CHAPTER 15. VECTOR FIELDS

Section 15.1 Vector and Scalar Fields (page 811)

1. $\mathbf{F} = x\mathbf{i} + x\mathbf{j}$.
The field lines satisfy $\dfrac{dx}{x} = \dfrac{dy}{x}$, i.e., $dy = dx$. The field lines are $y = x + C$, straight lines parallel to $y = x$.

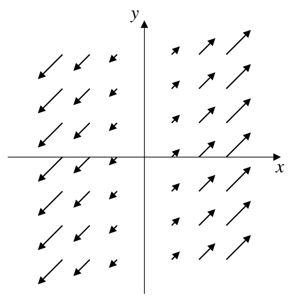

Fig. 15.1.1

2. $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$.
The field lines satisfy $\dfrac{dx}{x} = \dfrac{dy}{y}$.
Thus $\ln y = \ln x + \ln C$, or $y = Cx$. The field lines are straight half-lines emanating from the origin.

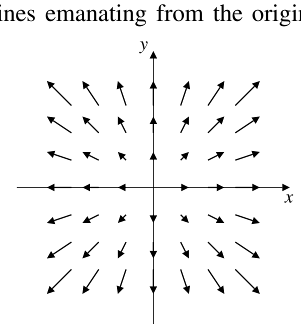

Fig. 15.1.2

3. $\mathbf{F} = y\mathbf{i} + x\mathbf{j}$.
The field lines satisfy $\dfrac{dx}{y} = \dfrac{dy}{x}$.
Thus $x\,dx = y\,dy$. The field lines are the rectangular hyperbolas (and their asymptotes) given by $x^2 - y^2 = C$.

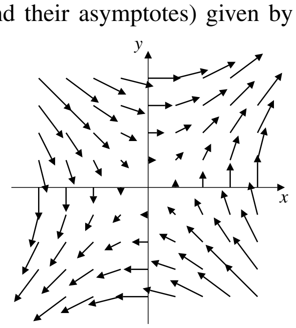

Fig. 15.1.3

4. $\mathbf{F} = \mathbf{i} + \sin x\,\mathbf{j}$.
The field lines satisfy $dx = \dfrac{dy}{\sin x}$.
Thus $\dfrac{dy}{dx} = \sin x$. The field lines are the curves $y = -\cos x + C$.

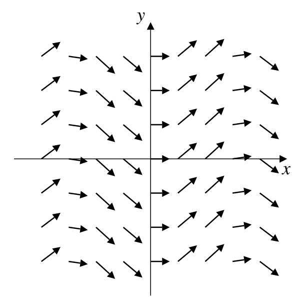

Fig. 15.1.4

5. $\mathbf{F} = e^x\mathbf{i} + e^{-x}\mathbf{j}$.
The field lines satisfy $\dfrac{dx}{e^x} = \dfrac{dy}{e^{-x}}$.
Thus $\dfrac{dy}{dx} = e^{-2x}$. The field lines are the curves $y = -\dfrac{1}{2}e^{-2x} + C$.

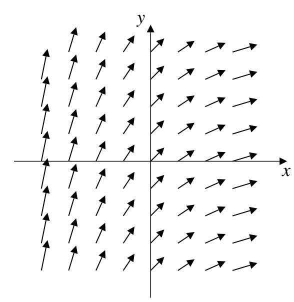

Fig. 15.1.5

6. $\mathbf{F} = \nabla(x^2 - y) = 2x\mathbf{i} - \mathbf{j}$.
The field lines satisfy $\dfrac{dx}{2x} = \dfrac{dy}{-1}$. They are the curves $y = -\dfrac{1}{2}\ln x + C$.

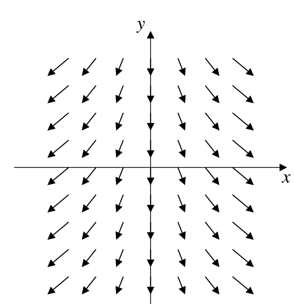

Fig. 15.1.6

7. $\mathbf{F} = \nabla \ln(x^2 + y^2) = \dfrac{2x\mathbf{i} + 2y\mathbf{j}}{x^2 + y^2}$.
The field lines satisfy $\dfrac{dx}{x} = \dfrac{dy}{y}$. Thus they are radial lines $y = Cx$ (and $x = 0$)

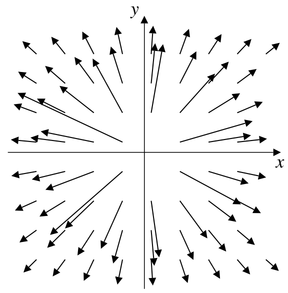

Fig. 15.1.7

8. $\mathbf{F} = \cos y\mathbf{i} - \cos x\mathbf{j}$.
The field lines satisfy $\dfrac{dx}{\cos y} = -\dfrac{dy}{\cos x}$, that is, $\cos x\,dx + \cos y\,dy = 0$. Thus they are the curves $\sin x + \sin y = C$.

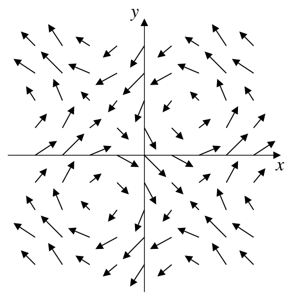

Fig. 15.1.8

9. $\mathbf{v}(x, y, z) = y\mathbf{i} - y\mathbf{j} - y\mathbf{k}$.
The streamlines satisfy $dx = -dy = -dz$. Thus $y + x = C_1$, $z + x = C_2$. The streamlines are straight lines parallel to $\mathbf{i} - \mathbf{j} - \mathbf{k}$.

10. $\mathbf{v}(x, y, z) = x\mathbf{i} + y\mathbf{j} - x\mathbf{k}$.
The streamlines satisfy $\dfrac{dx}{x} = \dfrac{dy}{y} = -\dfrac{dz}{x}$. Thus $z + x = C_1$, $y = C_2x$. The streamlines are straight half-lines emanating from the z-axis and perpendicular to the vector $\mathbf{i} + \mathbf{k}$.

11. $\mathbf{v}(x, y, z) = y\mathbf{i} - x\mathbf{j} + \mathbf{k}$.
The streamlines satisfy $\dfrac{dx}{y} = -\dfrac{dy}{x} = dz$. Thus $x\,dx + y\,dy = 0$, so $x^2 + y^2 = C_1^2$. Therefore,

$$\frac{dz}{dx} = \frac{1}{y} = \frac{1}{\sqrt{C_1^2 - x^2}}.$$

This implies that $z = \sin^{-1}\dfrac{x}{C_1} + C_2$. The streamlines are the spirals in which the surfaces $x = C_1 \sin(z - C_2)$ intersect the cylinders $x^2 + y^2 = C_1^2$.

12. $\mathbf{v} = \dfrac{x\mathbf{i} + y\mathbf{j}}{(1 + z^2)(x^2 + y^2)}$.
The streamlines satisfy $dz = 0$ and $\dfrac{dx}{x} = \dfrac{dy}{y}$. Thus $z = C_1$ and $y = C_2x$. The streamlines are horizontal half-lines emanating from the z-axis.

13. $\mathbf{v} = xz\mathbf{i} + yz\mathbf{j} + x\mathbf{k}$. The field lines satisfy

$$\frac{dx}{xz} = \frac{dy}{yz} = \frac{dz}{x},$$

or, equivalently, $dx/x = dy/y$ and $dx = z\,dz$. Thus the field lines have equations $y = C_1x$, $2x = z^2 + C_2$, and are therefore parabolas.

14. $\mathbf{v} = e^{xyz}(x\mathbf{i} + y^2\mathbf{j} + z\mathbf{k})$. The field lines satisfy

$$\frac{dx}{x} = \frac{dy}{y^2} = \frac{dz}{z},$$

so they are given by $z = C_1x$, $\ln|x| = \ln|C_2| - (1/y)$ (or, equivalently, $x = C_2e^{-1/y}$).

15. $\mathbf{v}(x, y) = x^2\mathbf{i} - y\mathbf{j}$. The field lines satisfy $dx/x^2 = -dy/y$, so they are given by $\ln|y| = (1/x) + \ln|C|$, or $y = Ce^{1/x}$.

16. $\mathbf{v}(x, y) = x\mathbf{i} + (x + y)\mathbf{j}$. The field lines satisfy

$$\frac{dx}{x} = \frac{dy}{x + y}$$
$$\frac{dy}{dx} = \frac{x + y}{x} \qquad \text{Let } y = xv(x)$$
$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$
$$v + x\frac{dv}{dx} = \frac{x(1 + v)}{x} = 1 + v.$$

Thus $dv/dx = 1/x$, and so $v(x) = \ln|x| + C$. The field lines have equations $y = x\ln|x| + Cx$.

17. $\mathbf{F} = \hat{\mathbf{r}} + r\hat{\boldsymbol{\theta}}$. The field lines satisfy $dr = d\theta$, so they are the spirals $r = \theta + C$.

18. $\mathbf{F} = \hat{\mathbf{r}} + \theta\hat{\boldsymbol{\theta}}$. The field lines satisfy $dr = r\,d\theta/\theta$, or $dr/r = d\theta/\theta$, so they are the spirals $r = C\theta$.

19. $\mathbf{F} = 2\hat{\mathbf{r}} + \theta\hat{\boldsymbol{\theta}}$. The field lines satisfy $dr/2 = r\,d\theta/\theta$, or $dr/r = 2d\theta/\theta$, so they are the spirals $r = C\theta^2$.

20. $\mathbf{F} = r\hat{\mathbf{r}} - \hat{\boldsymbol{\theta}}$. The field lines satisfy $dr/r = -r\,d\theta$, or $-dr/r^2 = d\theta$, so they are the spirals $1/r = \theta + C$, or $r = 1/(\theta + C)$.

Section 15.2 Conservative Fields (page 819)

1. $\mathbf{F} = x\mathbf{i} - 2y\mathbf{j} + 3z\mathbf{k}$, $F_1 = x$, $F_2 = -2y$, $F_3 = 3z$. We have

$$\frac{\partial F_1}{\partial y} = 0 = \frac{\partial F_2}{\partial x}, \quad \frac{\partial F_1}{\partial z} = 0 = \frac{\partial F_3}{\partial x}, \quad \frac{\partial F_2}{\partial z} = 0 = \frac{\partial F_3}{\partial y}.$$

Therefore, $\mathbf{F}$ may be conservative. If $\mathbf{F} = \nabla\phi$, then

$$\frac{\partial \phi}{\partial x} = x, \quad \frac{\partial \phi}{\partial y} = -2y, \quad \frac{\partial \phi}{\partial z} = 3z.$$

Evidently $\phi(x, y, z) = \dfrac{x^2}{2} - y^2 + \dfrac{3z^2}{2}$ is a potential for $\mathbf{F}$. Thus $\mathbf{F}$ is conservative on $\mathbb{R}^3$.

2. $\mathbf{F} = y\mathbf{i} + x\mathbf{j} + z^2\mathbf{k}$, $F_1 = y$, $F_2 = x$, $F_3 = z^2$. We have

$$\frac{\partial F_1}{\partial y} = 1 = \frac{\partial F_2}{\partial x}, \quad \frac{\partial F_1}{\partial z} = 0 = \frac{\partial F_3}{\partial x}, \quad \frac{\partial F_2}{\partial z} = 0 = \frac{\partial F_3}{\partial y}.$$

Therefore, $\mathbf{F}$ may be conservative. If $\mathbf{F} = \nabla\phi$, then

$$\frac{\partial \phi}{\partial x} = y, \quad \frac{\partial \phi}{\partial y} = x, \quad \frac{\partial \phi}{\partial z} = z^2.$$

Therefore,

$$\begin{aligned}
\phi(x, y, z) &= \int y\,dx = xy + C_1(y, z)\\
x &= \frac{\partial \phi}{\partial y} = x + \frac{\partial C_1}{\partial y} \Rightarrow \frac{\partial C_1}{\partial y} = 0\\
C_1(y, z) &= C_2(z), \qquad \phi(x, y, z) = xy + C_2(z)\\
z^2 &= \frac{\partial \phi}{\partial z} = C_2'(z) \Rightarrow C_2(z) = \frac{z^3}{3}.
\end{aligned}$$

Thus $\phi(x, y, z) = xy + \dfrac{z^3}{3}$ is a potential for $\mathbf{F}$, and $\mathbf{F}$ is conservative on $\mathbb{R}^3$.

3. $\mathbf{F} = \dfrac{x\mathbf{i} - y\mathbf{j}}{x^2 + y^2}$, $F_1 = \dfrac{x}{x^2 + y^2}$, $F_2 = -\dfrac{y}{x^2 + y^2}$. We have

$$\frac{\partial F_1}{\partial y} = -\frac{2xy}{(x^2 + y^2)^2}, \quad \frac{\partial F_2}{\partial x} = \frac{2xy}{(x^2 + y^2)^2}.$$

Thus $\mathbf{F}$ cannot be conservative.

4. $\mathbf{F} = \dfrac{x\mathbf{i} + y\mathbf{j}}{x^2 + y^2}$, $F_1 = \dfrac{x}{x^2 + y^2}$, $F_2 = \dfrac{y}{x^2 + y^2}$. We have

$$\frac{\partial F_1}{\partial y} = -\frac{2xy}{(x^2 + y^2)^2} = \frac{\partial F_2}{\partial x}.$$

Therefore, $\mathbf{F}$ may be conservative. If $\mathbf{F} = \nabla\phi$, then

$$\frac{\partial \phi}{\partial x} = \frac{x}{x^2 + y^2}, \quad \frac{\partial \phi}{\partial y} = \frac{y}{x^2 + y^2}.$$

Therefore,

$$\begin{aligned}
\phi(x, y) &= \int \frac{x}{x^2 + y^2}\,dx = \frac{\ln(x^2 + y^2)}{2} + C_1(y)\\
\frac{y}{x^2 + y^2} &= \frac{\partial \phi}{\partial y} = \frac{y}{x^2 + y^2} + c_1'(y) \Rightarrow c_1'(y) = 0.
\end{aligned}$$

Thus we can choose $C_1(y) = 0$, and

$$\phi(x, y) = \frac{1}{2}\ln(x^2 + y^2)$$

is a scalar potential for $\mathbf{F}$, and $\mathbf{F}$ is conservative everywhere on $\mathbb{R}^2$ except at the origin.

5. $\mathbf{F} = (2xy - z^2)\mathbf{i} + (2yz + x^2)\mathbf{j} - (2zx - y^2)\mathbf{k}$, $F_1 = 2xy - z^2$, $F_2 = 2yz + x^2$, $F_3 = y^2 - 2zx$. We have

$$\frac{\partial F_1}{\partial y} = 2x = \frac{\partial F_2}{\partial x}, \quad \frac{\partial F_1}{\partial z} = -2z = \frac{\partial F_3}{\partial x}, \quad \frac{\partial F_2}{\partial z} = 2y = \frac{\partial F_3}{\partial y}.$$

Therefore, $\mathbf{F}$ may be conservative. If $\mathbf{F} = \nabla\phi$, then

$$\frac{\partial \phi}{\partial x} = 2xy - z^2, \qquad \frac{\partial \phi}{\partial y} = 2yz + x^2, \qquad \frac{\partial \phi}{\partial z} = y^2 - 2zx.$$

Therefore,

$$\begin{aligned}
\phi(x, y, z) &= \int (2xy - z^2)\,dx = x^2 y - xz^2 + C_1(y, z)\\
2yz + x^2 &= \frac{\partial \phi}{\partial y} = x^2 + \frac{\partial C_1}{\partial y}\\
&\Rightarrow \frac{\partial C_1}{\partial y} = 2yz \Rightarrow C_1(y, z) = y^2 z + C_2(z)\\
\phi(x, y, z) &= x^2 y - xz^2 + y^2 z + C_2(z)\\
y^2 - 2zx &= \frac{\partial \phi}{\partial z} = -2xz + y^2 + C_2'(z)\\
&\Rightarrow C_2'(z) = 0.
\end{aligned}$$

Thus $\phi(x, y, z) = x^2 y - xz^2 + y^2 z$ is a scalar potential for $\mathbf{F}$, and $\mathbf{F}$ is conservative on $\mathbb{R}^3$.

6. $\mathbf{F} = e^{x^2+y^2+z^2}(xz\mathbf{i} + yz\mathbf{j} + xy\mathbf{k})$.
$F_1 = xze^{x^2+y^2+z^2}$, $F_2 = yze^{x^2+y^2+z^2}$,
$F_3 = xye^{x^2+y^2+z^2}$. We have

$$\begin{aligned} \frac{\partial F_1}{\partial y} &= 2xyze^{x^2+y^2+z^2} = \frac{\partial F_2}{\partial x}, \\ \frac{\partial F_1}{\partial z} &= (x + 2xz^2)e^{x^2+y^2+z^2}, \\ \frac{\partial F_3}{\partial x} &= (y + 2x^2y)e^{x^2+y^2+z^2} \neq \frac{\partial F_1}{\partial z}. \end{aligned}$$

Thus $\mathbf{F}$ cannot be conservative.

7. $$\begin{aligned} \phi(\mathbf{r}) &= \frac{1}{|\mathbf{r}-\mathbf{r}_0|^2} \\ \frac{\partial \phi}{\partial x} &= -\frac{2}{|\mathbf{r}-\mathbf{r}_0|^3}\frac{\partial}{\partial x}|\mathbf{r}-\mathbf{r}_0| \\ &= -\frac{2}{|\mathbf{r}-\mathbf{r}_0|^3}\frac{(\mathbf{r}-\mathbf{r}_0)\bullet\dfrac{\partial \mathbf{r}}{\partial x}}{|\mathbf{r}-\mathbf{r}_0|} \\ &= -\frac{2(x-x_0)}{|\mathbf{r}-\mathbf{r}_0|^4}. \end{aligned}$$

Since similar formulas hold for the other first partials of ϕ, we have

$$\begin{aligned} \mathbf{F} &= \nabla\phi \\ &= -\frac{2}{|\mathbf{r}-\mathbf{r}_0|^4}\Big[(x-x_0)\mathbf{i} + (y-y_0)\mathbf{j} + (z-z_0)\mathbf{k}\Big] \\ &= -2\frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^4}. \end{aligned}$$

This is the vector field whose scalar potential is ϕ.

8. $$\begin{aligned} \frac{\partial}{\partial x}\ln|\mathbf{r}| &= \frac{1}{|\mathbf{r}|}\frac{\mathbf{r}\bullet\dfrac{\partial \mathbf{r}}{\partial x}}{|\mathbf{r}|} = \frac{x}{|\mathbf{r}|^2} \\ \nabla\ln|\mathbf{r}| &= \frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{|\mathbf{r}|^2} = \frac{\mathbf{r}}{|\mathbf{r}|^2}. \end{aligned}$$

9. $\mathbf{F} = \dfrac{2x}{z}\mathbf{i} + \dfrac{2y}{z}\mathbf{j} - \dfrac{x^2+y^2}{z^2}\mathbf{k}$,
$F_1 = \dfrac{2x}{z}$, $F_2 = \dfrac{2y}{z}$, $F_3 = -\dfrac{x^2+y^2}{z^2}$. We have

$$\begin{aligned} \frac{\partial F_1}{\partial y} &= 0 = \frac{\partial F_2}{\partial x}, \\ \frac{\partial F_1}{\partial z} &= -\frac{2x}{z^2} = \frac{\partial F_3}{\partial x}, \\ \frac{\partial F_2}{\partial z} &= -\frac{2y}{z^2} = \frac{\partial F_3}{\partial y}. \end{aligned}$$

Therefore, $\mathbf{F}$ may be conservative in $\mathbb{R}^3$ except on the plane $z = 0$ where it is not defined. If $\mathbf{F} = \nabla\phi$, then

$$\frac{\partial\phi}{\partial x} = \frac{2x}{z}, \quad \frac{\partial\phi}{\partial y} = \frac{2y}{z}, \quad \frac{\partial\phi}{\partial z} = -\frac{x^2+y^2}{z^2}.$$

Therefore,

$$\begin{aligned} \phi(x,y,z) &= \int \frac{2x}{z}\,dx = \frac{x^2}{z} + C_1(y,z) \\ \frac{2y}{z} &= \frac{\partial\phi}{\partial y} = \frac{\partial C_1}{\partial y} \Rightarrow C_1(y,z) = \frac{y^2}{z} + C_2(z) \\ \phi(x,y,z) &= \frac{x^2+y^2}{z} + C_2(z) \\ -\frac{x^2+y^2}{z^2} &= \frac{\partial\phi}{\partial z} = -\frac{x^2+y^2}{z^2} + C_2'(z) \\ &\Rightarrow C_2(z) = 0. \end{aligned}$$

Thus $\phi(x,y,z) = \dfrac{x^2+y^2}{z}$ is a potential for $\mathbf{F}$, and $\mathbf{F}$ is conservative on $\mathbb{R}^3$ except on the plane $z = 0$.

The equipotential surfaces have equations

$$\frac{x^2+y^2}{z} = C, \quad \text{or} \quad Cz = x^2+y^2.$$

Thus the equipotential surfaces are circular paraboloids.

