CHAPTER 16. VECTOR CALCULUS

Section 16.1 Gradient, Divergence, and Curl (page 858)

1.
$$\mathbf{F} = x\mathbf{i} + y\mathbf{j}$$
$$\mathbf{div}\,\mathbf{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(0) = 1 + 1 = 2$$
$$\mathbf{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & 0 \end{vmatrix} = \mathbf{0}$$

2.
$$\mathbf{F} = y\mathbf{i} + x\mathbf{j}$$
$$\mathbf{div}\,\mathbf{F} = \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}(0) = 0 + 0 = 0$$
$$\mathbf{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x & 0 \end{vmatrix} = (1-1)\mathbf{k} = \mathbf{0}$$

3.
$$\mathbf{F} = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$$
$$\mathbf{div}\,\mathbf{F} = \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(z) + \frac{\partial}{\partial z}(x) = 0$$
$$\mathbf{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = -\mathbf{i} - \mathbf{j} - \mathbf{k}$$

4.
$$\mathbf{F} = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$$
$$\mathbf{div}\,\mathbf{F} = \frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial y}(xz) + \frac{\partial}{\partial z}(xy) = 0$$
$$\mathbf{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix}$$
$$= (x-x)\mathbf{i} + (y-y)\mathbf{j} + (z-z)\mathbf{k} = \mathbf{0}$$

5.
$$\mathbf{F} = x\mathbf{i} + x\mathbf{k}$$
$$\mathbf{div}\,\mathbf{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(x) = 1$$
$$\mathbf{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & 0 & x \end{vmatrix} = -\mathbf{j}$$

6.
$$\mathbf{F} = xy^2\mathbf{i} - yz^2\mathbf{j} + zx^2\mathbf{k}$$
$$\mathbf{div}\,\mathbf{F} = \frac{\partial}{\partial x}\left(xy^2\right) + \frac{\partial}{\partial y}\left(-yz^2\right) + \frac{\partial}{\partial z}\left(zx^2\right)$$
$$= y^2 - z^2 + x^2$$
$$\mathbf{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & -yz^2 & zx^2 \end{vmatrix}$$
$$= 2yz\mathbf{i} - 2xz\mathbf{j} - 2xy\mathbf{k}$$

7.
$$\mathbf{F} = f(x)\mathbf{i} + g(y)\mathbf{j} + h(z)\mathbf{k}$$
$$\mathbf{div}\,\mathbf{F} = \frac{\partial}{\partial x}f(x) + \frac{\partial}{\partial y}g(y) + \frac{\partial}{\partial z}h(z)$$
$$= f'(x) + g'(y) + h'(z)$$
$$\mathbf{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f(x) & g(y) & h(z) \end{vmatrix} = \mathbf{0}$$

8.
$$\mathbf{F} = f(z)\mathbf{i} - f(z)\mathbf{j}$$
$$\mathbf{div}\,\mathbf{F} = \frac{\partial}{\partial x}f(z) + \frac{\partial}{\partial y}\Big(-f(z)\Big) = 0$$
$$\mathbf{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f(z) & -f(z) & 0 \end{vmatrix} = f'(z)(\mathbf{i} + \mathbf{j})$$

9. Since $x = r\cos\theta$, and $y = r\sin\theta$, we have $r^2 = x^2 + y^2$, and so

$$\begin{aligned}
\frac{\partial r}{\partial x} &= \frac{x}{r} = \cos\theta \\
\frac{\partial r}{\partial y} &= \frac{y}{r} = \sin\theta \\
\frac{\partial}{\partial x}\sin\theta &= \frac{\partial}{\partial x}\frac{y}{r} = \frac{-xy}{r^3} = -\frac{\cos\theta\sin\theta}{r} \\
\frac{\partial}{\partial y}\sin\theta &= \frac{\partial}{\partial y}\frac{y}{r} = \frac{1}{r} - \frac{y^2}{r^3} \\
&= \frac{x^2}{r^3} = \frac{\cos^2\theta}{r} \\
\frac{\partial}{\partial x}\cos\theta &= \frac{\partial}{\partial x}\frac{x}{r} = \frac{1}{r} - \frac{x^2}{r^3} \\
&= \frac{y^2}{r^3} = \frac{\sin^2\theta}{r} \\
\frac{\partial}{\partial y}\cos\theta &= \frac{\partial}{\partial y}\frac{x}{r} = \frac{-xy}{r^3} = -\frac{\cos\theta\sin\theta}{r}.
\end{aligned}$$

(The last two derivatives are not needed for this exercise, but will be useful for the next two exercises.) For

$$\mathbf{F} = r\mathbf{i} + \sin\theta\mathbf{j},$$

we have

$$\mathbf{div}\,\mathbf{F} = \frac{\partial r}{\partial x} + \frac{\partial}{\partial y}\sin\theta = \cos\theta + \frac{\cos^2\theta}{r}$$
$$\mathbf{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ r & \sin\theta & 0 \end{vmatrix}$$
$$= \left(-\frac{\sin\theta\cos\theta}{r} - \sin\theta\right)\mathbf{k}.$$

10.
$$\mathbf{F} = \hat{\mathbf{r}} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$$
$$\mathbf{div}\,\mathbf{F} = \frac{\sin^2\theta}{r} + \frac{\cos^2\theta}{r} = \frac{1}{r} = \frac{1}{\sqrt{x^2+y^2}}$$
$$\mathbf{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \cos\theta & \sin\theta & 0 \end{vmatrix} = -\left(\frac{\cos\theta\sin\theta}{r} - \frac{\cos\theta\sin\theta}{r}\right)\mathbf{k} = \mathbf{0}$$

11.
$$\mathbf{F} = \hat{\boldsymbol{\theta}} = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}$$
$$\mathbf{div}\,\mathbf{F} = \frac{\cos\theta\sin\theta}{r} - \frac{\cos\theta\sin\theta}{r} = 0$$
$$\mathbf{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\sin\theta & \cos\theta & 0 \end{vmatrix} = \left(\frac{\sin^2\theta}{r} + \frac{\cos^2\theta}{r}\right)\mathbf{k} = \frac{1}{r}\mathbf{k} = \frac{1}{\sqrt{x^2+y^2}}\mathbf{k}$$

12. We use the Maclaurin expansion of **F**, as presented in the proof of Theorem 1:
$$\mathbf{F} = \mathbf{F}_0 + \mathbf{F}_1 x + \mathbf{F}_2 y + \mathbf{F}_3 z + \cdots,$$
where
$$\mathbf{F}_0 = \mathbf{F}(0,0,0)$$
$$\mathbf{F}_1 = \frac{\partial}{\partial x}\mathbf{F}(x,y,z)\Big|_{(0,0,0)} = \left(\frac{\partial F_1}{\partial x}\mathbf{i} + \frac{\partial F_2}{\partial x}\mathbf{j} + \frac{\partial F_3}{\partial x}\mathbf{k}\right)\Big|_{(0,0,0)}$$
$$\mathbf{F}_2 = \frac{\partial}{\partial y}\mathbf{F}(x,y,z)\Big|_{(0,0,0)} = \left(\frac{\partial F_1}{\partial y}\mathbf{i} + \frac{\partial F_2}{\partial y}\mathbf{j} + \frac{\partial F_3}{\partial y}\mathbf{k}\right)\Big|_{(0,0,0)}$$
$$\mathbf{F}_3 = \frac{\partial}{\partial z}\mathbf{F}(x,y,z)\Big|_{(0,0,0)} = \left(\frac{\partial F_1}{\partial z}\mathbf{i} + \frac{\partial F_2}{\partial z}\mathbf{j} + \frac{\partial F_3}{\partial z}\mathbf{k}\right)\Big|_{(0,0,0)}$$
and where $\cdots$ represents terms of degree 2 and higher in x, y, and z.
On the top of the box $B_{a,b,c}$, we have $z = c$ and $\hat{\mathbf{N}} = \mathbf{k}$. On the bottom of the box, we have $z = -c$ and $\hat{\mathbf{N}} = -\mathbf{k}$. On both surfaces $dS = dx\,dy$. Thus
$$\left(\iint_{\text{top}} + \iint_{\text{bottom}}\right)\mathbf{F}\bullet\hat{\mathbf{N}}\,dS$$
$$= \int_{-a}^{a} dx \int_{-b}^{b} dy \Big(c\mathbf{F}_3\bullet\mathbf{k} - c\mathbf{F}_3\bullet(-\mathbf{k})\Big) + \cdots$$
$$= 8abc\mathbf{F}_3\bullet\mathbf{k} + \cdots = 8abc\frac{\partial}{\partial z}F_3(x,y,z)\Big|_{(0,0,0)} + \cdots,$$
where $\cdots$ represents terms of degree 4 and higher in a, b, and c.
Similar formulas obtain for the two other pairs of faces, and the three formulas combine into
$$\oiint_{B_{a,b,c}} \mathbf{F}\bullet\hat{\mathbf{N}}\,dS = 8abc\,\mathbf{div}\,\mathbf{F}(0,0,0) + \cdots.$$
It follows that
$$\lim_{a,b,c\to 0+} \frac{1}{8abc}\oiint_{B_{a,b,c}} \mathbf{F}\bullet\hat{\mathbf{N}}\,dS = \mathbf{div}\,\mathbf{F}(0,0,0).$$

13. This proof just mimics that of Theorem 1. **F** can be expanded in Maclaurin series
$$\mathbf{F} = \mathbf{F}_0 + \mathbf{F}_1 x + \mathbf{F}_2 y + \cdots,$$
where
$$\mathbf{F}_0 = \mathbf{F}(0,0)$$
$$\mathbf{F}_1 = \frac{\partial}{\partial x}\mathbf{F}(x,y)\Big|_{(0,0)} = \left(\frac{\partial F_1}{\partial x}\mathbf{i} + \frac{\partial F_2}{\partial x}\mathbf{j}\right)\Big|_{(0,0)}$$
$$\mathbf{F}_2 = \frac{\partial}{\partial y}\mathbf{F}(x,y)\Big|_{(0,0)} = \left(\frac{\partial F_1}{\partial y}\mathbf{i} + \frac{\partial F_2}{\partial y}\mathbf{j}\right)\Big|_{(0,0)}$$
and where $\cdots$ represents terms of degree 2 and higher in x and y.
On the curve $\mathcal{C}_\epsilon$ of radius ϵ centred at $(0,0)$, we have $\hat{\mathbf{N}} = \frac{1}{\epsilon}(x\mathbf{i} + y\mathbf{j})$. Therefore,
$$\mathbf{F}\bullet\hat{\mathbf{N}} = \frac{1}{\epsilon}\Big(\mathbf{F}_0\bullet\mathbf{i}x + \mathbf{F}_0\bullet\mathbf{j}y + \mathbf{F}_1\bullet\mathbf{i}x^2 + \mathbf{F}_1\bullet\mathbf{j}xy + \mathbf{F}_2\bullet\mathbf{i}xy + \mathbf{F}_2\bullet\mathbf{j}y^2 + \cdots\Big)$$
where $\cdots$ represents terms of degree 3 or higher in x and y. Since
$$\oint_{\mathcal{C}_\epsilon} x\,ds = \oint_{\mathcal{C}_\epsilon} y\,ds = \oint_{\mathcal{C}_\epsilon} xy\,ds = 0$$
$$\oint_{\mathcal{C}_\epsilon} x^2\,ds = \oint_{\mathcal{C}_\epsilon} y^2\,ds = \int_0^{2\pi} \epsilon^2\cos^2\theta\,\epsilon\,d\theta = \pi\epsilon^3,$$
we have
$$\frac{1}{\pi\epsilon^2}\oint_{\mathcal{C}_\epsilon} \mathbf{F}\bullet\hat{\mathbf{N}}\,ds = \frac{1}{\pi\epsilon^2}\frac{\pi\epsilon^3}{\epsilon}(\mathbf{F}_1\bullet\mathbf{i} + \mathbf{F}_2\bullet\mathbf{j}) + \cdots = \mathbf{div}\,\mathbf{F}(0,0) + \cdots$$
where $\cdots$ represents terms of degree 1 or higher in ϵ. Therefore, taking the limit as $\epsilon \to 0$ we obtain
$$\lim_{\epsilon\to 0} \frac{1}{\pi\epsilon^2}\oint_{\mathcal{C}_\epsilon} \mathbf{F}\bullet\hat{\mathbf{N}}\,ds = \mathbf{div}\,\mathbf{F}(0,0).$$

14. We use the same Maclaurin expansion for $\mathbf{F}$ as in Exercises 12 and 13. On $\mathcal{C}_\epsilon$ we have

$$\begin{aligned}\mathbf{r} &= \epsilon\cos\theta\mathbf{i} + \epsilon\sin\theta\mathbf{j}, \quad (0\le\theta\le 2\pi)\\ d\mathbf{r} &= -\epsilon\sin\theta\mathbf{i} + \epsilon\cos\theta\mathbf{j}\\ \mathbf{F}\bullet d\mathbf{r} &= \Big(-\epsilon\sin\theta\mathbf{F}_0\bullet\mathbf{i} + \epsilon\cos\theta\mathbf{F}_0\bullet\mathbf{j}\\ &\quad - \epsilon^2\sin\theta\cos\theta\mathbf{F}_1\bullet\mathbf{i} + \epsilon^2\cos^2\theta\mathbf{F}_1\bullet\mathbf{j}\\ &\quad - \epsilon^2\sin^2\theta\mathbf{F}_2\bullet\mathbf{i} + \epsilon^2\sin\theta\cos\theta\mathbf{F}_2\bullet\mathbf{j} + \cdots\Big)\,ds,\end{aligned}$$

where $\cdots$ represents terms of degree 3 or higher in ϵ.
Since

$$\begin{aligned}&\int_0^{2\pi}\sin\theta\,d\theta = \int_0^{2\pi}\cos\theta\,d\theta = \int_0^{2\pi}\sin\theta\cos\theta\,d\theta = 0\\ &\int_0^{2\pi}\cos^2\theta\,d\theta = \int_0^{2\pi}\sin^2\theta\,d\theta = \pi,\end{aligned}$$

we have

$$\frac{1}{\pi\epsilon^2}\oint_{\mathcal{C}_\epsilon}\mathbf{F}\bullet d\mathbf{r} = \mathbf{F}_1\bullet\mathbf{j} - \mathbf{F}_2\bullet\mathbf{i} + \cdots,$$

where $\cdots$ represents terms of degree at least 1 in ϵ.
Hence

$$\begin{aligned}\lim_{\epsilon\to 0+}\frac{1}{\pi\epsilon^2}\oint_{\mathcal{C}_\epsilon}\mathbf{F}\bullet d\mathbf{r} &= \mathbf{F}_1\bullet\mathbf{j} - \mathbf{F}_2\bullet\mathbf{i}\\ &= \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\\ &= \mathbf{curl}\,\mathbf{F}\bullet\mathbf{k} = \mathbf{curl}\,\mathbf{F}\bullet\hat{\mathbf{N}}.\end{aligned}$$

Section 16.2 Some Identities Involving Grad, Div, and Curl (page 864)

1. Theorem 3(a):

$$\begin{aligned}\nabla(\phi\psi) &= \frac{\partial}{\partial x}(\phi\psi) + \frac{\partial}{\partial y}(\phi\psi) + \frac{\partial}{\partial z}(\phi\psi)\\ &= \left(\phi\frac{\partial\psi}{\partial x} + \frac{\partial\phi}{\partial x}\psi\right)\mathbf{i} + \cdots + \left(\phi\frac{\partial\psi}{\partial z} + \frac{\partial\phi}{\partial z}\psi\right)\mathbf{k}\\ &= \phi\nabla\psi + \psi\nabla\phi.\end{aligned}$$

