial^4 u}{\partial x^2\partial y^2} + \frac{\partial^4 u}{\partial x^2\partial z^2} + \frac{\partial^4 u}{\partial y^2\partial z^2}\right) = 0
\end{aligned}$$

by the equality of mixed partials.

If $u(x, y, z)$ is harmonic then the functions $xu(x, y, z)$, $yu(x, y, z)$, and $zu(x, y, z)$ are all biharmonic. The proof is almost identical to that given in Exercise 22.

27.
```
> f := x*y/(x^2+y^2);
```
$$f := \frac{xy}{x^2 + y^2}$$
```
> simplify(diff(f,x$4)  +
2*diff(f,x$2,y$2) + diff(f,y$4));
```
$$0$$

Section 12.5 The Chain Rule (page 671)

1. If $w = f(x, y, z)$ where $x = g(s, t)$, $y = h(s, t)$, and $z = k(s, t)$, then

$$\begin{aligned}
\frac{\partial w}{\partial t} &= f_1(x, y, z)g_2(s, t) + f_2(x, y, z)h_2(s, t)\\
&\quad + f_3(x, y, z)k_2(s, t).
\end{aligned}$$

2. If $w = f(x, y, z)$ where $x = g(s)$, $y = h(s, t)$ and $z = k(t)$, then

$$\frac{\partial w}{\partial t} = f_2(x, y, z)h_2(s, t) + f_3(x, y, z)k'(t).$$

3. If $z = g(x, y)$ where $y = f(x)$ and $x = h(u, v)$, then

$$\frac{\partial z}{\partial u} = g_1(x, y)h_1(u, v) + g_2(x, y)f'(x)h_1(u, v).$$

4. If $w = f(x, y)$ where $x = g(r, s)$, $y = h(r, t)$, $r = k(s, t)$ and $s = m(t)$, then

$$\begin{aligned}
\frac{dw}{dt} &= f_1(x, y)\Big[g_1(r, s)\Big(k_1(s, t)m'(t)\\
&\quad + k_2(s, t)\Big) + g_2(r, s)m'(t)\Big]\\
&\quad + f_2(x, y)\Big[h_1(r, t)\Big(k_1(s, t)m'(t)\\
&\quad + k_2(s, t)\Big) + h_2(r, t)\Big].
\end{aligned}$$

5. If $w = f(x, y, z)$ where $x = g(y, z)$ and $y = h(z)$, then

$$\begin{aligned}
\frac{dw}{dz} &= f_1(x, y, z)\Big[g_1(y, z)h'(z) + g_2(y, z)\Big]\\
&\quad + f_2(x, y, z)h'(z) + f_3(x, y, z)\\
\left.\frac{\partial w}{\partial z}\right|_x &= f_2(x, y, z)h'(z) + f_3(x, y, z)\\
\left.\frac{\partial w}{\partial z}\right|_{x,y} &= f_3(x, y, z).
\end{aligned}$$

6. If $u = \sqrt{x^2+y^2}$, where $x = e^{st}$ and $y = 1 + s^2 \cos t$, then

Method I.

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{x}{\sqrt{x^2+y^2}} se^{st} + \frac{y}{\sqrt{x^2+y^2}}(-s^2 \sin t)\\ &= \frac{xse^{st} - ys^2 \sin t}{\sqrt{x^2+y^2}}.\end{aligned}$$

Method II.

$$\begin{aligned}u &= \sqrt{e^{2st} + (1+s^2\cos t)^2}\\ \frac{\partial u}{\partial t} &= \frac{2se^{2st} - 2s^2 \sin t(1+s^2\cos t)}{2\sqrt{e^{2st}+(1+s^2\cos t)^2}}\\ &= \frac{x^2 s - ys^2 \sin t}{\sqrt{x^2+y^2}}.\end{aligned}$$

7. If $z = \tan^{-1}\dfrac{u}{v}$, where $u = 2x + y$ and $v = 3x - y$, then

Method I.

$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}\\ &= \frac{1}{1+\frac{u^2}{v^2}}\left(\frac{1}{v}\right)(2) + \frac{1}{1+\frac{u^2}{v^2}}\left(\frac{-u}{v^2}\right)(3)\\ &= \frac{2v-3u}{u^2+v^2} = -\frac{5y}{13x^2-2xy+2y^2}.\end{aligned}$$

Method II.

$$\begin{aligned}z &= \tan^{-1}\frac{2x+y}{3x-y}\\ \frac{\partial z}{\partial x} &= \frac{1}{1+\frac{(2x+y)^2}{(3x-y)^2}} \frac{(3x-y)(2)-(2x+y)(3)}{(3x-y)^2}\\ &= \frac{-5y}{(3x-y)^2+(2x+y)^2} = \frac{-5y}{13x^2-2xy+2y^2}.\end{aligned}$$

8. If $z = txy^2$, where $x = t + \ln(y + t^2)$ and $y = e^t$, then

Method I.

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial t} + \frac{\partial z}{\partial x}\left(\frac{\partial x}{\partial t} + \frac{\partial x}{\partial y}\frac{\partial y}{\partial t}\right)\\ &\quad + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}\\ &= xy^2 + ty^2\left(1+\frac{y+2t}{y+t^2}\right) + 2txy^2.\end{aligned}$$

Method II.

$$\begin{aligned}z &= t\Big(t+\ln(e^t+t^2)\Big)e^{2t}\\ \frac{\partial z}{\partial t} &= \Big(t+\ln(e^t+t^2)\Big)e^{2t} + te^{2t}\left(1+\frac{e^t+2t}{e^t+t^2}\right)\\ &\quad + 2te^{2t}\Big(t+\ln(e^t+t^2)\Big)\\ &= xy^2 + ty^2\left(1+\frac{y+2t}{y+t^2}\right) + 2txy^2.\end{aligned}$$

9. $\dfrac{\partial}{\partial x} f(2x, 3y) = 2f_1(2x, 3y)$.

10. $\dfrac{\partial}{\partial x} f(2y, 3x) = 3f_2(2y, 3x)$.

11. $\dfrac{\partial}{\partial x} f(y^2, x^2) = 2xf_2(y^2, x^2)$.

12.
$$\begin{aligned}&\frac{\partial}{\partial y} f\Big(yf(x,t),\, f(y,t)\Big)\\ &= f(x,t)f_1\Big(yf(x,t),\, f(y,t)\Big)\\ &\quad + f_1(y,t)f_2\Big(yf(x,t),\, f(y,t)\Big).\end{aligned}$$

13. $T = e^{-t}z$, where $z = f(t)$.

$$\frac{dT}{dt} = \frac{\partial T}{\partial t} + \frac{\partial T}{\partial z}\frac{dz}{dt} = -e^{-t}f(t) + e^{-t}f'(t).$$

If $f(t) = e^t$, then $f'(t) = e^t$ and $\dfrac{dT}{dt} = 0$. The temperature is rising with respect to depth at the same rate at which it is falling with respect to time.

14. If $E = f(x, y, z, t)$, where $x = \sin t$, $y = \cos t$ and $z = t$, then the rate of change of E is

$$\frac{dE}{dt} = \frac{\partial E}{\partial x}\cos t - \frac{\partial E}{\partial y}\sin t + \frac{\partial E}{\partial z} + \frac{\partial E}{\partial t}.$$

15. $z = f(x, y)$, where $x = 2s + 3t$ and $y = 3s - 2t$.

a) $$\begin{aligned}\frac{\partial^2 z}{\partial s^2} &= \frac{\partial}{\partial s}\Big(2f_1(x,y) + 3f_2(x,y)\Big)\\ &= 2(2f_{11} + 3f_{12}) + 3(2f_{21} + 3f_{22})\\ &= 4f_{11} + 12f_{12} + 9f_{22}\end{aligned}$$

b) $$\begin{aligned}\frac{\partial^2 z}{\partial s \partial t} = \frac{\partial^2 z}{\partial t \partial s} &= \frac{\partial}{\partial t}(2f_1 + 3f_2)\\ &= 2(3f_{11} - 2f_{12}) + 3(3f_{21} - 2f_{22})\\ &= 6f_{11} + 5f_{12} - 6f_{22}\end{aligned}$$

c) $$\begin{aligned}\frac{\partial^2 z}{\partial t^2} &= \frac{\partial}{\partial t}(3f_1 - 2f_2)\\ &= 3(3f_{11} - 2f_{12}) - 2(3f_{21} - 2f_{22})\\ &= 9f_{11} - 12f_{12} + 4f_{22}\end{aligned}$$

16. Let $u = \dfrac{x}{x^2+y^2}$, $v = -\dfrac{y}{x^2+y^2}$. Then

$$\frac{\partial u}{\partial x} = \frac{y^2-x^2}{(x^2+y^2)^2} \qquad \frac{\partial v}{\partial x} = \frac{2xy}{(x^2+y^2)^2}$$
$$\frac{\partial u}{\partial y} = -\frac{2xy}{(x^2+y^2)^2} \qquad \frac{\partial v}{\partial y} = \frac{y^2-x^2}{(x^2+y^2)^2}.$$

We have

$$\begin{aligned}
\frac{\partial}{\partial x}f(u,v) &= f_1(u,v)\frac{\partial u}{\partial x} + f_2(u,v)\frac{\partial v}{\partial x}\\
\frac{\partial}{\partial y}f(u,v) &= f_1(u,v)\frac{\partial u}{\partial y} + f_2(u,v)\frac{\partial v}{\partial y}\\
\frac{\partial^2}{\partial x^2}f(u,v) &= f_{11}\left(\frac{\partial u}{\partial x}\right)^2 + f_{12}\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + f_1\frac{\partial^2 u}{\partial x^2}\\
&\quad + f_{21}\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + f_{22}\left(\frac{\partial v}{\partial x}\right)^2 + f_2\frac{\partial^2 v}{\partial x^2}\\
\frac{\partial^2}{\partial y^2}f(u,v) &= f_{11}\left(\frac{\partial u}{\partial y}\right)^2 + f_{12}\frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + f_1\frac{\partial^2 u}{\partial y^2}\\
&\quad + f_{21}\frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + f_{22}\left(\frac{\partial v}{\partial y}\right)^2 + f_2\frac{\partial^2 v}{\partial y^2}.
\end{aligned}$$

Noting that

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \frac{1}{(x^2+y^2)^2} = \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2$$
$$\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} = 0,$$

we have

$$\begin{aligned}
&\frac{\partial^2}{\partial x^2}f(u,v) + \frac{\partial^2}{\partial y^2}f(u,v)\\
&\qquad = f_{11}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right]\\
&\qquad\quad + f_{22}\left[\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right]\\
&\qquad\quad + 2f_{12}\left[\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right]\\
&\qquad\quad + f_1\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] + f_2\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right]\\
&\qquad = f_1\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] + f_2\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right],
\end{aligned}$$

because we are given that f is harmonic, that is, $f_{11}(u,v) + f_{22}(u,v) = 0$.

Finally, u is harmonic by Exercise 10 of Section 3.4, and, by symmetry, so is v. Thus

$$\frac{\partial^2}{\partial x^2}f(u,v) + \frac{\partial^2}{\partial y^2}f(u,v) = 0$$

and $f\left(\dfrac{x}{x^2+y^2}, -\dfrac{y}{x^2+y^2}\right)$ is harmonic for $(x,y) \ne (0,0)$.

17. If $x = t\sin s$ and $y = t\cos s$, then

$$\begin{aligned}
\frac{\partial^2}{\partial s\partial t}f(x,y) &= \frac{\partial}{\partial s}\Big(\sin s f_1(x,y) + \cos s f_2(x,y)\Big)\\
&= \cos s f_1 + t\sin s\cos s f_{11} - t\sin^2 s f_{12}\\
&\qquad - \sin s f_2 + t\cos^2 s f_{12} - t\sin s\cos s f_{22}\\
&= \cos s f_1 - \sin s f_2 + t\cos s\sin s(f_{11} - f_{22})\\
&\qquad + t(\cos^2 s - \sin^2 s)f_{12},
\end{aligned}$$

where all partials of f are evaluated at $(t\sin s, t\cos s)$.

18.
$$\begin{aligned}
\frac{\partial^3}{\partial x\partial y^2}f(2x+3y,xy) &= \frac{\partial^2}{\partial x\partial y}(3f_1 + xf_2)\\
&= \frac{\partial}{\partial x}(9f_{11} + 3xf_{12} + 3xf_{21} + x^2 f_{22})\\
&= \frac{\partial}{\partial x}(9f_{11} + 6xf_{12} + x^2 f_{22})\\
&= 18f_{111} + 9yf_{112} + 6f_{12} + 12xf_{121} + 6xyf_{122}\\
&\qquad + 2xf_{22} + 2x^2 f_{221} + x^2 yf_{222}\\
&= 18f_{111} + (12x+9y)f_{112} + (6xy+2x^2)f_{122} + x^2 yf_{222}\\
&\qquad + 6f_{12} + 2xf_{22},
\end{aligned}$$

where all partials are evaluated at $(2x+3y, xy)$.

19.
$$\begin{aligned}
\frac{\partial^2}{\partial y\partial x}f(y^2,xy,-x^2) &= \frac{\partial}{\partial y}(yf_2 - 2xf_3)\\
&= f_2 + 2y^2 f_{21} + xyf_{22} - 4xyf_{31} - 2x^2 f_{32},
\end{aligned}$$

where all partials are evaluated at $(y^2, xy, -x^2)$.

20.
$$\begin{aligned}
\frac{\partial^3}{\partial t^2\partial s}f(s^2-t, s+t^2) &= \frac{\partial^2}{\partial t^2}(2sf_1 + f_2)\\
&= \frac{\partial}{\partial t}(-2sf_{11} + 4stf_{12} - f_{21} + 2tf_{22})\\
&= \frac{\partial}{\partial t}(-2sf_{11} + (4st-1)f_{12} + 2tf_{22})\\
&= 2sf_{111} - 4stf_{112} + 4sf_{12} - (4st-1)f_{121}\\
&\qquad + 2t(4st-1)f_{122} + 2f_{22} - 2tf_{221} + 4t^2 f_{222}\\
&= 2sf_{111} + (1-8st)f_{112} + 4t(2st-1)f_{122} + 4t^2 f_{222}\\
&\qquad + 4sf_{12} + 2f_{22},
\end{aligned}$$

where all partials are evaluated at $(s^2-t, s+t^2)$.

21. Let $g(x, y) = f(u, v)$, where $u = u(x, y)$, $v = v(x, y)$. Then

$$\begin{aligned}
g_1(x, y) &= f_1(u, v)u_1(x, y) + f_2(u, v)v_1(x, y)\\
g_2(x, y) &= f_1(u, v)u_2(x, y) + f_2(u, v)v_2(x, y)\\
g_{11}(x, y) &= f_1(u, v)u_{11}(x, y) + f_{11}(u, v)(u_1(x, y))^2\\
&\quad + f_{12}(u, v)u_1(x, y)v_1(x, y) + f_2(u, v)v_{11}(x, y)\\
&\quad + f_{21}(u, v)u_1(x, y)v_1(x, y) + f_{22}(u, v)(v_1(x, y))^2\\
g_{22}(x, y) &= f_1(u, v)u_{22}(x, y) + f_{11}(u, v)(u_2(x, y))^2\\
&\quad + f_{12}(u, v)u_2(x, y)v_2(x, y) + f_2(u, v)v_{22}(x, y)\\
&\quad + f_{21}(u, v)u_2(x, y)v_2(x, y) + f_{22}(u, v)(v_2(x, y))^2\\
g_{11}(x, y) &+ g_{22}(x, y)\\
&= f_1(u, v)[u_{11}(x, y) + u_{22}(x, y)]\\
&\quad + f_2(u, v)[v_{11}(x, y) + v_{22}(x, y)]\\
&\quad + [(u_1(x, y))^2 + (u_2(x, y))^2]f_{11}(u, v)\\
&\quad + [(v_1(x, y))^2 + (v_2(x, y))^2]f_{22}(u, v)\\
&\quad + 2[u_1(x, y)v_1(x, y) + u_2(x, y)v_2(x, y)]f_{12}(u, v).
\end{aligned}$$

The first two terms on the right are zero because u and v are harmonic. The next two terms simplify to $[(v_1)^2 + (v_2)^2][f_{11} + f_{22}] = 0$ because u and v satisfy the Cauchy-Riemann equations and f is harmonic. The last term is zero because u and v satisfy the Cauchy-Riemann equations. Thus g is harmonic.

22. If $r^2 = x^2 + y^2 + z^2$, then $2r\dfrac{\partial r}{\partial x} = 2x$, so $\dfrac{\partial r}{\partial x} = \dfrac{x}{r}$. Similarly, $\dfrac{\partial r}{\partial y} = \dfrac{y}{r}$ and $\dfrac{\partial r}{\partial z} = \dfrac{z}{r}$. If $u = \dfrac{1}{r}$, then

$$\begin{aligned}
\frac{\partial u}{\partial x} &= -\frac{1}{r^2}\frac{\partial r}{\partial x} = -\frac{x}{r^3}\\
\frac{\partial^2 u}{\partial x^2} &= -\frac{1}{r^3} + \frac{3x}{r^4}\frac{x}{r} = \frac{3x^2 - r^2}{r^5}.
\end{aligned}$$

Similarly,

$$\frac{\partial^2 u}{\partial y^2} = \frac{3y^2 - r^2}{r^5}, \qquad \frac{\partial^2 u}{\partial z^2} = \frac{3z^2 - r^2}{r^5}.$$

Adding these three expressions, we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0,$$

so u is harmonic except at $r = 0$.