The field lines of $\mathbf{F}$ satisfy

$$\frac{dx}{\dfrac{2x}{z}} = \frac{dy}{\dfrac{2y}{z}} = \frac{dz}{-\dfrac{x^2+y^2}{z^2}}.$$

From the first equation, $\dfrac{dx}{x} = \dfrac{dy}{y}$, so $y = Ax$ for an arbitrary constant A. Therefore

$$\frac{dx}{2x} = \frac{z\,dz}{-(x^2+y^2)} = \frac{z\,dz}{-x^2(1+A^2)},$$

so $-(1+A^2)x\,dx = 2z\,dz$. Hence

$$\frac{1+A^2}{2}x^2 + z^2 = \frac{B}{2},$$

or $x^2+y^2+2z^2 = B$, where B is a second arbitrary constant. The field lines of $\mathbf{F}$ are the ellipses in which the vertical planes containing the z-axis intersect the ellipsoids $x^2+y^2+2z^2 = B$. These ellipses are orthogonal to all the equipotential surfaces of $\mathbf{F}$.

10. $\mathbf{F} = \dfrac{2x}{z}\mathbf{i} + \dfrac{2y}{z}\mathbf{j} - \dfrac{x^2+y^2}{z^2}\mathbf{k} = \mathbf{G} + \mathbf{k}$,
where $\mathbf{G}$ is the vector field $\mathbf{F}$ of Exercise 9. Since $\mathbf{G}$ is conservative (except on the plane $z = 0$), so is $\mathbf{F}$, which has scalar potential

$$\phi(x,y,z) = \frac{x^2+y^2}{z} + z = \frac{x^2+y^2+z^2}{z},$$

since $\dfrac{x^2+y^2}{z}$ is a potential for $\mathbf{G}$ and z is a potential for the vector $\mathbf{k}$.

The equipotential surfaces of $\mathbf{F}$ are $\phi(x,y,z)=C$, or

$$x^2+y^2+z^2=Cz$$

which are spheres tangent to the xy-plane having centres on the z-axis.

The field lines of $\mathbf{F}$ satisfy

$$\frac{dx}{\frac{2x}{z}}=\frac{dy}{\frac{2y}{z}}=\frac{dz}{1-\frac{x^2+y^2}{z^2}}.$$

As in Exercise 9, the first equation has solutions $y=Ax$, representing vertical planes containing the z-axis. The remaining equations can then be written in the form

$$\frac{dz}{dx}=\frac{z^2-x^2-y^2}{2xz}=\frac{z^2-(1+A^2)x^2}{2zx}.$$

This first order DE is of homogeneous type (see Section 9.2), and can be solved by a change of dependent variable: $z=xv(x)$. We have

$$\begin{aligned}
v+x\frac{dv}{dx}&=\frac{dz}{dx}=\frac{x^2v^2-(1+A^2)x^2}{2x^2v}\\
x\frac{dv}{dx}&=\frac{v^2-(1+A^2)}{2v}-v=-\frac{v^2+(1+A^2)}{2v}\\
\frac{2v\,dv}{v^2+(1+A^2)}&=-\frac{dx}{x}\\
\ln\Big(v^2+(1+A^2)\Big)&=-\ln x+\ln B\\
v^2+1+A^2&=\frac{B}{x}\\
\frac{z^2}{x^2}+1+A^2&=\frac{B}{x}\\
z^2+x^2+y^2&=Bx.
\end{aligned}$$

These are spheres centred on the x-axis and passing through the origin. The field lines are the intersections of the planes $y=Ax$ with these spheres, so they are vertical circles passing through the origin and having centres in the xy-plane. (The technique used to find these circles excludes those circles with centres on the y-axis, but they are also field lines of $\mathbf{F}$.)

Note: In two dimensions, circles passing through the origin and having centres on the x-axis intersect perpendicularly circles passing through the origin and having centres on the y-axis. Thus the nature of the field lines of $\mathbf{F}$ can be determined geometrically from the nature of the equipotential surfaces.

11. The scalar potential for the two-source system is

$$\phi(x,y,z)=\phi(\mathbf{r})=-\frac{m}{|\mathbf{r}-\ell\mathbf{k}|}-\frac{m}{|\mathbf{r}+\ell\mathbf{k}|}.$$

Hence the velocity field is given by

$$\begin{aligned}
\mathbf{v}(\mathbf{r})&=\nabla\phi(\mathbf{r})\\
&=\frac{m(\mathbf{r}-\ell\mathbf{k})}{|\mathbf{r}-\ell\mathbf{k}|^3}+\frac{m(\mathbf{r}+\ell\mathbf{k})}{|\mathbf{r}+\ell\mathbf{k}|^3}\\
&=\frac{m(x\mathbf{i}+y\mathbf{j}+(z-\ell)\mathbf{k})}{[x^2+y^2+(z-\ell)^2]^{3/2}}+\frac{m(x\mathbf{i}+y\mathbf{j}+(z+\ell)\mathbf{k})}{[x^2+y^2+(z-\ell)^2]^{3/2}}.
\end{aligned}$$

Observe that $v_1=0$ if and only if $x=0$, and $v_2=0$ if and only if $y=0$. Also

$$\mathbf{v}(0,0,z)=m\left(\frac{z-\ell}{|z-\ell|^3}+\frac{z+\ell}{|z+\ell|^3}\right)\mathbf{k},$$

which is $\mathbf{0}$ if and only if $z=0$. Thus $\mathbf{v}=\mathbf{0}$ only at the origin.

At points in the xy-plane we have

$$\mathbf{v}(x,y,0)=\frac{2m(x\mathbf{i}+y\mathbf{j})}{(x^2+y^2+\ell^2)^{3/2}}.$$

The velocity is radially away from the origin in the xy-plane, as is appropriate by symmetry. The speed at $(x,y,0)$ is

$$v(x,y,0)=\frac{2m\sqrt{x^2+y^2}}{(x^2+y^2+\ell^2)^{3/2}}=\frac{2ms}{(s^2+\ell^2)^{3/2}}=g(s),$$

where $s=\sqrt{x^2+y^2}$. For maximum $g(s)$ we set

$$\begin{aligned}
0=g'(s)&=2m\frac{(s^2+\ell^2)^{3/2}-\frac{3}{2}s(s^2+\ell^2)^{1/2}2s}{(s^2+\ell^2)^3}\\
&=\frac{2m(\ell^2-2s^2)}{(s^2+\ell^2)^{5/2}}.
\end{aligned}$$

Thus, the speed in the xy-plane is greatest at points of the circle $x^2+y^2=\ell^2/2$.

12. The scalar potential for the source-sink system is

$$\phi(x,y,z)=\phi(\mathbf{r})=-\frac{2}{|\mathbf{r}|}+\frac{1}{|\mathbf{r}-\mathbf{k}|}.$$

Thus, the velocity field is

$$\begin{aligned}
\mathbf{v}=\nabla\phi&=\frac{2\mathbf{r}}{|\mathbf{r}|^3}-\frac{\mathbf{r}-\mathbf{k}}{|\mathbf{r}-\mathbf{k}|^3}\\
&=\frac{2(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})}{(x^2+y^2+z^2)^{3/2}}-\frac{x\mathbf{i}+y\mathbf{j}+(z-1)\mathbf{k}}{(x^2+y^2+(z-1)^2)^{3/2}}.
\end{aligned}$$

For vertical velocity we require

$$\frac{2x}{(x^2+y^2+z^2)^{3/2}} = \frac{x}{(x^2+y^2+(z-1)^2)^{3/2}},$$

and a similar equation for y. Both equations will be satisfied at all points of the z-axis, and also wherever

$$\begin{aligned} 2\Big(x^2+y^2+(z-1)^2\Big)^{3/2} &= \Big(x^2+y^2+z^2\Big)^{3/2} \\ 2^{2/3}\Big(x^2+y^2+(z-1)^2\Big) &= x^2+y^2+z^2 \\ x^2+y^2+(z-K)^2 &= K^2-K, \end{aligned}$$

where $K = 2^{2/3}/(2^{2/3}-1)$. This latter equation represents a sphere, S, since $K^2-K>0$. The velocity is vertical at all points on S, as well as at all points on the z-axis.

Since the source at the origin is twice as strong as the sink at $(0,0,1)$, only half the fluid it emits will be sucked into the sink. By symmetry, this half will the half emitted into the half-space $z>0$. The rest of the fluid emitted at the origin will flow outward to infinity. There is one point where $\mathbf{v}=\mathbf{0}$. This point (which is easily calculated to be $(0,0,2+\sqrt{2})$) lies inside S. Streamlines emerging from the origin parallel to the xy-plane lead to this point. Streamlines emerging into $z>0$ cross S and approach the sink. Streamlines emerging into $z<0$ flow to infinity. Some of these cross S twice, some others are tangent to S, some do not intersect S anywhere.

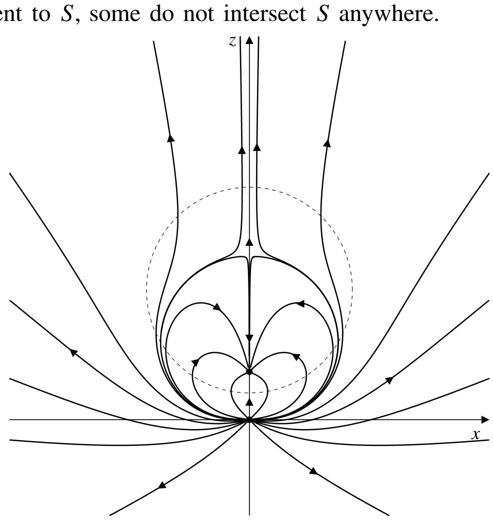

Fig. 15.2.12

13. Fluid emitted by interval Δz in time interval $[0,t]$ occupies, at time t, a cylinder of radius r, where

$$\pi r^2 \Delta Z = \text{vol. of cylinder} = 2\pi m t \Delta z.$$

Thus $r^2 = 2mt$, and $r\dfrac{dr}{dt} = m$. The surface of this cylinder is moving away from the z-axis at rate

$$\frac{dr}{dt} = \frac{m}{r} = \frac{m}{\sqrt{x^2+y^2}},$$

so the velocity at any point (x,y,z) is

$$\begin{aligned} \mathbf{v} &= \frac{m}{\sqrt{x^2+y^2}} \times \text{unit vector in direction } x\mathbf{i}+y\mathbf{j} \\ &= \frac{m(x\mathbf{i}+y\mathbf{j})}{x^2+y^2}. \end{aligned}$$

14. For $\mathbf{v}(x,y) = \dfrac{m(x\mathbf{i}+y\mathbf{j})}{x^2+y^2}$, we have

$$\frac{\partial v_1}{\partial y} = -\frac{2mxy}{(x^2+y^2)^2} = \frac{\partial v_2}{\partial x},$$

so $\mathbf{v}$ may be conservative, except at $(0,0)$. We have

$$\begin{aligned} \phi(x,y) &= m\int \frac{x\,dx}{x^2+y^2} = \frac{m}{2}\ln(x^2+y^2) + C_1(y) \\ \frac{my}{x^2+y^2} &= \frac{\partial \phi}{\partial y} = \frac{my}{x^2+y^2} + \frac{dC_1}{dy}. \end{aligned}$$

Thus we may take $C_1(y)=0$, and obtain

$$\phi(x,y) = \frac{m}{2}\ln(x^2+y^2) = m\ln|\mathbf{r}|,$$

as a scalar potential for the velocity field $\mathbf{v}$ of a line source of strength of m.

15. The two-dimensional dipole of strength μ has potential

$$\begin{aligned} &\phi(x,y) \\ &= \lim_{\substack{\ell\to 0\\ m\ell=\mu}} \frac{m}{2}\left[\ln\left(x^2+\left(y-\frac{\ell}{2}\right)^2\right) - \ln\left(x^2+\left(y+\frac{\ell}{2}\right)^2\right)\right] \\ &= \frac{\mu}{2}\lim_{\ell\to 0} \frac{\ln\left(x^2+\left(y-\frac{\ell}{2}\right)^2\right) - \ln\left(x^2+\left(y+\frac{\ell}{2}\right)^2\right)}{\ell} \\ &\qquad \text{(apply l'Hôpital's Rule)} \\ &= \frac{\mu}{2}\lim_{\ell\to 0} \frac{-\left(y-\frac{\ell}{2}\right)}{x^2+\left(y-\frac{\ell}{2}\right)^2} - \frac{\left(y+\frac{\ell}{2}\right)}{x^2+\left(y+\frac{\ell}{2}\right)^2} \\ &= -\frac{\mu y}{x^2+y^2} = -\frac{\mu y}{r^2}. \end{aligned}$$

Now

$$\begin{aligned} \frac{\partial \phi}{\partial x} &= \frac{2\mu y}{r^3}\frac{\partial r}{\partial x} = \frac{2\mu xy}{r^4} \\ \frac{\partial \phi}{\partial y} &= -\mu\frac{r^2 - 2yr\frac{y}{r}}{r^4} = \frac{\mu(y^2-x^2)}{r^4}. \end{aligned}$$

Thus

$$\mathbf{F} = \nabla\phi = \frac{\mu}{(x^2+y^2)^2}\Big(2xy\mathbf{i} + (y^2-x^2)\mathbf{j}\Big).$$

16. The equipotential curves for the two-dimensional dipole have equations $y = 0$ or

$$\begin{aligned}
-\frac{\mu y}{x^2+y^2} &= \frac{1}{C}\\
x^2 + y^2 + \mu C y &= 0\\
x^2 + \left(y + \frac{\mu C}{2}\right)^2 &= \frac{\mu^2 C^2}{4}.
\end{aligned}$$

These equipotentials are circles tangent to the x-axis at the origin.

17. All circles tangent to the y-axis at the origin intersect all circles tangent to the x-axis at the origin at right angles, so they must be the streamlines of the two-dimensional dipole.

As an alternative derivation of this fact, the streamlines must satisfy

$$\frac{dx}{2xy} = \frac{dy}{y^2 - x^2},$$

or, equivalently,

$$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}.$$

This homogeneous DE can be solved (as was that in Exercise 10) by a change in dependent variable. Let $y = xv(x)$. Then

$$\begin{aligned}
v + x\frac{dv}{dx} &= \frac{dy}{dx} = \frac{v^2x^2 - x^2}{2vx^2}\\
x\frac{dv}{dx} &= \frac{v^2-1}{2v} - v = -\frac{v^2+1}{2v}\\
\frac{2v\,dv}{v^2+1} &= -\frac{dx}{x}\\
\ln(v^2+1) &= -\ln x + \ln C\\
v^2 + 1 = \frac{C}{x} \quad &\Rightarrow \quad \frac{y^2}{x^2} + 1 = \frac{C}{x}\\
x^2 + y^2 &= Cx\\
(x - C)^2 + y^2 &= C^2.
\end{aligned}$$

These streamlines are circles tangent to the y-axis at the origin.

18. The velocity field for a point source of strength $m\,dt$ at $(0, 0, t)$ is

$$\mathbf{v}_t(x, y, z) = \frac{m\Big(x\mathbf{i} + y\mathbf{j} + (z-t)\mathbf{k}\Big)}{\Big(x^2 + y^2 + (z-t)^2\Big)^{3/2}}.$$

Hence we have

$$\begin{aligned}
&\int_{-\infty}^{\infty} \mathbf{v}_t(x, y, z)\,dt\\
&= m\int_{-\infty}^{\infty} \frac{x\mathbf{i} + y\mathbf{j} + (z-t)\mathbf{k}}{\Big(x^2+y^2+(z-t)^2\Big)^{3/2}}\,dt\\
&= m(x\mathbf{i} + y\mathbf{j})\int_{-\infty}^{\infty} \frac{dt}{\Big(x^2+y^2+(z-t)^2\Big)^{3/2}}\\
&\qquad \text{Let } z - t = \sqrt{x^2+y^2}\tan\theta\\
&\qquad -dt = \sqrt{x^2+y^2}\sec^2\theta\,d\theta\\
&= \frac{m(x\mathbf{i} + y\mathbf{j})}{x^2+y^2}\int_{-\pi/2}^{\pi/2}\cos\theta\,d\theta\\
&= \frac{2m(x\mathbf{i} + y\mathbf{j})}{x^2+y^2},
\end{aligned}$$

which is the velocity field of a line source of strength $2m$ along the z-axis.

The definition of strength of a point source in 3-space was made to ensure that the velocity field of a source of strength 1 had speed 1 at distance 1 from the source. This corresponds to fluid being emitted from the source at a volume rate of 4π. Similarly, the definition of strength of a line source guaranteed that a source of strength 1 gives rise to fluid speed of 1 at unit distance 1 from the line source. This corresponds to a fluid emission at a volume rate 2π per unit length along the line. Thus, the integral of a 3-dimensional source gives twice the volume rate of a 2-dimensional source, per unit length along the line.

The potential of a point source $m\,dt$ at $(0, 0, t)$ is

$$\phi(x, y, z) = -\frac{m}{\sqrt{x^2+y^2+(x-t)^2}}.$$

This potential cannot be integrated to give the potential for a line source along the z-axis because the integral

$$-m\int_{-\infty}^{\infty}\frac{dt}{\sqrt{x^2+y^2+(z-t)^2}}$$

does not converge, in the usual sense in which convergence of improper integrals was defined.

19. Since $x = r\cos\theta$ and $y = r\sin\theta$, we have

$$\begin{aligned}
\frac{\partial\phi}{\partial r} &= \cos\theta\frac{\partial\phi}{\partial x} + \sin\theta\frac{\partial\phi}{\partial y}\\
\frac{\partial\phi}{\partial\theta} &= -r\sin\theta\frac{\partial\phi}{\partial x} + r\cos\theta\frac{\partial\phi}{\partial y}.
\end{aligned}$$

Also,

$$\begin{aligned}
\hat{\mathbf{r}} &= \frac{x\mathbf{i} + y\mathbf{j}}{r} = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j}\\
\hat{\boldsymbol{\theta}} &= \frac{-y\mathbf{i} + x\mathbf{j}}{r} = -(\sin\theta)\mathbf{i} + (\cos\theta)\mathbf{j}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{\partial\phi}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{\boldsymbol{\theta}} \\
&= \left(\cos^2\theta\frac{\partial\phi}{\partial x} + \sin\theta\cos\theta\frac{\partial\phi}{\partial y}\right)\mathbf{i} \\
&\quad + \left(\cos\theta\sin\theta\frac{\partial\phi}{\partial x} + \sin^2\theta\frac{\partial\phi}{\partial y}\right)\mathbf{j} \\
&\quad + \left(\sin^2\theta\frac{\partial\phi}{\partial x} - \sin\theta\cos\theta\frac{\partial\phi}{\partial y}\right)\mathbf{i} \\
&\quad + \left(-\cos\theta\sin\theta\frac{\partial\phi}{\partial x} + \cos^2\theta\frac{\partial\phi}{\partial y}\right)\mathbf{j} \\
&= \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} = \nabla\phi.
\end{aligned}$$

20. If $\mathbf{F} = F_r(r,\theta)\hat{\mathbf{r}} + F_\theta(r,\theta)\hat{\boldsymbol{\theta}}$ is conservative, then $\mathbf{F} = \nabla\phi$ for some scalar field $\phi(r,\theta)$, and by Exercise 19,

$$\frac{\partial\phi}{\partial r} = F_r, \quad \frac{1}{r}\frac{\partial\phi}{\partial\theta} = F_\theta.$$

For the equality of the mixed second partial derivatives of ϕ, we require that

$$\frac{\partial F_r}{\partial\theta} = \frac{\partial}{\partial r}(rF_\theta) = F_\theta + r\frac{\partial F_\theta}{\partial r},$$

that is, $\dfrac{\partial F_r}{\partial\theta} - r\dfrac{\partial F_\theta}{\partial r} = F_\theta$.

21. If $\mathbf{F} = r\sin(2\theta)\hat{\mathbf{r}} + r\cos(2\theta)\hat{\boldsymbol{\theta}} = \nabla\phi(r,\theta)$, then we must have

$$\frac{\partial\phi}{\partial r} = r\sin(2\theta), \quad \frac{1}{r}\frac{\partial\phi}{\partial\theta} = r\cos(2\theta).$$

Both of these equations are satisfied by

$$\phi(r\theta) = \frac{1}{2}r^2\sin(2\theta) + C,$$

so $\mathbf{F}$ is conservative and this ϕ is a potential for it.

22. If $\mathbf{F} = r^2\cos\theta\hat{\mathbf{r}} + \alpha r^\beta\sin\theta\hat{\boldsymbol{\theta}} = \nabla\phi(r,\theta)$, then we must have

$$\frac{\partial\phi}{\partial r} = r^2\cos\theta, \quad \frac{1}{r}\frac{\partial\phi}{\partial\theta} = \alpha r^\beta\sin\theta.$$

From the first equation

$$\phi(r,\theta) = \frac{r^3}{3}\cos\theta + C(\theta).$$

The second equation then gives

$$C'(\theta) - \frac{r^3}{3}\sin\theta = \frac{\partial\phi}{\partial\theta} = \alpha r^{\beta+1}\sin\theta.$$

This equation can be solved for a function $C(\theta)$ independent of r only if $\alpha = -1/3$ and $\beta = 2$. In this case, $C(\theta) = C$ (a constant). $\mathbf{F}$ is conservative if α and β have these values, and a potential for it is $\phi = \frac{1}{3}r^3\cos\theta + C$.