2. Theorem 3(b):

$$\begin{aligned}\nabla\bullet(\phi\mathbf{F}) &= \frac{\partial}{\partial x}(\phi F_1) + \frac{\partial}{\partial y}(\phi F_2) + \frac{\partial}{\partial z}(\phi F_3)\\ &= \frac{\partial\phi}{\partial x}F_1 + \phi\frac{\partial F_1}{\partial x} + \cdots + \frac{\partial\phi}{\partial z}F_3 + \phi\frac{\partial F_3}{\partial z} + \cdots\\ &= \nabla\phi\bullet\mathbf{F} + \phi\nabla\bullet\mathbf{F}.\end{aligned}$$

3. Theorem 3(d):

$$\begin{aligned}\nabla\bullet(\mathbf{F}\times\mathbf{G}) &= \frac{\partial}{\partial x}(F_2G_3 - F_3G_2) + \cdots\\ &= \frac{\partial F_2}{\partial x}G_3 + F_2\frac{\partial G_3}{\partial x} - \frac{\partial F_3}{\partial x}G_2 - F_3\frac{\partial G_2}{\partial x} + \cdots\\ &= (\nabla\times\mathbf{F})\bullet\mathbf{G} - \mathbf{F}\bullet(\nabla\times\mathbf{G}).\end{aligned}$$

4. Theorem 3(f). The first component of $\nabla(\mathbf{F}\bullet\mathbf{G})$ is

$$\frac{\partial F_1}{\partial x}G_1 + F_1\frac{\partial G_1}{\partial x} + \frac{\partial F_2}{\partial x}G_2 + F_2\frac{\partial G_2}{\partial x} + \frac{\partial F_3}{\partial x}G_3 + F_3\frac{\partial G_3}{\partial x}.$$

We calculate the first components of the four terms on the right side of the identity to be proved.
The first component of $\mathbf{F}\times(\nabla\times\mathbf{G})$ is

$$F_2\left(\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y}\right) - F_3\left(\frac{\partial G_1}{\partial z} - \frac{\partial G_3}{\partial x}\right).$$

The first component of $\mathbf{G}\times(\nabla\times\mathbf{F})$ is

$$G_2\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) - G_3\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right).$$

The first component of $(\mathbf{F}\bullet\nabla)\mathbf{G}$ is

$$F_1\frac{\partial G_1}{\partial x} + F_2\frac{\partial G_1}{\partial y} + F_3\frac{\partial G_1}{\partial z}.$$

The first component of $(\mathbf{G}\bullet\nabla)\mathbf{F}$ is

$$G_1\frac{\partial F_1}{\partial x} + G_2\frac{\partial F_1}{\partial y} + G_3\frac{\partial F_1}{\partial z}.$$

When we add these four first components, eight of the fourteen terms cancel out and the six remaining terms are the six terms of the first component of $\nabla(\mathbf{F}\bullet\mathbf{G})$, as calculated above. Similar calculations show that the second and third components of both sides of the identity agree. Thus

$$\nabla(\mathbf{F}\bullet\mathbf{G}) = \mathbf{F}\times(\nabla\times\mathbf{G}) + \mathbf{G}\times(\nabla\times\mathbf{F}) + (\mathbf{F}\bullet\nabla)\mathbf{G} + (\mathbf{G}\bullet\nabla)\mathbf{F}.$$

5. Theorem 3(h). By equality of mixed partials,

$$\begin{aligned}\nabla\times\nabla\phi &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z}\end{vmatrix}\\ &= \left(\frac{\partial}{\partial y}\frac{\partial\phi}{\partial z} - \frac{\partial}{\partial z}\frac{\partial\phi}{\partial y}\right)\mathbf{i} + \cdots = \mathbf{0}.\end{aligned}$$

6. Theorem 3(i). We examine the first components of the terms on both sides of the identity

$$\nabla\times(\nabla\times\mathbf{F}) = \nabla(\nabla\bullet\mathbf{F}) - \nabla^2\mathbf{F}.$$

The first component of $\nabla \times (\nabla \times \mathbf{F})$ is

$$\frac{\partial}{\partial y}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)$$
$$= \frac{\partial^2 F_2}{\partial y \partial x} - \frac{\partial^2 F_1}{\partial y^2} - \frac{\partial^2 F_1}{\partial z^2} + \frac{\partial^2 F_3}{\partial z \partial x}.$$

The first component of $\nabla(\nabla \bullet \mathbf{F})$ is

$$\frac{\partial}{\partial x}\nabla \bullet \mathbf{F} = \frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_2}{\partial x \partial y} + \frac{\partial^2 F_3}{\partial x \partial z}.$$

The first component of $-\nabla^2 \mathbf{F}$ is

$$-\nabla^2 F_1 = -\frac{\partial^2 F_1}{\partial x^2} - \frac{\partial^2 F_1}{\partial y^2} - \frac{\partial^2 F_1}{\partial z^2}.$$

Evidently the first components of both sides of the given identity agree. By symmetry, so do the other components.

7. If the field lines of $\mathbf{F}(x, y, z)$ are parallel straight lines, in the direction of the constant nonzero vector $\mathbf{a}$ say, then

$$\mathbf{F}(x, y, z) = \phi(x, y, z)\mathbf{a}$$

for some scalar field ϕ, which we assume to be smooth. By Theorem 3(b) and (c) we have

$$\mathbf{div}\,\mathbf{F} = \mathbf{div}\,(\phi\mathbf{a}) = \nabla\phi \bullet \mathbf{a}$$
$$\mathbf{curl}\,\mathbf{F} = \mathbf{curl}\,(\phi\mathbf{a}) = \nabla\phi \times \mathbf{a}.$$

Since $\nabla\phi$ is an arbitrary gradient, $\mathbf{div}\,\mathbf{F}$ can have any value, but $\mathbf{curl}\,\mathbf{F}$ is perpendicular to $\mathbf{a}$, and thereforе to $\mathbf{F}$.

8. If $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $r = |\mathbf{r}|$, then

$$\nabla \bullet \mathbf{r} = 3, \qquad \nabla \times \mathbf{r} = \mathbf{0}, \qquad \nabla r = \frac{\mathbf{r}}{r}.$$

If $\mathbf{c}$ is a constant vector, then its divergence and curl are both zero. By Theorem 3(d), (e), and (f) we have

$$\nabla \bullet (\mathbf{c} \times \mathbf{r}) = (\nabla \times \mathbf{c}) \bullet \mathbf{r} - \mathbf{c} \bullet (\nabla \times \mathbf{r}) = \mathbf{0}$$
$$\nabla \times (\mathbf{c} \times \mathbf{r}) = (\nabla \bullet \mathbf{r})\mathbf{c} + (\mathbf{r} \bullet \nabla)\mathbf{c} - (\nabla \bullet \mathbf{c})\mathbf{r} - (\mathbf{c} \bullet \nabla)\mathbf{r}$$
$$= 3\mathbf{c} + \mathbf{0} - \mathbf{0} - \mathbf{c} = 2\mathbf{c}$$

$$\nabla(\mathbf{c} \bullet \mathbf{r}) = \mathbf{c} \times (\nabla \times \mathbf{r}) + \mathbf{r} \times (\nabla \times \mathbf{c}) + (\mathbf{c} \bullet \nabla)\mathbf{r} + (\mathbf{r} \bullet \nabla)\mathbf{c}$$
$$= \mathbf{0} + \mathbf{0} + \mathbf{c} + \mathbf{0} = \mathbf{c}.$$

9. $$\nabla \bullet \Big(f(r)\mathbf{r}\Big) = \Big(\nabla f(r)\Big) \bullet \mathbf{r} + f(r)(\nabla \bullet \mathbf{r})$$
$$= f'(r)\frac{\mathbf{r} \bullet \mathbf{r}}{r} + 3f(r)$$
$$= rf'(r) + 3f(r).$$

If $f(r)\mathbf{r}$ is solenoidal then $\nabla \bullet \Big(f(r)\mathbf{r}\Big) = 0$, so that $u = f(r)$ satisfies

$$r\,\frac{du}{dr} + 3u = 0$$
$$\frac{du}{u} = -\frac{3\,dr}{r}$$
$$\ln|u| = -3\ln|r| + \ln|C|$$
$$u = Cr^{-3}.$$

Thus $f(r) = Cr^{-3}$, for some constant C.

10. Given that $\mathbf{div}\,\mathbf{F} = 0$ and $\mathbf{curl}\,\mathbf{F} = \mathbf{0}$, Theorem 3(i) implies that $\nabla^2\mathbf{F} = 0$ too. Hence the components of $\mathbf{F}$ are harmonic functions.
If $\mathbf{F} = \nabla\phi$, then

$$\nabla^2\phi = \nabla \bullet \nabla\phi = \nabla \bullet \mathbf{F} = 0,$$

so ϕ is also harmonic.

11. By Theorem 3(e) and 3(f),

$$\nabla \times (\mathbf{F} \times \mathbf{r}) = (\nabla \bullet \mathbf{r})\mathbf{F} + (\mathbf{r} \bullet \nabla)\mathbf{F} - (\nabla \bullet \mathbf{F})\mathbf{r} - (\mathbf{F} \bullet \nabla)\mathbf{r}$$
$$\nabla(\mathbf{F} \bullet \mathbf{r}) = \mathbf{F} \times (\nabla \times \mathbf{r}) + \mathbf{r} \times (\nabla \times \mathbf{F})$$
$$+ (\mathbf{F} \bullet \nabla)\mathbf{r} + (\mathbf{r} \bullet \nabla)\mathbf{F}.$$

If $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, then $\nabla \bullet \mathbf{r} = 3$ and $\nabla \times \mathbf{r} = \mathbf{0}$. Also,

$$(\mathbf{F} \bullet \nabla)\mathbf{r} = F_1\frac{\partial \mathbf{r}}{\partial x} + F_2\frac{\partial \mathbf{r}}{\partial y} + F_3\frac{\partial \mathbf{r}}{\partial z} = \mathbf{F}.$$

Combining all these results, we obtain

$$\nabla \times (\mathbf{F} \times \mathbf{r}) - \nabla(\mathbf{F} \bullet \mathbf{r}) = 3\mathbf{F} - 2(\mathbf{F} \bullet \nabla)\mathbf{r}$$
$$- (\nabla \bullet \mathbf{F})\mathbf{r} - \mathbf{r} \times (\nabla \times \mathbf{F})$$
$$= \mathbf{F} - (\nabla \bullet \mathbf{F})\mathbf{r} - \mathbf{r} \times (\nabla \times \mathbf{F}).$$

In particular, if $\nabla \bullet \mathbf{F} = 0$ and $\nabla \times \mathbf{F} = \mathbf{0}$, then

$$\nabla \times (\mathbf{F} \times \mathbf{r}) - \nabla(\mathbf{F} \bullet \mathbf{r}) = \mathbf{F}.$$

12. If $\nabla^2\phi = 0$ and $\nabla^2\psi = 0$, then

$$\nabla \bullet (\phi\nabla\psi - \psi\nabla\phi)$$
$$= \nabla\phi \bullet \nabla\psi + \phi\nabla^2\psi - \nabla\psi \bullet \nabla\phi - \psi\nabla^2\phi = 0,$$

so $\phi\nabla\psi - \psi\nabla\phi$ is solenoidal.

13. By Theorem 3(c) and (h),

$$\nabla \times (\phi\nabla\psi) = \nabla\phi \times \nabla\psi + \phi\nabla \times \nabla\psi = \nabla\phi \times \nabla\psi$$
$$-\nabla \times (\psi\nabla\phi) = -\nabla\psi \times \nabla\phi - \psi\nabla \times \nabla\phi = \nabla\phi \times \nabla\psi.$$

14. By Theorem 3(b), (d), and (h), we have

$$\begin{aligned}&\nabla \bullet \Big(f(\nabla g \times \nabla h)\Big)\\ &= \nabla f \bullet (\nabla g \times \nabla h) + f\nabla \bullet (\nabla g \times \nabla h)\\ &= \nabla f \bullet (\nabla g \times \nabla h) + f\Big((\nabla \times \nabla g) \bullet \nabla h - \nabla g \bullet (\nabla \times \nabla h)\Big)\\ &= \nabla f \bullet (\nabla g \times \nabla h) + \mathbf{0} - \mathbf{0} = \nabla f \bullet (\nabla g \times \nabla h).\end{aligned}$$

15. If $\mathbf{F} = \nabla\phi$ and $\mathbf{G} = \nabla\psi$, then $\nabla \times \mathbf{F} = \mathbf{0}$ and $\nabla \times \mathbf{G} = \mathbf{0}$ by Theorem 3(h). Therefore, by Theorem 3(d) we have

$$\nabla \bullet (\mathbf{F} \times \mathbf{G}) = (\nabla \times \mathbf{F}) \bullet \mathbf{G} + \mathbf{F} \bullet (\nabla \times \mathbf{G}) = \mathbf{0}.$$

Thus $\mathbf{F} \times \mathbf{G}$ is solenoidal. By Exercise 13,

$$\nabla \times (\phi\nabla\psi) = \nabla\phi \times \nabla\psi = \mathbf{F} \times \mathbf{G},$$

so $\phi\nabla\psi$ is a vector potential for $\mathbf{F} \times \mathbf{G}$. (So is $-\psi\nabla\phi$.)

16. If $\nabla \times \mathbf{G} = \mathbf{F} = -y\mathbf{i} + x\mathbf{j}$, then

$$\begin{aligned}\frac{\partial G_3}{\partial y} - \frac{\partial G_2}{\partial z} &= -y\\ \frac{\partial G_1}{\partial z} - \frac{\partial G_3}{\partial x} &= x\\ \frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y} &= 0.\end{aligned}$$

As in Example 1, we try to find a solution with $G_2 = 0$. Then

$$G_3 = -\int y\,dy = -\frac{y^2}{2} + M(x, z).$$

Again we try $M(x, z) = 0$, so $G_3 = -\dfrac{y^2}{2}$. Thus $\dfrac{\partial G_3}{\partial x} = 0$ and

$$G_1 = \int x\,dz = xz + N(x, y).$$

Since $\dfrac{\partial G_1}{\partial y} = 0$ we may take $N(x, y) = 0$.
$\mathbf{G} = xz\mathbf{i} - \dfrac{1}{2}y^2\mathbf{k}$ is a vector potential for $\mathbf{F}$. (Of course, this answer is not unique.)

17. If $\mathbf{F} = xe^{2z}\mathbf{i} + ye^{2z}\mathbf{j} - e^{2z}\mathbf{k}$, then

$$\mathbf{div}\,\mathbf{F} = e^{2z} + e^{2z} - 2e^{2z} = 0,$$

so $\mathbf{F}$ is solenoidal.
If $\mathbf{F} = \nabla \times \mathbf{G}$, then

$$\begin{aligned}\frac{\partial G_3}{\partial y} - \frac{\partial G_2}{\partial z} &= xe^{2z}\\ \frac{\partial G_1}{\partial z} - \frac{\partial G_3}{\partial x} &= ye^{2z}\\ \frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y} &= -e^{2z}.\end{aligned}$$

Look for a solution with $G_2 = 0$. We have

$$G_3 = \int xe^{2z}\,dy = xye^{2z} + M(x, z).$$

Try $M(x, z) = 0$. Then $G_3 = xye^{2z}$, and

$$\frac{\partial G_1}{\partial z} = ye^{2z} + \frac{\partial G_3}{\partial x} = 2ye^{2z}.$$

Thus

$$G_1 = \int 2ye^{2z}\,dz = ye^{2z} + N(x, y).$$

Since

$$-e^{2z} = -\frac{\partial G_1}{\partial y} = -e^{2z} - \frac{\partial N}{\partial y},$$

we can take $N(x, y) = 0$.
Thus $\mathbf{G} = ye^{2z}\mathbf{i} + xye^{2z}\mathbf{k}$ is a vector potential for $\mathbf{F}$.