23. If $x = e^s \cos t$ and $y = e^s \sin t$, then

$$\begin{aligned}
\frac{\partial x}{\partial s} &= e^s \cos t & \frac{\partial y}{\partial s} &= e^s \sin t\\
\frac{\partial x}{\partial t} &= -e^x \sin t & \frac{\partial y}{\partial t} &= e^s \cos t.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
\frac{\partial z}{\partial s} &= e^s \cos t\frac{\partial z}{\partial x} + e^s \sin t\frac{\partial z}{\partial y}\\
\frac{\partial z}{\partial t} &= -e^s \sin t\frac{\partial z}{\partial x} + e^s \cos t\frac{\partial z}{\partial y}\\
\frac{\partial^2 z}{\partial s^2} &= e^s \cos t\frac{\partial z}{\partial x} + e^s \sin t\frac{\partial z}{\partial y}\\
&\quad + e^s \cos t\left(e^s \cos t\frac{\partial^2 z}{\partial x^2} + e^s \sin t\frac{\partial^2 z}{\partial y\partial x}\right)\\
&\quad + e^s \sin t\left(e^s \cos t\frac{\partial^2 z}{\partial x\partial y} + e^s \sin t\frac{\partial^2 z}{\partial y^2}\right)\\
\frac{\partial^2 z}{\partial t^2} &= -e^s \cos t\frac{\partial z}{\partial x} - e^s \sin t\frac{\partial z}{\partial y}\\
&\quad - e^s \sin t\left(-e^s \sin t\frac{\partial^2 z}{\partial x^2} + e^s \cos t\frac{\partial^2 z}{\partial y\partial x}\right)\\
&\quad + e^s \cos t\left(-e^s \sin t\frac{\partial^2 z}{\partial x\partial y} + e^s \cos t\frac{\partial^2 z}{\partial y^2}\right).
\end{aligned}$$

It follows that

$$\begin{aligned}
\frac{\partial^2 z}{\partial s^2} + \frac{\partial^2 z}{\partial t^2} &= e^{2s}(\cos^2 t + \sin^2 t)\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right)\\
&= (x^2 + y^2)\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right).
\end{aligned}$$

24. If $x = r\cos\theta$ and $y = r\sin\theta$, then $r^2 = x^2 + y^2$ and $\tan\theta = y/x$. Thus $2r\dfrac{\partial r}{\partial x} = 2x$, so $\dfrac{\partial r}{\partial x} = \dfrac{x}{r} = \cos\theta$, and similarly, $\dfrac{\partial r}{\partial y} = \dfrac{y}{r} = \sin\theta$. Also

$$\begin{aligned}
\sec^2\theta\frac{\partial\theta}{\partial x} &= -\frac{y}{x^2} & \sec^2\theta\frac{\partial\theta}{\partial y} &= \frac{1}{x}\\
\frac{\partial\theta}{\partial x} &= -\frac{y}{x^2 + y^2} & \frac{\partial\theta}{\partial x} &= \frac{x}{x^2 + y^2}\\
&= -\frac{\sin\theta}{r} & &= \frac{\cos\theta}{r}.
\end{aligned}$$

Now

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x} = \cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}$$
$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y} = \sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}$$
$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= \left(\frac{\partial}{\partial x}\cos\theta\right)\frac{\partial u}{\partial r} + \cos\theta\left(\cos\theta\frac{\partial^2 u}{\partial r^2} - \frac{\sin\theta}{r}\frac{\partial^2 u}{\partial\theta\partial r}\right)\\ &\quad - \left(\frac{\partial}{\partial x}\frac{\sin\theta}{r}\right)\frac{\partial u}{\partial \theta} - \frac{\sin\theta}{r}\left(\cos\theta\frac{\partial^2 u}{\partial r\partial\theta} - \frac{\sin\theta}{r}\frac{\partial^2 u}{\partial\theta^2}\right)\\ &= \frac{\sin^2\theta}{r}\frac{\partial u}{\partial r} + \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial\theta} + \cos^2\theta\frac{\partial^2 u}{\partial r^2}\\ &\quad - \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 u}{\partial\theta^2}\end{aligned}$$
$$\begin{aligned}\frac{\partial^2 u}{\partial y^2} &= \left(\frac{\partial}{\partial y}\sin\theta\right)\frac{\partial u}{\partial r} + \sin\theta\left(\sin\theta\frac{\partial^2 u}{\partial r^2} + \frac{\cos\theta}{r}\frac{\partial^2 u}{\partial\theta\partial r}\right)\\ &\quad + \left(\frac{\partial}{\partial y}\frac{\cos\theta}{r}\right)\frac{\partial u}{\partial \theta} + \frac{\cos\theta}{r}\left(\sin\theta\frac{\partial^2 u}{\partial r\partial\theta} + \frac{\cos\theta}{r}\frac{\partial^2 u}{\partial\theta^2}\right)\\ &= \frac{\cos^2\theta}{r}\frac{\partial u}{\partial r} - \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial\theta} + \sin^2\theta\frac{\partial^2 u}{\partial r^2}\\ &\quad + \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 u}{\partial\theta^2}.\end{aligned}$$

Therefore

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2},$$

as was to be shown.

25. If $u = r^2 \ln r$, where $r^2 = x^2 + y^2$, then, since $\partial r/\partial x = x/r$ and $\partial r/\partial y = y/r$, we have

$$\begin{aligned}\frac{\partial u}{\partial x} &= (2r\ln r + r)\frac{x}{r} = x(1 + 2\ln r)\\ \frac{\partial^2 u}{\partial x^2} &= 1 + 2\ln r + \frac{2x^2}{r^2}\\ \frac{\partial^2 u}{\partial y^2} &= 1 + 2\ln r + \frac{2y^2}{r^2} \quad \text{(similarly)}\\ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= 2 + 4\ln r + \frac{2(x^2+y^2)}{r^2} = 4 + 4\ln r.\end{aligned}$$

The constant 4 is harmonic, and so is $4\ln r$ by Exercise 11 of Section 3.4. Therefore $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}$ is harmonic, and so u is biharmonic.

26.
$$\begin{aligned}&f(tx, ty) = t^k f(x, y)\\ &xf_1(tx, ty) + yf_2(tx, ty) = kt^{k-1} f(x, y)\\ &x\Big(xf_{11}(tx, ty) + yf_{12}(tx, ty)\Big)\\ &\quad + y\Big(xf_{21}(tx, ty) + yf_{22}(tx, ty)\Big)\\ &\qquad = k(k-1)t^{k-2} f(x, y)\end{aligned}$$

Put $t = 1$ and get

$$x^2 f_{11}(x, y) + 2xyf_{12}(x, y) + y^2 f_{22}(x, y) = k(k-1) f(x, y).$$

27. If $f(x_1, \cdots, x_n)$ is positively homogeneous of degree k and has continuous partial derivatives of second order, then

$$\sum_{i,j=1}^{n} x_1 x_j f_{ij}(x_1, \cdots, x_n) = k(k-1) f(x_1, \cdots, x_n).$$

Proof: Differentiate $f(tx_1, \cdots, tx_n) = t^k f(x_1, \cdots, x_n)$ twice with respect to t:

$$\sum_{i=1}^{n} x_i f_i(tx_1, \cdots, tx_n) = kt^{k-1} f/xn$$
$$\sum_{i,j=1}^{n} x_i x_j f_{ij}(tx_1, \cdots, tx_n) = k(k-1)t^{k-2} f(x_1, \cdots, x_n),$$

and then put $t = 1$.

28. If $f(x_1, \cdots, x_n)$ is positively homogeneous of degree k and has continuous partial derivatives of mth order, then

$$\begin{aligned}&\sum_{i_1,\ldots,i_m=1}^{n} x_{i_1}\cdots x_{i_m} f_{i_1\ldots i_m}(x_1, \cdots, x_n)\\ &\qquad = k(k-1)\cdots(k-m+1) f(x_1, \cdots, x_n).\end{aligned}$$

The proof is identical to those of Exercises 26 or 27, except that you differentiate m times before putting $t = 1$.

29. $F(x, y) = \begin{cases} \dfrac{2xy(x^2 - y^2)}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0)\\ 0 & \text{if } (x, y) = (0, 0)\end{cases}$

a) For $(x, y) \neq (0, 0)$,

$$F(x, y) = \frac{2xy(x^2 - y^2)}{x^2 + y^2} = -\frac{2xy(y^2 - x^2)}{x^2 + y^2} = -F(y, x).$$

Since $0 = -0$, this holds for $(x, y) = (0, 0)$ also.

b) For $(x, y) \neq (0, 0)$,

$$\begin{aligned}F_1(x, y) &= \frac{\partial}{\partial x}F(x, y) = -\frac{\partial}{\partial x}F(y, x) = -F_2(y, x)\\ F_{12}(x, y) &= \frac{\partial}{\partial y}F_1(x, y) = -\frac{\partial}{\partial y}F_2(y, x) = -F_{21}(y, x).\end{aligned}$$

c) If $(x, y) \neq (0, 0)$, then $F_1(x, y) = \dfrac{2y(x^2 - y^2)}{x^2 + y^2} + 2xy\dfrac{\partial}{\partial x}\dfrac{x^2 - y^2}{x^2 + y^2}$. Thus $F_1(0, y) = -2y + 0 = -2y$ for $y \neq 0$. This result holds for $y = 0$ also, since $F_1(0, 0) = \lim_{h\to 0}(0 - 0)/h = 0.$.

d) By (b) and (c), $F_2(x,0) = -F_1(0,x) = 2x$, and $F_{21}(0,0) = 2$.

30. a) Since $F_{12}(x,y) = -F_{21}(y,x)$ for $(x,y) \neq (0,0)$, we have $F_{12}(x,x) = -F_{21}(x,x)$ for $x \neq 0$. However, all partial derivatives of the rational function F are continuous except possibly at the origin. Thus $F_{12}(x,x) = F_{21}(x,x)$ for $x \neq 0$. Therefore, $F_{12}(x,x) = 0$ for $x \neq 0$.

b) F_{12} cannot be continuous at $(0,0)$ because its value there (which is -2) differs from the value of $F_{21}(0,0)$ (which is 2). Alternatively, $F_{12}(0,0)$ is not the limit of $F_{12}(x,x)$ as $x \to 0$.

31. If $\xi = x + ct$, $\eta = x$, and $v(\xi,\eta) = v(x+ct,x) = u(x,t)$, then

$$\frac{\partial u}{\partial t} = \frac{\partial v}{\partial \xi}\frac{\partial \xi}{\partial t} = c\frac{\partial v}{\partial \xi}$$
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial v}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial v}{\partial \xi} + \frac{\partial v}{\partial \eta}.$$

If u satisfies $\dfrac{\partial u}{\partial t} = c\dfrac{\partial u}{\partial x}$, then v satisfies

$$c\frac{\partial v}{\partial \xi} = c\frac{\partial v}{\partial \xi} + c\frac{\partial v}{\partial \eta}, \qquad \text{that is,} \qquad \frac{\partial v}{\partial \eta} = 0.$$

Thus v is independent of η, so $v(\xi,\eta) = f(\xi)$ for an arbitrary differentiable function f of one variable. The original differential equation has solution

$$u(x,t) = f(x+ct).$$

32. If $w(r) = f(r) + g(s)$, where f and g are arbitrary twice differentiable functions, then

$$\frac{\partial^2 w}{\partial r \partial s} = \frac{\partial}{\partial r} g'(s) = 0.$$

33. If $r = x + ct$, $s = x - ct$, and $w(r,s) = w(x+ct, x-ct) = u(x,t)$, then

$$\frac{\partial u}{\partial t} = c\frac{\partial w}{\partial r} - c\frac{\partial w}{\partial s}$$
$$\frac{\partial^2 w}{\partial t^2} = c^2\frac{\partial^2 w}{\partial r^2} - 2c^2\frac{\partial^2 w}{\partial r \partial s} + c^2\frac{\partial^2 w}{\partial s^2}$$
$$\frac{\partial u}{\partial x} = \frac{\partial w}{\partial r} + \frac{\partial w}{\partial s}$$
$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 w}{\partial r^2} + 2\frac{\partial^2 w}{\partial r \partial s} + \frac{\partial^2 w}{\partial s^2}.$$

If u satisfies $\dfrac{\partial^2 u}{\partial t^2} = c^2\dfrac{\partial^2 u}{\partial x^2}$, then w satisfies

$$c^2\left(\frac{\partial^2 w}{\partial r^2} - 2\frac{\partial^2 w}{\partial r \partial s} + \frac{\partial^2 w}{\partial s^2}\right) = c^2\left(\frac{\partial^2 w}{\partial r^2} + 2\frac{\partial^2 w}{\partial r \partial s} + \frac{\partial^2 w}{\partial s^2}\right)$$

and hence

$$\frac{\partial^2 w}{\partial r \partial s} = 0.$$

By Exercise 38, $w(r,s) = f(r) + g(s)$, where f and g are arbitrary twice differentiable functions. Hence the original differential equation has solution

$$u(x,t) = f(x+ct) + g(x-ct).$$

34. By Exercise 39, the DE $u_t = c^2 u_{xx}$ has solution

$$u(x,t) = f(x+ct) + g(x-ct),$$

for arbitrary sufficiently smooth functions f and g. The initial conditions imply that

$$p(x) = u(x,0) = f(x) + g(x)$$
$$q(x) = u_t(x,0) = cf'(x) - cg'(x).$$

Integrating the second of these equations, we get

$$f(x) - g(x) = \frac{1}{c}\int_a^x q(s)\,ds,$$

where a is a constant. Solving the two equations for f and g we obtain

$$f(x) = \frac{1}{2}p(x) + \frac{1}{2c}\int_a^x q(s)\,ds$$
$$g(x) = \frac{1}{2}p(x) - \frac{1}{2c}\int_a^x q(s)\,ds.$$

Thus the solution to the initial-value problem is

$$u(x,t) = \frac{p(x+ct) + p(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} q(s)\,ds.$$

35.
```
> f := u(r*cos(t),r*sin(t)):
> simplify( diff(f,r$2) +
(1/r)*diff(f,r)
>       +(1/r^2)*diff(f,t$2));
```
$D_{2,2}(u)(r\cos(t), r\sin(t)) + D_{1,1}(u)(r\cos(t), r\sin(t))$

which confirms the identity.

36.
```
> g := f(x/(x^2+y^2),y/(x^2+y^2)):
> simplify(diff(g,x$2)+diff(g,y$2));
```
$$\frac{D_{1,1}(f)\left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right) + D_{2,2}(f)\left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)}{(x^2+y^2)^2}.$$

If f is harmonic, then the numerator is zero so g is harmonic.

37.
```
> simplify(diff(diff(
>   f(y^2,x*y,-x^2),x),y));
```
$$2y^2D_{1,2}(f) + xyD_{2,2}(f) + D_2(f) - 4xyD_{1,3}(f) - 2x^2D_{2,3}(f)$$

where all terms are evaluated at $(y^2, xy, -x^2)$.

38.
```
> simplify(diff(diff
>   (f(s^2-t,s+t^2),s),t$2));
```
$$2sD_{1,1,1}(f) - 8stD_{1,1,2}(f) + 8st^2D_{1,2,2}(f) + 4sD_{1,2}(f) + D_{1,1,2}(f) - 4tD_{1,2,2}(f) + 4t^2D_{2,2,2}(f) + 2D_{2,2}(f)$$

where all terms are evaluated at $(s^2 - t, s + t^2)$.

39.
```
> z := u(x,y):
> x  :=  exp(s)*cos(t): y  :=
exp(s)*sin(t):
> simplify(
>   (diff(z,s$2)+diff(z,t$2))/(x^2+y^2));
```
$$D_{2,2}(u)(e^s \cos t, e^s \sin t) + D_{1,1}(u)(e^s \cos t, e^s \sin t),$$

which confirms the identity in Exercise 23.

40.
```
> u := (x,t) -> (p(x-c*t)+p(x+c*t))/2
>      +(1/((2*c))*int(q(s),x=x-
c*t..x+c*t):
> simplify(diff(u(x,t),t$2)
>   -c^2*diff(u(x,t),x$2));
```
$$0$$
```
> simplify(u(x,0));
```
$$p(x)$$
```
>
simplify(subs(t=0,diff(u(x,t),t)));
```
$$q(x)$$

so u satisfies the PDE and initial conditions given in Exercise 34.

Section 12.6 Linear Approximations, Differentiability, and Differentials (page 679)

1. $f(x, y) = x^2y^3 \qquad f(3, 1) = 9$
$f_1(x, y) = 2xy^3 \qquad f_1(3, 1) = 6$
$f_2(x, y) = 3x^2y^2 \qquad f_2(3, 1) = 27$

$$\begin{aligned} f(3.1, 0.9) &= f(3 + 0.1, 1 - 0.1) \\ &\approx f(3, 1) + 0.1f_1(3, 1) - 0.1f_2(3, 1) \\ &= 9 + 0.6 - 2.7 = 6.9 \end{aligned}$$

2. $f(x, y) = \tan^{-1}\dfrac{y}{x} \qquad f(3, 3) = \dfrac{\pi}{4}$
$f_1(x, y) = -\dfrac{y}{x^2 + y^2} \qquad f_1(3, 3) = -\dfrac{1}{6}$
$f_2(x, y) = \dfrac{x}{x^2 + y^2} \qquad f_2(3, 3) = \dfrac{1}{6}$

$$\begin{aligned} f(3.01, 2.99) &= f(3 + 0.01, 3 - 0.01) \\ &\approx f(3, 3) + 0.01f_1(3, 3) - 0.01f_2(3, 3) \\ &= \frac{\pi}{4} - \frac{0.01}{6} - \frac{0.01}{6} = \frac{\pi}{4} - \frac{0.01}{3} \\ &\approx 0.7820648 \end{aligned}$$

3. $f(x, y) = \sin(\pi xy + \ln y),\ f(0, 1) = 0$
$f_1(x, y) = \pi y \cos(\pi xy + \ln y),\ f_1(0, 1) = \pi$
$f_2(x, y) = \left(\pi x + \dfrac{1}{y}\right)\cos(\pi xy + \ln y),\ f_2(0, 1) = 1$

$$\begin{aligned} f(0.01, 1.05) &\approx f(0, 1) + 0.01f_1(0, 1) + 0.05f_2(0, 1) \\ &= 0 + 0.01\pi + 0.05 \approx 0.081416 \end{aligned}$$

4. $f(x, y) = \dfrac{24}{x^2 + xy + y^2}$

$f_1(x, y) = \dfrac{-24(2x + y)}{(x^2 + xy + y^2)^2}, \qquad f_2(x, y) = \dfrac{-24(x + 2y)}{(x^2 + xy + y^2)^2}$

$f(2, 2) = 2, \qquad f_1(2, 2) = -1, \qquad f_2(2, 2) = -1$

$$\begin{aligned} f(2.1, 1.8) &\approx f(2, 2) + 0.1f_1(2, 2) - 0.2f_2(2, 2) \\ &= 2 - 0.1 + 0.2 = 2.1 \end{aligned}$$

5. $f(x, y, z) = \sqrt{x + 2y + 3z},\ f(2, 2, 1) = 3$

$f_1(x, y, z) = \dfrac{1}{2\sqrt{x + 2y + 3z}}, \qquad f_2(x, y, z) = \dfrac{1}{\sqrt{x + 2y + 3z}}$

$f_3(x, y, z) = \dfrac{3}{2\sqrt{x + 2y + 3z}}$

$$\begin{aligned} f(1.9, 1.8, 1.1) &\approx f(2, 2, 1) \\ &\quad - 0.1f_1(2, 2, 1) - 0.2f_2(2, 2, 1) + 0.1f_3(2, 2, 1) \\ &= 3 - \frac{0.1}{6} - \frac{0.2}{3} + \frac{0.1}{2} \approx 2.967 \end{aligned}$$

6. $f(x, y) = xe^{y+x^2} \qquad f(2, -4) = 2$
$f_1(x, y) = e^{y+x^2}(1 + 2x^2) \qquad f_1(2, -4) = 9$
$f_2(x, y) = xe^{y+x^2} \qquad f_2(2, -4) = 2$

$$\begin{aligned} f(2.05, -3.92) &\approx f(2, -4) + 0.05f_1(2, -4) + 0.08f_2(2, -4) \\ &= 2 + 0.45 + 0.16 = 2.61 \end{aligned}$$

7. If the edges are x, y, and z, and $\dfrac{dx}{x} = \dfrac{dy}{y} = \dfrac{dz}{z} = \dfrac{1}{100}$, then

a) $V = xyz \Rightarrow dV = yz\,dx + xz\,dy + xy\,dz$
$$\Rightarrow \frac{dV}{V} = \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = \frac{3}{100}.$$
The volume can be in error by about 3%.

b) $A = xy \Rightarrow dA = y\,dx + x\,dy$
$$\Rightarrow \frac{dA}{A} = \frac{dx}{x} + \frac{dy}{y} = \frac{2}{100}.$$
The area of a face can be in error by about 2%.

c) $D^2 = x^2 + y^2 + z^2 \Rightarrow 2D\,dD = 2x\,dx + 2y\,dy + 2z\,dz$
$$\Rightarrow \frac{dD}{D} = \frac{x^2}{D^2}\frac{dx}{x} + \frac{y^2}{D^2}\frac{dy}{y} + \frac{z^2}{D^2}\frac{dz}{z} = \frac{1}{100}.$$
The diagonal can be in error by about 1%.