Section 15.3 Line Integrals (page 824)

1. $\mathcal{C}$: $\mathbf{r} = a\cos t\sin t\mathbf{i} + a\sin^2 t\mathbf{j} + a\cos t\mathbf{k}$, $0 \le t \le \pi/2$. Since

$$|\mathbf{r}|^2 = a^2(\cos^2 t\sin^2 t + \sin^4 t + \cos^2 t) = a^2$$

for all t, $\mathcal{C}$ must lie on the sphere of radius a centred at the origin. We have

$$\begin{aligned}
ds &= a\sqrt{(\cos^2 t - \sin^2 t)^2 + 4\sin^2 t\cos^2 t + \sin^2 t}\,dt \\
&= a\sqrt{\cos^2 2t + \sin^2 2t + \sin^2 t}\,dt \\
&= a\sqrt{1 + \sin^2 t}\,dt.
\end{aligned}$$

Thus

$$\begin{aligned}
\int_{\mathcal{C}} z\,ds &= \int_0^{\pi/2} a\cos t\,a\sqrt{1+\sin^2 t}\,dt \quad \begin{array}{l}\text{Let } u = \sin t\\ du = \cos t\,dt\end{array} \\
&= a^2\int_0^1 \sqrt{1+u^2}\,du \quad \begin{array}{l}\text{Let } u = \tan\phi\\ du = \sec^2\phi\,d\phi\end{array} \\
&= a^2\int_0^{\pi/4}\sec^3\phi\,d\phi \\
&= \frac{a^2}{2}\Big[\sec\phi\tan\phi + \ln|\sec\phi + \tan\phi|\Big]\Big|_0^{\pi/4} \\
&= \frac{a^2}{2}\Big(\sqrt{2} + \ln(1+\sqrt{2})\Big).
\end{aligned}$$

2. $\mathcal{C}$: $x = t\cos t$, $y = t\sin t$, $z = t$, $(0 \le t \le 2\pi)$. We have

$$\begin{aligned}
ds &= \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 + 1}\,dt \\
&= \sqrt{2+t^2}\,dt.
\end{aligned}$$

Thus

$$\begin{aligned}
\int_{\mathcal{C}} z\,ds &= \int_0^{2\pi} t\sqrt{2+t^2}\,dt \quad \begin{array}{l}\text{Let } u = 2+t^2\\ du = 2t\,dt\end{array} \\
&= \frac{1}{2}\int_2^{2+4\pi^2} u^{1/2}\,du \\
&= \frac{1}{3}u^{3/2}\Big|_2^{2+4\pi^2} = \frac{(2+4\pi^2)^{3/2} - 2^{3/2}}{3}.
\end{aligned}$$

3. Wire: $\mathbf{r} = 3t\mathbf{i} + 3t^2\mathbf{j} + 2t^3\mathbf{k}, \quad (0 \le t \le 1)$

$\mathbf{v} = 3\mathbf{i} + 6t\mathbf{j} + 6t^2\mathbf{k}$

$v = 3\sqrt{1 + 4t^2 + 4t^4} = 3(1 + 2t^2).$

If the wire has density $\delta(t) = 1 + t$ g/unit length, then its mass is

$$\begin{aligned} m &= 3\int_0^1 (1 + 2t^2)(1 + t)\,dt \\ &= 3\left(t + \frac{t^2}{2} + \frac{2t^3}{3} + \frac{t^4}{2}\right)\Bigg|_0^1 = 8 \text{ g.} \end{aligned}$$

4. The wire of Example 3 lies in the first octant on the surfaces $z = x^2$ and $z = 2 - x^2 - 2y^2$, and, therefore, also on the surface $x^2 = 2 - x^2 - 2y^2$, or $x^2 + y^2 = 1$, a circular cylinder. Since it goes from $(1, 0, 1)$ to $(0, 1, 0)$ it can be parametrized

$$\begin{aligned} \mathbf{r} &= \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + \cos^2\mathbf{k}, \quad (0 \le t \le \pi/2) \\ \mathbf{v} &= -\sin t\,\mathbf{i} + \cos t\,\mathbf{j} - 2\cos t \sin t\,\mathbf{k} \\ v &= \sqrt{1 + \sin^2(2t)} = \sqrt{2 - \cos^2(2t)}. \end{aligned}$$

Since
the wire has density $\delta = xy = \sin t \cos t = \frac{1}{2}\sin(2t)$,
its mass is

$$\begin{aligned} m &= \frac{1}{2}\int_0^{\pi/2} \sqrt{2 - \cos^2(2t)}\,\sin(2t)\,dt \quad \begin{array}{l}\text{Let } v = \cos(2t) \\ dv = -2\sin(2t)\,dt\end{array} \\ &= \frac{1}{4}\int_{-1}^{1} \sqrt{2 - v^2}\,dv = \frac{1}{2}\int_0^1 \sqrt{2 - v^2}\,dv, \end{aligned}$$

which is the same integral obtained in Example 3, and has value $(\pi + 2)/8$.

5. $\mathcal{C}$: $\mathbf{r} = e^t\cos t\,\mathbf{i} + e^t \sin t\,\mathbf{j} + t\mathbf{k}$, $0 \le t \le 2\pi$).

$$\begin{aligned} ds &= \sqrt{e^{2t}(\cos t - \sin t)^2 + e^{2t}(\sin t + \cos t)^2 + 1}\,dt \\ &= \sqrt{1 + 2e^{2t}}\,dt. \end{aligned}$$

The moment of inertia of $\mathcal{C}$ about the z-axis is

$$\begin{aligned} I &= \delta\int_{\mathcal{C}} (x^2 + y^2)\,ds \\ &= \delta\int_0^{2\pi} e^{2t}\sqrt{1 + 2e^{2t}}\,dt \quad \begin{array}{l}\text{Let } u = 1 + 2e^{2t} \\ du = 4e^{2t}\,dt\end{array} \\ &= \frac{\delta}{4}\int_3^{1+2e^{4\pi}} \sqrt{u}\,du \\ &= \frac{\delta}{6}u^{3/2}\Big|_3^{1+2e^{4\pi}} = \frac{\delta}{6}\Big[(1 + 2e^{4\pi})^{3/2} - 3^{3/2}\Big]. \end{aligned}$$

6. $\mathcal{C}$ is the same curve as in Exercise 5. We have

$$\begin{aligned} \int_{\mathcal{C}} e^z\,ds &= \int_0^{2\pi} e^t\sqrt{1 + 2e^{2t}}\,dt \quad \begin{array}{l}\text{Let } \sqrt{2}e^t = \tan\theta \\ \sqrt{2}e^t\,dt = \sec^2\theta\,d\theta\end{array} \\ &= \frac{1}{\sqrt{2}}\int_{t=0}^{t=2\pi} \sec^3\theta\,d\theta \\ &= \frac{1}{2\sqrt{2}}\Big[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\Big]\Bigg|_{t=0}^{t=2\pi} \\ &= \frac{\sqrt{2}e^t\sqrt{1 + 2e^{2t}} + \ln(\sqrt{2}e^t + \sqrt{1 + 2e^{2t}})}{2\sqrt{2}}\Bigg|_0^{2\pi} \\ &= \frac{e^{2\pi}\sqrt{1 + 2e^{4\pi}} - \sqrt{3}}{2} \\ &\quad + \frac{1}{2\sqrt{2}}\ln\frac{\sqrt{2}e^{2\pi} + \sqrt{1 + 2e^{4\pi}}}{\sqrt{2} + \sqrt{3}}. \end{aligned}$$

7. The line of intersection of the planes $x - y + z = 0$ and $x + y + 2z = 0$ from $(0, 0, 0)$ to $(3, 1, -2)$ can be parametrized

$$\mathbf{r} = 3t\mathbf{i} + t\mathbf{j} - 2t\mathbf{k}, \qquad (0 \le t \le 1).$$

Thus $ds = \sqrt{14}\,dt$ and

$$\int_{\mathcal{C}} x^2\,ds = \sqrt{14}\int_0^1 9t^2\,dt = 3\sqrt{14}.$$

8. The curve $\mathcal{C}$ of intersection of $x^2 + z^2 = 1$ and $y = x^2$ can be parametrized

$$\mathbf{r} = \cos t\,\mathbf{i} + \cos^2 t\,\mathbf{j} + \sin t\,\mathbf{k}, \quad (0 \le t \le 2\pi).$$

Thus

$$ds = \sqrt{\sin^2 t + 4\sin^2 t\cos^2 t + \cos^2 t}\,dt = \sqrt{1 + \sin^2 2t}\,dt.$$

We have

$$\begin{aligned} &\int_{\mathcal{C}} \sqrt{1 + 4x^2z^2}\,ds \\ &= \int_0^{2\pi} \sqrt{1 + 4\cos^2 t\sin^2 t}\sqrt{1 + \sin^2 2t}\,dt \\ &= \int_0^{2\pi} (1 + \sin^2 2t)\,dt \\ &= \int_0^{2\pi} \left(1 + \frac{1 - \cos 4t}{2}\right)dt \\ &= \frac{3}{2}(2\pi) = 3\pi. \end{aligned}$$

9. $\mathbf{r} = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}, \quad (0 \le t \le 2\pi)$
$\mathbf{v} = -\sin t\mathbf{i} + \cos t\mathbf{j} + \mathbf{k}, \quad v = \sqrt{2}.$
If the density is $\delta = z = t$, then

$$\begin{aligned} m &= \sqrt{2}\int_0^{2\pi} t\,dt = 2\pi^2\sqrt{2} \\ M_{x=0} &= \sqrt{2}\int_0^{2\pi} t\cos t\,dt = 0 \\ M_{y=0} &= \sqrt{2}\int_0^{2\pi} t\sin t\,dt = -2\pi\sqrt{2} \\ M_{z=0} &= \sqrt{2}\int_0^{2\pi} t^2\,dt = \frac{8\pi^3\sqrt{2}}{3}. \end{aligned}$$

(We have omitted the details of the evaluation of these integrals.) The centre of mass is $\left(0, -\frac{1}{\pi}, \frac{4\pi}{3}\right)$.

10. Here the wire of Exercise 9 extends only from $t = 0$ to $t = \pi$:

$$\begin{aligned} m &= \sqrt{2}\int_0^{\pi} t\,dt = \frac{\pi^2\sqrt{2}}{2} \\ M_{x=0} &= \sqrt{2}\int_0^{\pi} t\cos t\,dt = -2\sqrt{2} \\ M_{y=0} &= \sqrt{2}\int_0^{\pi} t\sin t\,dt = \pi\sqrt{2} \\ M_{z=0} &= \sqrt{2}\int_0^{\pi} t^2\,dt = \frac{\pi^3\sqrt{2}}{3}. \end{aligned}$$

The centre of mass is $\left(-\frac{4}{\pi^2}, \frac{2}{\pi}, \frac{2\pi}{3}\right)$.

11.

$$\begin{aligned} \mathbf{r} &= e^t\mathbf{i} + \sqrt{2}t\mathbf{j} + e^{-t}\mathbf{k}, \quad (0 \le t \le 1) \\ \mathbf{v} &= e^t\mathbf{i} + \sqrt{2}\mathbf{j} - e^{-t}\mathbf{k} \\ v &= \sqrt{e^{2t} + 2 + e^{-2t}} = e^t + e^{-t} \\ \int_{\mathcal{C}} (x^2 + z^2)\,ds &= \int_0^1 (e^{2t} + e^{-2t})(e^t + e^{-t})\,dt \\ &= \int_0^1 (e^{3t} + e^t + e^{-t} + e^{-3t})\,dt \\ &= \frac{e^3}{3} + e - \frac{1}{e} - \frac{1}{3e^3}. \end{aligned}$$

12.

$$\begin{aligned} m &= \int_0^1 (e^t + e^{-t})\,dt = \frac{e^2 - 1}{e} \\ M_{x=0} &= \int_0^1 e^t(e^t + e^{-t})\,dt = \frac{e^2 + 1}{2} \\ M_{y=0} &= \int_0^1 \sqrt{2}t(e^t + e^{-t})\,dt = \frac{2\sqrt{2}(e - 1)}{e} \\ M_{z=0} &= \int_0^1 e^{-t}(e^t + e^{-t})\,dt = \frac{3e^2 - 1}{2e^2} \end{aligned}$$

The centroid is $\left(\frac{e^3 + e}{2e^2 - 2}, \frac{2\sqrt{2}}{e + 1}, \frac{3e^2 - 1}{2e^3 - 2e}\right)$.

13. The first octant part $\mathcal{C}$ of the curve $x^2 + y^2 = a^2$, $z = x$, can be parametrized

$$\mathbf{r} = a\cos t\mathbf{i} + a\sin t\mathbf{j} + a\cos t\mathbf{k}, \quad (0 \le t \le \pi/2).$$

We have $ds = a\sqrt{1 + \sin^2 t}\,dt$, so

$$\begin{aligned} \int_{\mathcal{C}} x\,ds &= a^2\int_0^{\pi/2} \cos t\sqrt{1 + \sin^2 t}\,dt \quad \begin{array}{l}\text{Let } \sin t = \tan\theta \\ \cos t\,dt = \sec^2\theta\,d\theta\end{array} \\ &= a^2\int_{t=0}^{t=\pi/2} \sec^3\theta\,d\theta \\ &= \frac{a^2}{2}\Big[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\Big]\Bigg|_{t=0}^{t=\pi/2} \\ &= \frac{a^2}{2}\Big[\sin t\sqrt{1 + \sin^2 t} + \ln|\sin t + \sqrt{1 + \sin^2 t}|\Big]\Bigg|_0^{\pi/2} \\ &= \frac{a^2}{2}\Big[\sqrt{2} + \ln(1 + \sqrt{2})\Big]. \end{aligned}$$

14. On $\mathcal{C}$, we have

$$z = \sqrt{1 - x^2 - y^2} = \sqrt{1 - x^2 - (1 - x)^2} = \sqrt{2(x - x^2)}.$$

Thus $\mathcal{C}$ can be parametrized

$$\mathbf{r} = t\mathbf{i} + (1 - t)\mathbf{j} + \sqrt{2(t - t^2)}\mathbf{k}, \quad (0 \le t \le 1).$$

Hence

$$ds = \sqrt{1 + 1 + \frac{(1 - 2t)^2}{2(t - t^2)}}\,dt = \frac{dt}{\sqrt{2(t - t^2)}}.$$

We have

$$\int_{\mathcal{C}} z\,ds = \int_0^1 \sqrt{2(t - t^2)}\frac{dt}{\sqrt{2(t - t^2)}} = 1.$$

15. The parabola $z^2 = x^2 + y^2$, $x + z = 1$, can be parametrized in terms of $y = t$ since

$$\begin{aligned} &(1 - x)^2 = z^2 = x^2 + y^2 = x^2 + t^2 \\ &\Rightarrow \quad 1 - 2x = t^2 \quad \Rightarrow \quad x = \frac{1 - t^2}{2} \\ &\Rightarrow \quad z = 1 - x = \frac{1 + t^2}{2}. \end{aligned}$$

Thus $ds = \sqrt{t^2 + 1 + t^2}\,dt = \sqrt{1 + 2t^2}\,dt$, and

$$\begin{aligned} \int_{\mathcal{C}} \frac{ds}{(2y^2 + 1)^{3/2}} &= \int_{-\infty}^{\infty} \frac{\sqrt{1 + 2t^2}}{(2t^2 + 1)^{3/2}}\,dt \\ &= 2\int_0^{\infty} \frac{dt}{1 + 2t^2} \\ &= \sqrt{2}\tan^{-1}(\sqrt{2}t)\Big|_0^{\infty} = \sqrt{2}\frac{\pi}{2} = \frac{\pi}{\sqrt{2}}. \end{aligned}$$

16. $\mathcal{C}$: $y = x^2$, $z = y^2$, from $(0,0,0)$ to $(2,4,16)$. Parametrize $\mathcal{C}$ by

$$\mathbf{r} = t\mathbf{i} + t^2\mathbf{j} + t^4\mathbf{k}, \qquad (0 \le t \le 2).$$

Since $ds = \sqrt{1 + 4t^2 + 16t^6}\,dt$, we have

$$\int_{\mathcal{C}} xyz\,ds = \int_0^2 t^7\sqrt{1 + 4t^2 + 16t^6}\,dt.$$

17. Helix: $x = a\cos t$, $y = b\sin t$, $z = ct$ $(0 < a < b)$.

$$\begin{aligned} ds &= \sqrt{a^2\sin^2 t + b^2\cos^2 t + c^2}\,dt \\ &= \sqrt{c^2 + b^2 - (b^2 - a^2)\sin^2 t}\,dt \\ &= \sqrt{b^2 + c^2}\sqrt{1 - k^2\sin^2 t}\,dt \qquad (k^2 = \frac{b^2 - a^2}{b^2 + c^2}). \end{aligned}$$

One complete revolution of the helix corresponds to $0 \le t \le 2\pi$, and has length

$$\begin{aligned} L &= \sqrt{b^2 + c^2}\int_0^{2\pi}\sqrt{1 - k^2\sin^2 t}\,dt \\ &= 4\sqrt{b^2 + c^2}\int_0^{\pi/2}\sqrt{1 - k^2\sin^2 t}\,dt \\ &= 4\sqrt{b^2 + c^2}E(k) = 4\sqrt{b^2 + c^2}E\left(\sqrt{\frac{b^2 - a^2}{b^2 + c^2}}\right) \text{ units.} \end{aligned}$$

The length of the part of the helix from $t = 0$ to $t = T < \pi/2$ is

$$\begin{aligned} L &= \sqrt{b^2 + c^2}\int_0^{T}\sqrt{1 - k^2\sin^2 t}\,dt \\ &= \sqrt{b^2 + c^2}E(k, T) = \sqrt{b^2 + c^2}E\left(\sqrt{\frac{b^2 - a^2}{b^2 + c^2}}, T\right) \text{ units.} \end{aligned}$$

18. The straight line L with equation $Ax + By = C$, $(C \ne 0)$, lies at distance $D = \sqrt{|C|}/\sqrt{A^2 + B^2}$ from the origin. So does the line L_1 with equation $y = D$. Since $x^2 + y^2$ depends only on distance from the origin, we have, by symmetry,

$$\begin{aligned} \int_L \frac{ds}{x^2 + y^2} &= \int_{L_1} \frac{ds}{x^2 + y^2} \\ &= \int_{-\infty}^{\infty} \frac{dx}{x^2 + D^2} \\ &= \frac{2}{D}\tan^{-1}\frac{x}{D}\bigg|_0^{\infty} = \frac{2}{D}\left(\frac{\pi}{2} - 0\right) \\ &= \frac{\pi}{D} = \frac{\pi\sqrt{A^2 + B^2}}{|C|}. \end{aligned}$$

Section 15.4 Line Integrals of Vector Fields (page 831)

1. $\mathbf{F} = xy\mathbf{i} - x^2\mathbf{j}$.
$\mathcal{C}:\quad \mathbf{r} = t\mathbf{i} + t^2\mathbf{j}, \quad (0 \le t \le 1)$.

$$\int_{\mathcal{C}} \mathbf{F}\bullet d\mathbf{r} = \int_0^1 [t^3 - t^2(2t)]\,dt = -\int_0^1 t^3\,dt = -\frac{1}{4}.$$

2. $\mathbf{F} = \cos x\mathbf{i} - y\mathbf{j} = \nabla\left(\sin x - \dfrac{y^2}{2}\right)$.