18. For (x, y, z) in D let $\mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. The line segment $\mathbf{r}(t) = t\mathbf{v}$, $(0 \le t \le 1)$, lies in D, so $\mathbf{div}\,\mathbf{F} = 0$ on the path. We have

$$\begin{aligned}\mathbf{G}(x, y, z) &= \int_0^1 t\mathbf{F}\Big(\mathbf{r}(t)\Big) \times \mathbf{v}\,dt\\ &= \int_0^1 t\mathbf{F}\Big(\xi(t), \eta(t), \zeta(t)\Big) \times \mathbf{v}\,dt\end{aligned}$$

where $\xi = tx$, $\eta = ty$, $\zeta = tz$. The first component of $\mathbf{curl}\,\mathbf{G}$ is

$$\begin{aligned}&(\mathbf{curl}\,\mathbf{G})_1\\ &= \int_0^1 t\Big(\mathbf{curl}\,(\mathbf{F} \times \mathbf{v})\Big)_1\,dt\\ &= \int_0^1 t\left(\frac{\partial}{\partial y}(\mathbf{F} \times \mathbf{v})_3 - \frac{\partial}{\partial z}(\mathbf{F} \times \mathbf{v})_2\right)dt\\ &= \int_0^1 t\left(\frac{\partial}{\partial y}(F_1 y - F_2 x) - \frac{\partial}{\partial z}(F_3 x - F_1 z)\right)dt\\ &= \int_0^1 \Big(tF_1 + t^2 y\frac{\partial F_1}{\partial \eta} - t^2 x\frac{\partial F_2}{\partial \eta} - t^2 x\frac{\partial F_3}{\partial \zeta}\\ &\qquad + tF_1 + t^2 z\frac{\partial F_1}{\partial \zeta}\Big)\,dt\\ &= \int_0^1 \left(2tF_1 + t^2 x\frac{\partial F_1}{\partial \xi} + t^2 y\frac{\partial F_1}{\partial \eta} + t^2 z\frac{\partial F_1}{\partial \zeta}\right)dt.\end{aligned}$$

To get the last line we used the fact that $div\mathbf{F} = 0$ to replace $-t^2x\dfrac{\partial F_2}{\partial \eta} - t^2x\dfrac{\partial F_3}{\partial \zeta}$ with $t^2x\dfrac{\partial F_1}{\partial \xi}$. Continuing the calculation, we have

$$\begin{aligned}(\mathbf{curl\,G})_1 &= \int_0^1 \frac{d}{dt}\Big(t^2F_1(\xi,\eta,\zeta)\Big)\,dt \\ &= t^2F_1(tx,ty,tz)\Big|_0^1 = F_1(x,y,z).\end{aligned}$$

Similarly, $(\mathbf{curl\,G})_2 = F_2$ and $(\mathbf{curl\,G})_3 = F_3$. Thus $\mathbf{curl\,G} = \mathbf{F}$, as required.

19. In the following we suppress output (which for some calculations can be quite lengthy) except for the final check on each inequality. You may wish to use semi-colons instead of colons to see what the output actually looks like.

```
> with(VectorCalculus):

>
SetCoordinates('cartesian'[x,y,z]):

> F := VectorField
(<u(x,y,z),v(x,y,z),w(x,y,z)>):

> G := VectorField
(<a(x,y,z),b(x,y,z),c(x,y,z)>):
```

(a)
```
LHS := Del(phi(x,y,z)*psi(x,y,z)):
RHS := phi(x,y,z)*Del(psi(x,y,z))
+ psi(x,y,z)*Del(phi(x,y,z)):
simplify(LHS - RHS);
```

$$0\,\bar{e}_x$$

(b)
```
LHS := Del . (F*phi(x,y,z)):
RHS := (Del(phi(x,y,z))).F +
phi(x,y,z)*(Del.F):
simplify(LHS - RHS);
```

$$0$$

(c)
```
LHS := Del &x (phi(x,y,z)*F):
RHS := RHS := (Del(phi(x,y,z))) &x
F + phi(x,y,z)*(Del &x F):
simplify(LHS - RHS);
```

$$0\,\bar{e}_x$$

(d)
```
LHS := Del . (F &x G):
RHS := (Del &x F) . G - F . (Del &x
G):
simplify(LHS - RHS);
```

$$0$$

(e)
```
LHS := Del &x (F &x G):
RHS1 := (Del . G)*F:
RHS2 := G[1]*diff(F,x)
+G[2]*diff(F,y)+G[3]*diff(F,z):
RHS3 := (Del . F)*G:
RHS4 := F[1]*diff(G,x)
+F[2]*diff(G,y)+F[3]*diff(G,z):
RHS := RHS1 + RHS2 - RHS3 - RHS4:
simplify(LHS - RHS);
```

$$0\,\bar{e}_x$$

(f)
```
LHS := Del(F . G):
RHS1 := F &x (Del &x G):
RHS2 := G &x (Del &x F):
RHS3 := F[1]*diff(G,x)
+F[2]*diff(G,y)+F[3]*diff(G,z):
RHS4 := G[1]*diff(F,x)
+G[2]*diff(F,y)+G[3]*diff(F,z):
RHS := RHS1 + RHS2 + RHS3 + RHS4:
simplify(LHS - RHS);
```

$$0\,\bar{e}_x$$

All these zero outputs indicate that the inequalities (a)–(f) of the theorem are valid.

Section 16.3 Green's Theorem in the Plane (page 868)

1. $$\begin{aligned}&\oint_{\mathcal{C}} (\sin x + 3y^2)\,dx + (2x - e^{-y^2})\,dy \\ &= \iint_R \left[\frac{\partial}{\partial x}(2x - e^{-y^2}) - \frac{\partial}{\partial y}(\sin x + 3y^2)\right] dA \\ &= \iint_R (2-6y)\,dA \\ &= \int_0^\pi d\theta \int_0^a (2 - 6r\sin\theta)r\,dr \\ &= \pi a^2 - 6\int_0^\pi \sin\theta\,d\theta \int_0^a r^2\,dr \\ &= \pi a^2 - 4a^3.\end{aligned}$$

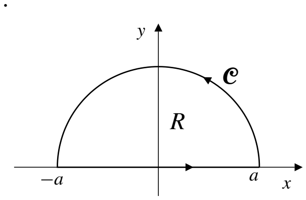

Fig. 16.3.1

2. $\oint_{\mathcal{C}} (x^2 - xy)\,dx + (xy - y^2)\,dy$

$$= -\iint_T \left[\frac{\partial}{\partial x}(xy - y^2) - \frac{\partial}{\partial y}(x^2 - xy)\right] dA$$
$$= -\iint_T (y + x)\,dA$$
$$= -(\bar{y} + \bar{x}) \times (\text{area of } T) = -\left(\frac{1}{3} + 1\right) \times 1 = -\frac{4}{3}.$$

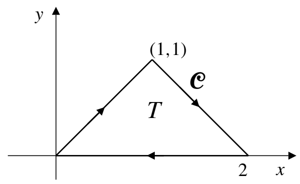

Fig. 16.3.2

3. $\oint_{\mathcal{C}} (x \sin y^2 - y^2)\,dx + (x^2 y \cos y^2 + 3x)\,dy$

$$= \iint_T \left[2xy \cos y^2 + 3 - (2xy \cos y^2 - 2y)\right] dA$$
$$= \iint_T (3 + 2y)\,dA = 3 \iint_T dA + 0 = 3 \times 3 = 9.$$

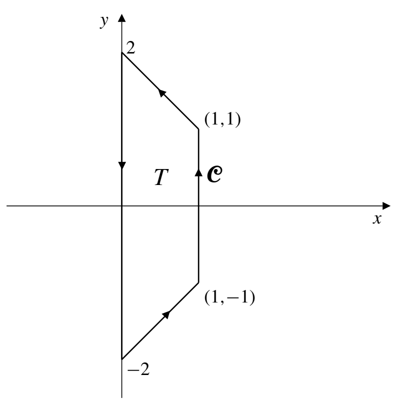

Fig. 16.3.3

4. Let D be the region $x^2 + y^2 \le 9$, $y \ge 0$. Since $\mathcal{C}$ is the clockwise boundary of D,

$$\oint_{\mathcal{C}} x^2 y\,dx - xy^2\,dy$$
$$= -\iint_D \left[\frac{\partial}{\partial x}(-xy^2) - \frac{\partial}{\partial y}(x^2 y)\right] dx\,dy$$
$$= \iint_D (y^2 + x^2)\,dA = \int_0^{\pi} d\theta \int_0^3 r^3\,dr = \frac{81\pi}{4}.$$

5. By Example 1,

$$\begin{aligned}
\text{Area } &= \frac{1}{2}\oint_{\mathcal{C}} x\,dy - y\,dx \\
&= \frac{1}{2}\int_0^{2\pi} \Big[a \cos^3 t\, 3b \sin^2 t\, \cos t \\
&\qquad - b \sin^3 t(-3a \cos^2 t\, \sin t)\Big]\, dt \\
&= \frac{3ab}{2}\int_0^{2\pi} \sin^2 t\, \cos^2 t\, dt \\
&= \frac{3ab}{2}\int_0^{2\pi} \frac{\sin^2(2t)}{4}\, dt = \frac{3\pi ab}{8}.
\end{aligned}$$

6. Let R, $\mathcal{C}$, and $\mathbf{F}$ be as in the statement of Green's Theorem. As noted in the proof of Theorem 7, the unit tangent $\hat{\mathbf{T}}$ to $\mathcal{C}$ and the unit exterior normal $\hat{\mathbf{N}}$ satisfy $\hat{\mathbf{N}} = \hat{\mathbf{T}} \times \mathbf{k}$. Let

$$\mathbf{G} = F_2(x, y)\mathbf{i} - F_1(x, y)\mathbf{j}.$$

Then $\mathbf{F} \bullet \hat{\mathbf{T}} = \mathbf{G} \bullet \hat{\mathbf{N}}$. Applying the 2-dimensional Divergence Theorem to $\mathbf{G}$, we obtain

$$\begin{aligned}
\int_{\mathcal{C}} F_1\,dx + F_2\,dy &= \int_{\mathcal{C}} \mathbf{F} \bullet \hat{\mathbf{T}}\,ds = \int_{\mathcal{C}} \mathbf{G} \bullet \hat{\mathbf{N}}\,ds \\
&= \iint_R \mathbf{div}\,\mathbf{G}\,dA \\
&= \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA
\end{aligned}$$

as required

7. $\mathbf{r} = \sin t\mathbf{i} + \sin 2t\mathbf{j}, \qquad (0 \le t \le 2\pi)$

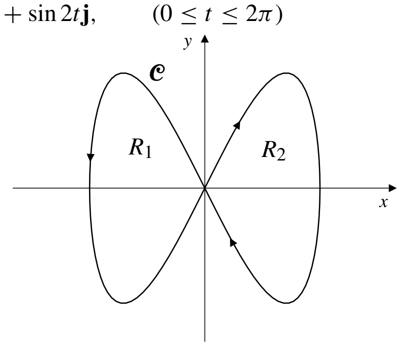

Fig. 16.3.7

$$\mathbf{F} = ye^{x^2}\mathbf{i} + x^3 e^y \mathbf{j}$$

$$\mathbf{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ye^{x^2} & x^3 e^y & 0 \end{vmatrix} = (3x^2 e^y - e^{x^2})\mathbf{k}.$$

Observe that $\mathcal{C}$ bounds two congruent regions, R_1 and R_2, one counterclockwise and the other clockwise. For R_1, $\hat{\mathbf{N}} = \mathbf{k}$; for R_2, $\hat{\mathbf{N}} = -\mathbf{k}$. Since R_1 and R_2 are mirror images of each other in the y-axis, and since $\mathbf{curl}\,\mathbf{F}$ is an even function of x, we have

$$\iint_{R_1} \mathbf{curl}\,\mathbf{F} \bullet \hat{\mathbf{N}}\,dS = -\iint_{R_2} \mathbf{curl}\,\mathbf{F} \bullet \hat{\mathbf{N}}\,dS.$$

Thus

$$\oint_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{r} = \left(\iint_{R_1} + \iint_{R_2} \right) \mathbf{curl}\, \mathbf{F} \bullet \hat{\mathbf{N}}\, dS = 0.$$

8. a) $\mathbf{F} = x^2 \mathbf{j}$

$$\oint_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{r} = \oint_{\mathcal{C}} x^2\, dy = \iint_R 2x\, dA = 2A\bar{x}.$$

b) $\mathbf{F} = xy\mathbf{i}$

$$\oint_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{r} = \oint_{\mathcal{C}} xy\, dx = -\iint_R x\, dA = -A\bar{x}.$$

c) $\mathbf{F} = y^2\mathbf{i} + 3xy\mathbf{j}$

$$\oint_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{r} = \oint_{\mathcal{C}} y^2\, dx + 3xy\, dy$$
$$= \iint_R (3y - 2y)\, dA = A\bar{y}.$$

9. The circle $\mathcal{C}_r$ of radius r and centre at $\mathbf{r}_0$ has parametrization

$$\mathbf{r} = \mathbf{r}_0 + r\cos t\,\mathbf{i} + r\sin t\,\mathbf{j}, \qquad (0 \le t \le 2\pi).$$

Note that $d\mathbf{r}/dt = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} = \hat{\mathbf{N}}$, the unit normal to $\mathcal{C}_r$ exterior to the disk D_r of which $\mathcal{C}_r$ is the boundary. The average value of $u(x, y)$ on $\mathcal{C}_r$ is

$$\bar{u}_r = \frac{1}{2\pi} \int_0^{2\pi} u(x_0 + r\cos t, y_0 + r\sin t)\, dt,$$

and so

$$\frac{d\bar{u}_r}{dr} = \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{\partial u}{\partial x} \cos t + \frac{\partial u}{\partial y} \sin t \right) dt$$
$$= \frac{1}{2\pi r} \oint_{\mathcal{C}_r} \nabla u \bullet \hat{\mathbf{N}}\, ds$$

since $ds = r\, dt$. By the (2-dimensional) divergence theorem, and since u is harmonic,

$$\frac{d\bar{u}_r}{dr} = \frac{1}{2\pi r} \iint_{D_r} \nabla \bullet \nabla u\, dx\, dy$$
$$= \frac{1}{2\pi r} \iint_{D_r} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) dx\, dy = 0.$$

Thus $\bar{u}_r = \lim_{r \to 0} \bar{u}_r = u(x_0, y_0)$.