8. $V = \frac{1}{3}\pi r^2 h \Rightarrow dV = \frac{2}{3}\pi r h\,dr + \frac{1}{3}\pi r^2\,dh$. If $r = 25$ ft, $h = 21$ ft, and $dr = dh = 0.5/12$ ft, then
$$dV = \frac{\pi}{3}(2 \times 25 \times 21 + 25^2)\frac{0.5}{12} \approx 73.08.$$
The calculated volume can be in error by about 73 cubic feet.

9. $S = \pi r\sqrt{r^2 + h^2}$, so
$$\begin{aligned} dS &= \left(\pi\sqrt{r^2+h^2} + \frac{\pi r^2}{\sqrt{r^2+h^2}}\right) dr + \frac{\pi r h}{\sqrt{r^2+h^2}}\, dh \\ &= \pi\left(\sqrt{25^2+21^2} + \frac{25^2 + 25\times 21}{\sqrt{25^2+21^2}}\right)\frac{0.5}{12} \approx 8.88. \end{aligned}$$
The surface area can be in error by about 9 square feet.

10. If the sides and contained angle of the triangle are x and y m and θ radians, then its area A satisfies
$$\begin{aligned} A &= \frac{1}{2}xy\sin\theta \\ dA &= \frac{1}{2}y\sin\theta\,dx + \frac{1}{2}x\sin\theta\,dy + \frac{1}{2}xy\cos\theta\,d\theta \\ \frac{dA}{A} &= \frac{dx}{x} + \frac{dy}{y} + \cot\theta\,d\theta. \end{aligned}$$
For $x = 224$, $y = 158$, $\theta = 64° = 64\pi/180$, $dx = dy = 0.4$, and $d\theta = 2° = 2\pi/180$, we have
$$\frac{dA}{A} = \frac{0.4}{224} + \frac{0.4}{158} + (\cot 64°)\frac{2\pi}{180} \approx 0.0213.$$
The calculated area of the plot can be in error by a little over 2%.

11. From the figure we have
$$\begin{aligned} h &= s\tan\theta \\ h &= (s+x)\tan\phi = \left(\frac{h}{\tan\theta} + x\right)\tan\phi. \end{aligned}$$
Solving the latter equation for h, we obtain
$$h = \frac{x\tan\phi\tan\theta}{\tan\theta - \tan\phi}.$$

We calculate the values of h and its first partials at $x = 100$, $\theta = 50°$, $\phi = 35°$:
$$\begin{aligned} h &\approx 170 \\ \frac{\partial h}{\partial x} &= \frac{\tan\phi\tan\theta}{\tan\theta - \tan\phi} \approx 1.70 \\ \frac{\partial h}{\partial \theta} &= x\tan\phi\,\frac{(\tan\theta - \tan\phi)\sec^2\theta - \tan\theta\sec^2\theta}{(\tan\theta - \tan\phi)^2} \\ &= -\frac{x\tan^2\phi\sec^2\theta}{(\tan\theta - \tan\phi)^2} \approx -491.12 \\ \frac{\partial h}{\partial \phi} &= \frac{x\tan^2\theta\sec^2\phi}{(\tan\theta-\tan\phi)^2} \approx 876.02. \end{aligned}$$
Thus $dh \approx 1.70\,dx - 491\,d\theta + 876\,d\phi$. For $dx = 0.1$ m and $|d\theta| = |d\phi| = 1° = \pi/180$, the largest value of dh will come from taking $d\theta$ negative and $d\phi$ positive:
$$dh \approx (1.70)(0.1) + (491 + 876)\frac{\pi}{180} \approx 24.03.$$
The calculated height of the tower is 170 m and can be in error by as much as 24 m. The calculation of the height is most sensitive to the accuracy of the measurement of ϕ.

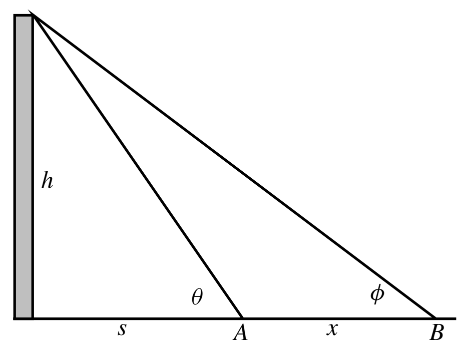


Fig. 12.6.11

12.
$$w = \frac{x^2y^3}{z^4} \qquad \frac{\partial w}{\partial x} = \frac{2xy^3}{z^4} = \frac{2w}{x}$$
$$\frac{\partial w}{\partial y} = \frac{3x^2y^2}{z^4} = \frac{3w}{y} \qquad \frac{\partial w}{\partial z} = -\frac{4x^2y^3}{z^5} = -\frac{4w}{x}.$$
$$\begin{aligned} dw &= \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz \\ \frac{dw}{w} &= 2\frac{dx}{x} + 3\frac{dy}{y} - 4\frac{dz}{z}. \end{aligned}$$
Since x increases by 1%, then $\frac{dx}{x} = \frac{1}{100}$. Similarly, $\frac{dy}{y} = \frac{2}{100}$ and $\frac{dz}{z} = \frac{3}{100}$. Therefore
$$\frac{\Delta w}{w} \approx \frac{dw}{w} = \frac{2 + 6 - 12}{100} = -\frac{4}{100},$$
and w decreases by about 4%.

13. $\mathbf{f}(r,\theta) = (r\cos\theta, r\sin\theta)$

$$D\mathbf{f}(r,\theta) = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

14. $\mathbf{f}(\rho,\phi,\theta) = (\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)$

$$D\mathbf{f}(\rho,\phi,\theta) = \begin{pmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{pmatrix}$$

15.
$$\mathbf{f}(x,y,z) = \begin{pmatrix} x^2+yz \\ y^2 - x\ln z \end{pmatrix}$$
$$D\mathbf{f}(x,y,z) = \begin{pmatrix} 2x & z & y \\ -\ln z & 2y & -x/z \end{pmatrix}$$
$$D\mathbf{f}(2,2,1) = \begin{pmatrix} 4 & 1 & 2 \\ 0 & 4 & -2 \end{pmatrix}$$
$$\mathbf{f}(1.98, 2.01, 1.03) \approx \mathbf{f}(2,2,1) + D\mathbf{f}(2,2,1)\begin{pmatrix} -0.02 \\ 0.01 \\ 0.03 \end{pmatrix}$$
$$= \begin{pmatrix} 6 \\ 4 \end{pmatrix} + \begin{pmatrix} -0.01 \\ -0.02 \end{pmatrix} = \begin{pmatrix} 5.99 \\ 3.98 \end{pmatrix}$$

16.
$$\mathbf{g}(r,s,t) = \begin{pmatrix} r^2 s \\ r^2 t \\ s^2 - t^2 \end{pmatrix}$$
$$D\mathbf{g}(r,s,t) = \begin{pmatrix} 2rs & r^2 & 0 \\ 2rt & 0 & r^2 \\ 0 & 2s & -2t \end{pmatrix}$$
$$D\mathbf{g}(1,3,3) = \begin{pmatrix} 6 & 1 & 0 \\ 6 & 0 & 1 \\ 0 & 6 & -6 \end{pmatrix}$$
$$\mathbf{g}(0.99, 3.02, 2.97) \approx \mathbf{g}(1,3,3) + D\mathbf{g}(1,3,3)\begin{pmatrix} -0.01 \\ 0.02 \\ -0.03 \end{pmatrix}$$
$$= \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix} + \begin{pmatrix} -0.04 \\ -0.09 \\ 0.30 \end{pmatrix} = \begin{pmatrix} 2.96 \\ 2.91 \\ 0.30 \end{pmatrix}$$

17. If f is differentiable at (a,b), then

$$\frac{f(a+h,b+k) - f(a,b) - hf_1(a,b) - kf_2(a,b)}{\sqrt{h^2+k^2}}$$

approaches 0 as $(h,k) \to (0,0)$. Since the denominator of this fraction approaches zero, the numerator must also approach 0 or the fraction would not have a limit. Since the terms $hf_1(a,b)$ and $kf_2(a,b)$ both approach 0, we must have

$$\lim_{(h,k)\to(0,0)} [f(a+h,b+k) - f(a,b)] = 0.$$

Thus f is continuous at (a,b).

18. Let $g(t) = f(a+th, b+tk)$. Then

$$g'(t) = hf_1(a+th, b+tk) + kf_2(a+th, b+tk).$$

If h and k are small enough that $(a+h, b+k)$ belongs to the disk referred to in the statement of the problem, then we can apply the (one-variable) Mean-Value Theorem to $g(t)$ on $[0,1]$ and obtain

$$g(1) = g(0) + g'(\theta),$$

for some θ satisfying $0 < \theta < 1$, i.e.,

$$f(a+h,b+k) = f(a,b) + hf_1(a+\theta h, b+\theta k) + kf_2(a+\theta h, b+\theta k).$$

19. Apply Taylor's Formula:

$$g(1) = g(0) + g'(0) + \frac{g''(\theta)}{2!}$$

for some θ between 0 and 1 to $g(t) = f(a+th, b+tk)$. We have

$$g'(t) = hf_1(a+th,b+tk) + kf_2(a+th,b+tk)$$
$$g'(0) = hf_1(a,b) + kf_2(a,b)$$
$$g''(t) = h^2 f_{11}(a+th,b+tk) + 2hkf_{12}(a+th,b+tk) + k^2 f_{22}(a+th,b+tk).$$

Thus

$$f(a+h,b+k) = f(a,b) + hf_1(a,b) + kf_2(a,b) + \frac{1}{2}\Big(h^2 f_{11}(a+\theta h,b+\theta k) + 2hkf_{12}(a+\theta h,b+\theta k) + k^2 f_{22}(a+\theta h,b+\theta k)\Big)$$

$$\Delta f = f(a+h,b+k) - f(a,b)$$
$$df = hf_1(a,b) + kf_2(a,b)$$
$$\left|\Delta f - df\right| \le \frac{1}{2}\Big|h^2 f_{11}(a+\theta h,b+\theta k) + 2hkf_{12}(a+\theta h,b+\theta k) + k^2 f_{22}(a+\theta h,b+\theta k)\Big|$$
$$\le K(h^2+k^2) \qquad (\text{since } 2hk \le h^2+k^2),$$

for some K depending on f, and valid in some disk $h^2+k^2 \le R^2$ of positive radius R.

Section 12.7 Gradients and Directional Derivatives (page 688)

1. $f(x,y) = x^2 - y^2$, $\quad f(2,-1) = 3$.
$\nabla f(x,y) = 2x\mathbf{i} - 2y\mathbf{j}$, $\quad \nabla f(2,-1) = 4\mathbf{i} + 2\mathbf{j}$.
Tangent plane to $z = f(x,y)$ at $(2,-1,3)$ has equation $4(x-2) + 2(y+1) = z - 3$, or $4x + 2y - z = 3$.
Tangent line to $f(x,y) = 3$ at $(2,-1)$ has equation $4(x-2) + 2(y+1) = 0$, or $2x + y = 3$.

2. $f(x,y)=\dfrac{x-y}{x+y}, \qquad f(1,1)=0.$
$\nabla f=\dfrac{2y\mathbf{i}-2x\mathbf{j}}{(x+y)^2},$
$\nabla f(1,1)=\dfrac{1}{2}(\mathbf{i}-\mathbf{j})$. Tangent plane to $z=f(x,y)$ at $(1,1,0)$ has equation $\frac{1}{2}(x-1)-\frac{1}{2}(y-1)=z$, or $x-y-2z=0$.
Tangent line to $f(x,y)=0$ at $(1,1)$ has equation $\frac{1}{2}(x-1)-\frac{1}{2}(y-1)$, or $x=y$.

3. $f(x,y)=\dfrac{x}{x^2+y^2},$
$f_1(x,y)=\dfrac{(x^2+y^2)(1)-x(2x)}{(x^2+y^2)^2}=\dfrac{y^2-x^2}{(x^2+y^2)^2},$
$f_2(x,y)=-\dfrac{2xy}{(x^2+y^2)^2}.$
$\nabla f(x,y)=\dfrac{1}{(x^2+y^2)^2}\Big((y^2-x^2)\mathbf{i}-2xy\mathbf{j}\Big),$
$\nabla f(1,2)=\frac{3}{25}\mathbf{i}-\frac{4}{25}\mathbf{j}.$

Tangent plane to $z=f(x,y)$ at $(1,2,\frac{1}{5})$ has equation $\dfrac{3}{25}(x-1)-\dfrac{4}{25}(y-2)=z-\dfrac{1}{5}$, or $3x-4y-25z=-10$.
Tangent line to $f(x,y)=1/5$ at $(1,2)$ has equation $\dfrac{3}{25}(x-1)-\dfrac{4}{25}(y-2)=0$, or $3x-4y=-5$.

4. $f(x,y)=e^{xy}, \quad \nabla f=ye^{xy}\mathbf{i}+xe^{xy}\mathbf{j},$
$\nabla f(2,0)=2\mathbf{j}$. Tangent plane to $z=f(x,y)$ at $(2,0,1)$ has equation $2y=z-1$, or $2y-z=-1$.
Tangent line to $f(x,y)=1$ at $(2,0)$ has equation $y=0$.

5. $f(x,y)=\ln(x^2+y^2), \quad \nabla f(x,y)=\dfrac{2x\mathbf{i}+2y\mathbf{j}}{x^2+y^2},$
$\nabla f(1,-2)=\frac{2}{5}\mathbf{i}-\frac{4}{5}\mathbf{j}$. Tangent plane to $z=f(x,y)$ at $(1,-2,\ln 5)$ has equation $\dfrac{2}{5}(x-1)-\dfrac{4}{5}(y+2)=z-\ln 5$, or $2x-4y-5z=10-5\ln 5$.
Tangent line to $f(x,y)=\ln 5$ at $(1,-2)$ has equation $\dfrac{2}{5}(x-1)-\dfrac{4}{5}(y+2)=0$, or $x-2y=5$.

6. $f(x,y)=\sqrt{1+xy^2}, \qquad f(2,-2)=3.$
$\nabla f(x,y)=\dfrac{y^2\mathbf{i}+2xy\mathbf{j}}{2\sqrt{1+xy^2}},$
$\nabla f(2,-2)=\dfrac{2}{3}\mathbf{i}-\dfrac{4}{3}\mathbf{j}.$
Tangent plane to $z=f(x,y)$ at $(2,-2,3)$ has equation $\dfrac{2}{3}(x-2)-\dfrac{4}{3}(y+2)=z-3$, or $2x-4y-3z=3$.
Tangent line to $f(x,y)=3$ at $(2,-2)$ has equation $\dfrac{2}{3}(x-2)-\dfrac{4}{3}(y+2)=0$, or $x-2y=6$.

7. $f(x,y,z)=x^2y+y^2z+z^2x, \qquad f(1,-1,1)=1.$
$\nabla f(x,y,z)=(2xy+z^2)\mathbf{i}+(x^2+2yz)\mathbf{j}+(y^2+2zx)\mathbf{k},$
$\nabla f(1,-1,1)=-\mathbf{i}-\mathbf{j}+3\mathbf{k}.$
Tangent plane to $f(x,y,z)=1$ at $(1,-1,1)$ has equation $-(x-1)-(y+1)+3(z-1)=0$, or $x+y-3z=-3$.

8. $f(x,y,z)=\cos(x+2y+3z),$
$f\left(\dfrac{\pi}{2},\pi,\pi\right)=\cos\dfrac{11\pi}{2}=0.$
$\nabla f(x,y,z)=-\sin(x+2y+3z)(\mathbf{i}+2\mathbf{j}+3\mathbf{k}),$
$\nabla f\left(\dfrac{\pi}{2},\pi,\pi\right)=-\sin\dfrac{11\pi}{2}(\mathbf{i}+2\mathbf{j}+3\mathbf{k})=\mathbf{i}+2\mathbf{j}+3\mathbf{k}.$
Tangent plane to $f(x,y,z)=0$ at $\left(\dfrac{\pi}{2},\pi,\pi\right)$ has equation

$$x-\frac{\pi}{2}+2(y-\pi)+3(z-\pi)=0,$$

or $x+2y+3z=\dfrac{11\pi}{2}$.

9. $f(x,y,z)=ye^{-x^2}\sin z, \qquad f(0,1,\pi/3)=\sqrt{3}/2.$
$\nabla f(x,y,z)=-2xye^{-x^2}\sin z\mathbf{i}+e^{-x^2}\sin z\mathbf{j}+ye^{-x^2}\cos z\mathbf{k},$
$\nabla f(0,1,\pi/3)=\dfrac{\sqrt{3}}{2}\mathbf{j}+\dfrac{1}{2}\mathbf{k}.$
The tangent plane to $f(x,y,z)=\dfrac{\sqrt{3}}{2}$ at $0,1,\pi/3)$ has equation

$$\frac{\sqrt{3}}{2}(y-1)+\frac{1}{2}\left(z-\frac{\pi}{3}\right)=0,$$

or $\sqrt{3}y+z=\sqrt{3}+\dfrac{\pi}{3}$.

10. $f(x,y)=3x-4y, \qquad \nabla f(0,2)=\nabla f(x,y)=3\mathbf{i}-4\mathbf{j},$
$D_{-\mathbf{i}}f(0,2)=-\mathbf{i}\bullet(3\mathbf{i}-4\mathbf{j})=-3.$

11. $f(x,y)=x^2y, \quad \nabla f=2xy\mathbf{i}+x^2\mathbf{j},$
$\nabla f(-1,-1)=2\mathbf{i}+\mathbf{j}.$
Rate of change of f at $(-1,-1)$ in the direction of $\mathbf{i}+2\mathbf{j}$ is

$$\frac{\mathbf{i}+2\mathbf{j}}{\sqrt{5}}\bullet(2\mathbf{i}+\mathbf{j})=\frac{4}{\sqrt{5}}.$$

12. $f(x,y)=\dfrac{x}{1+y}, \quad \nabla f(x,y)=\dfrac{1}{1+y}\mathbf{i}-\dfrac{x}{(1+y)^2}\mathbf{j},$
$\nabla f(0,0)=\mathbf{i}, \qquad \mathbf{u}=\dfrac{\mathbf{i}-\mathbf{j}}{\sqrt{2}},$
$D_{\mathbf{u}}f(0,0)=\mathbf{i}\bullet\left(\dfrac{\mathbf{i}-\mathbf{j}}{\sqrt{2}}\right)=\dfrac{1}{\sqrt{2}}.$

13. $f(x,y)=x^2+y^2, \quad \nabla f=2x\mathbf{i}+2y\mathbf{j},$
$\nabla f(1,-2)=2\mathbf{i}-4\mathbf{j}.$
A unit vector in the direction making a 60° angle with the positive x-axis is $\mathbf{u}=\frac{1}{2}\mathbf{i}+\frac{\sqrt{3}}{2}\mathbf{j}$.
The rate of change of f at $(1,-2)$ in the direction of $\mathbf{u}$ is $\mathbf{u}\bullet\nabla f(1,-2)=1-2\sqrt{3}$.