$\mathcal{C}:\quad y = \sin x$ from (0,0) to $(\pi, 0)$.

$$\int_{\mathcal{C}} \mathbf{F}\bullet d\mathbf{r} = \left(\sin x - \frac{y^2}{2}\right)\bigg|_{(0,0)}^{(\pi,0)} = 0.$$

3. $\mathbf{F} = y\mathbf{i} + z\mathbf{j} - x\mathbf{k}$.
$\mathcal{C}:\quad \mathbf{r} = t\mathbf{i} + t\mathbf{j} + t\mathbf{k}, \quad (0 \le t \le 1)$.

$$\int_{\mathcal{C}} \mathbf{F}\bullet d\mathbf{r} = \int_0^1 (t + t - t)\,dt = \frac{t^2}{2}\bigg|_0^1 = \frac{1}{2}.$$

4. $\mathbf{F} = z\mathbf{i} - y\mathbf{j} + 2x\mathbf{k}$.
$\mathcal{C}$: $\mathbf{r} = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$, $(0 \le t \le 1)$.

$$\begin{aligned} \int_{\mathcal{C}} \mathbf{F}\bullet d\mathbf{r} &= \int_0^1 [t^3 - t^2(2t) + 2t(3t^2)]\,dt \\ &= \int_0^1 5t^3\,dt = \frac{5t^4}{4}\bigg|_0^1 = \frac{5}{4}. \end{aligned}$$

5. $\mathbf{F} = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k} = \nabla(xyz)$.
$\mathcal{C}$: a curve from $(-1,0,0)$ to $(1,0,0)$. (Since $\mathbf{F}$ is conservative, it doesn't matter what curve.)

$$\int_{\mathcal{C}} \mathbf{F}\bullet d\mathbf{r} = xyz\bigg|_{(-1,0,0)}^{(1,0,0)} = 0 - 0 = 0.$$

6. $\mathbf{F} = (x - z)\mathbf{i} + (y - z)\mathbf{j} - (x + y)\mathbf{k}$
$= \nabla\left(\dfrac{x^2 + y^2}{2} - (x + y)z\right)$.
$\mathcal{C}$ is a given polygonal path from (0,0,0) to (1,1,1) (but any other piecewise smooth path from the first point to the second would do as well).

$$\int_{\mathcal{C}} \mathbf{F}\bullet d\mathbf{r} = \left(\frac{x^2 + y^2}{2} - (x + y)z\right)\bigg|_{(0,0,0)}^{(1,1,1)} = 1 - 2 = -1.$$

7. $\mathbf{F} = (x + y)\mathbf{i} + (x - z)\mathbf{j} + (z - y)\mathbf{k}$
$= \nabla\left(\dfrac{x^2 + z^2}{2} + y(x - z)\right)$.
The work done by $\mathbf{F}$ in moving an object from $(1, 0, -1)$ to $(0, -2, 3)$ is

$$\begin{aligned} W = \int_{\mathcal{C}} \mathbf{F}\bullet d\mathbf{r} &= \left(\frac{x^2 + z^2}{2} + y(x - z)\right)\bigg|_{(1,0,-1)}^{(0,-2,3)} \\ &= \frac{9}{2} - 2(-3) - (1 + 0) = \frac{19}{2} \text{ units.} \end{aligned}$$

8. $\mathcal{C}$ is made up of four segments as shown in the figure.
On $\mathcal{C}_1$, $y = 0$, $dy = 0$, and x goes from 0 to 1.
On $\mathcal{C}_2$, $x = 1$, $dx = 0$, and y goes from 0 to 1.
On $\mathcal{C}_3$, $y = 1$, $dy = 0$, and x goes from 1 to 0.
On $\mathcal{C}_4$, $x = 0$, $dx = 0$, and y goes from 1 to 0.
Thus

$$\int_{\mathcal{C}_1} x^2y^2\,dx + x^3y\,dy = 0$$
$$\int_{\mathcal{C}_2} x^2y^2\,dx + x^3y\,dy = \int_0^1 y\,dy = \frac{1}{2}$$
$$\int_{\mathcal{C}_3} x^2y^2\,dx + x^3y\,dy = \int_1^0 x^2\,dx = -\frac{1}{3}$$
$$\int_{\mathcal{C}_4} x^2y^2\,dx + x^3y\,dy = 0.$$

Finally, therefore,

$$\int_{\mathcal{C}} x^2y^2\,dx + x^3y\,dy = 0 + \frac{1}{2} - \frac{1}{3} + 0 = \frac{1}{6}.$$

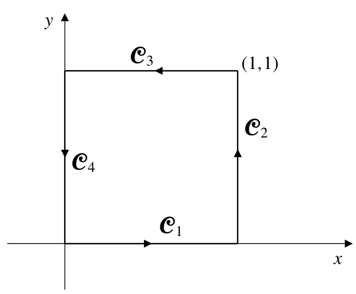

Fig. 15.4.8

9. Observe that if $\phi = e^{x+y}\sin(y+z)$, then

$$\nabla\phi = e^{x+y}\sin(y+z)\mathbf{i} + e^{x+y}\Big(\sin(y+z) + \cos(y+z)\Big)\mathbf{j} + e^{x+y}\cos(y+z)\mathbf{k}.$$

Thus, for any piecewise smooth path from $(0,0,0)$ to $\left(1, \frac{\pi}{4}, \frac{\pi}{4}\right)$, we have

$$\int_{\mathcal{C}} e^{x+y}\sin(y+z)\,dx + e^{x+y}\Big(\sin(y+z)+\cos(y+z)\Big)\,dy + e^{x+y}\cos(y+z)\,dz$$
$$= \int_{\mathcal{C}} \nabla\phi \bullet d\mathbf{r} = \phi(x,y,z)\Big|_{(0,0,0)}^{(1,\pi/4,\pi/4)} = e^{1+(\pi/4)}.$$

10. $\mathbf{F} = (axy+z)\mathbf{i} + x^2\mathbf{j} + (bx+2z)\mathbf{k}$ is conservative if

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x} \quad\Leftrightarrow\quad a = 2$$
$$\frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x} \quad\Leftrightarrow\quad b = 1$$
$$\frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y} \quad\Leftrightarrow\quad 0 = 0.$$

If $a = 2$ and $b = 1$, then $\mathbf{F} = \nabla\phi$ where

$$\phi = \int (2xy+z)\,dx = x^2y + xz + C_2(y,z)$$
$$\frac{\partial C_1}{\partial y} + x^2 = F_2 = x^2 \quad\Rightarrow\quad C_1(y,z) = C_2(z)$$
$$\frac{dC_2}{dz} + x = F_3 = x + 2z \quad\Rightarrow\quad C_2(z) = z^2 + C.$$

Thus $\phi = x^2y + xz + z^2 + C$ is a potential for $\mathbf{F}$.

11. $\mathbf{F} = Ax\ln z\mathbf{i} + By^2z\mathbf{j} + \left(\frac{x^2}{z} + y^3\right)\mathbf{k}$ is conservative if

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x} \quad\Leftrightarrow\quad 0 = 0$$
$$\frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x} \quad\Leftrightarrow\quad A = 2$$
$$\frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y} \quad\Leftrightarrow\quad B = 3.$$

If $A = 2$ and $B = 3$, then $\mathbf{F} = \nabla\phi$ where $\phi = x^2\ln z + y^3z$. If $\mathcal{C}$ is the straight line $x = t+1$, $y = 1$, $z = t+1$, $(0 \le t \le 1)$, from $(1,1,1)$ to $(2,1,2)$, then

$$\int_{\mathcal{C}} 2x\ln z\,dx + 2y^2z\,dy + y^3\,dz$$
$$= \int_{\mathcal{C}} \nabla\phi \bullet d\mathbf{r} - \int_{\mathcal{C}} y^2z\,dy + \frac{x^2}{z}\,dz$$
$$= (x^2\ln z + y^3z)\Big|_{(1,1,1)}^{(2,1,2)} - \int_0^1 [(t+1)(0) + (t+1)]\,dt$$
$$= 4\ln 2 + 2 - 1 - \left(\frac{t^2}{2} + t\right)\Big|_0^1 = 4\ln 2 - \frac{1}{2}.$$

12. $\mathbf{F} = (y^2\cos x + z^3)\mathbf{i} + (2y\sin x - 4)\mathbf{j} + (3xz^2 + 2)\mathbf{k}$
$= \nabla(y^2\sin x + xz^3 - 4y + 2z)$.
The curve $\mathcal{C}$: $x = \sin^{-1}t$, $y = 1 - 2t$, $z = 3t - 1$, $(0 \le t \le 1)$, goes from $(0,1,-1)$ to $(\pi/2,-1,2)$. The work done by $\mathbf{F}$ in moving a particle along $\mathcal{C}$ is

$$W = \int_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{r}$$
$$= (y^2\sin x + xz^3 - 4y + 2z)\Big|_{(0,1,-1)}^{(\pi/2,-1,2)}$$
$$= 1 + 4\pi + 4 + 4 - 0 - 0 + 4 + 2 = 15 + 4\pi.$$

13. For $z = \ln(1+x)$, $y = x$, from $x = 0$ to $x = 1$, we have

$$\begin{aligned}
&\int_{\mathcal{C}} \Big[(2x\sin(\pi y) - e^z)\,dx \\
&\qquad + (\pi x^2\cos(\pi y) - 3e^z)\,dy - xe^z\,dz\Big] \\
&= \int_{\mathcal{C}} \nabla\Big(x^2\sin(\pi y) - xe^z\Big) \bullet d\mathbf{r} - 3\int_{\mathcal{C}} e^z\,dy \\
&= \Big(x^2\sin(\pi y) - xe^z\Big)\Big|_{(0,0,0)}^{(1,1,\ln 2)} - 3\int_0^1 (1+x)\,dx \\
&= -2 - 3\left(x + \frac{x^2}{2}\right)\Big|_0^1 = -2 - \frac{9}{2} = -\frac{13}{2}.
\end{aligned}$$

14.
a) $S = \{(x,y) : x > 0, y \geq 0\}$ is a simply connected domain.

b) $S = \{(x,y) : x = 0, y \geq 0\}$ is not a domain. (It has empty interior.)

c) $S = \{(x,y) : x \neq 0, y > 0\}$ is a domain but is not connected. There is no path in S from $(-1,1)$ to $(1,1)$.

d) $S = \{(x,y,z) : x^2 > 1\}$ is a domain but is not connected. There is no path in S from $(-2,0,0)$ to $(2,0,0)$.

e) $S = \{(x,y,z) : x^2 + y^2 > 1\}$ is a connected domain but is not simply connected. The circle $x^2 + y^2 = 2$, $z = 0$ lies in S, but cannot be shrunk through S to a point since it surrounds the cylinder $x^2 + y^2 \leq 1$ which is outside S.

f) $S = \{(x,y,z) : x^2 + y^2 + z^2 > 1\}$ is a simply connected domain even though it has a ball-shaped "hole" in it.

15. $\mathcal{C}$ is the curve $\mathbf{r} = a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j}$, $(0 \leq t \leq 2\pi)$.

$$\oint_{\mathcal{C}} x\,dy = \int_0^{2\pi} a\cos t\, a\cos t\,dt = \pi a^2$$
$$\oint_{\mathcal{C}} y\,dx = \int_0^{2\pi} a\sin t(-a\sin t)\,dt = -\pi a^2.$$

16. $\mathcal{C}$ is the curve $\mathbf{r} = a\cos t\,\mathbf{i} + b\sin t\,\mathbf{j}$, $(0 \leq t \leq 2\pi)$.

$$\oint_{\mathcal{C}} x\,dy = \int_0^{2\pi} a\cos t\, b\cos t\,dt = \pi ab$$
$$\oint_{\mathcal{C}} y\,dx = \int_0^{2\pi} b\sin t(-a\sin t)\,dt = -\pi ab.$$

17. $\mathcal{C}$ consists of two parts:
On $\mathcal{C}_1$, $y = 0$, $dy = 0$, and x goes from $-a$ to a.
On $\mathcal{C}_2$, $x = a\cos t$, $y = a\sin t$, t goes from 0 to π.

$$\begin{aligned}
\oint_{\mathcal{C}} x\,dy &= \int_{\mathcal{C}_1} x\,dy + \int_{\mathcal{C}_2} x\,dy \\
&= 0 + \int_0^{\pi} a^2\cos^2 t\,dt = \frac{\pi a^2}{2}, \\
\oint_{\mathcal{C}} y\,dx &= \int_{\mathcal{C}_1} y\,dx + \int_{\mathcal{C}_2} y\,dx \\
&= 0 + \int_0^{\pi} (-a^2\cos^2 t)\,dt = -\frac{\pi a^2}{2}.
\end{aligned}$$

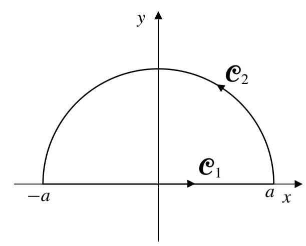

Fig. 15.4.17

18. $\mathcal{C}$ is made up of four segments as shown in the figure.
On $\mathcal{C}_1$, $y = 0$, $dy = 0$, and x goes from 0 to 1.
On $\mathcal{C}_2$, $x = 1$, $dx = 0$, and y goes from 0 to 1.
On $\mathcal{C}_3$, $y = 1$, $dy = 0$, and x goes from 1 to 0.
On $\mathcal{C}_4$, $x = 0$, $dx = 0$, and y goes from 1 to 0.

$$\begin{aligned}
\oint_{\mathcal{C}} x\,dy &= \int_{\mathcal{C}_1} + \int_{\mathcal{C}_2} + \int_{\mathcal{C}_3} + \int_{\mathcal{C}_4} \\
&= 0 + \int_0^1 dy + 0 + 0 = 1 \\
\oint_{\mathcal{C}} y\,dx &= \int_{\mathcal{C}_1} + \int_{\mathcal{C}_2} + \int_{\mathcal{C}_3} + \int_{\mathcal{C}_4} \\
&= 0 + 0 + \int_1^0 dx + 0 = -1.
\end{aligned}$$

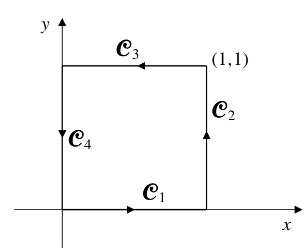

Fig. 15.4.18

19. $\mathcal{C}$ is made up of three segments as shown in the figure.
On $\mathcal{C}_1$, $y = 0$, $dy = 0$, and x goes from 0 to a.
On $\mathcal{C}_2$, $y = bt$, $x = a(1-t)$, and t goes from 0 to 1.
On $\mathcal{C}_3$, $x = 0$, $dx = 0$, and y goes from b to 0.

$$\begin{aligned}
\oint_{\mathcal{C}} x\,dy &= \int_{\mathcal{C}_1} + \int_{\mathcal{C}_2} + \int_{\mathcal{C}_3} \\
&= 0 + \int_0^1 a(1-t)\,b\,dt + 0 = \frac{ab}{2} \\
\oint_{\mathcal{C}} y\,dx &= \int_{\mathcal{C}_1} + \int_{\mathcal{C}_2} + \int_{\mathcal{C}_3} \\
&= 0 + \int_0^1 bt\,(-a\,dt) + 0 = -\frac{ab}{2}.
\end{aligned}$$

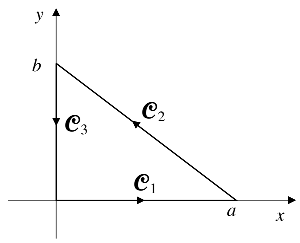

Fig. 15.4.19

20. Conjecture: If D is a domain in $\mathbb{R}^2$ whose boundary is a closed, non-self-intersecting curve $\mathcal{C}$, oriented counterclockwise, then

$$\begin{aligned}
&\oint_{\mathcal{C}} x\,dy = \text{area of } D, \\
&\oint_{\mathcal{C}} y\,dx = -\text{ area of } D.
\end{aligned}$$

Proof for a domain D that is x-simple and y-simple: Since D is x-simple, it can be specified by the inequalities

$$c \le y \le d, \qquad f(y) \le x \le g(y).$$

Let $\mathcal{C}$ consist of the four parts shown in the figure. On $\mathcal{C}_1$ and $\mathcal{C}_3$, $dy = 0$.
On $\mathcal{C}_2$, $x = g(y)$, where y goes from c to d.
On $\mathcal{C}_2$, $x = f(y)$, where y goes from d to c. Thus

$$\begin{aligned}
\oint_{\mathcal{C}} x\,dy &= \int_{\mathcal{C}_1} + \int_{\mathcal{C}_2} + \int_{\mathcal{C}_3} + \int_{\mathcal{C}_4} \\
&= 0 + \int_c^d g(y)\,dy + 0 + \int_d^c f(y)\,dy \\
&= \Big(g(y) - f(y)\Big)\,dy = \text{area of } D.
\end{aligned}$$

The proof that $\oint_{\mathcal{C}} y\,dx = -(\text{area of } D)$ is similar, and uses the fact that D is y-simple.

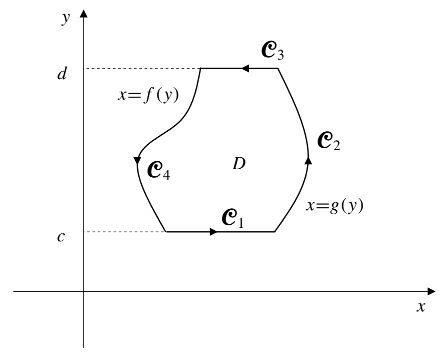

Fig. 15.4.20

21.
$$\begin{aligned}
\nabla(fg) &= +\left(f\frac{\partial g}{\partial x} + \frac{\partial f}{\partial x}g\right)\mathbf{i} + \left(f\frac{\partial g}{\partial y} + \frac{\partial f}{\partial y}g\right)\mathbf{j} \\
&\quad + \left(f\frac{\partial g}{\partial z} + \frac{\partial f}{\partial z}g\right)\mathbf{k} \\
&= g\nabla f + f\nabla g.
\end{aligned}$$

Thus, since $\mathcal{C}$ goes from P to Q,

$$\begin{aligned}
&\int_{\mathcal{C}} f\nabla g \bullet d\mathbf{r} + \int_{\mathcal{C}} g\nabla f \bullet d\mathbf{r} \\
&= \int_{\mathcal{C}} \nabla(fg) \bullet d\mathbf{r} = (fg)\Big|_P^Q \\
&= f(Q)g(Q) - f(P)g(P).
\end{aligned}$$

22. a) $\mathcal{C}$: $x = a\cos t$, $x = a\sin t$, $0 \le t \le 2\pi$.

$$\begin{aligned}
&\frac{1}{2\pi}\oint_{\mathcal{C}} \frac{x\,dy - y\,dx}{x^2 + y^2} \\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{a^2\cos^2 t + a^2\sin^2 t}{a^2\cos^2 t + a^2\sin^2 t}\,dt = 1.
\end{aligned}$$

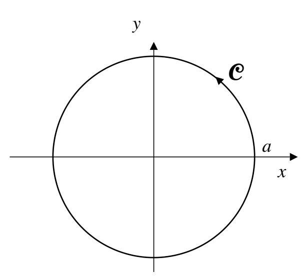

Fig. 15.4.22(a)

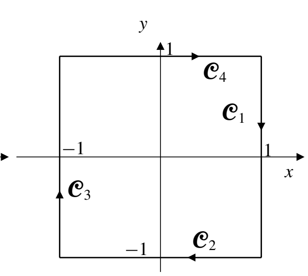

Fig. 15.4.22(b)

b) See the figure. $\mathcal{C}$ has four parts.
On $\mathcal{C}_1$, $x = 1$, $dx = 0$, y goes from 1 to -1.
On $\mathcal{C}_2$, $y = -1$, $dy = 0$, x goes from 1 to -1.
On $\mathcal{C}_3$, $x = -1$, $dx = 0$, y goes from -1 to 1.
On $\mathcal{C}_4$, $x = 1$, $dx = 0$, y goes from 1 to -1.

$$\begin{aligned}
&\frac{1}{2\pi}\oint_{\mathcal{C}} \frac{x\,dy - y\,dx}{x^2+y^2} \\
&= \frac{1}{2\pi}\left[\int_1^{-1} \frac{dy}{1+y^2} + \int_1^{-1} \frac{dx}{x^2+1}\right. \\
&\qquad \left. \int_{-1}^{1} \frac{-dy}{1+y^2} + \int_{-1}^{1} \frac{-dx}{x^2+1}\right] \\
&= -\frac{2}{\pi}\int_{-1}^{1} \frac{dt}{1+t^2} \\
&= -\frac{2}{\pi}\tan^{-1} t\Big|_{-1}^{1} = -\frac{2}{\pi}\left(\frac{\pi}{4}+\frac{\pi}{4}\right) = -1.
\end{aligned}$$

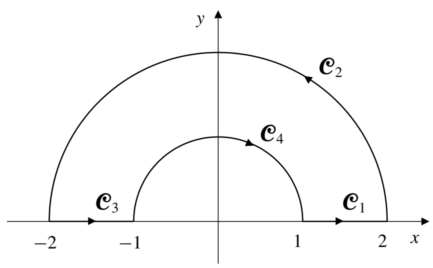

Fig. 15.4.22

c) See the figure. $\mathcal{C}$ has four parts.
On $\mathcal{C}_1$, $y = 0$, $dy = 0$, x goes from 1 to 2.
On $\mathcal{C}_2$, $x = 2\cos t$, $y = 2\sin t$, t goes from 0 to π.
On $\mathcal{C}_3$, $y = 0$, $dy = 0$, x goes from -2 to -1.
On $\mathcal{C}_4$, $x = \cos t$, $y = \sin t$, t goes from π to 0.

$$\begin{aligned}
&\frac{1}{2\pi}\oint_{\mathcal{C}} \frac{x\,dy - y\,dx}{x^2+y^2} \\
&= \frac{1}{2\pi}\left[0 + \int_0^{\pi} \frac{4\cos^2 t + 4\sin^2 t}{4\cos^2 t + 4\sin^2 t}\,dt\right. \\
&\qquad \left. +0 + \int_{\pi}^{0} \frac{\cos^2 t + \sin^2 t}{\cos^2 t + \sin^2 t}\,dt\right] \\
&= \frac{1}{2\pi}(\pi - \pi) = 0.
\end{aligned}$$

23. Although

$$\frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)$$

for all $(x, y) \neq (0, 0)$, Theorem 1 does not imply that $\oint_{\mathcal{C}} \frac{x\,dy - y\,dx}{x^2+y^2}$ is zero for all closed curves $\mathcal{C}$ in $\mathbb{R}^2$. The set consisting of points in $\mathbb{R}$ except the origin is *not simply connected*, and the vector field

$$\mathbf{F} = \frac{-y\mathbf{i} + x\mathbf{j}}{x^2+y^2}$$

is not conservative on any domain in $\mathbb{R}^2$ that contains the origin in its interior. (See Example 5.) However, the integral will be 0 for any closed curve that does not contain the origin in its interior. (An example is the curve in Exercise 22(c).)