Section 16.4 The Divergence Theorem in 3-Space (page 873)

1. In this exercise, the sphere $\mathcal{S}$ bounds the ball B of radius a centred at the origin.
If $\mathbf{F} = x\mathbf{i} - 2y\mathbf{j} + 4z\mathbf{k}$, then $\mathbf{div}\, \mathbf{F} = 1 - 2 + 4 = 3$. Thus

$$\oiint_{\mathcal{S}} \mathbf{F} \bullet \hat{\mathbf{N}}\, dS = \iiint_B 3\, dV = 4\pi a^3.$$

2. If $\mathbf{F} = ye^z\mathbf{i} + x^2e^z\mathbf{j} + xy\mathbf{k}$, then $\mathbf{div}\, \mathbf{F} = 0$, and

$$\oiint_{\mathcal{S}} \mathbf{F} \bullet \hat{\mathbf{N}}\, dS = \iiint_B 0\, dV = 0.$$

3. If $\mathbf{F} = (x^2 + y^2)\mathbf{i} + (y^2 - z^2)\mathbf{j} + z\mathbf{k}$, then $\mathbf{div}\, \mathbf{F} = 2x + 2y + 1$, and

$$\oiint_{\mathcal{S}} \mathbf{F} \bullet \hat{\mathbf{N}}\, dS = \iiint_B (2x+2y+1)\, dV = \iiint_B 1\, dV = \frac{4}{3}\pi a^3.$$

4. If $\mathbf{F} = x^3\mathbf{i} + 3yz^2\mathbf{j} + (3y^2z + x^2)\mathbf{k}$, then $\mathbf{div}\, \mathbf{F} = 3x^2 + 3z^2 + 3y^2$, and

$$\oiint_{\mathcal{S}} \mathbf{F} \bullet \hat{\mathbf{N}}\, dS = 3 \iiint_B (x^2 + y^2 + z^2)\, dV$$
$$= 3 \int_0^{2\pi} d\theta \int_0^{\pi} \sin\phi\, d\phi \int_0^a \rho^4\, d\rho$$
$$= \frac{12}{5}\pi a^5.$$

5. If $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$, then $\mathbf{div}\, \mathbf{F} = 2(x + y + z)$. Therefore the flux of $\mathbf{F}$ out of any solid region R is

$$\text{Flux} = \iiint_R \mathbf{div}\, \mathbf{F}\, dV$$
$$= 2 \iiint_R (x + y + z)\, dV = 2(\bar{x} + \bar{y} + \bar{z})V$$

where $(\bar{x}, \bar{y}, \bar{z})$ is the centroid of R and V is the volume of R.

If R is the ball $(x - 2)^2 + y^2 + (z - 3)^2 \le 9$, then $\bar{x} = 2$, $\bar{y} = 0$, $\bar{z} = 3$, and $V = (4\pi/3)3^3 = 36\pi$. The flux of $\mathbf{F}$ out of R is $2(2 + 0 + 3)(36\pi) = 360\pi$.

6. If $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$, then $\mathbf{div}\, \mathbf{F} = 2(x + y + z)$. Therefore the flux of $\mathbf{F}$ out of any solid region R is

$$\text{Flux} = \iiint_R \mathbf{div}\, \mathbf{F}\, dV$$
$$= 2 \iiint_R (x + y + z)\, dV = 2(\bar{x} + \bar{y} + \bar{z})V$$

where $(\bar{x}, \bar{y}, \bar{z})$ is the centroid of R and V is the volume of R.

If R is the ellipsoid $x^2 + y^2 + 4(z - 1)^2 \le 4$, then $\bar{x} = 0$, $\bar{y} = 0$, $\bar{z} = 1$, and $V = (4\pi/3)(2)(2)(1) = 16\pi/3$. The flux of $\mathbf{F}$ out of R is $2(0 + 0 + 1)(16\pi/3) = 32\pi/3$.

7. If $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$, then $\mathbf{div}\,\mathbf{F} = 2(x + y + z)$. Therefore the flux of $\mathbf{F}$ out of any solid region R is

$$\begin{aligned}\text{Flux} &= \iiint_R \mathbf{div}\,\mathbf{F}\,dV\\ &= 2\iiint_R (x+y+z)\,dV = 2(\bar{x}+\bar{y}+\bar{z})V\end{aligned}$$

where $(\bar{x}, \bar{y}, \bar{z})$ is the centroid of R and V is the volume of R.

If R is the tetrahedron with vertices $(3, 0, 0)$, $(0, 3, 0)$, $(0, 0, 3)$, and $(0, 0, 0)$, then $\bar{x} = \bar{y} = \bar{z} = 3/4$, and $V = (1/6)(3)(3)(3) = 9/2$. The flux of $\mathbf{F}$ out of R is $2((3/4) + (3/4) + (3/4))(9/2) = 81/4$.

8. If $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$, then $\mathbf{div}\,\mathbf{F} = 2(x + y + z)$. Therefore the flux of $\mathbf{F}$ out of any solid region R is

$$\begin{aligned}\text{Flux} &= \iiint_R \mathbf{div}\,\mathbf{F}\,dV\\ &= 2\iiint_R (x+y+z)\,dV = 2(\bar{x}+\bar{y}+\bar{z})V\end{aligned}$$

where $(\bar{x}, \bar{y}, \bar{z})$ is the centroid of R and V is the volume of R.

If R is the cylinder $x^2 + y^2 \le 2y$ (or, equivalently, $x^2 + (y-1)^2 \le 1$), $0 \le z \le 4$, then $\bar{x} = 0$, $\bar{y} = 1$, $\bar{z} = 2$, and $V = (\pi 1^2)(4) = 4\pi$. The flux of $\mathbf{F}$ out of R is $2(0 + 1 + 2)(4\pi) = 24\pi$.

9. If $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, then $\mathbf{div}\,\mathbf{F} = 3$. If C is any solid region having volume V, then

$$\iiint_C \mathbf{div}\,\mathbf{F}\,dV = 3V.$$

The region C described in the statement of the problem is the part of a solid cone with vertex at the origin that lies inside a ball of radius R with centre at the origin. The surface $\mathcal{S}$ of C consists of two parts, the conical wall $\mathcal{S}_1$, and the region D on the spherical boundary of the ball. At any point P on $\mathcal{S}_1$, the outward normal field $\hat{\mathbf{N}}$ is perpendicular to the line OP, that is, to $\mathbf{F}$, so $\mathbf{F} \bullet \hat{\mathbf{N}} = 0$. At any point P on D, $\hat{\mathbf{N}}$ is parallel to $\mathbf{F}$, in fact $\hat{\mathbf{N}} = \mathbf{F}/|\mathbf{F}| = \mathbf{F}/R$. Thus

$$\begin{aligned}\oiint_{\mathcal{S}} \mathbf{F} \bullet \hat{\mathbf{N}}\,dS &= \iint_{\mathcal{S}_1} \mathbf{F} \bullet \hat{\mathbf{N}}\,dS + \iint_D \mathbf{F} \bullet \hat{\mathbf{N}}\,dS\\ &= 0 + \iint_D \frac{\mathbf{F} \bullet \mathbf{F}}{R}\,dS = \frac{R^2}{R}\iint_D dS = AR\end{aligned}$$

where A is the area of D. By the Divergence Theorem, $3V = AR$, so $V = AR/3$.

10. The required surface integral,

$$I = \iint_{\mathcal{S}} \nabla\phi \bullet \hat{\mathbf{N}}\,dS,$$

can be calculated directly by the methods of Section 6.6. We will do it here by using the Divergence Theorem instead. $\mathcal{S}$ is one face of a tetrahedral domain D whose other faces are in the coordinate planes, as shown in the figure. Since $\phi = xy + z^2$, we have

$$\nabla\phi = y\mathbf{i} + x\mathbf{j} + 2z\mathbf{k}, \qquad \nabla \bullet \nabla\phi = \nabla^2\phi = 2.$$

Thus

$$\iiint_D \nabla \bullet \nabla\phi\,dV = 2 \times \frac{abc}{6} = \frac{abc}{3},$$

the volume of the tetrahedron D being $abc/6$ cubic units.

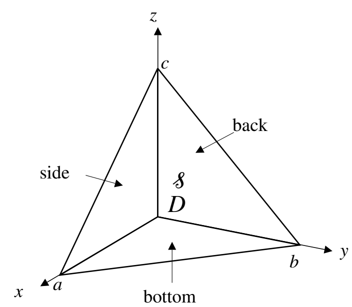

Fig. 16.4.10

The flux of $\nabla\phi$ out of D is the sum of its fluxes out of the four faces of the tetrahedron.

On the bottom, $\hat{\mathbf{N}} = -\mathbf{k}$ and $z = 0$, so $\nabla\phi \bullet \hat{\mathbf{N}} = 0$, and the flux out of the bottom face is 0.

On the side, $y = 0$ and $\hat{\mathbf{N}} = -\mathbf{j}$, so $\nabla\phi \bullet \hat{\mathbf{N}} = -x$. The flux out of the side face is

$$\iint_{\text{side}} \nabla\phi \bullet \hat{\mathbf{N}}\,dS = -\iint_{\text{side}} x\,dx\,dz = -\frac{ac}{2} \times \frac{a}{3} = -\frac{a^2c}{6}.$$

(We used the fact that $M_{x=0} = \text{area} \times \bar{x}$ and $\bar{x} = a/3$ for that face.)

On the back face, $x = 0$ and $\hat{\mathbf{N}} = -\mathbf{i}$, so the flux out of that face is

$$\iint_{\text{back}} \nabla\phi \bullet \hat{\mathbf{N}}\,dS = -\iint_{\text{back}} y\,dy\,dz = -\frac{bc}{2} \times \frac{b}{3} = -\frac{b^2c}{6}.$$

Therefore, by the Divergence Theorem

$$I - \frac{a^2c}{6} - \frac{b^2c}{6} + 0 = \frac{abc}{3},$$

so $\displaystyle\iint_{\mathcal{S}} \nabla\phi \bullet \hat{\mathbf{N}}\,dS = I = \frac{abc}{3} + \frac{c(a^2+b^2)}{6}.$

11. $\mathbf{F} = (x + y^2)\mathbf{i} + (3x^2y + y^3 - x^3)\mathbf{j} + (z + 1)\mathbf{k}$
$\mathbf{div}\,\mathbf{F} = 1 + 3(x^2 + y^2) + 1 = 2 + 3(x^2 + y^2)$.

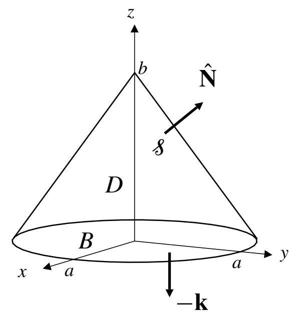

Fig. 16.4.11

Let D be the conical domain, $\mathcal{S}$ its conical surface, and B its base disk, as shown in the figure. We have

$$\begin{aligned}\iiint_D \mathbf{div}\,\mathbf{F}\,dV &= \int_0^{2\pi} d\theta \int_0^a r\,dr \int_0^{b(1-(r/a))} (2+3r^2)\,dz \\ &= 2\pi b \int_0^a r(2+3r^2)\left(1-\frac{r}{a}\right) dr \\ &= 2\pi b \int_0^a \left(2r+3r^3-\frac{2r^2}{a}-\frac{3r^4}{a}\right) dr \\ &= \frac{2\pi a^2 b}{3}+\frac{3\pi a^4 b}{10}.\end{aligned}$$

On B we have $z=0$, $\hat{\mathbf{N}}=-\mathbf{k}$, $\mathbf{F}\bullet\hat{\mathbf{N}}=-1$, so

$$\iint_B \mathbf{F}\bullet\hat{\mathbf{N}}\,dS = -\text{area of } B = -\pi a^2.$$

By the Divergence Theorem,

$$\iint_{\mathcal{S}} \mathbf{F}\bullet\hat{\mathbf{N}}\,dS + \iint_B \mathbf{F}\bullet\hat{\mathbf{N}}\,dS = \iiint_D \mathbf{div}\,\mathbf{F}\,dV,$$

so the flux of $\mathbf{F}$ upward through the conical surface $\mathcal{S}$ is

$$\iint_{\mathcal{S}} = \frac{2\pi a^2 b}{3}+\frac{3\pi a^4 b}{10}+\pi a^2.$$

12. $\mathbf{F}=(y+xz)\mathbf{i}+(y+yz)\mathbf{j}-(2x+z^2)\mathbf{k}$
$\mathbf{div}\,\mathbf{F}=z+(1+z)-2z=1$. Thus

$$\iiint_D \mathbf{div}\,\mathbf{F}\,dV = \text{volume of } D = \frac{\pi a^3}{6},$$

where D is the region in the first octant bounded by the sphere and the coordinate planes. The boundary of D consists of the spherical part $\mathcal{S}$ and the four planar parts, called the bottom, side, and back in the figure.

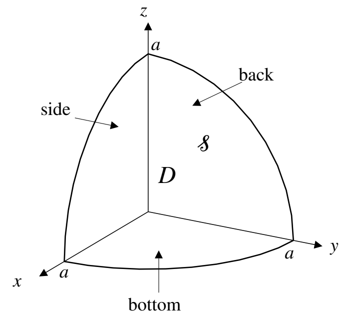

Fig. 16.4.12

On the side, $y=0$, $\hat{\mathbf{N}}=-\mathbf{j}$, $\mathbf{F}\bullet\hat{\mathbf{N}}=0$, so

$$\iint_{\text{side}} \mathbf{F}\bullet\hat{\mathbf{N}}\,dS = 0.$$

On the back, $x=0$, $\hat{\mathbf{N}}=-\mathbf{i}$, $\mathbf{F}\bullet\hat{\mathbf{N}}=-y$, so

$$\begin{aligned}\iint_{\text{back}} \mathbf{F}\bullet\hat{\mathbf{N}}\,dS &= -\int_0^{\pi/2} d\theta \int_0^a r\cos\theta\, r\,dr \\ &= -\sin\theta\Big|_0^{\pi/2} \times \frac{a^3}{3} = -\frac{a^3}{3}.\end{aligned}$$

On the bottom, $z=0$, $\hat{\mathbf{N}}=-\mathbf{k}$, $\mathbf{F}\bullet\hat{\mathbf{N}}=2x$, so

$$\iint_{\text{bottom}} \mathbf{F}\bullet\hat{\mathbf{N}}\,dS = 2\int_0^{\pi/2} d\theta \int_0^a r\cos\theta\, r\,dr = \frac{2a^3}{3}.$$

By the Divergence Theorem

$$\iint_{\mathcal{S}} \mathbf{F}\bullet\hat{\mathbf{N}}\,dS + 0 - \frac{a^3}{3} + \frac{2a^3}{3} = \frac{\pi a^3}{6}.$$

Hence the flux of $\mathbf{F}$ upward through $\mathcal{S}$ is

$$\iint_{\mathcal{S}} \mathbf{F}\bullet\hat{\mathbf{N}}\,dS = \frac{\pi a^3}{6} - \frac{a^3}{3}.$$

13. $\mathbf{F}=(x+yz)\mathbf{i}+(y-xz)\mathbf{j}+(z-e^x\sin y)\mathbf{k}$
$\mathbf{div}\,\mathbf{F}=1+1+1=3$.