14. $f(x, y) = \ln |\mathbf{r}|$, where $\mathbf{r} = x\mathbf{i}+y\mathbf{j}$. Since $|\mathbf{r}| = \sqrt{x^2 + y^2}$, we have

$$\nabla f(x, y) = \frac{1}{|\mathbf{r}|}\left(\frac{x}{|\mathbf{r}|}\mathbf{i} + \frac{y}{|\mathbf{r}|}\mathbf{j}\right) = \frac{\mathbf{r}}{|\mathbf{r}|^2}.$$

15. $f(x, y, z) = |\mathbf{r}|^{-n}$, where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Since $|\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$, we have

$$\begin{aligned}\nabla f(x, y, z) &= -n|\mathbf{r}|^{-n-1}\left(\frac{x}{|\mathbf{r}|}\mathbf{i} + \frac{y}{|\mathbf{r}|}\mathbf{j} + \frac{z}{|\mathbf{r}|}\mathbf{k}\right)\\ &= -\frac{n\mathbf{r}}{|\mathbf{r}|^{n+2}}.\end{aligned}$$

16. Since $x = r\cos\theta$ and $y = r\sin\theta$, we have

$$\begin{aligned}\frac{\partial f}{\partial r} &= \cos\theta\frac{\partial f}{\partial x} + \sin\theta\frac{\partial f}{\partial y}\\ \frac{\partial f}{\partial \theta} &= -r\sin\theta\frac{\partial f}{\partial x} + r\cos\theta\frac{\partial f}{\partial y}.\end{aligned}$$

Also,

$$\begin{aligned}\hat{\mathbf{r}} &= \frac{x\mathbf{i} + y\mathbf{j}}{r} = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j}\\ \hat{\boldsymbol{\theta}} &= \frac{-y\mathbf{i} + x\mathbf{j}}{r} = -(\sin\theta)\mathbf{i} + (\cos\theta)\mathbf{j}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}}\\ &= \left(\cos^2\theta\frac{\partial f}{\partial x} + \sin\theta\cos\theta\frac{\partial f}{\partial y}\right)\mathbf{i}\\ &\quad + \left(\cos\theta\sin\theta\frac{\partial f}{\partial x} + \sin^2\theta\frac{\partial f}{\partial y}\right)\mathbf{j}\\ &\quad + \left(\sin^2\theta\frac{\partial f}{\partial x} - \sin\theta\cos\theta\frac{\partial f}{\partial y}\right)\mathbf{i}\\ &\quad + \left(-\cos\theta\sin\theta\frac{\partial f}{\partial x} + \cos^2\theta\frac{\partial f}{\partial y}\right)\mathbf{j}\\ &= \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} = \nabla f.\end{aligned}$$

17. $f(x, y) = xy, \quad \nabla f(x, y) = y\mathbf{i} + x\mathbf{j}, \quad \nabla f(2, 0) = 2\mathbf{j}$.
Let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ be a unit vector. Thus $u_1^2 + u_2^2 = 1$. We have

$$-1 = D_{\mathbf{u}}f(2, 0)\mathbf{u} \bullet \nabla f(2, 0) = 2u_2$$

if $u_2 = -\frac{1}{2}$, and therefore $u_1 = \pm\frac{\sqrt{3}}{2}$. At $(2, 0)$, f has rate of change -1 in the directions $\pm\frac{\sqrt{3}}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}$.

If $-3 = D_{\mathbf{u}}f(2, 0) = 2u_2$, then $u_2 = -\frac{3}{2}$. This is not possible for a unit vector $\mathbf{u}$, so there is no direction at $(2, 0)$ in which f changes at rate -3.

If $-2 = D_{\mathbf{u}}f(2, 0) = 2u_2$, then $u_2 = -1$ and $u_1 = 0$. At $(2, 0)$, f has rate of change -2 in the direction $-\mathbf{j}$.

18. $f(x, y, z) = x^2 + y^2 - z^2$.
$\nabla f(a, b, c) = 2a\mathbf{i} + 2b\mathbf{j} - 2c\mathbf{k}$. The maximum rate of change of f at (a, b, c) is in the direction of $\nabla f(a, b, c)$, and is equal to $|\nabla f(a, b, c)|$.
Let $\mathbf{u}$ be a unit vector making an angle θ with $\nabla f(a, b, c)$. The rate of change of f at (a, b, c) in the direction of $\mathbf{u}$ will be half of the maximum rate of change of f at that point provided

$$\frac{1}{2}|\nabla f(a, b, c)| = \mathbf{u} \bullet \nabla f(a, b, c) = |\nabla f(a, b, c)|\cos\theta,$$

that is, if $\cos\theta = \frac{1}{2}$, which means $\theta = 60°$. At (a, b, c), f increases at half its maximal rate in all directions making $60°$ angles with the direction $a\mathbf{i} + b\mathbf{j} - c\mathbf{k}$.

19. Let $\nabla f(a, b) = u\mathbf{i} + v\mathbf{j}$. Then

$$\begin{aligned}3\sqrt{2} &= D_{(\mathbf{i}+\mathbf{j})/\sqrt{2}}f(a, b) = \frac{\mathbf{i} + \mathbf{j}}{\sqrt{2}} \bullet (u\mathbf{i} + v\mathbf{j}) = \frac{u + v}{\sqrt{2}}\\ 5 &= D_{(3\mathbf{i}-4\mathbf{j})/5}f(a, b) = \frac{3\mathbf{i} - 4\mathbf{j}}{5} \bullet (u\mathbf{i} + v\mathbf{j}) = \frac{3u - 4v}{5}.\end{aligned}$$

Thus $u + v = 6$ and $3u - 4v = 25$. This system has solution $u = 7$, $v = -1$. Thus $\nabla f(a, b) = 7\mathbf{i} - \mathbf{j}$.

20. Given the values $D_{\phi_1}f(a, b)$ and $D_{\phi_2}f(a, b)$, we can solve the equations

$$\begin{aligned}f_1(a, b)\cos\phi_1 + f_2(a, b)\sin\phi_1 &= D_{\phi_1}f(a, b)\\ f_1(a, b)\cos\phi_2 + f_2(a, b)\sin\phi_2 &= D_{\phi_2}f(a, b)\end{aligned}$$

for unique values of $f_1(a, b)$ and $f_2(a, b)$ (and hence determine $\nabla f(a, b)$ uniquely), provided the coefficients satisfy

$$0 \ne \begin{vmatrix}\cos\phi_1 & \sin\phi_1\\ \cos\phi_2 & \sin\phi_2\end{vmatrix} = \sin(\phi_2 - \phi_1).$$

Thus ϕ_1 and ϕ_2 must not differ by an integer multiple of π.

21. a) $T(x, y) = x^2 - 2y^2$.

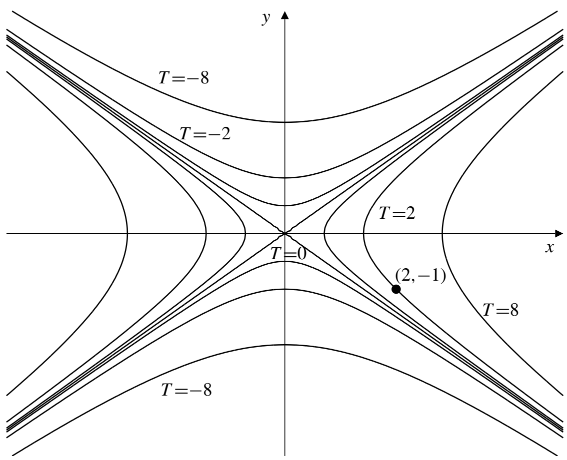


Fig. 12.7.21

b) $\nabla T = 2x\mathbf{i} - 4y\mathbf{j}$, $\nabla T(2,-1) = 4\mathbf{i} + 4\mathbf{j}$.
An ant at $(2,-1)$ should move in the direction of $-\nabla T(2,-1)$, that is, in the direction $-\mathbf{i} - \mathbf{j}$, in order to cool off as rapidly as possible.

c) If the ant moves at speed k in the direction $-\mathbf{i} - \mathbf{j}$, it will experience temperature decreasing at rate $|\nabla T(2,-1)|k = 4\sqrt{2}k$ degrees per unit time.

d) If the ant moves at speed k in the direction $-\mathbf{i} - 2\mathbf{j}$, it experiences temperature changing at rate

$$\frac{-\mathbf{i} - 2\mathbf{j}}{\sqrt{5}} \bullet (4\mathbf{i} + 4\mathbf{j})k = -\frac{12k}{\sqrt{5}},$$

that is, decreasing at rate $12k/\sqrt{5}$ degrees per unit time.

e) To continue to experience maximum rate of cooling, the ant should crawl along the curve $x = x(t)$, $y = y(t)$, which is everywhere tangent to $\nabla T(x, y)$. Thus we want

$$\frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} = \lambda(2x\mathbf{i} - 4y\mathbf{j}).$$

Thus $\dfrac{1}{y}\dfrac{dy}{dt} = -\dfrac{2}{x}\dfrac{dx}{dt}$, from which we obtain, on integration,

$$\ln|y(t)| = -2\ln|x(t)| + \ln|C|,$$

or $yx^2 = C$. Since the curve passes through $(2,-1)$, we have $yx^2 = -4$. Thus, the ant should crawl along the path $y = -4/x^2$.

22. Let the curve be $y = g(x)$. At (x, y) this curve has normal $\nabla\Big(g(x) - y\Big) = g'(x)\mathbf{i} - \mathbf{j}$.
A curve of the family $x^4 + y^2 = C$ has normal $\nabla(x^4 + y^2) = 4x^3\mathbf{i} + 2y\mathbf{j}$.
These curves will intersect at right angles if their normals are perpendicular. Thus we require that

$$0 = 4x^3 g'(x) - 2y = 4x^3 g'(x) - 2g(x),$$

or, equivalently,

$$\frac{g'(x)}{g(x)} = \frac{1}{2x^3}.$$

Integration gives $\ln|g(x)| = -\dfrac{1}{4x^2} + \ln|C|$,
or $g(x) = Ce^{-(1/4x^2)}$.
Since the curve passes through $(1, 1)$, we must have $1 = g(1) = Ce^{-1/4}$, so $C = e^{1/4}$.
The required curve is $y = e^{(1/4)-(1/4x^2)}$.

23. Let the curve be $y = f(x)$. At (x, y) it has normal $\dfrac{dy}{dx}\mathbf{i} - \mathbf{j}$.
The curve $x^2y^3 = K$ has normal $2xy^3\mathbf{i} + 3x^2y^2\mathbf{j}$.
These curves will intersect at right angles if their normals are perpendicular, that is, if

$$\begin{aligned} 2xy^3\frac{dy}{dx} - 3x^2y^2 &= 0 \\ \frac{dy}{dx} &= \frac{3x}{2y} \\ 2y\,dy &= 3x\,dx \\ y^2 &= \frac{3}{2}x^2 + C. \end{aligned}$$

Since the curve must pass through $(2,-1)$, we have $1 = 6 + C$, so $C = -5$.
The required curve is $3x^2 - 2y^2 = 10$.

24. Let $f(x, y) = e^{-(x^2+y^2)}$. Then

$$\nabla f(x, y) = -2e^{-(x^2+y^2)}(x\mathbf{i} + y\mathbf{j}).$$

The vector $\mathbf{u} = \dfrac{a\mathbf{i} + b\mathbf{j}}{\sqrt{a^2+b^2}}$ is a unit vector in the direction directly away from the origin at (a, b).
The first directional derivative of f at (x, y) in the direction of $\mathbf{u}$ is

$$\mathbf{u} \bullet \nabla f(x, y) = -\frac{2}{\sqrt{a^2+b^2}}(ax + by)e^{-(x^2+y^2)}.$$

The second directional derivative is

$$\begin{aligned} &\mathbf{u} \bullet \nabla\left(-\frac{2}{\sqrt{a^2+b^2}}(ax+by)e^{-(x^2+y^2)}\right) \\ &= -\frac{2}{a^2+b^2}(a\mathbf{i} + b\mathbf{j}) \bullet e^{-(x^2+y^2)} \\ &\qquad \Big[\Big(a - 2x(ax+by)\Big)\mathbf{i} + \Big(b - 2y(ax+by)\Big)\mathbf{j}\Big]. \end{aligned}$$

At (a, b) this second directional derivative is

$$-\frac{2e^{-(a^2+b^2)}}{a^2+b^2}\Big(a^2-2a^4-2a^2b^2+b^2-2a^2b^2-2b^4\Big)$$
$$=\frac{2}{a^2+b^2}\Big(2(a^2+b^2)^2-a^2-b^2\Big)e^{-(a^2+b^2)}$$
$$=2\Big(2(a^2+b^2)-1\Big)e^{-(a^2+b^2)}.$$

Remark: Since $f(x, y) = e^{-r^2}$ (expressed in terms of polar coordinates), the second directional derivative of f at (a, b) in the direction directly away from the origin (i.e., the direction of increasing r) can be more easily calculated as

$$\frac{d^2}{dr^2}e^{-r^2}\Big|_{r^2=a^2+b^2}.$$

25. $f(x, y, z) = xyz$, $\nabla f(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$.
The first directional derivative of f in the direction $\mathbf{i}-\mathbf{j}-\mathbf{k}$ is

$$\frac{\mathbf{i}-\mathbf{j}-\mathbf{k}}{\sqrt{3}}\bullet\nabla f(x,y,z)=\frac{1}{\sqrt{3}}(yz-xz-xy).$$

The second directional derivative in that direction is

$$\frac{\mathbf{i}-\mathbf{j}-\mathbf{k}}{\sqrt{3}}\bullet\frac{1}{\sqrt{3}}\nabla(yz-xz-xy)$$
$$=\frac{\mathbf{i}-\mathbf{j}-\mathbf{k}}{3}\bullet\Big[-(y+z)\mathbf{i}+(z-x)\mathbf{j}+(y-x)\mathbf{k}\Big]$$
$$=\frac{1}{3}\Big[-(y+z)-(z-x)-(y-x)\Big]=\frac{2x-2y-2z}{3}.$$

At $(2, 3, 1)$ this second directional derivative has value $-4/3$.

26. At $(1, -1, 1)$ the surface $x^2 + y^2 = 2$ has normal

$$\mathbf{n}_1=\nabla(x^2+y^2)\Big|_{(1,-1,1)}=2\mathbf{i}-2\mathbf{j},$$

and $y^2 + z^2 = 2$ has normal

$$\mathbf{n}_2=\nabla(y^2+z^2)\Big|_{(1,-1,1)}=-2\mathbf{j}+2\mathbf{k}.$$

A vector tangent to the curve of intersection of the two surfaces at $(1, -1, 1)$ must be perpendicular to both these normals. Since

$$(\mathbf{i}-\mathbf{j})\times(-\mathbf{j}+\mathbf{k})=-(\mathbf{i}+\mathbf{j}+\mathbf{k}),$$

the vector $\mathbf{i}+\mathbf{j}+\mathbf{k}$, or any scalar multiple of this vector, is tangent to the curve at the given point.

27. The vector $\mathbf{n}_1 = \mathbf{i}+\mathbf{j}+\mathbf{k}$ is normal to the plane $x + y + z = 6$ at $(1, 2, 3)$. A normal to the sphere $x^2 + y^2 + z^2 = 14$ at that point is

$$\mathbf{n}_2=\nabla(x^2+y^2+z^2)\Big|_{(1,2,3)}=2\mathbf{i}+4\mathbf{j}+6\mathbf{k}.$$

A vector tangent to the circle of intersection of the two surfaces at $(1, 2, 3)$ is

$$\mathbf{n}_1\times\mathbf{n}_2=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&1&1\\2&4&6\end{vmatrix}=2\mathbf{i}-4\mathbf{j}+2\mathbf{k}.$$

Any vector parallel to $\mathbf{i}-2\mathbf{j}+\mathbf{k}$ is tangent to the circle at $(1, 2, 3)$.

28. A vector tangent to the path of the fly at $(1, 1, 2)$ is given by

$$\mathbf{v}=\nabla(3x^2-y^2-z)\times\nabla(2x^2+2y^2-z^2)\Big|_{(1,1,2)}$$
$$=(6x\mathbf{i}-2y\mathbf{j}-\mathbf{k})\times(4x\mathbf{i}+4y\mathbf{j}-2z\mathbf{k})\Big|_{(1,1,2)}$$
$$=(6\mathbf{i}-2\mathbf{j}-\mathbf{k})\times(4\mathbf{i}+4\mathbf{j}-4\mathbf{k})$$
$$=4\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\6&-2&-1\\1&1&-1\end{vmatrix}=4(3\mathbf{i}+5\mathbf{j}+8\mathbf{k}).$$

The temperature $T = x^2 - y^2 + z^2 + xz^2$ has gradient at $(1, 1, 2)$ given by

$$\nabla T(1,1,2)=(2x+z^2)\mathbf{i}-2y\mathbf{j}+2z(1+x)\mathbf{k}\Big|_{(1,1,2)}$$
$$=6\mathbf{i}-2\mathbf{j}+8\mathbf{k}.$$

Thus the fly, passing through $(1, 1, 2)$ with speed 7, experiences temperature changing at rate

$$7\times\frac{\mathbf{v}}{|\mathbf{v}|}\bullet\nabla T(1,1,2)=7\frac{3\mathbf{i}+5\mathbf{j}+8\mathbf{k}}{\sqrt{98}}\bullet(6\mathbf{i}-2\mathbf{j}+8\mathbf{k})$$
$$=\frac{1}{\sqrt{2}}(18-10+64)=\frac{72}{\sqrt{2}}.$$

We don't know which direction the fly is moving along the curve, so all we can say is that it experiences temperature changing at rate $36\sqrt{2}$ degrees per unit time.

29. If $f(x, y, z)$ is differentiable at the point (a, b, c) and $\nabla f(a, b, c) \ne \mathbf{0}$, then $\nabla f(a, b, c)$ is normal to the level surface of f which passes through (a, b, c).

The proof is very similar to that of Theorem 6 of Section 3.7, modified to include the extra variable. The angle θ between $\nabla f(a,b,c)$ and the secant vector from (a,b,c) to a neighbouring point $(a+h, b+k, c+\ell)$ on the level surface of f passing through (a,b,c) satisfies

$$\begin{aligned}\cos\theta &= \frac{\nabla f(a,b,c) \bullet (h\mathbf{i}+k\mathbf{j}+\ell\mathbf{k})}{|\nabla f(a,b,c)|\sqrt{h^2+k^2+\ell^2}}\\ &= \frac{hf_1(a,b,c)+kf_2(a,b,c)+\ell f_3(a,b,c)}{|\nabla f(a,b,c)|\sqrt{h^2+k^2+\ell^2}}\\ &= \frac{-1}{|\nabla f(a,b,c)|\sqrt{h^2+k^2+\ell^2}}\Big[f(a+h,b+k,c+\ell)\\ &\qquad - f(a,b,c) - hf_1(a,b,c) - kf_2(a,b,c) - \ell f_3(a,b,c)\Big]\\ &\to 0 \quad \text{as } (h,k,\ell)\to(0,0,0)\end{aligned}$$

because f is differentiable at (a,b,c). Thus $\theta \to \dfrac{\pi}{2}$, and $\nabla f(a,b,c)$ is normal to the level surface of f through (a,b,c).