24. If $\mathcal{C}$ is a closed, piecewise smooth curve in $\mathbb{R}^2$ having equation $\mathbf{r} = \mathbf{r}(t)$, $a \le t \le b$, and if $\mathcal{C}$ does not pass through the origin, then the polar angle function $\theta = \theta\Big(x(t), y(t)\Big) = \theta(t)$ can be defined so as to vary continuously on $\mathcal{C}$. Therefore,

$$\theta(x, y)\Big|_{t=a}^{t=b} = 2\pi \times w(\mathcal{C}),$$

where $w(\mathcal{C})$ is the number of times $\mathcal{C}$ winds around the origin in a counterclockwise direction. For example, $w(\mathcal{C})$ equals 1, -1 and 0 respectively, for the curves $\mathcal{C}$ in parts (a), (b) and (c) of Exercise 22. Since

$$\begin{aligned}
\nabla\theta &= \frac{\partial\theta}{\partial x}\mathbf{i} + \frac{\partial\theta}{\partial y}\mathbf{j} \\
&= \frac{-y\mathbf{i} + x\mathbf{j}}{x^2+y^2},
\end{aligned}$$

we have

$$\begin{aligned}
\frac{1}{2\pi}\oint_{\mathcal{C}} \frac{x\,dy - y\,dx}{x^2+y^2} &= \frac{1}{2\pi}\oint_{\mathcal{C}} \nabla\theta \bullet d\mathbf{r} \\
&= \frac{1}{2\pi}\theta(x, y)\Big|_{t=a}^{t=b} = w(\mathcal{C}).
\end{aligned}$$

Section 15.5 Surfaces and Surface Integrals (page 842)

1. The polar curve $r = g(\theta)$ is parametrized by

$$x = g(\theta)\cos\theta, \qquad y = g(\theta)\sin\theta.$$

Hence its arc length element is

$$\begin{aligned}
ds &= \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\,d\theta \\
&= \sqrt{\Big(g'(\theta)\cos\theta - g(\theta)\sin\theta\Big)^2 + \Big(g'(\theta)\sin\theta + g(\theta)\cos\theta\Big)^2}\,d\theta \\
&= \sqrt{\Big(g(\theta)\Big)^2 + \Big(g'(\theta)\Big)^2}\,d\theta.
\end{aligned}$$

The area element on the vertical cylinder $r = g(\theta)$ is

$$dS = ds\,dz = \sqrt{\left(g(\theta)\right)^2 + \left(g'(\theta)\right)^2}\,d\theta\,dz.$$

2. The area element dS is bounded by the curves in which the coordinate planes at θ and $\theta + d\theta$ and the coordinate cones at ϕ and $\phi + d\phi$ intersect the sphere $R = a$. (See the figure.) The element is rectangular with sides $a\,d\phi$ and $a \sin\phi\,d\theta$. Thus

$$dS = a^2 \sin\phi\,d\phi\,d\theta.$$

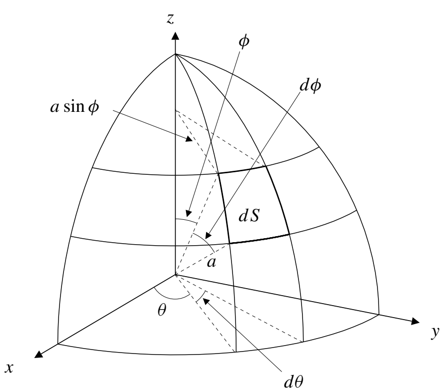

Fig. 15.5.2

3. The plane $Ax + By + Cz = D$ has normal $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$, and so an area element on it is given by

$$dS = \frac{|\mathbf{n}|}{|\mathbf{n} \bullet \mathbf{k}|}\,dx\,dy = \frac{\sqrt{A^2 + B^2 + C^2}}{|C|}\,dx\,dy.$$

Hence the area S of that part of the plane lying inside the elliptic cylinder

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is given by

$$\begin{aligned} S &= \iint_{\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1} \frac{\sqrt{A^2 + B^2 + C^2}}{|C|}\,dx\,dy \\ &= \frac{\pi ab\sqrt{A^2 + B^2 + C^2}}{|C|} \text{ sq. units.} \end{aligned}$$

4. One-quarter of the required area is shown in the figure. It lies above the semicircular disk R bounded by $x^2 + y^2 = 2ay$, or, in terms of polar coordinates, $r = 2a\sin\theta$. On the sphere $x^2 + y^2 + z^2 = 4a^2$, we have

$$2z\frac{\partial z}{\partial x} = -2x, \qquad \text{or} \qquad \frac{\partial z}{\partial x} = -\frac{x}{z}.$$

Similarly, $\dfrac{\partial z}{\partial y} = -\dfrac{y}{z}$, so the surface area element on the sphere can be written

$$dS = \sqrt{1 + \frac{x^2 + y^2}{z^2}}\,dx\,dy = \frac{2a\,dx\,dy}{\sqrt{4a^2 - x^2 - y^2}}.$$

The required area is

$$\begin{aligned} S &= 4\iint_R \frac{2a}{\sqrt{4a^2 - x^2 - y^2}}\,dx\,dy \\ &= 8a\int_0^{\pi/2} d\theta \int_0^{2a\sin\theta} \frac{r\,dr}{\sqrt{4a^2 - r^2}} \qquad \begin{array}{l}\text{Let } u = 4a^2 - r^2 \\ du = -2r\,dr\end{array} \\ &= 4a\int_0^{\pi/2} d\theta \int_{4a^2\cos^2\theta}^{4a^2} u^{-1/2}\,du \\ &= 8a\int_0^{\pi/2} (2a - 2a\cos\theta)\,d\theta \\ &= 16a^2(\theta - \sin\theta)\Big|_0^{\pi/2} = 8a^2(\pi - 2) \text{ sq. units.} \end{aligned}$$

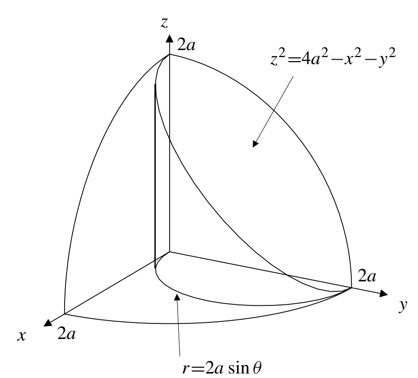

Fig. 15.5.4

5.
$$dS = \left|\frac{\nabla F(x, y, z)}{F_2(x, y, z)}\right| dx\,dz$$
$$dS = \left|\frac{\nabla F(x, y, z)}{F_1(x, y, z)}\right| dy\,dz$$

6. The cylinder $x^2 + y^2 = 2ay$ intersects the sphere $x^2 + y^2 + z^2 = 4a^2$ on the parabolic cylinder $2ay + z^2 = 4a^2$. By Exercise 5, the area element on $x^2 + y^2 - 2ay = 0$ is

$$\begin{aligned} dS &= \left|\frac{2x\mathbf{i} + (2y - 2a)\mathbf{j}}{2x}\right| dy\,dz \\ &= \sqrt{1 + \frac{(y - a)^2}{2ay - y^2}}\,dy\,dz \\ &= \sqrt{\frac{2ay - y^2 + y^2 - 2ay + a^2}{2ay - y^2}}\,dy\,dz = \frac{a}{\sqrt{2ay - y^2}}\,dy\,dz. \end{aligned}$$

The area of the part of the cylinder inside the sphere is 4 times the part shown in Figure 15.23 in the text, that is, 4 times the double integral of dS over the region $0 \le y \le 2a$, $0 \le z \le \sqrt{4a^2 - 2ay}$, or

$$\begin{aligned} S &= 4\int_0^{2a} \frac{a\,dy}{\sqrt{2ay - y^2}} \int_0^{\sqrt{4a^2-2ay}} dz \\ &= 4a\int_0^{2a} \frac{\sqrt{2a(2a-y)}}{\sqrt{y(2a-y)}}\,dy = 4\sqrt{2}a^{3/2}\int_0^{2a}\frac{dy}{\sqrt{y}} \\ &= 4\sqrt{2}a^{3/2}(2\sqrt{y})\Big|_0^{2a} = 16a^2 \text{ sq. units.} \end{aligned}$$

7. On the surface $\mathcal{S}$ with equation $z = x^2/2$ we have $\partial z/\partial x = x$ and $\partial z/\partial y = 0$. Thus

$$dS = \sqrt{1 + x^2}\,dx\,dy.$$

If R is the first quadrant part of the disk $x^2 + y^2 \le 1$, then the required surface integral is

$$\begin{aligned} \iint_{\mathcal{S}} x\,dS &= \iint_R x\sqrt{1+x^2}\,dx\,dy \\ &= \int_0^1 x\sqrt{1+x^2}\,dx \int_0^{\sqrt{1-x^2}} dy \\ &= \int_0^1 x\sqrt{1-x^4}\,dx \quad \text{Let } u = x^2,\ du = 2x\,dx \\ &= \frac{1}{2}\int_0^1 \sqrt{1-u^2}\,du = \frac{1}{2}\frac{\pi}{4} = \frac{\pi}{8}. \end{aligned}$$

8. The normal to the cone $z^2 = x^2 + y^2$ makes a 45° angle with the vertical, so $dS = \sqrt{2}\,dx\,dy$ is a surface area element for the cone. Both *nappes* (halves) of the cone pass through the interior of the cylinder $x^2 + y^2 = 2ay$, so the area of that part of the cone inside the cylinder is $2\sqrt{2}\pi a^2$ square units, since the cylinder has a circular cross-section of radius a.

9. One-quarter of the required area lies in the first octant. (See the figure.) In polar coordinates, the Cartesian equation $x^2 + y^2 = 2ay$ becomes $r = 2a\sin\theta$. The arc length element on this curve is

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta = 2a\,d\theta.$$

Thus $dS = \sqrt{x^2+y^2}\,ds = 2ar\,d\theta = 4a^2\sin\theta\,d\theta$ on the cylinder. The area of that part of the cylinder lying between the nappes of the cone is

$$4\int_0^{\pi/2} 4a^2\sin\theta\,d\theta = 16a^2 \text{ sq. units..}$$

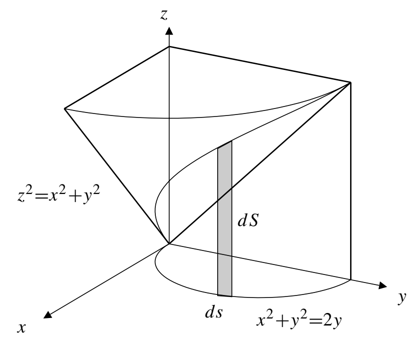

Fig. 15.5.9

10. One-eighth of the required area lies in the first octant, above the triangle T with vertices $(0,0,0)$, $(a,0,0)$ and $(a,a,0)$. (See the figure.)
The surface $x^2 + z^2 = a^2$ has normal $\mathbf{n} = x\mathbf{i} + z\mathbf{k}$, so an area element on it can be written

$$dS = \frac{|\mathbf{n}|}{|\mathbf{n} \bullet \mathbf{k}|}\,dx\,dy = \frac{a}{z}\,dx\,dy = \frac{a\,dx\,dy}{\sqrt{a^2 - x^2}}.$$

The area of the part of that cylinder lying inside the cylinder $y^2 + z^2 = a^2$ is

$$\begin{aligned} S &= 8\iint_T \frac{a\,dx\,dy}{\sqrt{a^2-x^2}} = 8a\int_0^a \frac{dx}{\sqrt{a^2-x^2}}\int_0^x dy \\ &= 8a\int_0^a \frac{x\,dx}{\sqrt{a^2-x^2}} \\ &= -8a\sqrt{a^2-x^2}\Big|_0^a = 8a^2 \text{ sq. units.} \end{aligned}$$

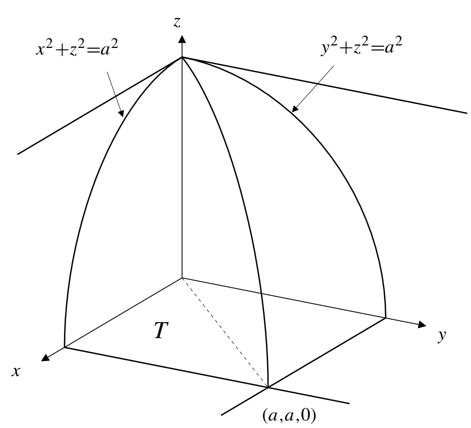

Fig. 15.5.10

11. Let the sphere be $x^2+y^2+z^2=R^2$, and the cylinder be $x^2+y^2=R^2$. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be the parts of the sphere and the cylinder, respectively, lying between the planes $z=a$ and $z=b$, where $-R\le a\le b\le R$.
Evidently, the area of $\mathcal{S}_2$ is $S_2=2\pi R(b-a)$ square units.
An area element on the sphere is given in terms of spherical coordinates by

$$dS=R^2\sin\phi\,d\phi\,d\theta.$$

On $\mathcal{S}_1$ we have $z=R\cos\phi$, so $\mathcal{S}_1$ lies between $\phi=\cos^{-1}(b/R)$ and $\phi=\cos^{-1}(a/R)$. Thus the area of $\mathcal{S}_1$ is

$$\begin{aligned}S_1&=R^2\int_0^{2\pi}d\theta\int_{\cos^{-1}(b/R)}^{\cos^{-1}(a/R)}\sin\phi\,d\phi\\&=2\pi R^2(-\cos\phi)\Big|_{\cos^{-1}(b/R)}^{\cos^{-1}(a/R)}=2\pi R(b-a)\text{ sq. units.}\end{aligned}$$

Observe that $\mathcal{S}_1$ and $\mathcal{S}_2$ have the same area.

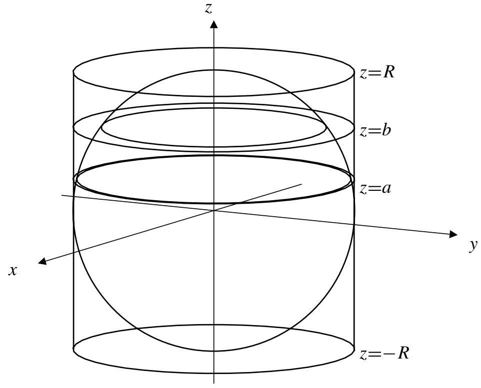

Fig. 15.5.11

12. We want to find A_1, the area of that part of the cylinder $x^2+z^2=a^2$ inside the cylinder $y^2+z^2=b^2$, and A_2, the area of that part of $y^2+z^2=b^2$ inside $x^2+z^2=a^2$. We have

$$A_1=8\times(\text{area of }\mathcal{S}_1),$$
$$A_2=8\times(\text{area of }\mathcal{S}_2),$$

where $\mathcal{S}_1$ and $\mathcal{S}_2$ are the parts of these surfaces lying in the first octant, as shown in the figure.
A normal to $\mathcal{S}_1$ is $\mathbf{n}_1=x\mathbf{i}+z\mathbf{k}$, and the area element on $\mathcal{S}_1$ is

$$dS_1=\frac{|\mathbf{n}_1|}{|\mathbf{n}_1\bullet\mathbf{i}|}\,dy\,dz=\frac{a\,dy\,dz}{\sqrt{a^2-z^2}}.$$

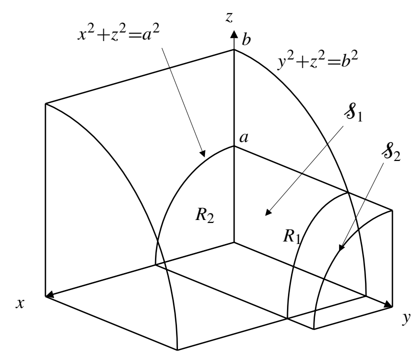

Fig. 15.5.12

A normal to $\mathcal{S}_2$ is $\mathbf{n}_2=x\mathbf{j}+z\mathbf{k}$, and the area element on $\mathcal{S}_2$ is

$$dS_2=\frac{|\mathbf{n}_2|}{|\mathbf{n}_2\bullet\mathbf{j}|}\,dx\,dz=\frac{b\,dx\,dz}{\sqrt{b^2-z^2}}.$$

Let R_1 be the region of the first quadrant of the yz-plane bounded by $y^2+z^2=b^2$, $y=0$, $z=0$, and $z=a$.
Let R_2 be the quarter-disk in the first quadrant of the xz-plane bounded by $x^2+z^2=a^2$, $x=0$, and $z=0$. Then

$$\begin{aligned}A_1&=8\iint_{R_1}dS_1=8a\int_0^a\frac{dz}{\sqrt{a^2-z^2}}\int_0^{\sqrt{b^2-z^2}}dy\\&=8a\int_0^a\frac{\sqrt{b^2-z^2}}{\sqrt{a^2-z^2}}\,dz\quad\text{Let }z=a\sin t\\&\qquad\qquad\qquad\qquad\qquad dz=a\cos t\,dt\\&=8a\int_0^{\pi/2}\sqrt{b^2-a^2\sin^2t}\,dt\\&=8ab\int_0^{\pi/2}\sqrt{1-\frac{a^2}{b^2}\sin^2t}\,dt\\&=8abE\left(\frac{a}{b}\right)\text{ sq. units.}\end{aligned}$$

$$\begin{aligned}A_2&=8\iint_{R_2}dS_2=8b\int_0^a\frac{dz}{\sqrt{b^2-z^2}}\int_0^{\sqrt{a^2-z^2}}dx\\&=8b\int_0^a\frac{\sqrt{a^2-z^2}}{\sqrt{b^2-z^2}}\,dz\quad\text{Let }z=b\sin t\\&\qquad\qquad\qquad\qquad\qquad dz=b\cos t\,dt\\&=8b\int_0^{\sin^{-1}(a/b)}\sqrt{a^2-b^2\sin^2t}\,dt\\&=8ab\int_0^{\sin^{-1}(a/b)}\sqrt{1-\frac{b^2}{a^2}\sin^2t}\,dt\\&=8abE\left(\frac{b}{a},\sin^{-1}\frac{a}{b}\right)\text{ sq. units.}\end{aligned}$$

13. The intersection of the plane $z = 1 + y$ and the cone $z = \sqrt{2(x^2 + y^2)}$ has projection onto the xy-plane the elliptic disk E bounded by

$$\begin{aligned}
(1+y)^2 &= 2(x^2+y^2)\\
1+2y+y^2 &= 2x^2+2y^2\\
2x^2+y^2-2y+1 &= 2\\
x^2+\frac{(y-1)^2}{2} &= 1.
\end{aligned}$$

Note that E has area $A = \pi(1)(\sqrt{2})$ and centroid $(0, 1)$. If $\mathcal{S}$ is the part of the plane lying inside the cone, then the area element on $\mathcal{S}$ is

$$dS = \sqrt{1+\left(\frac{\partial z}{\partial y}\right)^2}\,dx\,dy = \sqrt{2}\,dx\,dy.$$

Thus

$$\iint_{\mathcal{S}} y\,dS = \sqrt{2}\iint_E y\,dx\,dy = \sqrt{2}A\bar{y} = 2\pi.$$

14. Continuing the above solution, the cone $z = \sqrt{2(x^2 + y^2)}$ has area element

$$\begin{aligned}
dS &= \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dx\,dy\\
&= \sqrt{1+\frac{4(x^2+y^2)}{z^2}}\,dx\,dy = \sqrt{3}\,dx\,dy.
\end{aligned}$$

If $\mathcal{S}$ is the part of the cone lying below the plane $z = 1 + y$, then

$$\iint_{\mathcal{S}} y\,dS = \sqrt{3}\iint_E y\,dx\,dy = \sqrt{3}A\bar{y} = \sqrt{6}\pi.$$

15. If $\mathcal{S}$ is the part of $z = x^2$ in the first octant and inside (that is, below) $z = 1 - 3x^2 - y^2$, then $\mathcal{S}$ has projection E onto the xy-plane bounded by $x^2 = 1 - 3x^2 - y^2$, or $4x^2 + y^2 = 1$, an ellipse. Since $z = x^2$ has area element $dS = \sqrt{1+4x^2}\,dx\,dy$, we have

$$\begin{aligned}
\iint_{\mathcal{S}} xz\,dS &= \iint_E x^3\sqrt{1+4x^2}\,dx\,dy\\
&= \int_0^{1/2} x^3\sqrt{1+4x^2}\,dx\int_0^{\sqrt{1-4x^2}} dy\\
&= \int_0^{1/2} x^3\sqrt{1-16x^4}\,dx \quad \text{Let } u = 1-16x^4\\
&\qquad\qquad\qquad\qquad\qquad du = -64x^3\,dx\\
&= \frac{1}{64}\int_0^1 u^{1/2}\,du = \frac{1}{96}.
\end{aligned}$$

16. The surface $z = \sqrt{2xy}$ has area element

$$\begin{aligned}
dS &= \sqrt{1+\frac{y}{2x}+\frac{x}{2y}}\,dx\,dy\\
&= \sqrt{\frac{2xy+y^2+x^2}{2xy}}\,dx\,dy = \frac{|x+y|}{\sqrt{2xy}}\,dx\,dy.
\end{aligned}$$