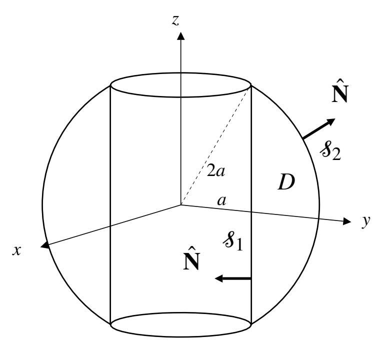

Fig. 16.4.13

a) The flux of $\mathbf{F}$ out of D through $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$ is

$$\begin{aligned}\oiint_{\mathcal{S}} \mathbf{F} \bullet \hat{\mathbf{N}}\, dS &= \iiint_D \mathbf{div}\,\mathbf{F}\, dV \\ &= 3\int_0^{2\pi} d\theta \int_a^{2a} r\, dr \int_0^{\sqrt{4a^2-r^2}} 2\, dz \\ &= 12\pi \int_a^{2a} r\sqrt{4a^2-r^2}\, dr \\ &\qquad \text{Let } u = 4a^2 - r^2 \\ &\qquad du = -2r\, dr \\ &= 6\pi \int_0^{3a^2} u^{1/2}\, du = 12\sqrt{3}\pi a^3.\end{aligned}$$

b) On $\mathcal{S}_1$, $\hat{\mathbf{N}} = -\dfrac{x\mathbf{i} + y\mathbf{j}}{a}$, $dS = a\, d\theta\, dz$. The flux of $\mathbf{F}$ out of D through $\mathcal{S}_1$ is

$$\begin{aligned}\iint_{\mathcal{S}_1} \mathbf{F} \bullet \hat{\mathbf{N}}\, dS &= \iint_{\mathcal{S}_1} \frac{-x^2 - xyz - y^2 + xyz}{a}\, a\, d\theta\, dz \\ &= -a^2 \int_0^{2\pi} d\theta \int_{-\sqrt{3}a}^{\sqrt{3}a} dz = -4\sqrt{3}\pi a^3.\end{aligned}$$

c) The flux of $\mathbf{F}$ out of D through the spherical part $\mathcal{S}_2$ is

$$\begin{aligned}\iint_{\mathcal{S}_2} \mathbf{F} \bullet \hat{\mathbf{N}}\, dS &= \oiint_{\mathcal{S}} \mathbf{F} \bullet \hat{\mathbf{N}}\, dS - \iint_{\mathcal{S}_1} \mathbf{F} \bullet \hat{\mathbf{N}}\, dS \\ &= 12\sqrt{3}\pi a^3 + 4\sqrt{3}\pi a^3 = 16\sqrt{3}\pi a^3.\end{aligned}$$

14. Let D be the domain bounded by $\mathcal{S}$, the coordinate planes, and the plane $x = 1$. If

$$\mathbf{F} = 3xz^2\mathbf{i} - x\mathbf{j} - y\mathbf{k},$$

then $\mathbf{div}\,\mathbf{F} = 3z^2$, so the total flux of $\mathbf{F}$ out of D is

$$\begin{aligned}\oiint_{\text{bdry of } D} \mathbf{F} \bullet \hat{\mathbf{N}}\, dS &= \iiint_D 3z^2\, dV \\ &= 3\int_0^1 dx \int_0^{\pi/2} d\theta \int_0^1 r^2\cos^2\theta\, r\, dr \\ &= 3 \times \frac{1}{4} \times \frac{\pi}{4} = \frac{3\pi}{16}.\end{aligned}$$

The boundary of D consists of the cylindrical surface $\mathcal{S}$ and four planar surfaces, the side, bottom, back, and front.

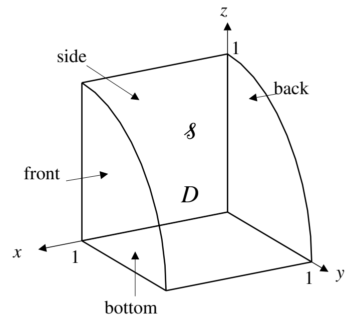

Fig. 16.4.14

On the side, $y = 0$, $\hat{\mathbf{N}} = -\mathbf{j}$, $\mathbf{F} \bullet \hat{\mathbf{N}} = x$, so

$$\iint_{\text{side}} \mathbf{F} \bullet \hat{\mathbf{N}}\, dS = \int_0^1 x\, dx \int_0^1 dz = \frac{1}{2}.$$

On the bottom, $z = 0$, $\hat{\mathbf{N}} = -\mathbf{k}$, $\mathbf{F} \bullet \hat{\mathbf{N}} = y$, so

$$\iint_{\text{bottom}} \mathbf{F} \bullet \hat{\mathbf{N}}\, dS = \int_0^1 y\, dy \int_0^1 dx = \frac{1}{2}.$$

On the back, $x = 0$, $\hat{\mathbf{N}} = -\mathbf{i}$, $\mathbf{F} \bullet \hat{\mathbf{N}} = 0$, so

$$\iint_{\text{back}} \mathbf{F} \bullet \hat{\mathbf{N}}\, dS = 0.$$

On the front, $x = 1$, $\hat{\mathbf{N}} = \mathbf{i}$, $\mathbf{F} \bullet \hat{\mathbf{N}} = 3z^2$, so

$$\iint_{\text{front}} \mathbf{F} \bullet \hat{\mathbf{N}}\, dS = 3\int_0^{\pi/2} d\theta \int_0^1 r^2\cos^2\theta\, r\, dr = \frac{3\pi}{16}.$$

Hence,

$$\iint_{\mathcal{S}} (3xz^2\mathbf{i} - x\mathbf{j} - y\mathbf{k}) \bullet \hat{\mathbf{N}}\, dS = \frac{3\pi}{16} - \frac{1}{2} - \frac{1}{2} - 0 - \frac{3\pi}{16} = -1.$$

15. $\mathbf{F} = (x^2 - x - 2y)\mathbf{i} + (2y^2 + 3y - z)\mathbf{j} - (z^2 - 4z + xy)\mathbf{k}$
$\mathbf{div}\,\mathbf{F} = 2x - 1 + 4y + 3 - 2z + 4 = 2x + 4y - 2z + 6.$

The flux of $\mathbf{F}$ out of R through its surface $\mathcal{S}$ is

$$\oiint_{\mathcal{S}} \mathbf{F} \bullet \hat{\mathbf{N}}\, dS = \iiint_R (2x + 4y - 2z + 6)\, dV.$$

Now $\iiint_R x\, dV = M_{x=0} = V\bar{x}$, where R has volume V and centroid $(\bar{x}, \bar{y}\, \bar{z})$. Similar formulas obtain for the other variables, so the required flux is

$$\oiint_{\mathcal{S}} \mathbf{F} \bullet \hat{\mathbf{N}}\, dS = 2V\bar{x} + 4V\bar{y} - 2V\bar{z} + 6V.$$

16. $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ implies that $\mathbf{div}\,\mathbf{F} = 3$. The total flux of $\mathbf{F}$ out of D is

$$\oiint_{\text{bdry of } D} \mathbf{F} \bullet \hat{\mathbf{N}}\, dS = 3\iiint_D dV = 12,$$

since the volume of D is half that of a cube of side 2, that is, 4 square units.
D has three triangular faces, three pentagonal faces, and a hexagonal face. By symmetry, the flux of $\mathbf{F}$ out of each triangular face is equal to that out of the triangular face T in the plane $z = 1$. Since $\mathbf{F} \bullet \hat{\mathbf{N}} = \mathbf{k} \bullet \mathbf{k} = 1$ on that face, these fluxes are

$$\iint_T dx\,dy = \text{area of } T = \frac{1}{2}.$$

Similarly, the flux of $\mathbf{F}$ out of each pentagonal face is equal to the flux out of the pentagonal face P in the plane $z = -1$, where $\mathbf{F} \bullet \hat{\mathbf{N}} = -\mathbf{k} \bullet (-\mathbf{k}) = 1$; that flux is

$$\iint_P dx\,dy = \text{area of } P = 4 - \frac{1}{2} = \frac{7}{2}.$$

Thus the flux of $\mathbf{F}$ out of the remaining hexagonal face H is

$$12 - 3 \times \left(\frac{1}{2} + \frac{7}{2}\right) = 0.$$

(This can also be seen directly, since $\mathbf{F}$ radiates from the origin, so is everywhere tangent to the plane of the hexagonal face, the plane $x + y + z = 0$.)

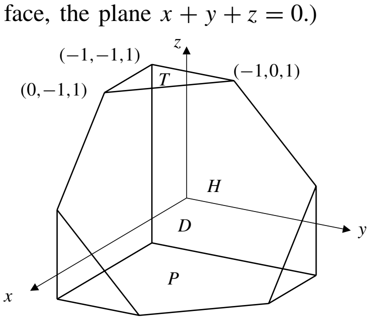

Fig. 16.4.16

17. The part of the sphere $\mathcal{S}$: $x^2 + y^2 + (z - a)^2 = 4a^2$ above $z = 0$ and the disk D: $x^2 + y^2 = 3a^2$ in the xy-plane form the boundary of a region R in 3-space. The outward normal from R on D is $-\mathbf{k}$. If

$$\mathbf{F} = (x^2 + y + 2 + z^2)\mathbf{i} + (e^{x^2} + y^2)\mathbf{j} + (3 + x)\mathbf{k},$$

then $div\mathbf{F} = 2x + 2y$. By the Divergence Theorem,

$$\iint_{\mathcal{S}} \mathbf{F} \bullet \hat{\mathbf{N}}\,dS + \iint_D \mathbf{F} \bullet (-\mathbf{k})\,dx\,dy = \iiint_R \mathbf{div}\,\mathbf{F}\,dV = 0$$

because R is symmetric about $x = 0$ and $y = 0$. Thus the flux of $\mathbf{F}$ outward across $\mathcal{S}$ is

$$\iint_{\mathcal{S}} \mathbf{F} \bullet \hat{\mathbf{N}}\,dS = \iint_D (3 + x)\,dx\,dy = 3\pi(3a^2) = 9\pi a^2.$$

18. $\phi = x^2 - y^2 + z^2$, $\mathbf{G} = \frac{1}{3}(-y^3\mathbf{i} + x^3\mathbf{j} + z^3\mathbf{k})$.
$\mathbf{F} = \nabla\phi + \mu\,\mathbf{curl}\,\mathbf{G}$.

Let R be the region of 3-space occupied by the sandpile. Then R is bounded by the upper surface $\mathcal{S}$ of the sandpile and by the disk D: $x^2 + y^2 \le 1$ in the plane $z = 0$. The outward (from R) normal on D is $-\mathbf{k}$. The flux of $\mathbf{F}$ out of R is given by

$$\iint_{\mathcal{S}} \mathbf{F} \bullet \hat{\mathbf{N}}\,dS + \iint_D \mathbf{F} \bullet (-\mathbf{k})\,dA = \iiint_R \mathbf{div}\,\mathbf{F}\,dV.$$

Now $\mathbf{div}\,\mathbf{curl}\,\mathbf{G} = 0$ by Theorem 3(g). Also
$\mathbf{div}\,\nabla\phi = \mathbf{div}\,(2x\mathbf{i} - 2y\mathbf{j} + 2z\mathbf{k}) = 2 - 2 + 2 = 2$. Therefore

$$\iiint_R \mathbf{div}\,\mathbf{F}\,dV = \iiint_R (2 + \mu \times 0)\,dV = 2(5\pi) = 10\pi.$$

In addition,

$$\mathbf{curl}\,\mathbf{G} = \frac{1}{3}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^3 & x^3 & z^3 \end{vmatrix} = 3(x^2 + y^2)\mathbf{k},$$

and $\nabla\phi \bullet \mathbf{k} = 2z = 0$ on D, so

$$\iint_D \mathbf{F} \bullet \mathbf{k}\,dA = 3\mu \int_0^{2\pi} d\theta \int_0^1 r^3\,dr = \frac{3\pi\mu}{2}.$$

The flux of $\mathbf{F}$ out of $\mathcal{S}$ is $10\pi + (3\pi\mu)/2$.

19. $\oiint_{\mathcal{S}} \mathbf{curl}\,\mathbf{F} \bullet \hat{\mathbf{N}}\,dS = \iiint_D \mathbf{div}\,\mathbf{curl}\,\mathbf{F} = 0$, by Theorem 3(g).

20. If $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, then $\mathbf{div}\,\mathbf{r} = 3$ and

$$\frac{1}{3}\oiint_{\mathcal{S}} \mathbf{r} \bullet \hat{\mathbf{N}}\,dS = \frac{1}{3}\iiint_D 3\,dV = V.$$

21. We use Theorem 7(b), the proof of which is given in Exercise 29. Taking $\phi(x, y, z) = x^2 + y^2 + z^2$, we have

$$\begin{aligned}\frac{1}{2V}\oiint_{\mathcal{S}} (x^2 + y^2 + z^2)\hat{\mathbf{N}}\,dS &= \frac{1}{2V}\oiint_{\mathcal{S}} \phi\hat{\mathbf{N}}\,dS \\ &= \frac{1}{2V}\iiint_D \mathbf{grad}\,\phi\,dV \\ &= \frac{1}{V}\iiint (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})\,dV \\ &= \bar{\mathbf{r}},\end{aligned}$$

since $\iint x\,dV = M_{x=0} = V\bar{x}$.

22. Taking $\mathbf{F} = \nabla\phi$ in the first identity in Theorem 7(a), we have

$$\oiint_{\mathcal{S}} \nabla\phi \times \hat{\mathbf{N}}\, dS = -\iiint_D \mathbf{curl}\, \nabla\phi\, dV = 0,$$

since $\nabla \times \nabla\phi = 0$ by Theorem 3(h).

23. $\mathbf{div}\,(\phi\mathbf{F}) = \phi\mathbf{div}\,\mathbf{F} + \nabla\phi \bullet \mathbf{F}$ by Theorem 3(b). Thus

$$\begin{aligned}\iiint_D \phi\mathbf{div}\,\mathbf{F}\, dV + \iiint_D \nabla\phi \bullet \mathbf{F}\, dV &= \iiint_D \mathbf{div}\,(\phi\mathbf{F})\, dV\\ &= \oiint_{\mathcal{S}} \phi\mathbf{F} \bullet \hat{\mathbf{N}}\, dS\end{aligned}$$

by the Divergence Theorem.

24. If $\mathbf{F} = \nabla\phi$ in the previous exercise, then $\mathbf{div}\,\mathbf{F} = \nabla^2\phi$ and

$$\iiint_D \phi\nabla^2\phi\, dV + \iiint_D |\nabla\phi|^2\, dV = \oiint_{\mathcal{S}} \phi\nabla\phi \bullet \hat{\mathbf{N}}\, dS.$$

If $\nabla^2\phi = 0$ in D and $\phi = 0$ on $\mathcal{S}$, then

$$\iiint_D |\nabla\phi|^2\, dV = 0.$$

Since ϕ is assumed to be smooth, $\nabla\phi = 0$ throughout D, and therefore ϕ is constant on each connected component of D. Since $\phi = 0$ on $\mathcal{S}$, these constants must all be 0, and $\phi = 0$ on D.

25. If u and v are two solutions of the given Dirichlet problem, and $\phi = u - v$, then

$$\begin{aligned}\nabla^2\phi &= \nabla^2 u - \nabla^2 v = f - f = 0 \text{ on } D\\ \phi &= u - v = g - g = 0 \text{ on } \mathcal{S}.\end{aligned}$$

By the previous exercise, $\phi = 0$ on D, so $u = v$ on D. That is, solutions of the Dirichlet problem are unique.

26. Re-examine the solution to Exercise 24 above. If $\nabla^2\phi = 0$ in D and $\partial\phi/\partial n = \nabla\phi \bullet \hat{\mathbf{N}} = 0$ on $\mathcal{S}$, then we can again conclude that

$$\iiint_D |\nabla\phi|\, dV = 0$$

and $\nabla\phi = 0$ throughout D. Thus ϕ is constant on the connected components of D. (We can't conclude the constant is 0 because we don't know the value of ϕ on $\mathcal{S}$.) If u and v are solutions of the given Neumann problem, then $\phi = u - v$ satisfies

$$\begin{aligned}\nabla^2\phi &= \nabla^2 u - \nabla^2 v = f - f = 0 \text{ on } D\\ \frac{\partial\phi}{\partial n} &= \frac{\partial u}{\partial n} - \frac{\partial v}{\partial n} = g - g = 0 \text{ on } \mathcal{S},\end{aligned}$$

so ϕ is constant on any connected component of $\mathcal{S}$, and u and v can only differ by a constant on $\mathcal{S}$.