30. The level surface of $f(x,y,z) = \cos(x+2y+3z)$ through (π,π,π) has equation $\cos(x+2y+3z) = \cos(6\pi) = 1$, which simplifies to $x+2y+3z = 6\pi$. This level surface is a plane, and is therefore its own tangent plane. We cannot determine this plane by the method used to find the tangent plane to the level surface of f through $(\pi/2,\pi,\pi)$ in Exercise 10, because $\nabla f(\pi,\pi,\pi) = \mathbf{0}$, so the gradient does not provide a usable normal vector to define the tangent plane.

31. By the version of the Mean-Value Theorem in Exercise 18 of Section 3.6,

$$f(x,y) = f(0,0) + xf_1(\theta x,\theta y) + yf_2(\theta x,\theta y)$$

for some θ between 0 and 1. Since ∇f is assumed to vanish throughout the disk $x^2+y^2<r^2$, this implies that $f(x,y) = f(0,0)$ throughout the disk, that is, f is constant there. (Note that Theorem 3 of Section 3.6 can be used instead of Exercise 18 of Section 3.6 in this argument.)

32. Let $f(x,y) = x^3 - y^2$. Then $\nabla f(x,y) = 3x^2\mathbf{i} - 2y\mathbf{j}$ exists everywhere, but equals $\mathbf{0}$ at $(0,0)$. The level curve of f passing through $(0,0)$ is $y^2 = x^3$, which has a cusp at $(0,0)$, so is not smooth there.

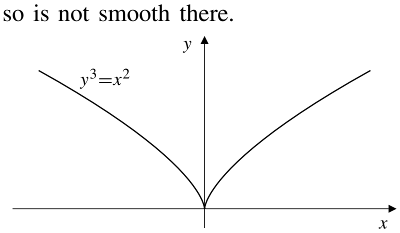


Fig. 12.7.32

33. Let $\mathbf{v} = v_1\mathbf{i}+v_2\mathbf{j}+v_3\mathbf{k}$. Thus

$$\begin{aligned}D_{\mathbf{v}}f &= v_1\frac{\partial f}{\partial x}+v_2\frac{\partial f}{\partial y}+v_3\frac{\partial f}{\partial z}\\ \nabla(D_{\mathbf{v}}f) &= \left(v_1\frac{\partial^2 f}{\partial x^2}+v_2\frac{\partial^2 f}{\partial x\partial y}+v_3\frac{\partial^2 f}{\partial x\partial z}\right)\mathbf{i}\\ &\quad+\left(v_1\frac{\partial^2 f}{\partial y\partial x}+v_2\frac{\partial^2 f}{\partial y^2}+v_3\frac{\partial^2 f}{\partial y\partial z}\right)\mathbf{j}\\ &\quad+\left(v_1\frac{\partial^2 f}{\partial x\partial z}+v_2\frac{\partial^2 f}{\partial y\partial z}+v_3\frac{\partial^2 f}{\partial z^2}\right)\mathbf{k}\\ D_{\mathbf{v}}(D_{\mathbf{v}}f) &= \mathbf{v}\bullet\nabla(D_{\mathbf{v}}f)\\ &= v_1^2\frac{\partial^2 f}{\partial x^2}+2v_1v_2\frac{\partial^2 f}{\partial x\partial y}+2v_1v_3\frac{\partial^2 f}{\partial x\partial z}\\ &\quad+v_2^2\frac{\partial^2 f}{\partial y^2}+2v_2v_3\frac{\partial^2 f}{\partial y\partial z}+v_3^2\frac{\partial^2 f}{\partial z^2}\end{aligned}$$

(assuming all second partials are continuous).

$D_{\mathbf{v}}(D_{\mathbf{v}}f)$ gives the second time derivative of the quantity f as measured by an observer moving with constant velocity $\mathbf{v}$.

34. $T = T(x,y,z)$. As measured by the observer,

$$\begin{aligned}\frac{dT}{dt} &= D_{\mathbf{v}(t)}T = \mathbf{v}(t)\bullet\nabla T\\ \frac{d^2T}{dt^2} &= \mathbf{a}(t)\bullet\nabla T+\mathbf{v}(t)\bullet\frac{d}{dt}\nabla T\\ &= D_{\mathbf{a}(t)}T+\left(v_1(t)\frac{d}{dt}\frac{\partial T}{\partial x}+\cdots\right)\\ &= D_{\mathbf{a}(t)}T+\left(v_1(t)\mathbf{v}(t)\bullet\nabla\frac{\partial T}{\partial x}+\cdots\right)\\ &= D_{\mathbf{a}(t)}T+\left(\big(v_1(t)\big)^2\frac{\partial^2 T}{\partial x^2}+v_1(t)v_2(t)\frac{\partial^2 T}{\partial y\partial x}+\cdots\right)\\ &= D_{\mathbf{a}(t)}T+D_{\mathbf{v}(t)}(D_{\mathbf{v}(t)}T)\end{aligned}$$

(as in Exercise 37 above).

35. $T = T(x,y,z,t)$. The calculation is similar to that of Exercise 38, but produces a few more terms because of the dependence of T explicitly on time t. We continue to use ∇ to denote the gradient with respect to the spatial variables only. Using the result of Exercise 38, we have

$$\begin{aligned}\frac{dT}{dt} &= \frac{\partial T}{\partial t}+\mathbf{v}(t)\bullet\nabla T\\ \frac{d^2T}{dt^2} &= \frac{d}{dt}\frac{\partial T}{\partial t}+\frac{d}{dt}\mathbf{v}(t)\bullet\nabla T\\ &= \frac{\partial^2 T}{\partial t^2}+\mathbf{v}(t)\bullet\frac{\partial T}{\partial t}\\ &\quad+\mathbf{v}(t)\bullet\frac{\partial}{\partial t}\nabla T+D_{\mathbf{a}(t)}T+D_{\mathbf{v}(t)}(D_{\mathbf{v}(t)}T)\\ &= \frac{\partial^2 T}{\partial t^2}+2D_{\mathbf{v}(t)}\left(\frac{\partial T}{\partial t}\right)+D_{\mathbf{a}(t)}T+D_{\mathbf{v}(t)}(D_{\mathbf{v}(t)}T).\end{aligned}$$

36. $f(x,y) = \begin{cases} \dfrac{\sin(xy)}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}.$

a) $f_1(0,0) = \lim_{h\to 0} \dfrac{0-0}{h} = 0 = f_2(0,0)$. Thus $\nabla f(0,0) = \mathbf{0}$.

b) If $\mathbf{u} = (\mathbf{i}+\mathbf{j})/\sqrt{2}$, then

$$D_{\mathbf{u}} f(0,0) = \lim_{h\to 0+} \frac{1}{h} \frac{\sin(h^2/2)}{\sqrt{h^2}} = \frac{1}{2}.$$

c) f cannot be differentiable at $(0,0)$; if it were, then the directional derivative obtained in part (b) would have been $\mathbf{u} \bullet \nabla f(0,0) = 0$.

37. $f(x,y) = \begin{cases} \dfrac{2x^2y}{x^4+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}.$

Let $\mathbf{u} = u\mathbf{i} + v\mathbf{j}$ be a unit vector. If $v \neq 0$, then

$$\begin{aligned} D_{\mathbf{u}} f(0,0) &= \lim_{h\to 0+} \frac{1}{h} \frac{2(h^2u^2)(hv)}{h^4u^4 + h^2v^2} \\ &= \lim_{h\to 0+} \frac{2u^2v}{h^2u^4+v^2} = \frac{2u^2}{v}. \end{aligned}$$

If $v = 0$, then $u = \pm 1$ and

$$D_{\mathbf{u}} f(0,0) = \lim_{h\to 0+} = \frac{1}{h}\frac{0}{h^2} = 0.$$

Thus f has a directional derivative in every direction at the origin even though it is not continuous there.

Section 12.8 Implicit Functions (page 698)

1. $xy^3 + x^4y = 2$ defines x as a function of y.

$$y^3 \frac{dx}{dy} + 3xy^2 + 4x^3y\frac{dx}{dy} + x^4 = 0$$

$$\frac{dx}{dy} = -\frac{x^4+3xy^2}{y^3+4x^3y}.$$

The given equation has a solution $x = x(y)$ with this derivative near any point where $y^3 + 4x^3y \neq 0$, i.e., $y \neq 0$ and $y^2 + 4x^3 \neq 0$.

2. $xy^3 = y - z$: $\quad x = x(y,z)$

$$y^3\frac{\partial x}{\partial y} + 3xy^2 = 1$$

$$\frac{\partial x}{\partial y} = \frac{1-3xy^2}{y^3}.$$

The given equation has a solution $x = x(y,z)$ with this partial derivative near any point where $y \neq 0$.

3. $z^2 + xy^3 = \dfrac{xz}{y}$: $\quad z = z(x,y)$

$$2z\frac{\partial z}{\partial y} + 3xy^2 = \frac{x}{y}\frac{\partial z}{\partial y} - \frac{xz}{y^2}$$

$$\frac{\partial z}{\partial y} = \frac{\dfrac{xz}{y^2} + 3xy^2}{\dfrac{x}{y} - 2z} = \frac{xz+3xy^4}{xy - 2y^2z}.$$

The given equation has a solution $z = z(x,y)$ with this derivative near any point where $y \neq 0$ and $x \neq 2yz$.

4. $e^{yz} - x^2z\ln y = \pi$: $\quad y = y(x,z)$

$$e^{yz}\left(z\frac{\partial y}{\partial z} + y\right) - x^2\ln y - \frac{x^2z}{y}\frac{\partial y}{\partial z} = 0$$

$$\frac{\partial y}{\partial z} = \frac{x^2\ln y - ye^{yz}}{ze^{yz} - \dfrac{x^2z}{y}} = \frac{x^2y\ln y - y^2e^{yz}}{yze^{yz} - x^2z}.$$

The given equation has a solution $y = y(x,z)$ with this derivative near any point where $y > 0$, $z \neq 0$, and $ye^{yz} \neq x^2$.

5. $x^2y^2 + y^2z^2 + z^2t^2 + t^2w^2 - xw = 0$: $\quad x = x(y,z,t,w)$

$$2xy^2\frac{\partial x}{\partial w} + 2t^2w - w\frac{\partial x}{\partial w} - x = 0$$

$$\frac{\partial x}{\partial w} = \frac{x - 2t^2w}{2xy^2 - w}.$$

The given equation has a solution with this derivative wherever $w \neq 2xy^2$.

6. $F(x,y,x^2-y^2) = 0$: $\quad y = y(x)$

$$F_1 + F_2\frac{dy}{dx} + F_3\left(2x - 2y\frac{dy}{dx}\right) = 0$$

$$\frac{dy}{dx} = \frac{F_1(x,y,x^2-y^2) + 2xF_3(x,y,x^2-y^2)}{2yF_3(x,y,x^2-y^2) - F_2(x,y,x^2-y^2)}.$$

The given equation has a solution with this derivative near any point where $F_2(x,y,x^2-y^2) \neq 2yF_3(x,y,x^2-y^2)$.

7. $G(x,y,z,u,v) = 0$: $\quad u = u(x,y,z,v)$

$$G_1 + G_4\frac{\partial u}{\partial x}$$

$$\frac{\partial u}{\partial x} = -\frac{G_1(x,y,z,u,v)}{G_4(x,y,z,u,v)}.$$

The given equation has a solution with this derivative near any point where $G_4(x,y,z,u,v) \neq 0$.

8. $F(x^2-z^2, y^2+xz) = 0$: $\quad z = z(x,y)$

$$F_1\left(2x - 2z\frac{\partial z}{\partial x}\right) + F_2\left(x\frac{\partial z}{\partial x} + z\right) = 0$$

$$\frac{\partial z}{\partial x} = \frac{2xF_1(x^2-z^2, y^2+xz) + zF_2(x^2-z^2, y^2+xz)}{2zF_1(x^2-z^2, y^2+xz) - xF_2(x^2-z^2, y^2+xz)}.$$

The given equation has a solution with this derivative near any point where $xF_2(x^2-z^2, y^2+xz) \neq 2zF_1(x^2-z^2, y^2+xz)$.

9. $H(u^2w, v^2t, wt) = 0$: $\quad w = w(u, v, t)$

$$H_1 u^2 \frac{\partial w}{\partial t} + H_2 v^2 + H_3 \left(t \frac{\partial w}{\partial t} + w \right) = 0$$

$$\frac{\partial w}{\partial t} = -\frac{H_2(u^2w, v^2t, wt)v^2 + H_3(u^2w, v^2t, wt)w}{H_1(u^2w, v^2t, wt)u^2 + H_3(u^2w, v^2t, wt)t}.$$

The given equation has a solution with this derivative near any point where $tH_3(u^2w, v^2t, wt) \neq -u^2H_1(u^2w, v^2t, wt)$.

10. $\begin{cases} xyuv = 1 \\ x + y + u + v = 0 \end{cases} \Rightarrow \begin{cases} y = y(x, u) \\ v = v(x, u) \end{cases}$

Differentiate the given equations with respect to x:

$$\begin{aligned} yuv + xuv\frac{\partial y}{\partial x} + xyu\frac{\partial v}{\partial x} &= 0 \\ 1 + \frac{\partial y}{\partial x} + \frac{\partial v}{\partial x} &= 0 \end{aligned}$$

Multiply the last equation by xyu and subtract the two equations:

$$yuv - xyu + (xuv - xyu)\frac{\partial y}{\partial x} = 0$$

$$\left(\frac{\partial y}{\partial x}\right)_u = \frac{y(x - v)}{x(v - y)}.$$

The given equations have a solution of the indicated form with this derivative near any point where $u \neq 0$, $x \neq 0$ and $y \neq v$.

11. $\begin{cases} x^2 + y^2 + z^2 + w^2 = 1 \\ x + 2y + 3z + 4w = 2 \end{cases} \Rightarrow \begin{cases} x = x(y, z) \\ w = w(y, z) \end{cases}$

$$\begin{aligned} 2x\frac{\partial x}{\partial y} + 2y + 2w\frac{\partial w}{\partial y} &= 0 \quad \times 2 \\ \frac{\partial x}{\partial y} + 2 + 4\frac{\partial w}{\partial y} &= 0 \quad \times w \end{aligned}$$

$$(4x - w)\frac{\partial x}{\partial y} + 4y - 2w = 0$$

$$\left(\frac{\partial x}{\partial y}\right)_z = \frac{2w - 4y}{4x - w}.$$

The given equations have a solution of the indicated form with this derivative near any point where $w \neq 4x$.

12. $\begin{cases} x^2y + y^2u - u^3 = 0 \\ x^2 + yu = 1 \end{cases} \Rightarrow \begin{cases} u = u(x) \\ y = y(x) \end{cases}$

$$\begin{aligned} 2xy + (x^2 + 2yu)\frac{dy}{dx} + (y^2 - 3u^2)\frac{du}{dx} &= 0 \\ 2x + u\frac{dy}{dx} + y\frac{du}{dx} &= 0 \end{aligned}$$

Multiply the first equation by u and the second by $x^2 + 2yu$ and subtract:

$$2x(x^2 + yu) + (x^2y + y^2u + 3u^3)\frac{du}{dx} = 0$$

$$\frac{du}{dx} = -\frac{2x(x^2 + yu)}{3u^3 + x^2y + y^2u} = -\frac{x}{2u^3}.$$

The given equations have a solution with the indicated derivative near any point where $u \neq 0$.

13. $\begin{cases} x = u^3 + v^3 \\ y = uv - v^2 \end{cases} \Rightarrow \begin{cases} u = u(x, y) \\ v = v(x, y) \end{cases}$

Take partials with respect to x:

$$\begin{aligned} 1 &= 3u^2\frac{\partial u}{\partial x} + 3v^2\frac{\partial v}{\partial x} \\ 0 &= v\frac{\partial u}{\partial x} + (u - 2v)\frac{\partial v}{\partial x}. \end{aligned}$$

At $u = v = 1$ we have

$$\begin{aligned} 1 &= 3\frac{\partial u}{\partial x} + 3\frac{\partial v}{\partial x} \\ 0 &= \frac{\partial u}{\partial x} - \frac{\partial v}{\partial x}. \end{aligned}$$

Thus $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial x} = \dfrac{1}{6}$.

Similarly, differentiating the given equations with respect to y and putting $u = v = 1$, we get

$$\begin{aligned} 0 &= 3\frac{\partial u}{\partial y} + 3\frac{\partial v}{\partial y} \\ 1 &= \frac{\partial u}{\partial y} - \frac{\partial v}{\partial y}. \end{aligned}$$

Thus $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial y} = \dfrac{1}{2}$.

Finally,

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{6}.$$

14. $\begin{cases} x = r^2 + 2s \\ y = s^2 - 2r \end{cases}$

$$\frac{\partial(x, y)}{\partial(r, s)} = \begin{vmatrix} 2r & 2 \\ -2 & 2s \end{vmatrix} = 4(rs + 1).$$

The given system can be solved for r and s as functions of x and y near any point (r, s) where $rs \neq -1$.

We have

$$\begin{aligned} 1 &= 2r\frac{\partial r}{\partial x} + 2\frac{\partial s}{\partial x} \\ 0 &= -2\frac{\partial r}{\partial x} + 2s\frac{\partial s}{\partial x} \\ 0 &= 2r\frac{\partial r}{\partial y} + 2\frac{\partial s}{\partial y} \\ 1 &= -2\frac{\partial r}{\partial y} + 2s\frac{\partial s}{\partial y}. \end{aligned}$$

Thus

$$\frac{\partial r}{\partial x} = \frac{s}{2(rs+1)} \qquad \frac{\partial r}{\partial y} = -\frac{1}{2(rs+1)}$$
$$\frac{\partial s}{\partial x} = \frac{1}{2(rs+1)} \qquad \frac{\partial s}{\partial y} = \frac{r}{2(rs+1)}.$$

15. $x = r\cos\theta, \qquad y = r\sin\theta$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r.$$

The transformation is one-to-one (and hence invertible) near any point where $r \neq 0$, that is, near any point except the origin.

16. $x = \rho\sin\phi\cos\theta,\ y = \rho\sin\phi\sin\theta,\ z = \rho\cos\phi.$

$$\begin{aligned}
\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} &= \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix} \\
&= \cos\phi \begin{vmatrix} \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix} \\
&\quad + \rho\sin\phi \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix} \\
&= \rho^2\cos\phi\Big[\cos\phi\sin\phi\cos^2\theta + \sin\phi\cos\phi\sin^2\theta\Big] \\
&\quad + \rho^2\sin\phi\Big[\sin^2\phi\cos^2\theta + \sin^2\phi\sin^2\theta\Big] \\
&= \rho^2\cos^2\phi\sin\phi + \rho^2\sin^3\phi = \rho^2\sin\phi.
\end{aligned}$$

The transformation is one-to-one (and invertible) near any point where $\rho^2\sin\phi \neq 0$, that is, near any point not on the z-axis.