If its density is kz, the mass of the specified part of the surface is

$$\begin{aligned}
m &= \int_0^5 dx\int_0^2 k\sqrt{2xy}\,\frac{x+y}{\sqrt{2xy}}\,dy\\
&= k\int_0^5 dx\int_0^2 (x+y)\,dy\\
&= k\int_0^5 (2x+2)\,dx = 35k \text{ units.}
\end{aligned}$$

17. The surface $\mathcal{S}$ is given by $x = e^u\cos v$, $y = e^u\sin v$, $z = u$, for $0 \le u \le 1$, $0 \le v \le \pi$. Since

$$\begin{aligned}
\frac{\partial(y,z)}{\partial(u,v)} &= \begin{vmatrix} e^u\sin v & e^u\cos v\\ 1 & 0\end{vmatrix} = -e^u\cos v\\
\frac{\partial(z,x)}{\partial(u,v)} &= \begin{vmatrix} 1 & 0\\ e^u\cos v & -e^u\sin v\end{vmatrix} = -e^u\sin v\\
\frac{\partial(x,y)}{\partial(u,v)} &= \begin{vmatrix} e^u\cos v & -e^u\sin v\\ e^u\sin v & e^u\cos v\end{vmatrix} = e^{2u}
\end{aligned}$$

the area element on $\mathcal{S}$ is

$$dS = \sqrt{e^{2u}\cos^2 v + e^{2u}\sin^2 v + e^{4u}}\,du\,dv = e^u\sqrt{1+e^{2u}}\,du\,dv.$$

If the charge density on $\mathcal{S}$ is $\sqrt{1+e^{2u}}$, then the total charge is

$$\begin{aligned}
\iint_S \sqrt{1+e^{2u}}\,dS &= \int_0^1 e^u(1+e^{2u})\,du\int_0^\pi dv\\
&= \pi\left(e^u+\frac{e^{3u}}{3}\right)\Bigg|_0^1 = \frac{\pi}{3}(3e+e^3-4).
\end{aligned}$$

18. The upper half of the spheroid $\dfrac{x^2}{a^2}+\dfrac{y^2}{a^2}+\dfrac{z^2}{c^2} = 1$ has a circular disk of radius a as projection onto the xy-plane. Since

$$\frac{2x}{a^2}+\frac{2z}{c^2}\frac{\partial z}{\partial x} = 0 \quad\Rightarrow\quad \frac{\partial z}{\partial x} = -\frac{c^2x}{a^2z},$$

and, similarly, $\dfrac{\partial z}{\partial y} = -\dfrac{c^2 y}{a^2 z}$, the area element on the spheroid is

$$\begin{aligned}
dS &= \sqrt{1 + \frac{c^4}{a^4}\,\frac{x^2+y^2}{z^2}}\,dx\,dy \\
&= \sqrt{1 + \frac{c^2}{a^2}\,\frac{x^2+y^2}{a^2-x^2-y^2}}\,dx\,dy \\
&= \sqrt{\frac{a^4+(c^2-a^2)r^2}{a^2(a^2-r^2)}}\,r\,dr\,d\theta
\end{aligned}$$

in polar coordinates. Thus the area of the spheroid is

$$\begin{aligned}
S &= \frac{2}{a}\int_0^{2\pi} d\theta \int_0^a \sqrt{\frac{a^4+(c^2-a^2)r^2}{a^2-r^2}}\,r\,dr \\
&\qquad \text{Let } u^2 = a^2 - r^2 \\
&\qquad u\,du = -r\,dr \\
&= \frac{4\pi}{a}\int_0^a \sqrt{a^4+(c^2-a^2)(a^2-u^2)}\,du \\
&= \frac{4\pi}{a}\int_0^a \sqrt{a^2c^2-(c^2-a^2)u^2}\,du \\
&= 4\pi c\int_0^a \sqrt{1-\frac{c^2-a^2}{a^2c^2}\,u^2}\,du.
\end{aligned}$$

For the case of a prolate spheroid $0 < a < c$, let $k^2 = \dfrac{c^2-a^2}{a^2c^2}$. Then

$$\begin{aligned}
S &= 4\pi c\int_0^a \sqrt{1-k^2u^2}\,du \qquad \text{Let } ku = \sin v \\
&\qquad\qquad\qquad\qquad\qquad\quad k\,du = \cos v\,dv \\
&= \frac{4\pi c}{k}\int_0^{\sin^{-1}(ka)} \cos^2 v\,dv \\
&= \frac{2\pi c}{k}(v + \sin v\cos v)\Big|_0^{\sin^{-1}(ka)} \\
&= \frac{2\pi ac^2}{\sqrt{c^2-a^2}}\sin^{-1}\frac{\sqrt{c^2-a^2}}{c} + 2\pi a^2 \text{ sq. units.}
\end{aligned}$$

19. We continue from the formula for the surface area of a spheroid developed part way through the solution above. For the case of an oblate spheroid $0 < c < a$, let $k^2 = \dfrac{a^2-c^2}{a^2c^2}$. Then

$$\begin{aligned}
S &= 4\pi c\int_0^a \sqrt{1+k^2u^2}\,du \qquad \text{Let } ku = \tan v \\
&\qquad\qquad\qquad\qquad\qquad\quad k\,du = \sec^2 v\,dv \\
&= \frac{4\pi c}{k}\int_0^{\tan^{-1}(ka)} \sec^3 v\,dv \\
&= \frac{2\pi c}{k}\Big(\sec v\tan v + \ln(\sec v + \tan v)\Big)\Big|_0^{\tan^{-1}(ka)} \\
&= \frac{2\pi ac^2}{\sqrt{a^2-c^2}}\left[\frac{a\sqrt{a^2-c^2}}{c^2} + \ln\left(\frac{a}{c} + \frac{\sqrt{a^2-c^2}}{c}\right)\right] \\
&= 2\pi a^2 + \frac{2\pi ac^2}{\sqrt{a^2-c^2}}\ln\left(\frac{a+\sqrt{a^2-c^2}}{c}\right) \text{ sq. units.}
\end{aligned}$$

20. $x = au\cos v,\quad y = au\sin v,\quad z = bv,$ $(0 \le u \le 1,\quad 0 \le v \le 2\pi)$. This surface is a spiral (helical) ramp of radius a and height $2\pi b$, wound around the z-axis. (It's like a circular staircase with a ramp instead of stairs.) We have

$$\begin{aligned}
\frac{\partial(x,y)}{\partial(u,v)} &= \begin{vmatrix} a\cos v & -au\sin v \\ a\sin v & au\cos v \end{vmatrix} = a^2u \\
\frac{\partial(y,z)}{\partial(u,v)} &= \begin{vmatrix} a\sin v & au\cos v \\ 0 & b \end{vmatrix} = ab\sin v \\
\frac{\partial(z,x)}{\partial(u,v)} &= \begin{vmatrix} 0 & b \\ a\cos v & -au\sin v \end{vmatrix} = -ab\cos v \\
dS &= \sqrt{a^4u^2 + a^2b^2\sin^2 v + a^2b^2\cos^2 v}\,du\,dv \\
&= a\sqrt{a^2u^2+b^2}\,du\,dv.
\end{aligned}$$

The area of the ramp is

$$\begin{aligned}
A &= a\int_0^1 \sqrt{a^2u^2+b^2}\,du\int_0^{2\pi} dv \\
&= 2\pi a\int_0^1 \sqrt{a^2u^2+b^2}\,du \qquad \text{Let } au = b\tan\theta \\
&\qquad\qquad\qquad\qquad\qquad\qquad a\,du = b\sec^2\theta\,d\theta \\
&= 2\pi b^2\int_{u=0}^{u=1} \sec^3\theta\,d\theta \\
&= \pi b^2\Big(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\Big)\Big|_{u=0}^{u=1} \\
&= \pi b^2\left(\frac{au\sqrt{a^2u^2+b^2}}{b^2} + \ln\left|\frac{au+\sqrt{a^2u^2+b^2}}{b}\right|\right)\Bigg|_0^1 \\
&= \pi a\sqrt{a^2+b^2} + \pi b^2\ln\left(\frac{a+\sqrt{a^2+b^2}}{b}\right) \text{ sq. units.}
\end{aligned}$$

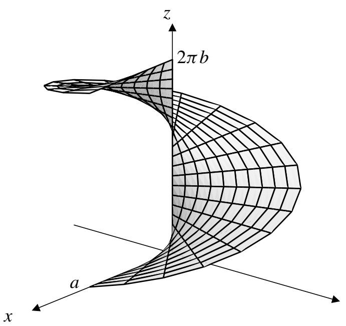

Fig. 15.5.20

21. The distance from the origin to the plane $\mathcal{P}$ with equation $Ax + By + Cz = D$, $(D \neq 0)$ is

$$\delta = \frac{|D|}{\sqrt{A^2 + B^2 + C^2}}.$$

If $\mathcal{P}_1$ is the plane $z = \delta$, then, since the integrand depends only on distance from the origin, we have

$$\begin{aligned}
&\iint_{\mathcal{P}} \frac{dS}{(x^2+y^2+z^2)^{3/2}} \\
&= \iint_{\mathcal{P}_1} \frac{dS}{(x^2+y^2+z^2)^{3/2}} \\
&\qquad = \int_0^{2\pi} d\theta \int_0^\infty \frac{r\,dr}{(r^2+\delta^2)^{3/2}} \qquad \begin{array}{l}\text{Let } u = r^2 + \delta^2 \\ du = 2r\,dr\end{array} \\
&\qquad = 2\pi \times \frac{1}{2} \int_{\delta^2}^\infty \frac{du}{u^{3/2}} \\
&\qquad = \pi \left(-\frac{2}{\sqrt{u}} \right)\Bigg|_{\delta^2}^{\infty} \\
&\qquad = \frac{2\pi}{\delta} = \frac{2\pi\sqrt{A^2+B^2+C^2}}{|D|}.
\end{aligned}$$

22. Use spherical coordinates. The area of the eighth-sphere $\mathcal{S}$ is

$$A = \frac{1}{8}(4\pi a^2) = \frac{\pi a^2}{2} \text{ sq. units.}$$

The moment about $z = 0$ is

$$\begin{aligned}
M_{z=0} &= \iint_{\mathcal{S}} z\,dS \\
&= \int_0^{\pi/2} d\theta \int_0^{\pi/2} a\cos\phi\, a^2 \sin\phi\, d\phi \\
&= \frac{\pi a^3}{2} \int_0^{\pi/2} \frac{\sin 2\phi}{2}\, d\phi = \frac{\pi a^3}{4}.
\end{aligned}$$

Thus $\bar{z} = \dfrac{M_{z=0}}{A} = \dfrac{a}{2}$. By symmetry, $\bar{x} = \bar{y} = \bar{z}$, so the centroid of that part of the surface of the sphere $x^2 + y^2 + z^2 = a^2$ lying in the first octant is $\left(\dfrac{a}{2}, \dfrac{a}{2}, \dfrac{a}{2}\right)$.

23. The cone $z = h\left(1 - \dfrac{\sqrt{x^2+y^2}}{a}\right)$ has normal

$$\begin{aligned}
\mathbf{n} &= -\frac{\partial z}{\partial x}\mathbf{i} - \frac{\partial z}{\partial y}\mathbf{j} + \mathbf{k} \\
&= -\frac{h}{a}\left(\frac{x\mathbf{i} + y\mathbf{j}}{\sqrt{x^2+y^2}}\right) + \mathbf{k},
\end{aligned}$$

so its surface area element is

$$dS = \sqrt{\frac{h^2}{a^2} + 1}\, dx\,dy = \frac{\sqrt{a^2+h^2}}{a}\, dx\,dy.$$

The mass of the conical shell is

$$m = \sigma \iint_{x^2+y^2 \le a^2} dS = \frac{\sigma\sqrt{a^2+h^2}}{a}(\pi a^2) = \pi\sigma a\sqrt{a^2+h^2}.$$

The moment about $z = 0$ is

$$\begin{aligned}
M_{z=0} &= \sigma \iint_{x^2+y^2 \le a^2} h\left(1 - \frac{\sqrt{x^2+y^2}}{a}\right) \frac{\sqrt{a^2+h^2}}{a}\, dx\,dy \\
&= \frac{2\pi\sigma h\sqrt{a^2+h^2}}{a} \int_0^a \left(1 - \frac{r}{a}\right) r\,dr \\
&= \frac{\pi\sigma h a\sqrt{a^2+h^2}}{3}.
\end{aligned}$$

Thus $\bar{z} = \dfrac{h}{3}$. By symmetry, $\bar{x} = \bar{y} = 0$. The centre of mass is on the axis of the cone, one-third of the way from the base towards the vertex.

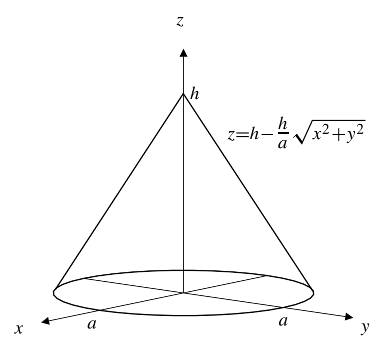

Fig. 15.5.23

24. By symmetry, the force of attraction of the hemisphere shown in the figure on the mass m at the origin is vertical. The vertical component of the force exerted by area element $dS = a^2 \sin\phi\, d\phi\, d\theta$ at the position with spherical coordinates (a, ϕ, θ) is

$$dF = \frac{km\sigma\, dS}{a^2}\cos\phi = km\sigma \sin\phi\cos\phi\, d\phi\, d\theta.$$

Thus, the total force on m is

$$F = km\sigma \int_0^{2\pi} d\theta \int_0^{\pi/2} \sin\phi \cos\phi \, d\phi = \pi km\sigma \text{ units.}$$

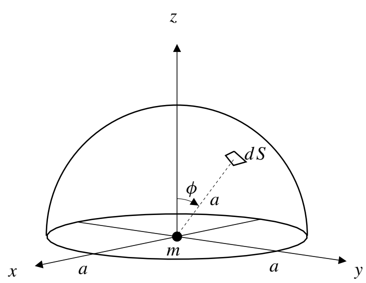

Fig. 15.5.24

25. The surface element $dS = a\, d\theta\, dz$ at the point with cylindrical coordinates (a, θ, z) attracts mass m at point $(0, 0, b)$ with a force whose vertical component (see the figure) is

$$\begin{aligned} dF &= \frac{km\sigma\, dS}{D^2} \cos\psi = \frac{km\sigma a(b-z)\, d\theta\, dz}{D^3} \\ &= \frac{km\sigma a(b-z)\, d\theta\, dz}{\left(a^2 + (b-z)^2\right)^{3/2}}. \end{aligned}$$

The total force exerted by the cylindrical surface on the mass m is

$$\begin{aligned} F &= -\int_0^{2\pi} d\theta \int_0^h \frac{km\sigma a(b-z)\, dz}{\left(a^2 + (b-z)^2\right)^{3/2}} \quad \begin{array}{l} \text{Let } b - z = a\tan t \\ -dz = a\sec^2 t\, dt \end{array} \\ &= 2\pi km\sigma a \int_{z=0}^{z=h} \frac{a\tan t\, a\sec^2 t\, dt}{a^3 \sec^3 t} \\ &= 2\pi km\sigma \int_{z=0}^{z=h} \sin t\, dt \\ &= 2\pi km\sigma (-\cos t)\Big|_{z=0}^{z=h} \\ &= 2\pi km\sigma \frac{a}{\sqrt{a^2 + (b-z)^2}}\Bigg|_0^h \\ &= 2\pi km\sigma a \left(\frac{1}{\sqrt{a^2 + (b-h)^2}} - \frac{1}{\sqrt{a^2 + b^2}} \right). \end{aligned}$$

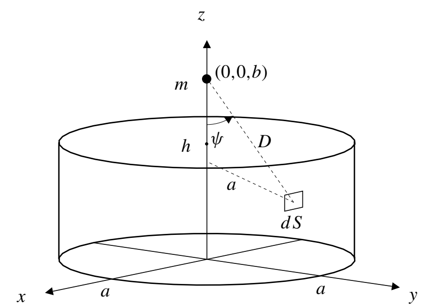

Fig. 15.5.25

26. $\mathcal{S}$ is the cylindrical surface $x^2 + y^2 = a^2$, $0 \le z \le h$, with areal density σ. Its mass is $m = 2\pi ah\sigma$. Since all surface elements are at distance a from the z-axis, the radius of gyration of the cylindrical surface about the z-axis is $\bar{D} = a$. Therefore the moment of inertia about that axis is

$$I = m\bar{D}^2 = ma^2 = 2\pi\sigma a^3 h.$$

27. $\mathcal{S}$ is the spherical shell, $x^2 + y^2 + z^2 = a^2$, with areal density σ. Its mass is $4\pi\sigma a^2$. Its moment of inertia about the z-axis is

$$\begin{aligned} I &= \sigma \iint_{\mathcal{S}} (x^2 + y^2)\, dS \\ &= \sigma \int_0^{2\pi} d\theta \int_0^{\pi} a^2 \sin^2\phi\, a^2 \sin\phi\, d\phi \\ &= 2\pi\sigma a^4 \int_0^{\pi} \sin\phi (1 - \cos^2\phi)\, d\phi \quad \begin{array}{l} \text{Let } u = \cos\phi \\ du = -\sin\phi\, d\phi \end{array} \\ &= 2\pi\sigma a^4 \int_{-1}^{1} (1 - u^2)\, du = \frac{8\pi\sigma a^4}{3}. \end{aligned}$$

The radius of gyration is $\bar{D} = \sqrt{I/m} = \sqrt{\dfrac{2}{3}}a$.

28. The surface area element for a conical surface $\mathcal{S}$,

$$z = h\left(1 - \frac{\sqrt{x^2 + y^2}}{a}\right),$$

having base radius a and height h, was determined in the solution to Exercise 23 to be

$$dS = \frac{\sqrt{a^2 + h^2}}{a}\, dx\, dy.$$

The mass of $\mathcal{S}$, which has areal density σ, was also determined in that exercise: $m = \pi\sigma a\sqrt{a^2+h^2}$. The moment of inertia of $\mathcal{S}$ about the z-axis is

$$\begin{aligned} I &= \sigma \iint_{\mathcal{S}} (x^2+y^2)\,dS \\ &= \frac{\sigma\sqrt{a^2+h^2}}{a}\int_0^{2\pi} d\theta \int_0^a r^2\, r\,dr \\ &= \frac{2\pi\sigma\sqrt{a^2+h^2}}{a}\,\frac{a^4}{4} = \frac{\pi\sigma a^3\sqrt{a^2+h^2}}{2}. \end{aligned}$$

The radius of gyration is $\bar{D} = \sqrt{I/m} = \dfrac{a}{\sqrt{2}}$.