27. Apply the Divergence Theorem to $\mathbf{F} = \nabla\phi$:

$$\begin{aligned}\iiint_D \nabla^2\phi\, dV &= \iiint_D \nabla \bullet \nabla\phi\, dV\\ &= \oiint_{\mathcal{S}} \nabla\phi \bullet \hat{\mathbf{N}}\, dS = \oiint_{\mathcal{S}} \frac{\partial\phi}{\partial n}\, dS.\end{aligned}$$

28. By Theorem 3(b),

$$\begin{aligned}&\mathbf{div}\,(\phi\nabla\psi - \psi\nabla\phi)\\ &\quad = \nabla\phi \bullet \nabla\psi + \phi\nabla^2\psi - \nabla\psi \bullet \nabla\phi - \psi\nabla^2\phi\\ &\quad = \phi\nabla^2\psi - \psi\nabla^2\phi.\end{aligned}$$

Hence, by the Divergence Theorem,

$$\begin{aligned}\iiint_D (\phi\nabla^2\psi - \psi\nabla^2\phi)\, dV &= \iiint_D \mathbf{div}\,(\phi\nabla\psi - \psi\nabla\phi)\, dV\\ &= \oiint_{\mathcal{S}} (\phi\nabla\psi - \psi\nabla\phi) \bullet \hat{\mathbf{N}}\, dS\\ &= \oiint_{\mathcal{S}} \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right) dS.\end{aligned}$$

29. If $\mathbf{F} = \phi\mathbf{c}$, where $\mathbf{c}$ is an arbitrary, constant vector, then $\mathbf{div}\,\mathbf{F} = \nabla\phi \bullet \mathbf{c}$, and by the Divergence Theorem,

$$\begin{aligned}\mathbf{c} \bullet \iiint_D \nabla\phi\, dV &= \iiint_D \mathbf{div}\,\mathbf{F}\, dV\\ &= \oiint_{\mathcal{S}} \mathbf{F} \bullet \hat{\mathbf{N}}\, dS\\ &= \oiint_{\mathcal{S}} \phi\mathbf{c} \bullet \hat{\mathbf{N}}\, dS = \mathbf{c} \bullet \oiint_{\mathcal{S}} \phi\hat{\mathbf{N}}\, dS.\end{aligned}$$

Thus

$$\mathbf{c} \bullet \left(\iiint_D \nabla\phi\, dV - \oiint_{\mathcal{S}} \phi\hat{\mathbf{N}}\, dS\right) = 0.$$

Since $\mathbf{c}$ is arbitrary, the vector in the large parentheses must be the zero vector. Hence

$$\iiint_D \nabla\phi\, dV = \oiint_{\mathcal{S}} \phi\hat{\mathbf{N}}\, dS.$$

30. $$\frac{1}{\text{vol}(D_\epsilon)}\oiint_{\mathcal{S}_\epsilon}\mathbf{F}\bullet\hat{\mathbf{N}}\,dS = \frac{1}{\text{vol}(D_\epsilon)}\iiint_{D_\epsilon}\mathbf{div}\,\mathbf{F}\,dV$$
$$= \frac{1}{\text{vol}(D_\epsilon)}\left[\iiint_{D_\epsilon}\mathbf{div}\,\mathbf{F}(P_0)\,dV + \iiint_{D_\epsilon}\Big(\mathbf{div}\,\mathbf{F} - \mathbf{div}\,\mathbf{F}(P_0)\Big)\,dV\right]$$
$$= \mathbf{div}\,\mathbf{F}(P_0) + \frac{1}{\text{vol}(D_\epsilon)}\iiint_{D_\epsilon}\Big(\mathbf{div}\,\mathbf{F} - \mathbf{div}\,\mathbf{F}(P_0)\Big)\,dV.$$

Thus

$$\left|\frac{1}{\text{vol}(D_\epsilon)}\oiint_{\mathcal{S}_\epsilon}\mathbf{F}\bullet\hat{\mathbf{N}}\,dS - \mathbf{div}\,\mathbf{F}(P_0)\right|$$
$$\le \frac{1}{\text{vol}(D_\epsilon)}\iiint_{D_\epsilon}|\mathbf{div}\,\mathbf{F} - \mathbf{div}\,\mathbf{F}(P_0)|\,dV$$
$$\le \max_{P \text{ in } D_\epsilon}|\mathbf{div}\,\mathbf{F} - \mathbf{div}\,\mathbf{F}(P_0)|$$
$$\to 0 \text{ as } \epsilon \to 0+ \text{ assuming } \mathbf{div}\,\mathbf{F} \text{ is continuous.}$$
$$\lim_{\epsilon\to 0+}\frac{1}{\text{vol}(D_\epsilon)}\oiint_{\mathcal{S}_\epsilon}\mathbf{F}\bullet\hat{\mathbf{N}}\,dS = \mathbf{div}\,\mathbf{F}(P_0).$$

Section 16.5 Stokes's Theorem (page 878)

1. The triangle T lies in the plane $x + y + z = 1$. We use the downward normal

$$\hat{\mathbf{N}} = -\frac{\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{3}}$$

on T, because of the given orientation of its boundary. If $\mathbf{F} = xy\mathbf{i} + yz\mathbf{j} + zx\mathbf{k}$, then

$$\mathbf{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & yz & zx \end{vmatrix} = -y\mathbf{i} - z\mathbf{j} - x\mathbf{k}.$$

Therefore

$$\oint_{\mathcal{C}} xy\,dx + yz\,dz + zx\,dz = \oint_{\mathcal{C}}\mathbf{F}\bullet d\mathbf{r}$$
$$= \iint_T \mathbf{curl}\,\mathbf{F}\bullet\hat{\mathbf{N}}\,dS = \iint_T \frac{y+z+x}{\sqrt{3}}\,dS$$
$$= \frac{1}{\sqrt{3}}\iint_T dS = \frac{1}{\sqrt{3}}\times(\text{area of } T)$$
$$= \frac{1}{\sqrt{3}}\times\left(\frac{1}{2}\times\sqrt{2}\times\frac{\sqrt{3}}{\sqrt{2}}\right) = \frac{1}{2}.$$

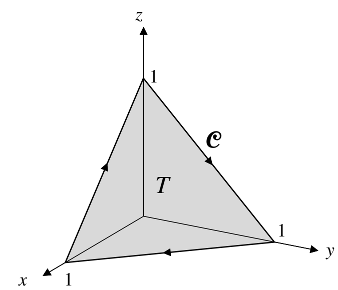

Fig. 16.5.1

2. Let $\mathcal{S}$ be the part of the surface $z = y^2$ lying inside the cylinder $x^2 + y^2 = 4$, and having upward normal $\hat{\mathbf{N}}$. Then $\mathcal{C}$ is the oriented boundary of $\mathcal{S}$. Let D be the disk $x^2 + y^2 \le 4$, $z = 0$, that is, the projection of $\mathcal{S}$ onto the xy-plane.

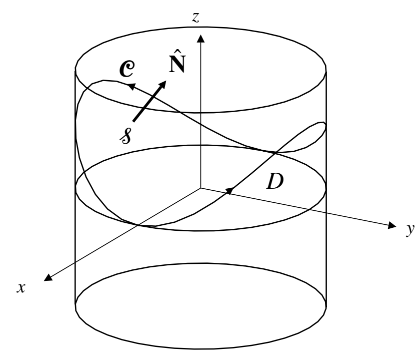

Fig. 16.5.2

If $\mathbf{F} = y\mathbf{i} - x\mathbf{j} + z^2\mathbf{k}$, then

$$\mathbf{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & z^2 \end{vmatrix} = -2\mathbf{k}.$$

Since $dS = \dfrac{dx\,dy}{\mathbf{k}\bullet\hat{\mathbf{N}}}$ on $\mathcal{S}$, we have

$$\oint_{\mathcal{C}} y\,dx - x\,dy + z^2\,dz = \oint_{\mathcal{C}}\mathbf{F}\bullet d\mathbf{r} = \iint_{\mathcal{S}}\mathbf{curl}\,\mathbf{F}\bullet\hat{\mathbf{N}}\,dS$$
$$= \iint_D -2\mathbf{k}\bullet\hat{\mathbf{N}}\,\frac{dx\,dy}{\mathbf{k}\bullet\hat{\mathbf{N}}} = -8\pi.$$

3. Let $\mathcal{C}$ be the circle $x^2 + y^2 = a^2$, $z = 0$, oriented counterclockwise as seen from the positive z-axis. Let D be the disk bounded by $\mathcal{C}$, with normal $\mathbf{k}$. We have

$$\mathbf{F} = 3y\mathbf{i} - 2xz\mathbf{j} + (x^2 - y^2)\mathbf{k}$$
$$\mathbf{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3y & -2xz & x^2 - y^2 \end{vmatrix}$$
$$= 2(x - y)\mathbf{i} - 2x\mathbf{j} - (2z + 3)\mathbf{k}.$$

Applying Stokes's Theorem (twice) we calculate

$$\begin{aligned}\iint_{\mathcal{S}} &= \oint_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{r} = \iint_D \mathbf{curl\,F} \bullet \mathbf{k}\, dA \\ &= -\iint_D 3\, dA = -3\pi a^2.\end{aligned}$$

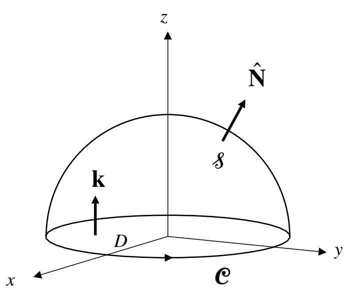

Fig. 16.5.3

4. The surface $\mathcal{S}$ with equation

$$x^2 + y^2 + 2(z-1)^2 = 6, \qquad z \ge 0,$$

with outward normal $\hat{\mathbf{N}}$, is that part of an ellipsoid of revolution about the z-axis, centred at $(0,0,1)$, and lying above the xy-plane. The boundary of $\mathcal{S}$ is the circle $\mathcal{C}$: $x^2+y^2=4$, $z=0$, oriented counterclockwise as seen from the positive z-axis. $\mathcal{C}$ is also the oriented boundary of the disk $x^2+y^2 \le 4$, $z=0$, with normal $\hat{\mathbf{N}} = \mathbf{k}$.
If $\mathbf{F} = (xz - y^3\cos z)\mathbf{i} + x^3e^z\mathbf{j} + xyze^{x^2+y^2+z^2}\mathbf{k}$, then, on $z=0$, we have

$$\begin{aligned}\mathbf{curl\,F} \bullet \mathbf{k} &= \left(\frac{\partial}{\partial x}x^3e^z - \frac{\partial}{\partial y}(xz - y^3\cos z)\right)\bigg|_{z=0} \\ &= \left(3x^2e^z + 3y^2\cos z\right)\bigg|_{z=0} = 3(x^2+y^2).\end{aligned}$$

Thus

$$\begin{aligned}\iint_{\mathcal{S}} \mathbf{curl\,F} \bullet \hat{\mathbf{N}}\, dS &= \oint_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{r} = \iint_D \mathbf{curl\,F} \bullet \mathbf{k}\, dA \\ &= \int_0^{2\pi} d\theta \int_0^2 3r^2\, r\, dr = 24\pi.\end{aligned}$$

5. The circle $\mathcal{C}$ of intersection of $x^2+y^2+z^2=a^2$ and $x+y+z=0$ is the boundary of a circular disk of radius a in the plane $x+y+z=0$.
If $\mathbf{F} = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$, then

$$\mathbf{curl\,F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = -(\mathbf{i}+\mathbf{j}+\mathbf{k}).$$

If $\mathcal{C}$ is oriented so that D has normal

$$\hat{\mathbf{N}} = -\frac{\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{3}},$$

then $\mathbf{curl\,F} \bullet \hat{\mathbf{N}} = \sqrt{3}$ on D, so

$$\begin{aligned}\oint_{\mathcal{C}} y\,dx + z\,dy + x\,dz &= \oint_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{r} = \iint_D \mathbf{curl\,F} \bullet \hat{\mathbf{N}}\, dS \\ &= \sqrt{3}\iint_D dS = \sqrt{3}\pi a^2,\end{aligned}$$

since D has area πa^2.

6. The curve $\mathcal{C}$:

$$\mathbf{r} = \cos t\mathbf{i} + \sin t\mathbf{j} + \sin 2t\mathbf{k}, \qquad 0 \le t \le 2\pi,$$

lies on the surface $z = 2xy$, since $\sin 2t = 2\cos t \sin t$. It also lies on the cylinder $x^2+y^2=1$, so it is the boundary of that part of $z=2xy$ lying inside that cylinder. Since $\mathcal{C}$ is oriented counterclockwise as seen from high on the z-axis, $\mathcal{S}$ should be oriented with upward normal,

$$\hat{\mathbf{N}} = \frac{-2y\mathbf{i} - 2x\mathbf{j} + \mathbf{k}}{\sqrt{1+4(x^2+y^2)}},$$

and has area element

$$dS = \sqrt{1+4(x^2+y^2)}\, dx\, dy.$$

If $\mathbf{F} = (e^x - y^3)\mathbf{i} + (e^y + x^3)\mathbf{j} + e^z\mathbf{k}$, then

$$\mathbf{curl\,F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x - y^3 & e^y + x^3 & e^z \end{vmatrix} = 3(x^2+y^2)\mathbf{k}.$$

If D is the disk $x^2+y^2 \le 1$ in the xy-plane, then

$$\begin{aligned}\oint_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{r} &= \iint_{\mathcal{S}} \mathbf{curl\,F} \bullet \hat{\mathbf{N}}\, dS = \iint_D 3(x^2+y^2)\, dx\, dy \\ &= 3\int_0^{2\pi} d\theta \int_0^1 r^2\, r\, dr = \frac{3\pi}{2}.\end{aligned}$$

7. The part of the paraboloid $z = 9 - x^2 - y^2$ lying above the xy-plane having upward normal $\hat{\mathbf{N}}$ has boundary the circle $\mathcal{C}$: $x^2+y^2=9$, oriented counterclockwise as seen from above. $\mathcal{C}$ is also the oriented boundary of the plane disk $x^2+y^2 \le 9$, $z=0$, oriented with normal field $\hat{\mathbf{N}} = \mathbf{k}$.

If $\mathbf{F} = -y\mathbf{i} + x^2\mathbf{j} + z\mathbf{k}$, then

$$\mathbf{curl\,F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x^2 & z \end{vmatrix} = (2x+1)\mathbf{k}.$$

By Stokes's Theorem, the circulation of $\mathbf{F}$ around $\mathcal{C}$ is

$$\oint_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{r} = \iint_D (\mathbf{curl}\,\mathbf{F} \bullet \mathbf{k})\, dA$$
$$= \iint_D (2x+1)\, dA = 0 + \pi(3^2) = 9\pi.$$

8. The closed curve

$$\mathbf{r} = (1+\cos t)\mathbf{i} + (1+\sin t)\mathbf{j} + (1-\cos t - \sin t)\mathbf{k},$$

$(0 \le t \le 2\pi)$, lies in the plane $x+y+z=3$ and is oriented counterclockwise as seen from above. Therefore it is the boundary of a region $\mathcal{S}$ in that plane with normal field $\hat{\mathbf{N}} = (\mathbf{i}+\mathbf{j}+\mathbf{k})/\sqrt{3}$. The projection of $\mathcal{S}$ onto the xy-plane is the circular disk D of radius 1 with centre at $(1,1)$.

If $\mathbf{F} = ye^x\mathbf{i} + (x^2+e^x)\mathbf{j} + z^2e^z\mathbf{k}$, then

$$\mathbf{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ ye^x & x^2+e^x & z^2+e^z \end{vmatrix} = 2x\mathbf{k}.$$

By Stokes's Theorem,

$$\oint_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{r} = \iint_{\mathcal{S}} \mathbf{curl}\,\mathbf{F} \bullet \hat{\mathbf{N}}\, dS$$
$$= \iint_{\mathcal{S}} \frac{2x}{\sqrt{3}}\, dS = \iint_D \frac{2x}{\sqrt{3}}(\sqrt{3})\, dx\, dy$$
$$= 2\bar{x}A = 2\pi,$$

where $\bar{x} = 1$ is the x-coordinate of the centre of D, and $A = \pi 1^2 = \pi$ is the area of D.