17. Let $F(x,y,z,u,v) = xy^2 + zu + v^2 - 3$
$G(x,y,z,u,v) = x^3z + 2y - uv - 2$
$H(x,y,z,u,v) = xu + yv - xyz - 1.$
Then

$$\frac{\partial(F,G,H)}{\partial(x,y,z)} = \begin{vmatrix} y^2 & 2xy & u \\ 3x^2z & 2 & x^3 \\ u-yz & v-xz & -xy \end{vmatrix}.$$

At point P_0 where $x = y = z = u = v = 1$, we have

$$\frac{\partial(F,G,H)}{\partial(x,y,z)} = \begin{vmatrix} 1 & 2 & 1 \\ 3 & 2 & 1 \\ 0 & 0 & -1 \end{vmatrix} = 4.$$

Since this Jacobian is not zero, the equations $F = G = H = 0$ can be solved for x, y, and z as functions of u and v near P_0. Also,

$$\begin{aligned}
\left(\frac{\partial y}{\partial u}\right)_v\bigg|_{(1,1)} &= -\frac{1}{4}\frac{\partial(F,G,H)}{\partial(x,u,z)}\bigg|_{P_0} \\
&= -\frac{1}{4}\begin{vmatrix} y^2 & z & u \\ 3x^2z & -v & x^3 \\ u-yz & x & -xy \end{vmatrix}\Bigg|_{P_0} \\
&= -\frac{1}{4}\begin{vmatrix} 1 & 1 & 1 \\ 3 & -1 & 1 \\ 0 & 1 & -1 \end{vmatrix} = -\frac{3}{2}.
\end{aligned}$$

18. Let $F(x,y,z,u,v) = xe^y + uz - \cos v - 2$
$G(x,y,z,u,v) = u\cos y + x^2v - yz^2 - 1.$
If P_0 is the point where $(x,y,z) = (2,0,1)$ and $(u,v) = (1,0)$, then

$$\begin{aligned}
\frac{\partial(F,G)}{\partial(u,v)}\bigg|_{P_0} &= \begin{vmatrix} z & \sin v \\ \cos y & x^2 \end{vmatrix}\Bigg|_{P_0} \\
&= \begin{vmatrix} 1 & 0 \\ 1 & 4 \end{vmatrix} = 4.
\end{aligned}$$

Since this Jacobian is not zero, the equations $F = G = 0$ can be solved for u, and v in terms of x, y and z near P_0. Also,

$$\begin{aligned}
\left(\frac{\partial u}{\partial z}\right)_{x,y}\bigg|_{(2,0,1)} &= -\frac{1}{4}\frac{\partial(F,G)}{\partial(z,v)}\bigg|_{P_0} \\
&= -\frac{1}{4}\begin{vmatrix} u & \sin v \\ -2yz & x^2 \end{vmatrix}\Bigg|_{P_0} \\
&= -\frac{1}{4}\begin{vmatrix} 1 & 0 \\ 0 & 4 \end{vmatrix} = -1.
\end{aligned}$$

19. $$\begin{cases} F(x,y,z,w) = 0 \\ G(x,y,z,w) = 0 \\ H(x,y,x,w) = 0 \end{cases} \Rightarrow \begin{cases} x = \text{x(y)} \\ z = \text{z(y)} \\ w = \text{w(y)} \end{cases}$$

$$\begin{aligned}
F_1\frac{dx}{dy} + F_2 + F_3\frac{dz}{dy} + F_4\frac{dw}{dy} &= 0 \\
G_1\frac{dx}{dy} + G_2 + G_3\frac{dz}{dy} + G_4\frac{dw}{dy} &= 0 \\
H_1\frac{dx}{dy} + H_2 + H_3\frac{dz}{dy} + H_4\frac{dw}{dy} &= 0
\end{aligned}$$

By Cramer's Rule,

$$\frac{dx}{dy} = -\frac{\dfrac{\partial(F,G,H)}{\partial(y,z,w)}}{\dfrac{\partial(F,G,H)}{\partial(x,z,w)}}.$$

20. $F(x, y, z, u, v) = 0$
$G(x, y, z, u, v) = 0$
$H(x, y, z, u, v) = 0$
To calculate $\frac{\partial x}{\partial y}$ we require that x be one of three dependent variables, and y be one of two independent variables. The other independent variable can be z or u or v. The possible interpretations for this partial, and their values, are

$$\left(\frac{\partial x}{\partial y}\right)_z = -\frac{\frac{\partial(F,G,H)}{\partial(y,u,v)}}{\frac{\partial(F,G,H)}{\partial(x,u,v)}}$$
$$\left(\frac{\partial x}{\partial y}\right)_u = -\frac{\frac{\partial(F,G,H)}{\partial(y,z,v)}}{\frac{\partial(F,G,H)}{\partial(x,z,v)}}$$
$$\left(\frac{\partial x}{\partial y}\right)_v = -\frac{\frac{\partial(F,G,H)}{\partial(y,z,u)}}{\frac{\partial(F,G,H)}{\partial(x,z,u)}}.$$

21. $F(x_1, x_2, \dots, x_8) = 0$
$G(x_1, x_2, \dots, x_8) = 0$
$H(x_1, x_2, \dots, x_8) = 0$
To find $\frac{\partial x_1}{\partial x_2}$ we require that x_1 be one of three dependent variables, and that x_2 be one of five independent variables. The other four independent variables must be chosen from among the six remaining variables. This can be done in

$$\binom{6}{4} = \frac{6!}{4!2!} = 15 \text{ ways.}$$

There are 15 possible interpretations for $\frac{\partial x_1}{\partial x_2}$.
We have

$$\left(\frac{\partial x_1}{\partial x_2}\right)_{x_4x_6x_7x_8} = -\frac{\frac{\partial(F,G,H)}{\partial(x_2,x_3,x_5)}}{\frac{\partial(F,G,H)}{\partial(x_1,x_3,x_5)}}.$$

22. If $F(x, y, z) = 0 \quad \Rightarrow \quad z = z(x, y)$, then

$$F_1 + F_3\frac{\partial z}{\partial x} = 0, \qquad F_2 + F_3\frac{\partial z}{\partial y} = 0$$
$$F_{11} + F_{13}\frac{\partial z}{\partial x} + F_{31}\frac{\partial z}{\partial x} + F_{33}\left(\frac{\partial z}{\partial x}\right)^2 + F_3\frac{\partial^2 z}{\partial x^2} = 0.$$

Thus

$$\begin{aligned}\frac{\partial^2 z}{\partial x^2} &= -\frac{1}{F_3}\left[F_{11} + 2F_{13}\left(-\frac{F_1}{F_3}\right) + F_{33}\left(-\frac{F_1}{F_3}\right)^2\right]\\ &= -\frac{1}{F_3^3}\Big[F_{11}F_3^2 - 2F_1F_3F_{13} + F_1^2F_{33}\Big].\end{aligned}$$

Similarly,

$$\frac{\partial^2 z}{\partial y^2} = -\frac{1}{F_3^3}\Big[F_{22}F_3^2 - 2F_2F_3F_{23} + F_2^2F_{33}\Big].$$

Also,

$$F_{12} + F_{13}\frac{\partial z}{\partial y} + \left(F_{32} + F_{33}\frac{\partial z}{\partial y}\right)\frac{\partial z}{\partial x} + F_3\frac{\partial^2 z}{\partial y\partial x}.$$

Therefore

$$\begin{aligned}\frac{\partial^2 z}{\partial x\partial y} &= -\frac{1}{F_3}\Bigg[F_{12} + F_{13}\left(-\frac{F_2}{F_3}\right) + F_{23}\left(-\frac{F_1}{F_3}\right)\\ &\qquad + F_{33}\left(\frac{F_1F_2}{F_3^2}\right)\Bigg]\\ &= -\frac{1}{F_3^2}\Big[F_3^2F_{12} - F_2F_3F_{13} - F_1F_3F_{23} + F_1F_2F_{33}\Big].\end{aligned}$$

23. $x = u + v,\ y = uv,\ z = u^2 + v^2.$
The first two equations define u and v as functions of x and y, and therefore derivatives of z with respect to x and y can be determined by the Chain Rule.
Differentiate the first two equations with respect to x:

$$\begin{aligned}1 &= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}\\ 0 &= v\frac{\partial u}{\partial x} + u\frac{\partial v}{\partial x}.\end{aligned}$$

Thus $\frac{\partial u}{\partial x} = \frac{u}{u-v}$ and $\frac{\partial v}{\partial x} = \frac{v}{v-u}$, and

$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}\\ &= 2u\frac{u}{u-v} + 2v\frac{v}{v-u} = \frac{2(u^2-v^2)}{u-v} = 2(u+v) = 2x.\end{aligned}$$

Similarly, differentiating the first two of the given equations with respect to y, we get

$$\begin{aligned}0 &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\\ 1 &= v\frac{\partial u}{\partial y} + u\frac{\partial v}{\partial y}.\end{aligned}$$

Thus $\frac{\partial u}{\partial y} = \frac{1}{v-u}$ and $\frac{\partial v}{\partial y} = \frac{1}{u-v}$, and

$$\frac{\partial z}{\partial y} = \frac{2u}{v-u} + \frac{2v}{u-v} = \frac{2(u-v)}{v-u} = -2$$
$$\frac{\partial^2 z}{\partial x\partial y} = 0.$$

24. $pV = T - \dfrac{4p}{T^2}, \qquad T = T(p, V)$

a) $V = \dfrac{\partial T}{\partial p} - \dfrac{4}{T^2} + \dfrac{8p}{T^3}\dfrac{\partial T}{\partial p}$

$p = \dfrac{\partial T}{\partial V} + \dfrac{8p}{T^3}\dfrac{\partial T}{\partial V}.$

Putting $p = V = 1$ and $T = 2$, we obtain

$$2\frac{\partial T}{\partial p} = 2, \qquad 2\frac{\partial T}{\partial V} = 1,$$

so $\dfrac{\partial T}{\partial p} = 1$ and $\dfrac{\partial T}{\partial V} = \dfrac{1}{2}$.

b) $dT = \dfrac{\partial T}{\partial p}\,dp + \dfrac{\partial T}{\partial V}\,dV.$

If $p = 1$, $|dp| \le 0.001$, $V = 1$, and $|dV| \le 0.002$, then $T = 2$ and

$$|dT| \le (1)(0.001) + \frac{1}{2}(0.002) = 0.002.$$

The approximate maximum error in T is 0.002.

25. $F(x, y, z) = 0$

$$F_1\left(\frac{\partial x}{\partial y}\right)_z + F_2 = 0, \quad \Rightarrow \quad \left(\frac{\partial x}{\partial y}\right)_z = -\frac{F_2}{F_1}.$$

Similarly, $\left(\dfrac{\partial y}{\partial z}\right)_x = -\dfrac{F_3}{F_2}$, and $\left(\dfrac{\partial z}{\partial x}\right)_y = -\dfrac{F_1}{F_3}$. Hence

$$\left(\frac{\partial x}{\partial y}\right)_z\left(\frac{\partial y}{\partial z}\right)_x\left(\frac{\partial z}{\partial x}\right)_y = (-1)^3 = -1.$$

For $F(x, y, z, u) = 0$ we have, similarly,

$$\left(\frac{\partial x}{\partial y}\right)_{z,u}\left(\frac{\partial y}{\partial z}\right)_{u,x}\left(\frac{\partial z}{\partial u}\right)_{x,y}\left(\frac{\partial u}{\partial x}\right)_{y,z} = (-1)^4 = 1.$$

For $F(x, y, z, u, v) = 0$ we have, similarly,

$$\left(\frac{\partial x}{\partial y}\right)_{z,u,v}\left(\frac{\partial y}{\partial z}\right)_{u,v,x}\left(\frac{\partial z}{\partial u}\right)_{v,x,y}\left(\frac{\partial u}{\partial v}\right)_{x,y,z}\left(\frac{\partial v}{\partial x}\right)_{y,z,u}$$
$$= (-1)^5 = -1.$$

In general, if $F(x_1, x_2, \ldots, x_n) = 0$, then

$$\left(\frac{\partial x_1}{\partial x_2}\right)_{x_3,\ldots,x_n}\left(\frac{\partial x_2}{\partial x_3}\right)_{x_4,\ldots,x_n,x_1}\cdots\left(\frac{\partial x_n}{\partial x_1}\right)_{x_2,\ldots,x_{n-1}}$$
$$= (-1)^n.$$

26. Given $F(x, y, u, v) = 0, \qquad G(x, y, u, v) = 0$, let

$$\Delta = \frac{\partial(F, G)}{\partial(x, y)} = \frac{\partial F}{\partial x}\frac{\partial G}{\partial y} - \frac{\partial F}{\partial y}\frac{\partial G}{\partial x}.$$

Then, regarding the given equations as defining x and y as functions of u and v, we have

$$\frac{\partial x}{\partial u} = -\frac{1}{\Delta}\frac{\partial(F, G)}{\partial(u, y)} \qquad \frac{\partial y}{\partial u} = -\frac{1}{\Delta}\frac{\partial(F, G)}{\partial(x, u)}$$
$$\frac{\partial x}{\partial v} = -\frac{1}{\Delta}\frac{\partial(F, G)}{\partial(v, y)} \qquad \frac{\partial y}{\partial v} = -\frac{1}{\Delta}\frac{\partial(F, G)}{\partial(x, v)}.$$

Therefore,

$$\begin{aligned}
\frac{\partial(x, y)}{\partial(u, v)} &= \frac{1}{\Delta^2}\left(\frac{\partial(F, G)}{\partial(u, y)}\frac{\partial(F, G)}{\partial(x, v)} - \frac{\partial(F, G)}{\partial(v, y)}\frac{\partial(F, G)}{\partial(x, u)}\right)\\
&= \frac{1}{\Delta^2}\left[\left(\frac{\partial F}{\partial u}\frac{\partial G}{\partial y} - \frac{\partial F}{\partial y}\frac{\partial G}{\partial u}\right)\left(\frac{\partial F}{\partial x}\frac{\partial G}{\partial v} - \frac{\partial F}{\partial v}\frac{\partial G}{\partial x}\right)\right.\\
&\quad\left. - \left(\frac{\partial F}{\partial v}\frac{\partial G}{\partial y} - \frac{\partial F}{\partial y}\frac{\partial G}{\partial v}\right)\left(\frac{\partial F}{\partial x}\frac{\partial G}{\partial u} - \frac{\partial F}{\partial u}\frac{\partial G}{\partial x}\right)\right]\\
&= \frac{1}{\Delta^2}\left[\frac{\partial F}{\partial u}\frac{\partial G}{\partial y}\frac{\partial F}{\partial x}\frac{\partial G}{\partial v} - \frac{\partial F}{\partial y}\frac{\partial G}{\partial u}\frac{\partial F}{\partial x}\frac{\partial G}{\partial v}\right.\\
&\quad - \frac{\partial F}{\partial u}\frac{\partial G}{\partial y}\frac{\partial F}{\partial v}\frac{\partial G}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial G}{\partial u}\frac{\partial F}{\partial v}\frac{\partial G}{\partial x}\\
&\quad - \frac{\partial F}{\partial v}\frac{\partial G}{\partial y}\frac{\partial F}{\partial x}\frac{\partial G}{\partial u} + \frac{\partial F}{\partial v}\frac{\partial G}{\partial y}\frac{\partial F}{\partial u}\frac{\partial G}{\partial x}\\
&\quad\left. + \frac{\partial F}{\partial y}\frac{\partial G}{\partial v}\frac{\partial F}{\partial x}\frac{\partial G}{\partial u} - \frac{\partial F}{\partial y}\frac{\partial G}{\partial v}\frac{\partial F}{\partial u}\frac{\partial G}{\partial x}\right]\\
&= \frac{1}{\Delta^2}\left[\frac{\partial F}{\partial u}\frac{\partial G}{\partial y}\frac{\partial F}{\partial x}\frac{\partial G}{\partial v} + \frac{\partial F}{\partial y}\frac{\partial G}{\partial u}\frac{\partial F}{\partial v}\frac{\partial G}{\partial x}\right.\\
&\quad\left. - \frac{\partial F}{\partial v}\frac{\partial G}{\partial y}\frac{\partial F}{\partial x}\frac{\partial G}{\partial u} - \frac{\partial F}{\partial y}\frac{\partial G}{\partial v}\frac{\partial F}{\partial u}\frac{\partial G}{\partial x}\right]\\
&= \frac{1}{\Delta^2}\left(\frac{\partial F}{\partial x}\frac{\partial G}{\partial y} - \frac{\partial F}{\partial y}\frac{\partial G}{\partial x}\right)\left(\frac{\partial F}{\partial u}\frac{\partial G}{\partial v} - \frac{\partial F}{\partial v}\frac{\partial G}{\partial u}\right)\\
&= \frac{1}{\Delta^2}\frac{\partial(F, G)}{\partial(x, y)}\frac{\partial(F, G)}{\partial(u, v)}\\
&= \frac{1}{\Delta}\frac{\partial(F, G)}{\partial(u, v)} = \frac{\partial(F, G)}{\partial(u, v)}\bigg/\frac{\partial(F, G)}{\partial(x, y)}.
\end{aligned}$$

27. By Exercise 26, with the roles of (x, y) and (u, v) reversed, we have

$$\frac{\partial(u, v)}{\partial(x, y)} = \frac{\partial(F, G)}{\partial(x, y)}\bigg/\frac{\partial(F, G)}{\partial(u, v)}.$$

Apply this with

$$F(x, y, u, v) = f(u, v) - x = 0$$
$$G(x, y, u, v) = g(u, v) - y = 0$$

so that

$$\frac{\partial(F, G)}{\partial(x, y)} = \begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} = 1$$

and

$$\frac{\partial(F, G)}{\partial(u, v)} = \frac{\partial(f, g)}{\partial(u, v)} = \frac{\partial(x, y)}{\partial(u, v)}$$

and we obtain

$$\frac{\partial(u,v)}{\partial(x,y)} = 1\bigg/\frac{\partial(x,y)}{\partial(u,v)}.$$

28. By the Chain Rule,

$$\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial s} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial s} \end{pmatrix}$$
$$= \begin{pmatrix} \frac{\partial x}{\partial u}\frac{\partial u}{\partial r} + \frac{\partial x}{\partial v}\frac{\partial v}{\partial r} & \frac{\partial x}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial x}{\partial v}\frac{\partial v}{\partial s} \\ \frac{\partial y}{\partial u}\frac{\partial u}{\partial r} + \frac{\partial y}{\partial v}\frac{\partial v}{\partial r} & \frac{\partial y}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial y}{\partial v}\frac{\partial v}{\partial s} \end{pmatrix}$$
$$= \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial s} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial s} \end{pmatrix}.$$

Since the determinant of a product of matrices is the product of their determinants, we have

$$\frac{\partial(x,y)}{\partial(r,s)} = \frac{\partial(x,y)}{\partial(u,v)}\frac{\partial(u,v)}{\partial(r,s)}.$$

29. If $f(x,y) = k\Big(g(x,y)\Big)$, then

$$\frac{\partial f}{\partial x} = k'\Big(g(x,y)\Big)\frac{\partial g}{\partial x}, \qquad \frac{\partial f}{\partial y} = k'\Big(g(x,y)\Big)\frac{\partial g}{\partial y}.$$

Therefore,

$$\frac{\partial(f,g)}{\partial(x,y)} = k'\Big(g(x,y)\Big)\frac{\partial(g,g)}{\partial(r,s)} = 0.$$

30. Let $u = f(x,y)$ and $v = g(x,y)$, and suppose that

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(f,g)}{\partial(x,y)} = 0$$

for all (x,y). Thus

$$\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x} = 0.$$

Now consider the equations $u = f(x,y)$ and $v = g(x,y)$ as defining u and y as functions of x and v. Holding v constant and differentiating with respect to x, we get

$$\frac{\partial g}{\partial x} + \frac{\partial g}{\partial y}\frac{\partial y}{\partial x} = 0,$$

and

$$\begin{aligned}\left(\frac{\partial u}{\partial x}\right)_v &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial x} \\ &= \frac{1}{\frac{\partial g}{\partial y}}\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}\right) = 0.\end{aligned}$$

This says that $u = u(x,v)$ is independent of x, and so depends only on v: $u = k(v)$ for some function k of one variable. Thus $f(x,y) = k\Big(g(x,y)\Big)$, so f and g are functionally dependent.