29. By Exercise 27, the moment of inertia of a spherical shell of radius a about its diameter is $I = \frac{2}{3}ma^2$. Following the argument given in Example 4(b) of Section 5.7, the kinetic energy of the sphere, rolling with speed v down a plane inclined at angle α above the horizontal (and therefore rotating with angular speed $\Omega = v/a$) is

$$\begin{aligned} K.E. &= \frac{1}{2}mv^2 + \frac{1}{2}I\Omega^2 \\ &= \frac{1}{2}mv^2 + \frac{1}{2}\,\frac{2}{3}ma^2\,\frac{v^2}{a^2} \\ &= \frac{5}{6}mv^2. \end{aligned}$$

The potential energy is $P.E. = mgh$, so, by conservation of total energy,

$$\frac{5}{6}mv^2 + mgh = \text{constant}.$$

Differentiating with respect to time t, we get

$$0 = \frac{5}{6}m\,2v\,\frac{dv}{dt} + mg\frac{dh}{dt} = \frac{5}{3}mv\frac{dv}{dt} + mgv\sin\alpha.$$

Thus the sphere rolls with acceleration

$$\frac{dv}{dt} = \frac{3}{5}g\sin\alpha.$$

Section 15.6 Oriented Surfaces and Flux Integrals (page 848)

1. $\mathbf{F} = x\mathbf{i} + z\mathbf{j}$.
The surface $\mathcal{S}$ of the tetrahedron has four faces:
On $\mathcal{S}_1$, $x = 0$, $\hat{\mathbf{N}} = -\mathbf{i}$, $\mathbf{F}\bullet\hat{\mathbf{N}} = 0$.
On $\mathcal{S}_2$, $y = 0$, $\hat{\mathbf{N}} = -\mathbf{j}$, $\mathbf{F}\bullet\hat{\mathbf{N}} = -z$, $dS = dx\,dz$.
On $\mathcal{S}_3$, $z = 0$, $\hat{\mathbf{N}} = -\mathbf{k}$, $\mathbf{F}\bullet\hat{\mathbf{N}} = 0$.
On $\mathcal{S}_4$, $x+2y+3z = 6$, $\hat{\mathbf{N}} = \dfrac{\mathbf{i}+2\mathbf{j}+3\mathbf{k}}{\sqrt{14}}$, $\mathbf{F}\bullet\hat{\mathbf{N}} = \dfrac{x+2z}{\sqrt{14}}$,
$dS = \dfrac{dx\,dy}{|\hat{\mathbf{N}}\bullet\mathbf{j}|} = \dfrac{\sqrt{14}}{2}\,dx\,dz$.
We have

$$\begin{aligned} \iint_{\mathcal{S}_1} \mathbf{F}\bullet\hat{\mathbf{N}}\,dS &= \iint_{\mathcal{S}_3} \mathbf{F}\bullet\hat{\mathbf{N}}\,dS = 0 \\ \iint_{\mathcal{S}_2} \mathbf{F}\bullet\hat{\mathbf{N}}\,dS &= -\int_0^2 z\,dz\int_0^{6-3z} dx \\ &= -\int_0^2 (6z-3z^2)\,dz = -4 \\ \iint_{\mathcal{S}_4} \mathbf{F}\bullet\hat{\mathbf{N}}\,dS &= \frac{\sqrt{14}}{2}\,\frac{1}{\sqrt{14}}\int_0^2 dz\int_0^{6-3z}(x+2z)\,dx \\ &= \frac{1}{2}\int_0^2\left(\frac{(6-3z)^2}{2} + 2z(6-3z)\right)dz \\ &= \frac{1}{4}\int_0^2 (6-3z)(6+z)\,dz \\ &= \frac{1}{4}(36z-6z^2-z^3)\Big|_0^2 = 10. \end{aligned}$$

The flux of $\mathbf{F}$ out of the tetrahedron is

$$\iint_{\mathcal{S}} \mathbf{F}\bullet\hat{\mathbf{N}}\,dS = 0-4+0+10 = 6.$$

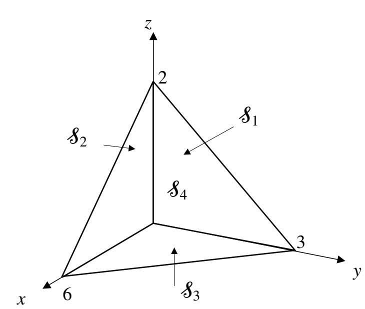

Fig. 15.6.1

2. On the sphere $\mathcal{S}$ with equation $x^2+y^2+z^2 = a^2$ we have

$$\hat{\mathbf{N}} = \frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{a}.$$

If $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, then $\mathbf{F} \bullet \hat{\mathbf{N}} = a$ on $\mathcal{S}$. Thus the flux of $\mathbf{F}$ out of $\mathcal{S}$ is

$$\iint_{\mathcal{S}} \mathbf{F} \bullet \hat{\mathbf{N}}\, dS = a \times 4\pi a^2 = 4\pi a^3.$$

3. $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.
The box has six faces. $\mathbf{F} \bullet \hat{\mathbf{N}} = 0$ on the three faces $x = 0$, $y = 0$, and $z = 0$. On the face $x = a$, we have $\hat{\mathbf{N}} = \mathbf{i}$, so $\mathbf{F} \bullet \hat{\mathbf{N}} = a$. Thus the flux of $\mathbf{F}$ out of that face is

$$a \times (\text{area of the face}) = abc.$$

By symmetry, the flux of $\mathbf{F}$ out of the faces $y = b$ and $z = c$ are also each abc. Thus the total flux of $\mathbf{F}$ out of the box is $3abc$.

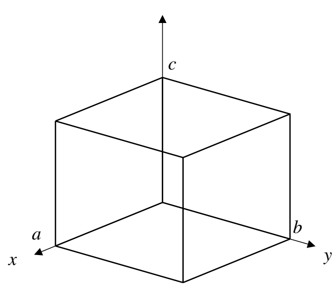

Fig. 15.6.3

4. $\mathbf{F} = y\mathbf{i} + z\mathbf{k}$. Let $\mathcal{S}_1$ be the conical surface and $\mathcal{S}_2$ be the base disk. The flux of $\mathbf{F}$ outward through the surface of the cone is

$$\iint_{\mathcal{S}} \mathbf{F} \bullet \hat{\mathbf{N}} = \iint_{\mathcal{S}_1} + \iint_{\mathcal{S}_2}.$$

On $\mathcal{S}_1$: $\hat{\mathbf{N}} = \dfrac{1}{\sqrt{2}}\left(\dfrac{x\mathbf{i} + y\mathbf{j}}{\sqrt{x^2+y^2}} + \mathbf{k}\right)$, $dS = \sqrt{2}\, dx\, dy$.

Thus

$$\begin{aligned}
&\iint_{\mathcal{S}_1} \mathbf{F} \bullet \hat{\mathbf{N}}\, dS \\
&= \iint_{x^2+y^2\le 1} \left(\frac{xy}{\sqrt{x^2+y^2}} + 1 - \sqrt{x^2+y^2}\right) dx\, dy \\
&= 0 + \pi \times 1^2 - \int_0^{2\pi} d\theta \int_0^1 r^2\, dr \\
&= \pi - \frac{2\pi}{3} = \frac{\pi}{3}.
\end{aligned}$$

On $\mathcal{S}_2$: $\hat{\mathbf{N}} = -\mathbf{k}$ and $z = 0$, so $\mathbf{F} \bullet \hat{\mathbf{N}} = 0$. Thus, the total flux of $\mathbf{F}$ out of the cone is $\pi/3$.

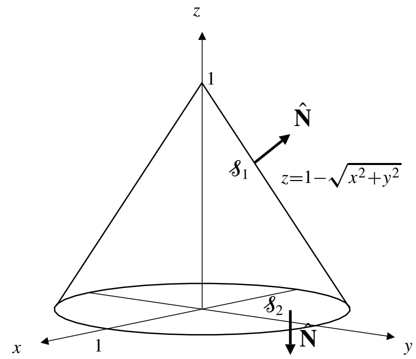

Fig. 15.6.4

5. The part $\mathcal{S}$ of $z = a - x^2 - y^2$ lying above $z = b < a$ lies inside the vertical cylinder $x^2 + y^2 = a - b$. For $z = a - x^2 - y^2$, the upward vector surface element is

$$\hat{\mathbf{N}}\, dS = \frac{2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}}{1}\, dx\, dy.$$

Thus the flux of $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ upward through $\mathcal{S}$ is

$$\begin{aligned}
&\iint_{\mathcal{S}} \mathbf{F} \bullet \hat{\mathbf{N}}\, dS \\
&= \iint_{x^2+y^2\le a-b} [2(x^2+y^2) + a - x^2 - y^2]\, dx\, dy \\
&= \int_0^{2\pi} d\theta \int_0^{\sqrt{a-b}} (r^2 + a) r\, dr \\
&= 2\pi\left(\frac{(a-b)^2}{4} + \frac{a(a-b)}{2}\right) = \frac{\pi}{2}(a-b)(3a-b).
\end{aligned}$$

6. For $z = x^2 - y^2$ the upward surface element is

$$\hat{\mathbf{N}}\, dS = \frac{-2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}}{1}\, dx\, dy.$$

The flux of $\mathbf{F} = x\mathbf{i} + x\mathbf{j} + \mathbf{k}$ upward through $\mathcal{S}$, the part of $z = x^2 - y^2$ inside $x^2 + y^2 = a^2$ is

$$\begin{aligned}
\iint_{\mathcal{S}} \mathbf{F} \bullet \hat{\mathbf{N}}\, dS &= \iint_{x^2+y^2\le a^2} (-2x^2 + 2xy + 1)\, dx\, dy \\
&= -2\int_0^{2\pi} \cos^2\theta\, d\theta \int_0^a r^3\, dr + 0 + \pi a^2 \\
&= \pi a^2 - 2(\pi)\frac{a^4}{4} = \frac{\pi}{2} a^2(2 - a^2).
\end{aligned}$$

7. The part $\mathcal{S}$ of $z = 4 - x^2 - y^2$ lying above $z = 2x + 1$ has projection onto the xy-plane the disk D bounded by

$$2x + 1 = 4 - x^2 - y^2, \quad \text{or } (x+1)^2 + y^2 = 4.$$

Note that D has area 4π and centroid $(-1, 0)$. For $z = 4 - x^2 - y^2$, the downward vector surface element is

$$\hat{\mathbf{N}}\, dS = \frac{-2x\mathbf{i} - 2y\mathbf{j} - \mathbf{k}}{1}\, dx\, dy.$$

Thus the flux of $\mathbf{F} = y^3\mathbf{i} + z^2\mathbf{j} + x\mathbf{k}$ downward through $\mathcal{S}$ is

$$\begin{aligned}\iint_{\mathcal{S}} \mathbf{F} \bullet \hat{\mathbf{N}}\, dS &= -\iint_D \Big(2xy^3 + 2y(4 - x^2 - y^2)^2 + x\Big)\, dx\, dy \\ &\qquad \text{(use the symmetry of } D \text{ about the } x\text{-axis)} \\ &= -\iint_D x\, dA = -(4\pi)(-1) = 4\pi.\end{aligned}$$

8. The upward vector surface element on the top half of $x^2 + y^2 + z^2 = a^2$ is

$$\hat{\mathbf{N}}\, dS = \frac{2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}}{2z}\, dx\, dy = \left(\frac{x\mathbf{i} + y\mathbf{j}}{z} + \mathbf{k}\right) dx\, dy.$$

The flux of $\mathbf{F} = z^2\mathbf{k}$ upward through the first octant part $\mathcal{S}$ of the sphere is

$$\iint_{\mathcal{S}} \mathbf{F} \bullet \hat{\mathbf{N}}\, dS = \int_0^{\pi/2} d\theta \int_0^a (a^2 - r^2) r\, dr = \frac{\pi a^4}{8}.$$

9. The upward vector surface element on $z = 2 - x^2 - 2y^2$ is

$$\hat{\mathbf{N}}\, dS = \frac{2x\mathbf{i} + 4y\mathbf{j} + \mathbf{k}}{1}\, dx\, dy.$$

If E is the elliptic disk bounded by $\dfrac{x^2}{2} + y^2 = 1$, then the flux of $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$ through the required surface $\mathcal{S}$ is

$$\begin{aligned}&\iint_{\mathcal{S}} \mathbf{F} \bullet \hat{\mathbf{N}}\, dS \\ &= \iint_E (2x^2 + 4y^2)\, dx\, dy \quad \begin{array}{l}\text{Let } x = \sqrt{2}u,\ \ y = v \\ dx\, dy = \sqrt{2}\, du\, dv\end{array} \\ &= 4\sqrt{2} \iint_{u^2+v^2 \le 1} (u^2 + v^2)\, du\, dv \quad \text{(now use polars)} \\ &= 4\sqrt{2} \int_0^{2\pi} d\theta \int_0^1 r^3\, dr = 2\sqrt{2}\pi.\end{aligned}$$

10. $\mathcal{S}$: $\mathbf{r} = u^2v\mathbf{i} + uv^2\mathbf{j} + v^3\mathbf{k}$, $(0 \le u \le 1,\ 0 \le v \le 1)$, has upward surface element

$$\begin{aligned}\hat{\mathbf{N}}\, dS &= \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\, du\, dv \\ &= (2uv\mathbf{i} + v^2\mathbf{j}) \times (u^2\mathbf{i} + 2uv\mathbf{j} + 3v^2\mathbf{k})\, du\, dv \\ &= (3v^4\mathbf{i} - 6uv^3\mathbf{j} + 3u^2v^2\mathbf{k})\, du\, dv.\end{aligned}$$

The flux of $\mathbf{F} = 2x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ upward through $\mathcal{S}$ is

$$\begin{aligned}&\iint_{\mathcal{S}} \mathbf{F} \bullet \hat{\mathbf{N}}\, dS \\ &= \int_0^1 du \int_0^1 (6u^2v^5 - 6u^2v^5 + 3u^2v^5)\, dv \\ &= \frac{1}{2} \int_0^1 u^2\, du = \frac{1}{6}.\end{aligned}$$

11. $\mathcal{S}$: $\mathbf{r} = u\cos v\mathbf{i} + u\sin v\mathbf{j} + u\mathbf{k}$, $(0 \le u \le 2,\ 0 \le v \le \pi)$, has upward surface element

$$\begin{aligned}\hat{\mathbf{N}}\, dS &= \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\, du\, dv \\ &= (-u\cos v\mathbf{i} - u\sin v\mathbf{j} + u\mathbf{k})\, du\, dv.\end{aligned}$$

The flux of $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z^2\mathbf{k}$ upward through $\mathcal{S}$ is

$$\begin{aligned}&\iint_{\mathcal{S}} \mathbf{F} \bullet \hat{\mathbf{N}}\, dS \\ &= \int_0^2 du \int_0^\pi (-u^2\cos^2 v - u^2\sin^2 v + u^3)\, dv \\ &= \int_0^2 (u^3 - u^2)\, du \int_0^\pi dv = \frac{4\pi}{3}.\end{aligned}$$

12. $\mathcal{S}$: $\mathbf{r} = e^u\cos v\mathbf{i} + e^u\sin v\mathbf{j} + u\mathbf{k}$, $(0 \le u \le 1,\ 0 \le v \le \pi)$, has upward surface element

$$\begin{aligned}\hat{\mathbf{N}}\, dS &= \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\, du\, dv \\ &= (-e^u\cos v\mathbf{i} - e^u\sin v\mathbf{j} + e^{2u}\mathbf{k})\, du\, dv.\end{aligned}$$

The flux of $\mathbf{F} = yz\mathbf{i} - xz\mathbf{j} + (x^2 + y^2)\mathbf{k}$ upward through $\mathcal{S}$ is

$$\begin{aligned}&\iint_{\mathcal{S}} \mathbf{F} \bullet \hat{\mathbf{N}}\, dS \\ &= \int_0^1 du \int_0^\pi (-ue^{2u}\sin v\cos v + ue^{2u}\sin v\cos v + e^{4u})\, dv \\ &= \int_0^1 e^{4u}\, du \int_0^\pi dv = \pi\, \frac{(e^4 - 1)}{4}.\end{aligned}$$

13. $\mathbf{F} = \dfrac{m\mathbf{r}}{|\mathbf{r}|^3} = \dfrac{m(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})}{(x^2 + y^2 + z^2)^{3/2}}$.

By symmetry, the flux of $\mathbf{F}$ out of the cube $-a \le x, y, z \le a$ is 6 times the flux out of the top face, $z = a$, where $\hat{\mathbf{N}} = \mathbf{k}$ and $dS = dx\, dy$. The total flux is

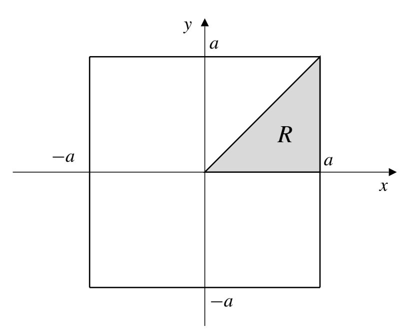

Fig. 15.6.13

$$\begin{aligned}
&6ma \int_{\substack{-a\le x\le a\\ -a\le y\le a}} \frac{dx\,dy}{(x^2+y^2+a^2)^{3/2}}\\
&= 48ma \iint_R \frac{r\,dr\,d\theta}{(r^2+a^2)^{3/2}}\\
&\qquad (R \text{ as shown in the figure})\\
&= 48ma \int_0^{\pi/4} d\theta \int_0^{a\sec\theta} \frac{r\,dr}{(r^2+a^2)^{3/2}}\\
&\qquad \text{Let } u = r^2+a^2\\
&\qquad du = 2r\,dr\\
&= 24ma \int_0^{\pi/4} d\theta \int_{a^2}^{a^2(1+\sec^2\theta)} \frac{du}{u^{3/2}}\\
&= 48ma \int_0^{\pi/4} \left(\frac{1}{a} - \frac{1}{a\sqrt{1+\sec^2\theta}}\right) d\theta\\
&= 48m \left(\frac{\pi}{4} - \int_0^{\pi/4} \frac{\cos\theta\,d\theta}{\sqrt{\cos^2\theta+1}}\right)\\
&= 48m \left(\frac{\pi}{4} - \int_0^{\pi/4} \frac{\cos\theta\,d\theta}{\sqrt{2-\sin^2\theta}}\right)\\
&\qquad \text{Let } \sqrt{2}\sin v = \sin\theta\\
&\qquad \sqrt{2}\cos v\,dv = \cos\theta\,d\theta\\
&= 48m \left(\frac{\pi}{4} - \int_0^{\pi/6} \frac{\sqrt{2}\cos v\,dv}{\sqrt{2}\cos v}\right)\\
&= 48m \left(\frac{\pi}{4} - \frac{\pi}{6}\right) = 4\pi m.
\end{aligned}$$

14. The flux of $\mathbf{F} = \dfrac{m\mathbf{r}}{|\mathbf{r}|^3}$ out of the cube $1 \le x, y, z \le 2$ is equal to three times the total flux out of the pair of opposite faces $z = 1$ and $z = 2$, which have outward normals $-\mathbf{k}$ and $\mathbf{k}$ respectively. This latter flux is $2mI_2 - mI_1$, where

$$\begin{aligned}
I_k &= \int_1^2 dx \int_1^2 \frac{dy}{(x^2+y^2+k^2)^{3/2}}\\
&\qquad \text{Let } y = \sqrt{x^2+k^2}\tan u\\
&\qquad dy = \sqrt{x^2+k^2}\sec^2 u\,du\\
&= \int_1^2 \frac{dx}{x^2+k^2} \int_{y=1}^{y=2} \cos u\,du\\
&= \int_1^2 \frac{dx}{x^2+k^2} \left(\sin u\right)\Big|_{y=1}^{y=2}\\
&= \int_1^2 \frac{dx}{x^2+k^2} \left(\frac{y}{\sqrt{x^2+y^2+k^2}}\Bigg|_1^2\right) = J_{k2} - J_{k1},
\end{aligned}$$

where

$$\begin{aligned}
J_{kn} &= n \int_1^2 \frac{dx}{(x^2+k^2)\sqrt{x^2+n^2+k^2}}\\
&\qquad \text{Let } x = \sqrt{n^2+k^2}\tan v\\
&\qquad dx = \sqrt{n^2+k^2}\sec^2 v\,dv\\
&= n \int_{x=1}^{x=2} \frac{\sec^2 v\,dv}{\left[(n^2+k^2)\tan^2 v + k^2\right]\sec v}\\
&= n \int_{x=1}^{x=2} \frac{\cos v\,dv}{(n^2+k^2)\sin^2 v + k^2\cos^2 v}\\
&= n \int_{x=1}^{x=2} \frac{\cos v\,dv}{k^2+n^2\sin^2 v} \qquad \text{Let } w = n\sin v\\
&\qquad\qquad dw = n\cos v\,dv\\
&= \int_{x=1}^{x=2} \frac{dw}{k^2+w^2} = \frac{1}{k}\tan^{-1}\frac{w}{k}\Bigg|_{x=1}^{x=2}\\
&= \frac{1}{k}\tan^{-1}\frac{n\sin v}{k}\Bigg|_{x=1}^{x=2}\\
&= \frac{1}{k}\tan^{-1}\frac{nx}{k\sqrt{x^2+n^2+k^2}}\Bigg|_1^2\\
&= \frac{1}{k}\left(\tan^{-1}\frac{2n}{k\sqrt{4+n^2+k^2}} - \tan^{-1}\frac{n}{k\sqrt{1+n^2+k^2}}\right).
\end{aligned}$$

Thus

$$\begin{aligned}
I_k = \frac{1}{k}\bigg[&\tan^{-1}\frac{4}{k\sqrt{8+k^2}} - 2\tan^{-1}\frac{2}{k\sqrt{5+k^2}}\\
&+ \tan^{-1}\frac{1}{k\sqrt{2+k^2}}\bigg].
\end{aligned}$$

The contribution to the total flux from the pair of surfaces $z = 1$ and $z = 2$ of the cube is

$$\begin{aligned}
&2mI_2 - mI_1\\
&= m\bigg[\tan^{-1}\frac{1}{\sqrt{3}} - 2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{2\sqrt{6}}\\
&\qquad - \tan^{-1}\frac{4}{3} + 2\tan^{-1}\frac{2}{\sqrt{6}} - \tan^{-1}\frac{1}{\sqrt{3}}\bigg].
\end{aligned}$$

Using the identities

$$\begin{aligned}
2\tan^{-1}a &= \tan^{-1}\frac{2a}{1-a^2}, \quad \text{and}\\
\tan^{-1}a &= \frac{\pi}{2} - \tan^{-1}\frac{1}{a},
\end{aligned}$$

we calculate

$$\begin{aligned}
-2\tan^{-1}\frac{1}{3} &= -\tan^{-1}\frac{3}{4} = -\frac{\pi}{2} + \tan^{-1}\frac{4}{3}\\
2\tan^{-1}\frac{2}{\sqrt{6}} &= \tan^{-1}\frac{12}{\sqrt{6}} = \frac{\pi}{2} - \tan^{-1}\frac{1}{2\sqrt{6}}.
\end{aligned}$$

Thus the net flux out of the pair of opposite faces is 0. By symmetry this holds for each pair, and the total flux out of the cube is 0. (You were warned this would be a difficult calculation!)