9. If $\mathcal{S}_1$ and $\mathcal{S}_2$ are two surfaces joining $\mathcal{C}_1$ to $\mathcal{C}_2$, each having upward normal, then the closed surface $\mathcal{S}_3$ consisting of $\mathcal{S}_1$ and $-\mathcal{S}_2$ (that is, $\mathcal{S}_2$ with downward normal) bound a region R in 3-space. Then

$$\iint_{\mathcal{S}_1} \mathbf{F} \bullet \hat{\mathbf{N}}\, dS - \iint_{\mathcal{S}_2} \mathbf{F} \bullet \hat{\mathbf{N}}\, dS$$
$$= \iint_{\mathcal{S}_1} \mathbf{F} \bullet \hat{\mathbf{N}}\, dS + \iint_{-\mathcal{S}_2} \mathbf{F} \bullet \hat{\mathbf{N}}\, dS$$
$$= \oiint_{\mathcal{S}_3} \mathbf{F} \bullet \hat{\mathbf{N}}\, dS = \pm \iiint_R \mathbf{div}\,\mathbf{F}\, dV = 0,$$

provided that $\mathbf{div}\,\mathbf{F} = 0$ identically. Since

$$\mathbf{F} = (\alpha x^2 - z)\mathbf{i} + (xy + y^3 + z)\mathbf{j} + \beta y^2(z+1)\mathbf{k},$$

we have $\mathbf{div}\,\mathbf{F} = 2\alpha x + x + 3y^2 + \beta y^2 = 0$ if $\alpha = -1/2$ and $\beta = -3$. In this case we can evaluate $\iint_{\mathcal{S}} \mathbf{F} \bullet \hat{\mathbf{N}}\, dS$ for any such surface $\mathcal{S}$ by evaluating the special case where S is the half-disk H: $x^2+y^2 \le 1$, $z=0$, $y \ge 0$, with upward normal $\hat{\mathbf{N}} = \mathbf{k}$. We have

$$\iint_{\mathcal{S}} \mathbf{F} \bullet \hat{\mathbf{N}}\, dS = -3 \iint_H y^2\, dx\, dy$$
$$= -3 \int_0^{\pi} \sin^2\theta\, d\theta \int_0^1 r^3\, dr = -\frac{3\pi}{8}.$$

10. The curve $\mathcal{C}$: $(x-1)^2 + 4y^2 = 16$, $2x+y+z=3$, oriented counterclockwise as seen from above, bounds an elliptic disk $\mathcal{S}$ on the plane $2x+y+z=3$. $\mathcal{S}$ has normal $\hat{\mathbf{N}} = (2\mathbf{i}+\mathbf{j}+\mathbf{k})/\sqrt{6}$. Since its projection onto the xy-plane is an elliptic disk with centre at $(1,0,0)$ and area $\pi(4)(2) = 8\pi$, therefore $\mathcal{S}$ has area $8\sqrt{6}\pi$ and centroid $(1,0,1)$. If

$$\mathbf{F} = (z^2+y^2+\sin x^2)\mathbf{i} + (2xy+z)\mathbf{j} + (xz+2yz)\mathbf{k},$$

then

$$\mathbf{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ z^2+y^2+\sin x^2 & 2xy+z & xz+2yz \end{vmatrix}$$
$$= (2z-1)\mathbf{i} + z\mathbf{j}.$$

By Stokes's Theorem,

$$\oint_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{r} = \iint_{\mathcal{S}} \mathbf{curl}\,\mathbf{F} \bullet \hat{\mathbf{N}}\, dS$$
$$= \frac{1}{\sqrt{6}} \iint_{\mathcal{S}} (2(2z-1)+z)\, dS$$
$$= \frac{5\bar{z}-2}{\sqrt{6}}(8\sqrt{6}\pi) = 24\pi.$$

11. As was shown in Exercise 13 of Section 7.2,

$$\nabla \times (\phi\nabla\psi) = -\nabla \times (\psi \times \phi) = \nabla\phi \times \nabla\psi.$$

Thus, by Stokes's Theorem,

$$\oint_{\mathcal{C}} \phi\nabla\psi = \iint_{\mathcal{S}} \nabla \times (\phi\nabla\psi) \bullet \hat{\mathbf{N}}\, dS$$
$$= \iint_{\mathcal{S}} (\nabla\phi \times \nabla\psi) \bullet \hat{\mathbf{N}}\, dS$$
$$-\oint_{\mathcal{C}} \psi\nabla\phi = \iint_{\mathcal{S}} -\nabla \times (\psi\nabla\phi) \bullet \hat{\mathbf{N}}\, dS$$
$$= \iint_{\mathcal{S}} (\nabla\phi \times \nabla\psi) \bullet \hat{\mathbf{N}}\, dS.$$

$\nabla\phi \times \nabla\psi$ is solenoidal, with potential $\phi\nabla\psi$, or $-\psi\nabla\phi$.

12. We are given that $\mathcal{C}$ bounds a region R in a plane P with unit normal $\hat{\mathbf{N}} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$. Therefore, $a^2 + b^2 + c^2 = 1$.
If $\mathbf{F} = (bz - cy)\mathbf{i} + (cx - az)\mathbf{j} + (ay - bx)\mathbf{k}$, then

$$\begin{aligned}\mathbf{curl}\,\mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ bz - cy & cx - az & ay - bx \end{vmatrix} \\ &= 2a\mathbf{i} + 2b\mathbf{j} + 2c\mathbf{k}.\end{aligned}$$

Hence $\mathbf{curl}\,\mathbf{F} \bullet \hat{\mathbf{N}} = 2(a^2 + b^2 + c^2) = 2$. We have

$$\begin{aligned}&\frac{1}{2}\oint_{\mathcal{C}} (bz - cy)\,dx + (cx - az)\,dy + (ay - bx)\,dz \\ &\qquad = \frac{1}{2}\oint_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{r} = \frac{1}{2}\iint_R \mathbf{curl}\,\mathbf{F} \bullet \hat{\mathbf{N}}\,dS \\ &\qquad = \frac{1}{2}\iint_R 2\,dS = \text{area of } R.\end{aligned}$$

13. The circle $\mathcal{C}_\epsilon$ of radius ϵ centred at P is the oriented boundary of the disk $\mathcal{S}_\epsilon$ of area $\pi\epsilon^2$ having constant normal field $\hat{\mathbf{N}}$. By Stokes's Theorem,

$$\begin{aligned}\oint_{\mathcal{C}_\epsilon} \mathbf{F} \bullet d\mathbf{r} &= \iint_{\mathcal{S}_\epsilon} \mathbf{curl}\,\mathbf{F} \bullet \hat{\mathbf{N}}\,dS \\ &= \iint_{\mathcal{S}_\epsilon} \mathbf{curl}\,\mathbf{F}(P) \bullet \hat{\mathbf{N}}\,dS \\ &\qquad + \iint_{\mathcal{S}_\epsilon} \Big(\mathbf{curl}\,\mathbf{F} - \mathbf{curl}\,\mathbf{F}(P)\Big) \bullet \hat{\mathbf{N}}\,dS \\ &= \pi\epsilon^2 \mathbf{curl}\,\mathbf{F}(P) \bullet \hat{\mathbf{N}} \\ &\qquad + \iint_{\mathcal{S}_\epsilon} \Big(\mathbf{curl}\,\mathbf{F} - \mathbf{curl}\,\mathbf{F}(P)\Big) \bullet \hat{\mathbf{N}}\,dS.\end{aligned}$$

Since $\mathbf{F}$ is assumed smooth, its curl is continuous at P. Therefore

$$\begin{aligned}&\left|\frac{1}{\pi\epsilon^2}\oint_{\mathcal{C}_\epsilon} \mathbf{F} \bullet d\mathbf{r} - \mathbf{curl}\,\mathbf{F}(P) \bullet \hat{\mathbf{N}}\right| \\ &\le \frac{1}{\pi\epsilon^2}\iint_{\mathcal{S}_\epsilon} \left|\Big(\mathbf{curl}\,\mathbf{F} - \mathbf{curl}\,\mathbf{F}(P)\Big) \bullet \hat{\mathbf{N}}\right|\,dS \\ &\le \max_{Q \text{ on } \mathcal{S}_\epsilon} |\mathbf{curl}\,\mathbf{F}(Q) - \mathbf{curl}\,\mathbf{F}(P)| \\ &\to 0 \text{ as } \epsilon \to 0+.\end{aligned}$$

Thus $\displaystyle\lim_{\epsilon \to 0+} \oint_{\mathcal{C}_\epsilon} \mathbf{F} \bullet d\mathbf{r} = \mathbf{curl}\,\mathbf{F}(P) \bullet \hat{\mathbf{N}}$.

Section 16.6 Some Physical Applications of Vector Calculus (page 885)

1. a) If we measure depth in the liquid by $-z$, so that the z-axis is vertical and $z = 0$ at the surface, then the pressure at depth $-z$ is $p = -\delta gz$, where δ is the density of the liquid. Thus

$$\nabla p = -\delta g\mathbf{k} = \delta\mathbf{g},$$

where $\mathbf{g} = -g\mathbf{k}$ is the constant downward vector acceleration of gravity.
The force of the liquid on surface element dS of the solid with outward (from the solid) normal $\hat{\mathbf{N}}$ is

$$d\mathbf{B} = -p\hat{\mathbf{N}}\,dS = -(-\delta gz)\hat{\mathbf{N}}\,dS = \delta gz\hat{\mathbf{N}}\,dS.$$

Thus, the total force of the liquid on the solid (the buoyant force) is

$$\begin{aligned}\mathbf{B} &= \oiint_{\mathcal{S}} \delta gz\hat{\mathbf{N}}\,dS \\ &= \iiint_R \nabla(\delta gz)\,dV \quad \text{(see Theorem 7)} \\ &= -\iiint_R \delta\mathbf{g}\,dV = -M\mathbf{g},\end{aligned}$$

where $M = \displaystyle\iiint_R \delta\,dV$ is the mass of the liquid which would occupy the same space as the solid. Thus $\mathbf{B} = -\mathbf{F}$, where $\mathbf{F} = M\mathbf{g}$ is the weight of the liquid displaced by the solid.

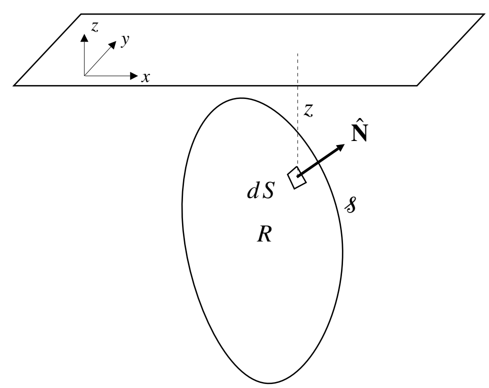

Fig. 16.6.1

b) The above argument extends to the case where the solid is only partly submerged. Let R^* be the part of the region occupied by the solid that is below the surface of the liquid. Let $\mathcal{S}^* = \mathcal{S}_1 \cup \mathcal{S}_2$ be the boundary of R^*, with $\mathcal{S}_1 \subset \mathcal{S}$ and $\mathcal{S}_2$ in the plane of the surface of the liquid. Since $p = -\delta gz = 0$ on $\mathcal{S}_2$, we have

$$\iint_{\mathcal{S}_2} \delta gz\hat{\mathbf{N}}\,dS = 0.$$

Therefore the buoyant force on the solid is

$$\begin{aligned}\mathbf{B} &= \iint_{\mathcal{S}_1} \delta g z \hat{\mathbf{N}}\, dS \\ &= \iint_{\mathcal{S}_1} \delta g z \hat{\mathbf{N}}\, dS + \iint_{\mathcal{S}_2} \delta g z \hat{\mathbf{N}}\, dS \\ &= \oiint_{\mathcal{S}^*} \delta g z \hat{\mathbf{N}}\, dS \\ &= -\iiint_{R^*} \delta \mathbf{g}\, dV = -M^*\mathbf{g},\end{aligned}$$

where $M^* = \iiint_{R^*} \delta\, dV$ is the mass of the liquid which would occupy R^*. Again we conclude that the buoyant force is the negative of the weight of the liquid displaced.

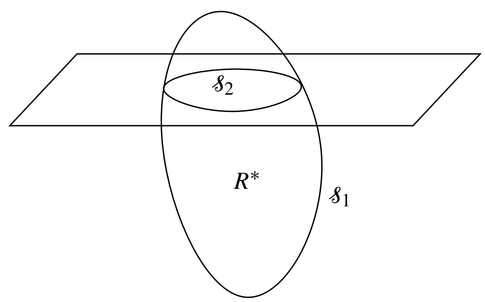

Fig. 16.6.1

2. The first component of $\mathbf{F}(\mathbf{G} \bullet \hat{\mathbf{N}})$ is $(F_1\mathbf{G}) \bullet \hat{\mathbf{N}}$. Applying the Divergence Theorem and Theorem 3(b), we obtain

$$\begin{aligned}\oiint_{\mathcal{S}} (F_1\mathbf{G}) \bullet \hat{\mathbf{N}}\, dS &= \iiint_D \mathbf{div}\,(F_1\mathbf{G})\, dV \\ &= \iiint_D \Big(\nabla F_1 \bullet \mathbf{G} + F_1 \nabla \bullet \mathbf{G}\Big)\, dS.\end{aligned}$$

But $\nabla F_1 \bullet \mathbf{G}$ is the first component of $(\mathbf{G} \bullet \nabla)\mathbf{F}$, and $F_1\nabla \bullet \mathbf{G}$ is the first component of $\mathbf{F}\mathbf{div}\,\mathbf{G}$. Similar results obtain for the other components, so

$$\oiint_{\mathcal{S}} \mathbf{F}(\mathbf{G} \bullet \hat{\mathbf{N}})\, dS = \iiint_D \Big(\mathbf{F}\mathbf{div}\,\mathbf{G} + (\mathbf{G} \bullet \nabla)\mathbf{F}\Big)\, dV.$$

3. Suppose the closed surface $\mathcal{S}$ bounds a region R in which charge is distributed with density ρ. Since the electric field $\mathbf{E}$ due to the charge satisfies $\mathbf{div}\,\mathbf{E} = k\rho$, the total flux of $\mathbf{E}$ out of R through $\mathcal{S}$ is, by the Divergence Theorem,

$$\oiint_{\mathcal{S}} \mathbf{E} \bullet \hat{\mathbf{N}}\, dS = \iiint_R \mathbf{div}\,\mathbf{E}\, dV = k \iiint_R \rho\, dV = kQ,$$

where $Q = \iiint_R \rho\, dV$ is the total charge in R.