Section 12.9 Taylor Series and Approximations (page 704)

1. Since the Maclaurin series for $\dfrac{1}{1+t}$ is

$$1 - t + t^2 - \cdots = \sum_{n=0}^{\infty}(-1)^n t^n,$$

the Taylor series for

$$f(x,y) = \frac{1}{2+xy^2} = \frac{1}{2}\,\frac{1}{1+\frac{xy^2}{2}}$$

about $(0,0)$ is $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^n y^{2n}}{2^{n+1}}$.

2. Since

$$\begin{aligned} f(x,y) &= \ln(1+x+y+xy) \\ &= \ln\Big((1+x)(1+y)\Big) \\ &= \ln(1+x) + \ln(1+y), \end{aligned}$$

the Taylor series for f about $(0,0)$ is

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n+y^n}{n}.$$

3. Since $f(x,y) = \tan^{-1}(x+xy) = \tan^{-1}(ux)$, where $u = y+1$, the Taylor series for f about $(0,-1)$ is

$$\sum_{n=0}^{\infty}(-1)^n\frac{(ux)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}(1+y)^{2n+1}}{2n+1}.$$

4. Let $u = x-1$, $v = y+1$. Thus

$$\begin{aligned} f(x,y) &= x^2 + xy + y^3 \\ &= (u+1)^2 + (u+1)(v-1) + (v-1)^3 \\ &= 1 + 2u + u^2 - 1 + v - u + uv + v^3 - 3v^2 + 3v - 1 \\ &= -1 + u + 4v + u^2 + uv - 3v^2 + v^3 \\ &= -1 + (x-1) + 4(y+1) + (x-1)^2 \\ &\quad + (x-1)(y+1) - 3(y+1)^2 + (y+1)^3. \end{aligned}$$

This is the Taylor series for f about $(1,-1)$.

5. $$\begin{aligned} f(x,y) &= e^{x^2+y^2} \\ &= \sum_{n=0}^{\infty} \frac{(x^2+y^2)^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} x^{2j} y^{2n-2j} \\ &= \sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{x^{2j} y^{2n-2j}}{j!(n-j)!}. \end{aligned}$$
This is the Taylor series for f about $(0,0)$.

6. $$f(x,y) = \sin(2x+3y) = \sum_{n=0}^{\infty} (-1)^n \frac{(2x+3y)^{2n+1}}{(2n+1)!}$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \sum_{j=0}^{2n+1} \frac{(2n+1)!}{j!(2n+1-j)!} (2x)^j (3y)^{2n+1-j}$$
$$= \sum_{n=0}^{\infty} \sum_{j=0}^{2n+1} \frac{(-1)^n 2^j 3^{2n+1-j}}{j!(2n+1-j)!} x^j y^{2n+1-j}.$$
This is the Taylor series for f about $(0,0)$.

7. Let $u = x-2$, $v = y-1$. Then

$$\begin{aligned} f(x,y) &= \frac{1}{2+x-2y} = \frac{1}{2+(2+u)-2(v+1)} \\ &= \frac{1}{2+u-2v} = \frac{1}{2\left(1+\dfrac{u-2v}{2}\right)} \\ &= \frac{1}{2}\left[1 - \frac{u-2v}{2} + \left(\frac{u-2v}{2}\right)^2 - \left(\frac{u-2v}{2}\right)^3 + \cdots\right] \\ &= \frac{1}{2} - \frac{u}{4} + \frac{v}{2} + \frac{u^2}{8} - \frac{uv}{2} \\ &\quad + \frac{v^2}{2} - \frac{u^3}{16} + \frac{3u^2v}{8} - \frac{3uv^2}{4} + \frac{v^3}{2} + \cdots. \end{aligned}$$

The Taylor polynomial of degree 3 for f about $(2,1)$ is

$$\begin{aligned} &\frac{1}{2} - \frac{x-2}{4} + \frac{y-1}{2} + \frac{(x-1)^2}{8} \\ &- \frac{(x-2)(y-1)}{2} + \frac{(y-1)^2}{2} - \frac{(x-2)^3}{16} \\ &+ \frac{3(x-2)^2(y-1)}{8} - \frac{3(x-2)(y-1)^2}{4} + \frac{(y-1)^3}{2}. \end{aligned}$$

8. Let $u = x-1$. Then

$$\begin{aligned} f(x,y) &= \ln(x^2+y^2) = \ln(1+2u+u^2+y^2) \\ &= (2u+u^2+y^2) - \frac{(2u+u^2+y^2)^2}{2} \\ &\quad + \frac{(2u+u^2+y^2)^3}{3} - \cdots \\ &= 2u + u^2 + y^2 - 2u^2 - 2u^3 - 2uy^2 + \frac{8u^3}{3} + \cdots. \end{aligned}$$

The Taylor polynomial of degree 3 for f near $(1,0)$ is

$$\begin{aligned} &2(x-1) - (x-1)^2 + y^2 - 2(x-1)^3 \\ &\qquad - 2(x-1)y^2 + \frac{8}{3}(x-1)^3. \end{aligned}$$

9. $$\begin{aligned} f(x,y) &= \int_0^{x+y^2} e^{-t^2}\,dt \\ &= \int_0^{x+y^2} \left(1 - t^2 + \cdots\right) dt \\ &= \left(t - \frac{t^3}{3} + \cdots\right)\Bigg|_0^{x+y^2} \\ &= x + y^2 - \frac{1}{3}(x+y^2)^3 + \cdots \\ &= x + y^2 - \frac{x^3}{3} + \cdots. \end{aligned}$$
The Taylor polynomial of degree 3 for f near $(0,0)$ is

$$x + y^2 - \frac{x^3}{3}.$$

10. $$\begin{aligned} f(x,y) &= \cos(x+\sin y) \\ &= 1 - \frac{(x+\sin y)^2}{2!} + \frac{(x+\sin y)^4}{4!} - \cdots \\ &= 1 - \frac{\left(x + y - \dfrac{y^3}{6} + \cdots\right)^2}{2} + \frac{(x+y-\cdots)^4}{4} - \cdots \\ &= 1 - \frac{1}{2}\left(x^2 + y^2 + 2xy - \frac{xy^3}{3} - \frac{y^4}{3} + \cdots\right) \\ &\quad + \frac{1}{4}(x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 + \cdots). \end{aligned}$$
The Taylor polynomial of degree 4 for f near $(0,0)$ is

$$\begin{aligned} &1 - \frac{x^2}{2} - xy - \frac{y^2}{2} + \frac{x^4}{4} + x^3y \\ &\qquad + \frac{3x^2y^2}{2} + \frac{7xy^3}{6} + \frac{5y^4}{12}. \end{aligned}$$

11. Let $u = x - \dfrac{\pi}{2}$, $v = y-1$. Then

$$\begin{aligned} f(x,y) &= \frac{\sin x}{y} = \frac{\sin(u+\pi/2)}{1+v} = \frac{\cos u}{1+v} \\ &= \left(1 - \frac{u^2}{2} + \cdots\right)(1 - v + v^2 - \cdots) \\ &= 1 - v - \frac{u^2}{2} + v^2 + \cdots. \end{aligned}$$

The Taylor polynomial of degree 2 for f near $(\pi/2, 1)$ is

$$1 - (y-1) - \frac{1}{2}\left(x - \frac{\pi}{2}\right)^2 + (y-1)^2.$$

12. $f(x,y) = \dfrac{1+x}{1+x^2+y^4}$
$= (1+x)\Big(1-(x^2+y^4)+\cdots\Big)$
$= 1+x-x^2-\cdots.$

The Taylor polynomial of degree 2 for f near $(0,0)$ is

$$1+x-x^2.$$

13. The equation $x\sin y = y+\sin x$ can be written $F(x,y)=0$ where $F(x,y)=x\sin y - y - \sin x$. Since $F(0,0)=0$, and $F_2(0,0)=-1\neq 0$, the given equation has a solution of the form $y=f(x)$ where $f(0)=0$.
Try $y=a_1x+a_2x^2+a_3x^3+a_4x^4+\cdots$. Then

$$\begin{aligned}\sin y &= y-\frac{1}{6}y^3+\cdots\\ &= a_1x+a_2x^2+a_3x^3+a_4x^4+\cdots-\frac{1}{6}(a_1x+\cdots)^3+\cdots.\end{aligned}$$

Substituting into the given equation we obtain

$$\begin{aligned}&a_1x^2+a_2x^3+\left(a_3-\frac{1}{6}a_1^3\right)x^4+\cdots\\ &\quad= a_1x+a_2x^2+a_3x^3+a_4x^4+\cdots+x-\frac{1}{6}x^3+\cdots.\end{aligned}$$

Comparing coefficients of various powers of x on both sides, we get

$$a_1+1=0,\quad a_2=a_1,\quad a_3-\frac{1}{6}=a_2.$$

Thus $a_1=-1$, $a_2=-1$, and $a_3=-5/6$. The required solution is

$$y=-x-x^2-\frac{5}{6}x^3+\cdots.$$

14. The equation $\sqrt{1+xy}=1+x+\ln(1+y)$ can be rewritten $F(x,y)=0$, where $F(x,y)=\sqrt{1+xy}-1-x-\ln(1+y)$. Since $F(0,0)=0$ and $F_2(0,0)=-1\neq 0$, the given equation has a solution of the form $y=f(x)$ where $f(0)=0$.

Try $y=a_1x+a_2x^2+a_3x^3+a_4x^4+\cdots$. We have

$$\begin{aligned}&\sqrt{1+xy}\\ &=\sqrt{1+a_1x^2+a_2x^3+a_3x_4+\cdots}\\ &=1+\frac{1}{2}(a_1x^2+a_2x^3+a_3x^4+\cdots)\\ &\qquad-\frac{1}{8}(a_1x^2+\cdots)^2+\cdots\\ &1+x+\ln(1+y)\\ &=1+x+(a_1x+a_2x^2+a_3x^3+a_4x^4+\cdots)\\ &\quad-\frac{1}{2}(a_1x+a_2x^2+a_3x^3+\cdots)^2+\frac{1}{3}(a_1x+a_2x^2\cdots)^3-\cdots\end{aligned}$$

Thus we must have

$$\begin{aligned}&0=1+a_1\\ &\frac{1}{2}a_1=a_2-\frac{1}{2}a_1^2\\ &\frac{1}{2}a_2=a_3-a_1a_2+\frac{1}{3}a_1^3\\ &\frac{1}{2}a_3-\frac{1}{8}a_1^2=a_4-\frac{1}{2}a_2^2-a_1a_3+a_1^2a_2,\end{aligned}$$

and $a_1=-1$, $a_2=0$, $a_3=\dfrac{1}{3}$, $a_4=-\dfrac{7}{24}$. The required solution is

$$y=-x+\frac{1}{3}x^3-\frac{7}{24}x^4+\cdots.$$

15. The equation $x+2y+z+e^{2z}=1$ can be written $F(x,y,z)=0$, where $F(x,y,z)=x+2y+z+e^{2z}-1$. Since $F(0,0,0)=0$ and $F_3(0,0,0)=3\neq 0$, the given equation has a solution of the form $z=f(x,y)$, where $f(0,0)=0$.

Try $z=Ax+By+Cx^2+Dxy+Ey^2+\cdots$. Then

$$\begin{aligned}&x+2y+Ax+By+Cx^2+Dxy+Ey^2+\cdots\\ &\qquad+1+2(Ax+By+Cx^2+Dxy+Ey^2+\cdots)\\ &\qquad+2(Ax+By+\cdots)^2+\cdots=1.\end{aligned}$$

Thus

$$\begin{aligned}1+A+2A=0&\quad\Rightarrow\quad A=-1/3\\ 2+B+2B=0&\quad\Rightarrow\quad B=-2/3\\ C+2C+2A^2=0&\quad\Rightarrow\quad C=-2/27\\ D+2D+4AB=0&\quad\Rightarrow\quad D=-8/27\\ E+2E+2B^2=0&\quad\Rightarrow\quad E=-8/27.\end{aligned}$$

The Taylor polynomial of degree 2 for z is

$$-\frac{1}{3}x-\frac{2}{3}y-\frac{2}{27}x^2-\frac{8}{27}xy-\frac{8}{27}y^2.$$

16. The coefficient of x^2y in the Taylor series for $f(x,y)=\tan^{-1}(x+y)$ about $(0,0)$ is

$$\frac{1}{2!1!}f_{112}(0,0)=\frac{1}{2}f_{112}(0,0).$$

But

$$\begin{aligned}\tan^{-1}(x+y)&=x+y-\frac{1}{3}(x+y)^3+\cdots\\ &=x+y-\frac{1}{3}(x^3+3x^2y+3xy^2+y^3)+\cdots\end{aligned}$$

so the coefficient of x^2y is -1. Hence $f_{112}(0,0) = -2$.

17. Let $f(x,y) = \dfrac{1}{1+x^2+y^2}$.
The coefficient of $x^{2n}y^{2n}$ in the Taylor series for $f(x,y)$ about $(0,0)$ is

$$\frac{1}{(2n)!(2n)!}\frac{\partial^{4n}}{\partial x^{2n}\partial y^{2n}}f(x,y)\bigg|_{(0,0)}.$$

However,

$$\begin{aligned} f(x,y) &= \sum_{j=0}^{\infty}(-1)^j(x^2+y^2)^j \\ &= \sum_{j=0}^{\infty}(-1)^j\sum_{k=0}^{j}\frac{j!}{k!(j-k)!}x^{2k}y^{2j-2k}. \end{aligned}$$

The coefficient of $x^{2n}y^{2n}$ is

$$(-1)^{2n}\frac{(2n)!}{n!n!} = \frac{(2n)!}{(n!)^2}.$$

Thus $\dfrac{\partial^{4n}}{\partial x^{2n}\partial y^{2n}}f(x,y)\bigg|_{(0,0)} = \dfrac{[(2n)!]^3}{(n!)^2}$.

Review Exercises 12 (page 704)

1. $x + \dfrac{4y^2}{x} = C$

$x^2 + 4y^2 = Cx$

$\dfrac{\left(x-(C/2)\right)^2}{(C/2)^2} + \dfrac{y^2}{(C/4)^2} = 1$

Ellipse: centre $((C/2),0)$, semi-axes: $C/2$, $C/4$, with the origin deleted.

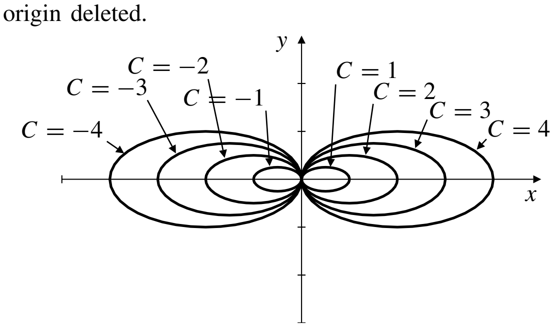


Fig. R-12.1

2. $T = \dfrac{140+30x^2-60x+120y^2}{8+x^2-2x+4y^2}$

$= 30 - \dfrac{100}{(x-1)^2+4y^2+7}$

Ellipses: centre $(1,0)$, values of T between $30-(100/7)$ (minimum) at $(1,0)$ and 30 (at infinite distance from $(1,0)$).

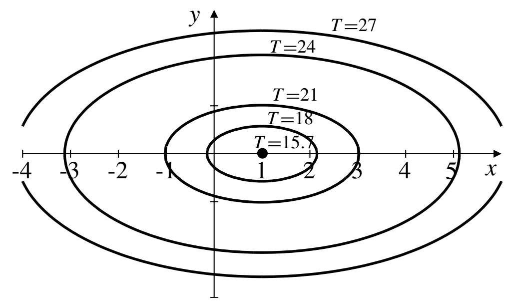


Fig. R-12.2

3. The graph is a saddle-like surface with downward slopes for legs and a tail, thus *monkey saddle*.

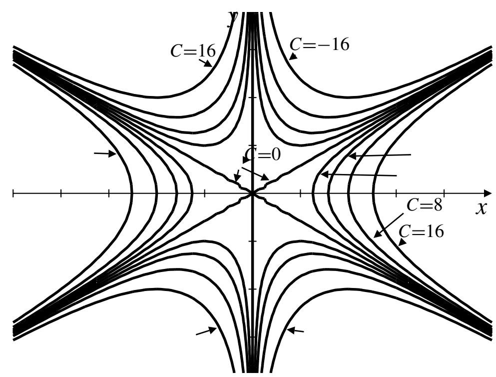


Fig. R-12.3

4. $f(x,y) = \begin{cases} x^3/(x^2+y^2) & \text{if } (x,y)\neq(0,0) \\ 0 & \text{if } (x,y)=(0,0) \end{cases}.$

$$\begin{aligned} f_1(0,0) &= \lim_{h\to 0}\frac{(h^3-0)/h^2}{h} = 1 \\ f_2(0,0) &= \lim_{k\to 0}\frac{0-0}{k} = 0. \end{aligned}$$

For $(x,y)\neq(0,0)$, we have

$$\begin{aligned} f_1(x,y) &= \frac{x^4+3x^2y^2}{(x^2+y^2)^2} \\ f_2(x,y) &= -\frac{2x^3y}{(x^2+y^2)^2} \\ f_{12}(0,0) &= \lim_{k\to 0}\frac{f_1(0,k)-f_1(0,0)}{k} = \lim_{k\to 0}\frac{0-1}{k} \text{ does not exist} \\ f_{21}(0,0) &= \lim_{h\to 0}\frac{f_2(h,0)-f_2(0,0)}{h} = \lim_{h\to 0}\frac{0-0}{h} = 0. \end{aligned}$$

5. $f(x,y) = \dfrac{x^3-y^3}{x^2-y^2} = \dfrac{(x-y)(x^2+xy+y^2)}{(x-y)(x+y)}.$

f is continuous except on the lines $x = y$ and $x = -y$ where it is not defined. It has a continuous extension, namely $\dfrac{x^2+xy+y^2}{x+y}$, to all points of $x = y$ except the origin. It cannot be extended so as to be continuous at the origin. For example, if $(x, y) \to (0, 0)$ along the curve $y = -x + x^4$, then

$$f(x,y) = \frac{x^2 - x^2 + x^5 + (x^4 - x)^2}{x^4} = \frac{x^6 - x^3 + 1}{x^2},$$

which $\to \infty$ as $x \to 0$.

If we define $f(0, 0) = 0$, then

$$f_1(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{h}{h} = 1$$
$$f_2(0,0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k} = \lim_{k\to 0} \frac{k}{k} = 1.$$

6. $f(x,y) = e^{x^2-2x-4y^2+5}$ $\quad f(1,-1) = 1$

$f_1(x,y) = 2(x-1)e^{x^2-2x-4y^2+5}$ $\quad f_1(1,-1) = 0$

$f_2(x,y) = -8ye^{x^2-2x-4y^2+5}$ $\quad f_2(1,-1) = 8.$

a) The tangent plane to $z = f(x, y)$ at $(1, -1, 1)$ has equation $z = 1 + 8(y + 1)$, or $z = 8y + 9$.

b) $f(x,y) = C \Rightarrow (x-1)^2 - 4y^2 + 4 = \ln C$
$\Rightarrow (x-1)^2 - 4y^2 = \ln C - 4.$
These are hyperbolas with centre $(1, 0)$ and asymptotes $x = 1 \pm 2y$.