15. The flux of the plane vector field $\mathbf{F}$ across the piecewise smooth curve $\mathcal{C}$, in the direction of the unit normal $\hat{\mathbf{N}}$ to the curve, is

$$\int_{\mathcal{C}} \mathbf{F} \bullet \mathbf{n}\, ds.$$

The flux of $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$ outward across

a) the circle $x^2 + y^2 = a^2$ is

$$\oint_{\mathcal{C}} \mathbf{F} \bullet \left(\frac{x\mathbf{i} + y\mathbf{j}}{a}\right) ds = \frac{a^2}{a} \times 2\pi a = 2\pi a^2.$$

b) the boundary of the square $-1 \le x, y \le 1$ is

$$4\int_{-1}^{1} (\mathbf{i} + y\mathbf{j}) \bullet \mathbf{i}\, dy = 4\int_{-1}^{1} dy = 8.$$

16. $\mathbf{F} = -\dfrac{x\mathbf{i} + y\mathbf{j}}{x^2 + y^2}$.

a) The flux of $\mathbf{F}$ inward across the circle of Exercise 7(a) is

$$\begin{aligned} -\oint_{\mathcal{C}} \left(-\frac{x\mathbf{i} + y\mathbf{j}}{a^2}\right) \bullet \frac{x\mathbf{i} + y\mathbf{j}}{a}\, ds \\ = \oint_{\mathcal{C}} \frac{a^2}{a^3}\, ds = \frac{1}{a} \times 2\pi a = 2\pi. \end{aligned}$$

b) The flux of $\mathbf{F}$ inward across the boundary of the square of Exercise 7(b) is four times the flux inward across the edge $x = 1$, $-1 \le y \le 1$. Thus it is

$$\begin{aligned} -4\int_{-1}^{1} \left(-\frac{\mathbf{i} + y\mathbf{j}}{1 + y^2}\right) \bullet \mathbf{i}\, dy &= 4\int_{-1}^{1} \frac{dy}{1 + y^2} \\ &= 4\tan^{-1} y \Big|_{-1}^{1} = 2\pi. \end{aligned}$$

17. The flux of $\hat{\mathbf{N}}$ across $\mathcal{S}$ is

$$\iint_{\mathcal{S}} \hat{\mathbf{N}} \bullet \hat{\mathbf{N}}\, dS = \iint_{\mathcal{S}} dS = \text{area of } \mathcal{S}.$$

18. Let $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ be a constant vector field.

a) If R is a rectangular box, we can choose the origin and coordinate axes in such a way that the box is $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$. On the faces $x = 0$ and $x = a$ we have $\hat{\mathbf{N}} = -\mathbf{i}$ and $\hat{\mathbf{N}} = \mathbf{i}$ respectively. Since F_1 is constant, the total flux out of the box through these two faces is

$$\iint_{\substack{0 \le y \le b \\ 0 \le z \le c}} (F_1 - F_1)\, dy\, dz = 0.$$

The flux out of the other two pairs of opposite faces is also 0. Thus the total flux of $\mathbf{F}$ out of the box is 0.

b) If $\mathcal{S}$ is a sphere of radius a we can choose the origin so that $\mathcal{S}$ has equation $x^2 + y^2 + z^2 = a^2$, and so its outward normal is

$$\hat{\mathbf{N}} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{a}.$$

Thus the flux out of $\mathcal{S}$ is

$$\frac{1}{a}\iint_{\mathcal{S}} (F_1 x + F_2 y + F_3 z)\, ds = 0,$$

since the sphere $\mathcal{S}$ is symmetric about the origin.

Review Exercises 15 (page 848)

1. $\mathcal{C}:\quad x = t,\ y = 2e^t,\ z = e^{2t}, \quad (-1 \le t \le 1)$

$$\begin{aligned} v &= \sqrt{1 + 4e^{2t} + 4e^{4t}} = 1 + 2e^{2t} \\ \int_{\mathcal{C}} \frac{ds}{y} &= \int_{-1}^{1} \frac{1 + 2e^{2t}}{2e^t}\, dt \\ &= \left(-\frac{e^{-t}}{2} + e^t\right)\Bigg|_{-1}^{1} = \frac{3(e^2 - 1)}{2e}. \end{aligned}$$

2. $\mathcal{C}$ can be parametrized $x = t$, $y = 2t$, $z = t + 4t^2$, $(0 \le t \le 2)$. Thus

$$\begin{aligned} &\int_{\mathcal{C}} 2y\, dx + x\, dy + 2\, dz \\ &\quad = \int_0^2 [4t(1) + t(2) + 2(1 + 8t)]\, dt \\ &\quad = \int_0^2 (22t + 2)\, dt = 48. \end{aligned}$$

3. The cone $z = \sqrt{x^2 + y^2}$ has area element

$$dS = \sqrt{1 + \frac{x^2 + y^2}{z^2}}\, dx\, dy = \sqrt{2}\, dx\, dy.$$

If $\mathcal{S}$ is the part of the cone in the region $0 \le x \le 1 - y^2$ (which itself lies between $y = -1$ and $y = 1$), then

$$\begin{aligned} \iint_{\mathcal{S}} x\, dS &= \sqrt{2}\int_{-1}^{1} dy \int_0^{1-y^2} x\, dx \\ &= 2\sqrt{2}\int_0^1 \frac{1 - 2y^2 + y^4}{2}\, dy = \frac{8\sqrt{2}}{15}. \end{aligned}$$

4. The plane $x+y+z=1$ has area element $dS=\sqrt{3}\,dx\,dy$. If $\mathcal{S}$ is the part of the plane in the first octant, then the projection of $\mathcal{S}$ on the xy-plane is the triangle $0\le x\le 1$, $0\le y\le 1-x$. Thus

$$\begin{aligned}\iint_{\mathcal{S}} xyz\,dS &= \sqrt{3}\int_0^1 x\,dx\int_0^{1-x} y(1-x-y)\,dy\\ &= \sqrt{3}\int_0^1 \frac{x(1-x)^3}{6}\,dx \quad \begin{array}{l}\text{Let } u=1-x\\ du=-dx\end{array}\\ &= \frac{\sqrt{3}}{6}\int_0^1 u^3(1-u)\,du = \frac{\sqrt{3}}{6}\left(\frac14-\frac15\right)=\frac{\sqrt{3}}{120}.\end{aligned}$$

5. For $z=xy$, the upward vector surface element is

$$\hat{\mathbf{N}}\,dS = \frac{-y\mathbf{i}-x\mathbf{j}+\mathbf{k}}{1}\,dx\,dy.$$

The flux of $\mathbf{F}=x^2y\mathbf{i}-10xy^2\mathbf{j}$ upward through $\mathcal{S}$, the part of $z=xy$ satisfying $0\le x\le 1$ and $0\le y\le 1$ is

$$\begin{aligned}\iint_{\mathcal{S}}\mathbf{F}\bullet\hat{\mathbf{N}}\,dS &= \int_0^1 dx\int_0^1(-x^2y^2+10x^2y^2)\,dy\\ &= \int_0^1 3x^2\,dx\int_0^1 3y^2\,dy = 1.\end{aligned}$$

6. The plane $x+2y+3z=6$ has downward vector surface element

$$\hat{\mathbf{N}}\,dS = \frac{-\mathbf{i}-2\mathbf{j}-3\mathbf{k}}{3}\,dx\,dy.$$

If $\mathcal{S}$ is the part of the plane in the first octant, then the projection of $\mathcal{S}$ on the xy-plane is the triangle $0\le y\le 3$, $0\le x\le 6-2y$. Thus

$$\begin{aligned}&\iint_{\mathcal{S}}(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})\bullet\hat{\mathbf{N}}\,dS\\ &\quad= -\frac13\int_0^3 dy\int_0^{6-2y}(x+2y+6-x-2y)\,dx\\ &\quad= -2\int_0^3(6-2y) = -36+18=-18.\end{aligned}$$

7. $\mathbf{r}=a\sin t\mathbf{i}+a\cos t\mathbf{j}+bt\mathbf{k}$, $(0\le t\le 6\pi)$
$\mathbf{r}(0)=a\mathbf{j}$, $\mathbf{r}(6\pi)=a\mathbf{j}+6\pi b\mathbf{k}$.

a) The force $\mathbf{F}=-mg\mathbf{k}=-\nabla(mgz)$ is conservative, so the work done by $\mathbf{F}$ as the bead moves from $\mathbf{r}(6\pi)$ to $\mathbf{r}(0)$ is

$$W=\int_{t=6\pi}^{t=0}\mathbf{F}\bullet d\mathbf{r} = -mgz\Big|_{z=6\pi b}^{z=0} = 6\pi mgb.$$

b) $\mathbf{v}=a\cos t\mathbf{i}-a\sin t\mathbf{j}+b\mathbf{k}$, $|\mathbf{v}|=\sqrt{a^2+b^2}$. A force of constant magnitude R opposing the motion of the bead is in the direction of $-\mathbf{v}$, so it is

$$\mathbf{F}=-R\frac{\mathbf{v}}{|\mathbf{v}|}=-\frac{R}{\sqrt{a^2+b^2}}\mathbf{v}.$$

Since $d\mathbf{r}=\mathbf{v}\,dt$, the work done against the resistive force is

$$W=\int_0^{6\pi}\frac{R}{\sqrt{a^2+b^2}}|\mathbf{v}|^2\,dt = 6\pi R\sqrt{a^2+b^2}.$$

8. $\int_{\mathcal{C}}\mathbf{F}\bullet d\mathbf{r}$ can be determined using only the endpoints of $\mathcal{C}$, provided

$$\mathbf{F}=(axy+3yz)\mathbf{i}+(x^2+3xz+by^2z)\mathbf{j}+(bxy+cy^3)\mathbf{k}$$

is conservative, that is, if

$$\begin{aligned}ax+3z &= \frac{\partial F_1}{\partial y}=\frac{\partial F_2}{\partial x}=2x+3z\\ 3y &= \frac{\partial F_1}{\partial z}=\frac{\partial F_3}{\partial x}=by\\ 3x+by^2 &= \frac{\partial F_2}{\partial z}=\frac{\partial F_3}{\partial y}=bx+3cy^2.\end{aligned}$$

Thus we need $a=2$, $b=3$, and $c=1$.
With these values, $\mathbf{F}=\nabla(x^2y+3xyz+y^3z)$. Thus

$$\int_{\mathcal{C}}\mathbf{F}\bullet d\mathbf{r} = (x^2y+3xyz+y^3z)\Big|_{(0,1,-1)}^{(2,1,1,)} = 11-(-1)=12.$$

9. $\mathbf{F}=(x^2/y)\mathbf{i}+y\mathbf{j}+\mathbf{k}$.
The field lines satisfy $\dfrac{y\,dx}{x^2}=\dfrac{dy}{y}=dz$. Thus $dx/x^2=dy/y^2$ and the field lines are given by

$$\frac1x=\frac1y+C_1,\qquad \ln y = z+C_2.$$

The field line passes through $(1,1,0)$ provided $C_1=0$ and $C_2=0$. In this case the field line also passes through $(e,e,1)$, and the segment from $(1,1,0)$ to $(e,e,1)$ can be parametrized $\mathbf{r}(t)=e^t\mathbf{i}+e^t\mathbf{j}+t\mathbf{k}$, $(0\le t\le 1)$. Then

$$\begin{aligned}\int_{\mathcal{C}}\mathbf{F}\bullet d\mathbf{r} &= \int_0^1(e^{2t}+e^{2t}+1)\,dt\\ &= (e^{2t}+t)\Big|_0^1 = e^2.\end{aligned}$$

10. a) $\mathbf{F}=(1+x)e^{x+y}\mathbf{i}+(xe^{x+y}+2y)\mathbf{j}-2z\mathbf{k}$
$=\nabla(xe^{x+y}+y^2-z^2)$.
Thus $\mathbf{F}$ is conservative.

b) $\mathbf{G} = (1+x)e^{x+y}\mathbf{i} + (xe^{x+y} + 2z)\mathbf{j} - 2y\mathbf{k}$
$= \mathbf{F} + 2(z-y)(\mathbf{j}+\mathbf{k})$.
$\mathcal{C} : \mathbf{r} = (1-t)e^t\mathbf{i} + t\mathbf{j} + 2t\mathbf{k}, \quad (0 \le t \le 1)$.
$\mathbf{r}(0) = (1,0,0), \quad \mathbf{r}(1) = (0,1,2)$. Thus

$$\begin{aligned}\int_{\mathcal{C}} \mathbf{G} \bullet d\mathbf{r} &= \int_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{r} + \int_{\mathcal{C}} 2(z-y)(\mathbf{j}+\mathbf{k}) \bullet d\mathbf{r} \\ &= (xe^{x+y} + y^2 - z^2)\Big|_{(1,0,0)}^{(0,1,2)} \\ &\quad + 2\int_0^1 (2t-t)(1+2)\,dt \\ &= -3 - e + 3t^2\Big|_0^1 = -e.\end{aligned}$$

11. Since the field lines of $\mathbf{F}$ are $xy = C$, and so satisfy

$$y\,dx + x\,dy = 0, \quad \text{or } \frac{dx}{x} = -\frac{dy}{y},$$

thus $\mathbf{F} = \lambda(x,y)(x\mathbf{i} - y\mathbf{j})$. Since $|\mathbf{F}(x,y)| = 1$ if $(x,y) \ne (0,0)$, $\lambda(x,y) = \pm 1/\sqrt{x^2+y^2}$, and

$$\mathbf{F}(x,y) = \pm\frac{x\mathbf{i} - y\mathbf{j}}{\sqrt{x^2+y^2}}.$$

Since $\mathbf{F}(1,1) = (\mathbf{i}-\mathbf{j})/\sqrt{2}$, we need the plus sign. Thus

$$\mathbf{F}(x,y) = \frac{x\mathbf{i} - y\mathbf{j}}{\sqrt{x^2+y^2}},$$

which is continuous everywhere except at $(0,0)$.

12. The first octant part of the cylinder $y^2 + z^2 = 16$ has outward vector surface element

$$\hat{\mathbf{N}}\,dS = \frac{2y\mathbf{j} + 2z\mathbf{k}}{2z}\,dx\,dy = \left(\frac{y}{\sqrt{16-y^2}}\mathbf{j} + \mathbf{k}\right)dx\,dy.$$

The flux of $3z^2x\mathbf{i} - x\mathbf{j} - y\mathbf{k}$ outward through the specified surface $\mathcal{S}$ is

$$\begin{aligned}\mathbf{F} \bullet \hat{\mathbf{N}}\,dS &= \int_0^5 dx \int_0^4 \left(0 - \frac{xy}{\sqrt{16-y^2}} - y\right) dy \\ &= \int_0^5 \left(x\sqrt{16-y^2} - \frac{y^2}{2}\right)\Big|_{y=0}^{y=4} dx \\ &= -\int_0^5 (4x+8)\,dx = -90.\end{aligned}$$

Challenging Problems 15 (page 849)

1. Given: $x = (2+\cos v)\cos u$, $y = (2+\cos v)\sin u$, $z = \sin v$ for $0 \le u \le 2\pi$, $0 \le v \le \pi$.
The cylindrical coordinate r satisfies

$$\begin{aligned}r^2 &= x^2 + y^2 = (2+\cos v)^2 \\ r &= 2 + \cos v \\ (r-2)^2 &+ z^2 = 1.\end{aligned}$$

This equation represents the surface of a torus, obtained by rotating about the z-axis the circle of radius 1 in the xz-plane centred at $(2,0,0)$. Since $0 \le v \le \pi$ implies that $z \ge 0$, the given surface is only the top half of the toroidal surface.
By symmetry, $\bar{x} = 0$ and $\bar{y} = 0$.
A ring-shaped strip on the surface at angular position v with width dv has radius $2+\cos v$, and so its surface area is $dS = 2\pi(2+\cos v)\,dv$. The area of the whole given surface is

$$S = \int_0^\pi 2\pi(2+\cos v)\,dv = 4\pi^2.$$

The strip has moment $z\,dS = 2\pi(2+\cos v)\sin v\,dv$ about $z = 0$, so the moment of the whole surface about $z = 0$ is

$$\begin{aligned}M_{z=0} &= 2\pi\int_0^\pi (2+\cos v)\sin v\,dv \\ &= 2\pi\left(-2\cos v - \frac{1}{4}\cos(2v)\right)\Big|_0^\pi = 8\pi.\end{aligned}$$

Thus $\bar{z} = \dfrac{8\pi}{4\pi^2} = \dfrac{2}{\pi}$. The centroid is $(0,0,2/\pi)$.

2. This is a trick question. Observe that the given parametrization $\mathbf{r}(u,v)$ satisfies

$$\mathbf{r}(u+\pi, v) = \mathbf{r}(u, -v).$$

Therefore the surface $\mathcal{S}$ is traced out twice as u goes from 0 to 2π. (It is a Möbius band. See Figure 15.28 in the text.) If $\mathcal{S}_1$ is the part of the surface corresponding to $0 \le u \le \pi$, and $\mathcal{S}_2$ is the part corresponding to $\pi \le u \le 2\pi$, then $\mathcal{S}_1$ and $\mathcal{S}_2$ coincide as point sets, but their normals are oppositely oriented: $\hat{\mathbf{N}}_2 = -\hat{\mathbf{N}}_1$ at corresponding points on the two surfaces. Hence

$$\iint_{\mathcal{S}_1} \mathbf{F} \bullet \hat{\mathbf{N}}_1\,dS = -\iint_{\mathcal{S}_2} \mathbf{F} \bullet \hat{\mathbf{N}}_2\,dS,$$

for any smooth vector field, and

$$\iint_{\mathcal{S}} \mathbf{F} \bullet \hat{\mathbf{N}}\, dS = \iint_{\mathcal{S}_1} \mathbf{F} \bullet \hat{\mathbf{N}}_1\, dS + \iint_{\mathcal{S}_2} \mathbf{F} \bullet \hat{\mathbf{N}}_2\, dS = 0.$$

3.

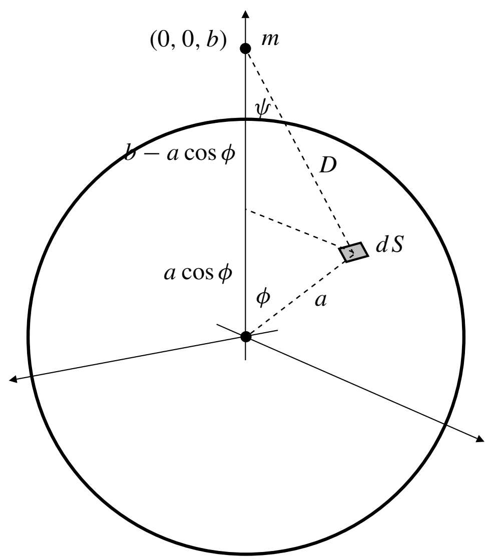

Fig. C-15.3

The mass element $\sigma\, dS$ at position $[a, \phi, \theta]$ on the sphere is at distance $D = \sqrt{a^2 + b^2 - 2ab\cos\phi}$ from the mass m located at $(0, 0, b)$, and thus it attracts m with a force of magnitude $dF = km\sigma dS/D^2$. By symmetry, the horizontal components of dF coresponding to mass elements on opposite sides of the sphere (i.e., at $[a, \phi, \theta]$ and $[a, \phi, \theta + \pi]$) cancel, but the vertical components

$$dF \cos\psi = \frac{km\sigma\, dS}{D^2}\, \frac{b - a\cos\phi}{D}$$

reinforce. The total force on the mass m is the sum of all such vertical components. Since $dS = a^2 \sin\phi\, d\phi\, d\theta$, it is

$$\begin{aligned} F &= km\sigma a^2 \int_0^{2\pi} d\theta \int_0^{\pi} \frac{(b - a\cos\phi)\sin\phi\, d\phi}{(a^2 + b^2 - 2ab\cos\phi)^{3/2}} \\ &= 2\pi km\sigma a^2 \int_{-1}^{1} \frac{(b - at)dt}{(a^2 - 2abt + b^2)^{3/2}}. \end{aligned}$$

We have made the change of variable $t = \cos\phi$ to get the last integral. This integral can be evaluated by using another substitution. Let $u = \sqrt{a^2 - 2abt + b^2}$. Thus

$$t = \frac{a^2 + b^2 - u^2}{2ab}, \quad dt = -\frac{u\, du}{ab}, \quad b - at = \frac{u^2 + b^2 - a^2}{2b}.$$

When $t = -1$ and $t = 1$ we have $u = a + b$ and $u = |a - b|$ respectively. Therefore

$$\begin{aligned} F &= 2\pi km\sigma a^2 \int_{a+b}^{|a-b|} \frac{u^2 + b^2 - a^2}{2bu^3} \left(-\frac{u\, du}{ab}\right) \\ &= \frac{\pi km\sigma a}{b^2} \int_{|a-b|}^{a+b} \left(1 + \frac{b^2 - a^2}{u^2}\right) du \\ &= \frac{\pi km\sigma a}{b^2} \left(u - \frac{b^2 - a^2}{u}\right)\Bigg|_{|a-b|}^{a+b}. \end{aligned}$$

There are now two cases to consider. If the mass m is *outside* the sphere, so that $b > a$ and $|a - b| = b - a$, then

$$F = \frac{\pi km\sigma a}{b^2} \Big((a+b) - (b-a) - (b-a) + (b+a)\Big) = 4\pi km\sigma \frac{a^2}{b^2}.$$

However, if m is *inside* the sphere, so that $b < a$ and $|a - b| = a - b$, then

$$F = \frac{\pi km\sigma a}{b^2} \Big((a + b) + (a - b) - (a - b) - (a + b)\Big) = 0.$$