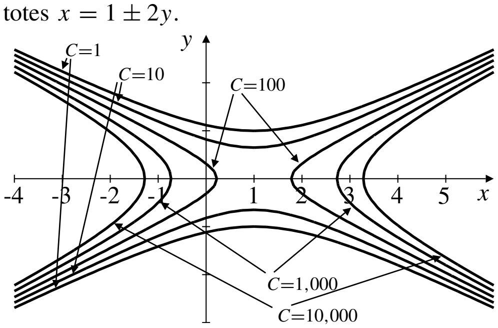


Fig. R-12.6

7. Let $f(x, y, z) = x^2 + y^2 + 4z^2$. Then S has equation $f(x, y, z) = 16$.

a) $\nabla f(a, b, c) = 2a\mathbf{i} + 2b\mathbf{j} + 8c\mathbf{k}$. The tangent plane to S at (a, b, c) has equation

$$2a(x-a) + 2b(y-b) + 4c(z-c) = 0, \quad \text{or}$$
$$ax + by + 4cz = a^2 + b^2 + 4c^2 = 16.$$

b) The tangent plane $ax+by+4cz = 16$ passes through $(0, 0, 4)$ if $16c = 16$, that is, if $c = 1$. In this case $a^2+b^2 = 16-4c^2 = 12$. These points (a, b, c) lie on a horizontal circle of radius $\sqrt{12}$ centred at $(0, 0, 1)$ in the plane $z = 1$.

c) The tangent plane of part (a) is parallel to the plane $x + y + 2\sqrt{2}z = 97$ if

$$a\mathbf{i} + b\mathbf{j} + 4c\mathbf{k} = t(\mathbf{i} + \mathbf{j} + 2\sqrt{2}\mathbf{k}),$$

that is, $a = t$, $b = t$, $c = t/\sqrt{2}$. Then $16 = a^2 + b^2 + 4c^2 = 4t^2$, so $t = \pm 2$. The two points on S where the tangent plane is parallel to $x + y + 2\sqrt{2}z = 97$ are $(2, 2, \sqrt{2})$ and $(-2, -2, -\sqrt{2})$.

8. $\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2}$

$-\dfrac{1}{R^2}\,dR = -\dfrac{1}{R_1^2}\,dR_1 - \dfrac{1}{R_2^2}\,dR_2$

If $R_1 = 100$ and $R_2 = 25$, so that $R = 20$, and if $|dR_1/R_1| = 5/100$ and $|dR_2/R_2| = 2/100$, then

$$\frac{1}{20}\left|\frac{dR}{R}\right| \le \frac{1}{100}\cdot\frac{5}{100} + \frac{1}{25}\cdot\frac{2}{100} = \frac{13}{100^2}.$$

Thus $|dR/R| \le 13/500$; R can be in error by about 2.6%.

9. The measured sides of the field are $x = 150$ m and $y = 200$ m with $|dx| = 1$ and $|dy| = 1$, and the contained angle between them is $\theta = 30°$ with $|d\theta| = 2° = \pi/90$ rad. The area A of the field satisfies

$$A = \frac{1}{2}xy\sin\theta \approx 7,500$$
$$dA = \frac{y}{2}\sin\theta\,dx + \frac{x}{2}\sin\theta\,dy + \frac{xy}{2}\cos\theta\,d\theta$$
$$= \frac{175}{2} + 15,000\frac{\sqrt{3}}{2}\cdot\frac{\pi}{90} \approx 541.$$

The area is 7,500 m^2, accurate to within about 540 m^2 for a percentage error of about 7.2%.

10. $T = x^3y + y^3z + z^3x$.

a) $\nabla T = (3x^2y + z^3)\mathbf{i} + (3y^2z + x^3)\mathbf{j} + (3z^2x + y^3)\mathbf{k}$
$\nabla T(2, -1, 0) = -12\mathbf{i} + 8\mathbf{j} - \mathbf{k}$.
A unit vector in the direction from $(2, -1, 0)$ towards $(1, 1, 2)$ is $\mathbf{u} = (-\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})/3$. The directional derivative of T at $(2, -1, 0)$ in the direction of $\mathbf{u}$ is

$$\mathbf{u} \bullet \nabla T(2,-1,0) = \frac{12 + 16 - 2}{3} = \frac{26}{3}.$$

b) Since $\nabla(2x^2 + 3y^2 + z^2) = 4x\mathbf{i} + 6y\mathbf{j} + 2z\mathbf{k}$, at $t = 0$ the fly is at $(2, -1, 0)$ and is moving in the direction $\pm(8\mathbf{i} - 6\mathbf{j})$, so its velocity is

$$\pm 5\frac{8\mathbf{i} - 6\mathbf{j}}{10} = \pm(4\mathbf{i} - 3\mathbf{j}).$$

Since the fly is moving in the direction of increasing T, the rate at which it experiences T increasing is

$$\frac{dT}{dt} = |(4\mathbf{i} - 3\mathbf{j}) \bullet (-12\mathbf{i} + 8\mathbf{j} - \mathbf{k})| = 48 + 24 = 72.$$

11. $f(x, y, z) = x^2y + yz + z^2$.

a) $\nabla f(x, y, z) = 2xy\mathbf{i} + (x^2 + z)\mathbf{j} + (y + 2z)\mathbf{k}$
$\nabla f(1, -1.1) = -2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$.
The directional derivative of f in the direction $\mathbf{i} + \mathbf{k}$ at $(1, -1, 1)$ is

$$\frac{\mathbf{i} + \mathbf{k}}{\sqrt{2}} \bullet (-2\mathbf{i} + 2\mathbf{j} + \mathbf{k}) = -\frac{1}{\sqrt{2}}.$$

b) The plane $x+y+z = 1$ intersects the level surface of f through $(1, -1, 1)$ in a curve whose tangent vector at $(1, -1, 1)$ is perpendicular to both $\nabla f(1, -1, 1)$ and the normal vector $\mathbf{i}+\mathbf{j}+\mathbf{k}$ to the plane. Thus the ant is crawling in the direction of the cross product of these vectors:

$$\pm \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix} = \pm(\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}).$$

c) The second ant is crawling in the direction of the vector projection of $\nabla f(1, -1, 1)$ onto the plane $x+y+z = 1$, which is $\nabla f(1, -1, 1)$ minus its vector projection onto the normal to that plane:

$$\begin{aligned} &\nabla f(1, -1, 1) - \frac{\nabla f(1, -1, 1) \bullet (\mathbf{i} + \mathbf{j} + \mathbf{k})}{|\mathbf{i} + \mathbf{j} + \mathbf{k}|^2}(\mathbf{i} + \mathbf{j} + \mathbf{k}) \\ &\quad = -2\mathbf{i} + 2\mathbf{j} + \mathbf{k} - \frac{1}{3}(\mathbf{i} + \mathbf{j} + \mathbf{k}) = \frac{-7\mathbf{i} + 5\mathbf{j} + 2\mathbf{k}}{3}, \end{aligned}$$

that is, in the direction $-7\mathbf{i} + 5\mathbf{j} + 2\mathbf{k}$.

12. $f(x, y, z) = (x^2 + z^2)\sin\frac{\pi xy}{2} + yz^2$, $P_0 = (1, 1, -1)$.

a)
$$\begin{aligned} \nabla f &= \left(2x\sin\frac{\pi xy}{2} + \frac{\pi y}{2}(x^2 + z^2)\cos\frac{\pi xy}{2}\right)\mathbf{i} \\ &\quad + \left(\frac{\pi x}{2}(x^2 + z^2)\cos\frac{\pi xy}{2} + z^2\right)\mathbf{j} \\ &\quad + 2z\left(\sin\frac{\pi xy}{2} + y\right)\mathbf{k} \end{aligned}$$
$\nabla f(P_0) = 2\mathbf{i} + \mathbf{j} - 4\mathbf{k}$.

b) Since $f(P_0) = 2 + 1 = 3$, the linearization of f at P_0 is

$$L(x, y, z) = 3 + 2(x - 1) + (y - 1) - 4(z + 1).$$

c) The tangent plane at P_0 to the level surface of f through P_0 has equation

$$\begin{aligned} &\nabla f(P_0) \bullet \Big((x - 1)\mathbf{i} + (y - 1)\mathbf{j} + (z + 1)\mathbf{k}\Big) = 0 \\ &2(x - 1) + (y - 1) - 4(z + 1) = 0 \\ &2x + y - 4z = 7. \end{aligned}$$

d) The bird is flying in direction

$$(2 - 1)\mathbf{i} + (-1 - 1)\mathbf{j} + (1 + 1)\mathbf{k} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k},$$

a vector of length 3. Since the bird's speed is 5, its velocity is
$$\mathbf{v} = \frac{5}{3}(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}).$$
The rate of change of f as experienced by the bird is

$$\frac{df}{dt} = \mathbf{v} \bullet \nabla f(P_0) = \frac{5}{3}(2 - 2 - 8) = -\frac{40}{3}.$$

e) To experience the greatest rate of increase of f while flying through P_0 at speed 5, the bird should fly in the direction of $\nabla f(P_0)$, that is, $2\mathbf{i} + \mathbf{j} - 4\mathbf{k}$.

13.
$$\begin{aligned} u &= k\left(\ln\cos\frac{x}{k} - \ln\cos\frac{y}{k}\right) \\ u_x &= k\left(-\frac{1}{k}\tan\frac{x}{k}\right) = -\tan\frac{x}{k} \\ u_y &= k\left(\frac{1}{k}\tan\frac{y}{k}\right) = \tan\frac{y}{k} \\ u_{xx} &= -\frac{1}{k}\sec^2\frac{x}{k} \\ u_{yy} &= \frac{1}{k}\sec^2\frac{y}{k} \\ u_{xy} &= 0 \end{aligned}$$
$$\begin{aligned} &(1+u_x^2)u_{yy} - uu_xu_yu_{xy} + (1 + u_y^2)u_{xx} \\ &\quad = \frac{1}{k}\sec^2\frac{x}{k}\sec^2\frac{y}{k} - 0 - \frac{1}{k}\sec^2\frac{y}{k}\sec^2\frac{x}{k} = 0. \end{aligned}$$

14. If $F(x, y, z) = 0$, $G(x, y, z) = 0$ are solved for $x = x(y)$, $z = z(y)$, then

$$\begin{aligned} F_1\frac{dx}{dy} + F_2 + F_3\frac{dz}{dy} &= 0 \\ G_1\frac{dx}{dy} + G_2 + G_3\frac{dz}{dy} &= 0. \end{aligned}$$

Eliminating dz/dy from these equations, we obtain

$$\frac{dx}{dy} = -\frac{F_2G_3 - F_3G_2}{F_1G_3 - F_3G_1}.$$

Similarly, if the equations are solved for $x = x(z)$, $y = y(z)$, then

$$\frac{dy}{dz} = -\frac{F_3G_1 - F_1G_3}{F_2G_1 - F_1G_2},$$

and if the equations are solved for $y = y(x)$, $z = z(x)$, then

$$\frac{dz}{dx} = -\frac{F_1G_2 - F_2G_1}{F_3G_2 - F_2G_3}.$$

Hence

$$\frac{dx}{dy} \cdot \frac{dy}{dz} \cdot \frac{dz}{dx} = -\frac{F_2G_3 - F_3G_2}{F_1G_3 - F_3G_1} \cdot \frac{F_3G_1 - F_1G_3}{F_2G_1 - F_1G_2} \cdot \frac{F_1G_2 - F_2G_1}{F_3G_2 - F_2G_3} = 1.$$

15. $x = u^3 - uv$
$y = 3uv + 2v^2$
Assume these equations define $u = u(x, y)$ and $v = v(x, y)$ near the point P where $(u, v, x, y) = (-1, 2, 1, 2)$.

a) Differentiating both equations with respect to x, we get

$$1 = 3u^2\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}$$
$$0 = 3v\frac{\partial u}{\partial x} + 3u\frac{\partial v}{\partial x} + 4v\frac{\partial v}{\partial x}.$$

At P, these equations become

$$1 = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}, \quad 0 = 6\frac{\partial u}{\partial x} + 5\frac{\partial v}{\partial x},$$

from which we obtain $\partial u/\partial x\Big|_P = -5$.
Similarly, differentiating the given equations with respect to y leads to

$$0 = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}, \quad 1 = 6\frac{\partial u}{\partial y} + 5\frac{\partial v}{\partial y},$$

from which we obtain $\partial u/\partial y\Big|_P = 1$.

b) Since $u(1, 2) = -1$, we have

$$u(1.02, 1.97) \approx -1 + \frac{\partial u}{\partial x}\Big|_P(0.02) + \frac{\partial u}{\partial y}\Big|_P(-0.03)$$
$$= -1 - 5(0.02) + 1(-0.03) = -1.13.$$

16. $u = x^2 + y^2$
$v = x^2 - 2xy^2$
Assume these equations define $x = x(u, v)$ and $y = y(u, v)$ near the point $(u, v) = (5, -7)$, with $x = 1$ and $y = 2$ at that point.

a) Differentiate the given equations with respect to u to obtain

$$1 = 2x\frac{\partial x}{\partial u} + 2y\frac{\partial y}{\partial u}$$
$$0 = 2(x - y^2)\frac{\partial x}{\partial u} - 4xy\frac{\partial y}{\partial u}.$$

At $x = 1$, $y = 2$,

$$2\frac{\partial x}{\partial u} + 4\frac{\partial y}{\partial u} = 1$$
$$-6\frac{\partial x}{\partial u} - 8\frac{\partial y}{\partial u} = 0,$$

from which we obtain $\partial x/\partial u = -1$ and $\partial y/\partial u = 3/4$ at $(5, -7)$.

b) If $z = \ln(y^2 - x^2)$, then

$$\frac{\partial z}{\partial u} = \frac{1}{y^2 - x^2}\left[-2x\frac{\partial x}{\partial u} + 2y\frac{\partial y}{\partial u}\right].$$

At $(u, v) = (5, -7)$, we have $(x, y) = (1, 2)$, and so

$$\frac{\partial z}{\partial u} = \frac{1}{3}\left[-2(-1) + 4\left(\frac{3}{4}\right)\right] = \frac{5}{3}.$$

Challenging Problems 12 (page 705)

1. a) If f is differentiable at (a, b), then its graph has a nonvertical tangent plane at $(a, b, f(a, b))$. Any line through that point, part of which lies on the surface $z = f(x, y)$ near (a, b), must be tangent to that surface at (a, b), so must lie in the tangent plane.

b) The surface S with equation $z = y\,g(x/y)$ has the property that if $P = (x_0, y_0, z_0)$ is any point on it, then all points other than the origin on the line joining P_0 to the origin also lie on S. Specifically, if $t \neq 0$, then (tx_0, ty_0, tz_0) lies on S, because

$$tz_0 = ty_0\,g\left(\frac{tx_0}{ty_0}\right) \Leftrightarrow z_0 = y_0\,g\left(\frac{x_0}{y_0}\right).$$

Thus S consists entirely of lines through the origin; it is some kind of "cone" with vertex at the origin. By part (a), all tangent planes to S contain the lines on S through the points of contact, so all tangent planes must pass through the origin.

2. Let the position vector of the particle at time t be $\mathbf{r} = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. Then the velocity of the particle is

$$\mathbf{v} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}.$$

This velocity must be parallel to

$$\nabla f(x, y, z) = -2x\mathbf{i} - 4y\mathbf{j} + 6z\mathbf{k}$$

at every point of the path, that is,

$$\frac{dx}{dt} = -2tx, \quad \frac{dy}{dt} = -4ty, \quad \frac{dz}{dt} = 6tz,$$

so that $\frac{dx}{-2x} = \frac{dy}{-4y} = \frac{dz}{6z}$. Integrating these equations, we get

$$\ln|y| = 2\ln|x| + C_1, \quad \ln|z| = -3\ln|x| + C_2.$$

Since the path passes through $(1, 1, 8)$, C_1 and C_2 are determined by

$$\ln 1 = 2\ln 1 + C_1, \quad \ln 8 = -3\ln 1 + C_2.$$

Thus $C_1 = 0$ and $C_2 = \ln 8$. The path therefore has equations $y = x^2$, $z = 8/x^3$. Evidently $(2, 4, 1)$ lies on the path, and $(3, 7, 0)$ does not.

3. We used Maple V to verify the stated identity. Using `r`, `p`, and `t` to represent ρ, ϕ, and θ, respectively, we defined

```
>  v := (r,p,t) ->
>   u(r*sin(p)*cos(t),
r*sin(p)*sin(t),
>   r*cos(p));
```

and then asked Maple to calculate and simplify the left side of the identity:

```
>  simplify(diff(v(r,,p,t),r$2)
>   +(2/r)*diff(v(r,p,t),r)
>   +(cot(p)/r^2)*diff(v(r,p,t),p)
>   +(1/r^2)*diff(v(r,p,t),p$2)
>   +(1/(r*sin(p))^2)*diff(v(r,p,t),t$2));
```

Maple responded with

$$D_{1,1}(u) + D_{3,3}(u) + D_{2,2}(u),$$

with all three terms evaluated at
$(r\sin(p)\cos(t), r\sin(p)\sin(t), r\cos(p))$, thus confirming the identity.

4. If $u(x, y, z, t) = v(\rho, t) = \frac{f(\rho - ct)}{\rho}$ is independent of θ and ϕ, then

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{\partial^2 v}{\partial \rho^2} + \frac{2}{\rho}\frac{\partial v}{\partial \rho}$$

by Problem 3. We have

$$\frac{\partial v}{\partial \rho} = \frac{f'(\rho - ct)}{\rho} - \frac{f(\rho - ct)}{\rho^2}$$
$$\frac{\partial^2 v}{\partial \rho^2} = \frac{f''(\rho - ct)}{\rho} - \frac{2f'(\rho - ct)}{\rho^2} + \frac{2f(\rho - ct)}{\rho^3}$$
$$\frac{\partial v}{\partial t} = -\frac{cf'(\rho - ct)}{\rho}$$

$$\frac{\partial^2 v}{\partial t^2} = \frac{c^2 f''(\rho - ct)}{\rho}$$
$$\begin{aligned}\frac{\partial^2 v}{\partial \rho^2} + \frac{2}{\rho}\frac{\partial v}{\partial \rho} &= \frac{f''(\rho - ct)}{\rho} - \frac{2f'(\rho - ct)}{\rho^2} + \frac{2f(\rho - ct)}{\rho^3} \\ &\quad + \frac{2f'(\rho - ct)}{\rho^2} - \frac{2f(\rho - ct)}{\rho^3} \\ &= \frac{f''(\rho - ct)}{\rho} \\ &= \frac{1}{c^2}\frac{\partial^2 v}{\partial t^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}.\end{aligned}$$

The function $f(\rho - ct)/\rho$ represents the shape of a symmetrical wave travelling uniformly away from the origin at speed c. Its amplitude at distance ρ from the origin decreases as ρ increases; it is proportional to the reciprocal of ρ.