4. If f is continuous and vanishes outside a bounded region (say the ball of radius R centred at $\mathbf{r}$), then $|f(\xi, \eta, \zeta)| \le K$, and, if (ρ, ϕ, θ) denote spherical coordinates centred at $\mathbf{r}$, then

$$\begin{aligned}\iiint_{\mathbb{R}^3} \frac{|f(\mathbf{s})|}{|\mathbf{r}-\mathbf{s}|}\, dV_s &\le K \int_0^{2\pi} d\theta \int_0^{\pi} \sin\phi\, d\phi \int_0^R \frac{\rho^2}{\rho}\, d\rho \\ &= 2\pi K R^2 \quad \text{a constant.}\end{aligned}$$

5. This derivation is similar to that of the continuity equation for fluid motion given in the text. If $\mathcal{S}$ is an (imaginary) surface bounding an arbitrary region D, then the rate of change of total charge in D is

$$\frac{\partial}{\partial t} \iiint_D \rho\, dV = \iiint_D \frac{\partial \rho}{\partial t}\, dV,$$

where ρ is the charge density. By conservation of charge, this rate must be equal to the rate at which charge is crossing $\mathcal{S}$ into D, that is, to

$$\oiint_{\mathcal{S}} (-\mathbf{J}) \bullet \hat{\mathbf{N}}\, dS = -\iiint_D \mathbf{div}\,\mathbf{J}\, dV.$$

(The negative sign occurs because $\hat{\mathbf{N}}$ is the outward (from D) normal on $\mathcal{S}$.) Thus we have

$$\iiint_D \left(\frac{\partial \rho}{\partial t} + \mathbf{div}\,\mathbf{J}\right) dV = 0.$$

Since D is arbitrary and we are assuming the integrand is continuous, it must be 0 at every point:

$$\frac{\partial \rho}{\partial t} + \mathbf{div}\,\mathbf{J} = 0.$$

6. Since $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, we have

$$\begin{aligned}|\mathbf{r}-\mathbf{b}|^2 &= (x-b_1)^2 + (y-b_2)^2 + (z-b_3)^2 \\ 2|\mathbf{r}-\mathbf{b}| \frac{\partial}{\partial x}|\mathbf{r}-\mathbf{b}| &= 2(x-b_1) \\ \frac{\partial}{\partial x}|\mathbf{r}-\mathbf{b}| &= \frac{x-b_1}{|\mathbf{r}-\mathbf{b}|}.\end{aligned}$$

Similar formulas hold for the other first partials of $|\mathbf{r}-\mathbf{b}|$, so

$$\begin{aligned}&\nabla\left(\frac{1}{|\mathbf{r}-\mathbf{b}|}\right) \\ &= \frac{-1}{|\mathbf{r}-\mathbf{b}|^2}\left(\frac{\partial}{\partial x}|\mathbf{r}-\mathbf{b}|\mathbf{i} + \cdots + \frac{\partial}{\partial z}|\mathbf{r}-\mathbf{b}|\mathbf{k}\right) \\ &= \frac{-1}{|\mathbf{r}-\mathbf{b}|^2} \frac{(x-b_1)\mathbf{i} + (y-b_2)\mathbf{j} + (z-b_3)\mathbf{k}}{|\mathbf{r}-\mathbf{b}|} \\ &= -\frac{\mathbf{r}-\mathbf{b}}{|\mathbf{r}-\mathbf{b}|^3}.\end{aligned}$$

7. Using the result of Exercise 4 and Theorem 3(d) and (h), we calculate, for constant $\mathbf{a}$,

$$\begin{aligned}&\mathbf{div}\left(\mathbf{a}\times\frac{\mathbf{r}-\mathbf{b}}{|\mathbf{r}-\mathbf{b}|^3}\right)\\&=-\mathbf{div}\left(\mathbf{a}\times\nabla\frac{1}{|\mathbf{r}-\mathbf{b}|}\right)\\&=-(\nabla\times\mathbf{a})\bullet\nabla\frac{1}{|\mathbf{r}-\mathbf{b}|}+\mathbf{a}\bullet\nabla\times\nabla\frac{1}{|\mathbf{r}-\mathbf{b}|}=0+0=0.\end{aligned}$$

8. For any element $d\mathbf{s}$ on the filament $\mathcal{F}$, we have

$$\mathbf{div}\left(d\mathbf{s}\times\frac{\mathbf{r}-\mathbf{s}}{|\mathbf{r}-\mathbf{s}|^3}\right)=0$$

by Exercise 5, since the divergence is taken with respect to $\mathbf{r}$, and so $\mathbf{s}$ and $d\mathbf{s}$ can be regarded as constant. Hence

$$\mathbf{div}\oint_{\mathcal{F}}\frac{d\mathbf{s}\times(\mathbf{r}-\mathbf{s})}{|\mathbf{r}-\mathbf{s}|^3}=\oint_{\mathcal{F}}\mathbf{div}\left(d\mathbf{s}\times\frac{\mathbf{r}-\mathbf{s}}{|\mathbf{r}-\mathbf{s}|^3}\right)=0.$$

9. By the result of Exercise 4 and Theorem 3(e), we calculate

$$\begin{aligned}&\mathbf{curl}\left(\mathbf{a}\times\frac{\mathbf{r}-\mathbf{b}}{|\mathbf{r}-\mathbf{b}|^3}\right)\\&\quad=-\mathbf{curl}\left(\mathbf{a}\times\nabla\frac{1}{|\mathbf{r}-\mathbf{b}|}\right)\\&\quad=-\left(\nabla\bullet\nabla\frac{1}{|\mathbf{r}-\mathbf{b}|}\right)\mathbf{a}-\left(\nabla\frac{1}{|\mathbf{r}-\mathbf{b}|}\bullet\nabla\right)\mathbf{a}\\&\qquad+(\nabla\bullet\mathbf{a})\nabla\frac{1}{|\mathbf{r}-\mathbf{b}|}+(\mathbf{a}\bullet\nabla)\nabla\frac{1}{|\mathbf{r}-\mathbf{b}|}.\end{aligned}$$

Observe that $\nabla\bullet\nabla\frac{1}{|\mathbf{r}-\mathbf{b}|}=0$ for $\mathbf{r}\neq\mathbf{b}$, either by direct calculation or by noting that $\nabla\frac{1}{|\mathbf{r}-\mathbf{b}|}$ is the field of a point source at $\mathbf{r}=\mathbf{b}$ and applying the result of Example 3 of Section 7.1.
Also $-\left(\nabla\frac{1}{|\mathbf{r}-\mathbf{b}|}\bullet\nabla\right)\mathbf{a}=\mathbf{0}$ and $\nabla\bullet\mathbf{a}=0$, since $\mathbf{a}$ is constant. Therefore we have

$$\begin{aligned}\mathbf{curl}\left(\mathbf{a}\times\frac{\mathbf{r}-\mathbf{b}}{|\mathbf{r}-\mathbf{b}|^3}\right)&=(\mathbf{a}\bullet\nabla)\nabla\frac{1}{|\mathbf{r}-\mathbf{b}|}\\&=-(\mathbf{a}\bullet\nabla)\frac{\mathbf{r}-\mathbf{b}}{|\mathbf{r}-\mathbf{b}|^3}.\end{aligned}$$

10. The first component of $(d\mathbf{s}\bullet\nabla)\mathbf{F}(s)$ is $\nabla F_1(s)\bullet d\mathbf{s}$. Since $\mathcal{F}$ is closed and ∇F_1 is conservative,

$$\mathbf{i}\bullet\oint_{\mathcal{F}}(d\mathbf{s}\bullet\nabla)\mathbf{F}(s)=\oint_{\mathcal{F}}\nabla F_1(s)\bullet d\mathbf{s}=0.$$

Similarly, the other components have zero line integrals, so

$$\oint_{\mathcal{F}}(d\mathbf{s}\bullet\nabla)\mathbf{F}(s)=\mathbf{0}.$$

11. Using the results of Exercises 7 and 8, we have

$$\mathbf{curl}\oint_{\mathcal{F}}\frac{d\mathbf{s}\times(\mathbf{r}-\mathbf{s})}{|\mathbf{r}-\mathbf{s}|^3}=\oint_{\mathcal{F}}\mathbf{curl}\left(d\mathbf{s}\times\frac{\mathbf{r}-\mathbf{s}}{|\mathbf{r}-\mathbf{s}|^3}\right)=\mathbf{0}$$

for $\mathbf{r}$ not on $\mathcal{F}$. (Again, this is because the curl is taken with respect to $\mathbf{r}$, so $\mathbf{s}$ and $d\mathbf{s}$ can be regarded as constant for the calculation of the curl.)

12. By analogy with the filament case, the current in volume element dV at position $\mathbf{s}$ is $\mathbf{J}(\mathbf{s})\,dV$, which gives rise at position $\mathbf{r}$ to a magnetic field

$$d\mathbf{H}(\mathbf{r})=\frac{1}{4\pi}\frac{\mathbf{J}(\mathbf{s})\times(\mathbf{r}-\mathbf{s})}{|\mathbf{r}-\mathbf{s}|^3}\,dV.$$

If R is a region of 3-space outside which $\mathbf{J}$ is identically zero, then at any point $\mathbf{r}$ in 3-space, the total magnetic field is

$$\mathbf{H}(\mathbf{r})=\frac{1}{4\pi}\iiint_R\frac{\mathbf{J}(\mathbf{s})\times(\mathbf{r}-\mathbf{s})}{|\mathbf{r}-\mathbf{s}|^3}\,dV.$$

Now $\mathbf{A}(\mathbf{r})$ was defined to be

$$\mathbf{A}(\mathbf{r})=\frac{1}{4\pi}\iiint_R\frac{\mathbf{J}(\mathbf{s})}{|\mathbf{r}-\mathbf{s}|}\,dV.$$

We have

$$\begin{aligned}\mathbf{curl}\,\mathbf{A}(\mathbf{r})&=\frac{1}{4\pi}\iiint_R\nabla_{\mathbf{r}}\times\left(\frac{1}{|\mathbf{r}-\mathbf{s}|}\mathbf{J}(\mathbf{s})\right)dV\\&=\frac{1}{4\pi}\iiint_R\nabla_{\mathbf{r}}\frac{1}{|\mathbf{r}-\mathbf{s}|}\times\mathbf{J}(\mathbf{s})\,dV\\&\qquad\text{(by Theorem 3(c))}\\&=-\frac{1}{4\pi}\iiint_R\frac{(\mathbf{r}-\mathbf{s})\times\mathbf{J}(\mathbf{s})}{|\mathbf{r}-\mathbf{s}|^3}\,dV\\&\qquad\text{(by Exercise 4)}\\&=\mathbf{H}(\mathbf{r}).\end{aligned}$$

13.

$$\begin{aligned}\mathbf{A}(\mathbf{r})&=\frac{I}{4\pi}\oint_{\mathcal{F}}\frac{d\mathbf{s}}{|\mathbf{r}-\mathbf{s}|}\\\mathbf{div}\,\mathbf{A}(\mathbf{r})&=\frac{I}{4\pi}\oint_{\mathcal{F}}\mathbf{div}_{\mathbf{r}}\left(\frac{1}{|\mathbf{r}-\mathbf{s}|}\,d\mathbf{s}\right)\\&=\frac{I}{4\pi}\oint_{\mathcal{F}}\nabla\left(\frac{1}{|\mathbf{r}-\mathbf{s}|}\right)\bullet d\mathbf{s}\\&\qquad\text{(by Theorem 3(b))}\\&=0\quad\text{for } \mathbf{r} \text{ not on } \mathcal{F},\end{aligned}$$

since $\nabla(1/|\mathbf{r}-\mathbf{s}|)$ is conservative.

14. $\mathbf{A}(\mathbf{r}) = \dfrac{1}{4\pi}\iiint_R \dfrac{\mathbf{J}(\mathbf{s})\,dV}{|\mathbf{r}-\mathbf{s}|}$, where R is a region of 3-space such that $\mathbf{J}(\mathbf{s}) = \mathbf{0}$ outside R. We assume that $\mathbf{J}(\mathbf{s})$ is continuous, so $\mathbf{J}(\mathbf{s}) = \mathbf{0}$ on the surface $\mathcal{S}$ of R.
In the following calculations we use subscripts $\mathbf{s}$ and $\mathbf{r}$ to denote the variables with respect to which derivatives are taken. By Theorem 3(b),

$$\begin{aligned}\mathbf{div}_{\mathbf{s}}\frac{\mathbf{J}(\mathbf{s})}{|\mathbf{r}-\mathbf{s}|} &= \left(\nabla_{\mathbf{s}}\frac{1}{|\mathbf{r}-\mathbf{s}|}\right)\bullet\mathbf{J}(\mathbf{s}) + \frac{1}{|\mathbf{r}-\mathbf{s}|}\nabla_{\mathbf{s}}\bullet\mathbf{J}(\mathbf{s})\\ &= -\nabla_{\mathbf{r}}\left(\frac{1}{|\mathbf{r}-\mathbf{s}|}\right)\bullet\mathbf{J}(\mathbf{s}) + 0\end{aligned}$$

because $\nabla_{\mathbf{r}}|\mathbf{r}-\mathbf{s}| = -\nabla_{\mathbf{s}}|\mathbf{r}-\mathbf{s}|$, and because $\nabla\bullet\mathbf{J} = \nabla\bullet(\nabla\times\mathbf{H}) = 0$ by Theorem 3(g). Hence

$$\begin{aligned}\mathbf{div}\,\mathbf{A}(\mathbf{r}) &= \frac{1}{4\pi}\iiint_R \left(\nabla_{\mathbf{r}}\frac{1}{|\mathbf{r}-\mathbf{s}|}\right)\bullet\mathbf{J}(\mathbf{s})\,dV\\ &= -\frac{1}{4\pi}\iiint_R \nabla_{\mathbf{s}}\bullet\frac{\mathbf{J}(\mathbf{s})}{|\mathbf{r}-\mathbf{s}|}\,dV\\ &= -\frac{1}{4\pi}\oiint_{\mathcal{S}}\frac{\mathbf{J}(\mathbf{s})}{|\mathbf{r}-\mathbf{s}|}\bullet\hat{\mathbf{N}}\,dS = 0\end{aligned}$$

since $\mathbf{J}(\mathbf{s}) = \mathbf{0}$ on $\mathcal{S}$.

By Theorem 3(i),

$$\mathbf{J} = \nabla\times\mathbf{H} = \nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\bullet\mathbf{A}) - \nabla^2\mathbf{A} = -\nabla^2\mathbf{A}.$$

15. By Maxwell's equations, since $\rho = 0$ and $\mathbf{J} = \mathbf{0}$,

$$\begin{aligned}&\mathbf{div}\,\mathbf{E} = 0 &\qquad &\mathbf{div}\,\mathbf{H} = 0\\ &\mathbf{curl}\,\mathbf{E} = -\mu_0\frac{\partial\mathbf{H}}{\partial t} & &\mathbf{curl}\,\mathbf{H} = \epsilon_0\frac{\partial\mathbf{E}}{\partial t}\end{aligned}$$

Therefore,

$$\begin{aligned}&\mathbf{curl}\,\mathbf{curl}\,\mathbf{E} = \mathbf{grad}\,\mathbf{div}\,\mathbf{E} - \nabla^2\mathbf{E} = -\nabla^2\mathbf{E}\\ &\nabla^2\mathbf{E} = -\mathbf{curl}\,\mathbf{curl}\,\mathbf{E} = \mu_0\frac{\partial}{\partial t}\mathbf{curl}\,\mathbf{H} = \mu_0\epsilon_0\frac{\partial^2\mathbf{E}}{\partial t^2}.\end{aligned}$$

Similarly,

$$\nabla^2\mathbf{H} = \mu_0\epsilon_0\frac{\partial^2\mathbf{H}}{\partial t^2}.$$

Thus $\mathbf{U} = \mathbf{E}$ and $\mathbf{U} = \mathbf{H}$ both satisfy the wave equation

$$\frac{\partial^2\mathbf{U}}{\partial t^2} = c^2\nabla^2\mathbf{U}, \quad \text{where} \quad c^2 = \frac{1}{\mu_0\epsilon_0}.$$

16. The heat content of an arbitrary region R (with surface $\mathcal{S}$) at time t is

$$H(t) = \delta c\iiint_R T(x,y,z,t)\,dV.$$

This heat content increases at (time) rate

$$\frac{dH}{dt} = \delta c\iiint_R \frac{\partial T}{\partial t}\,dV.$$

If heat is not "created" or "destroyed" (by chemical or other means) within R, then the increase in heat content must be due to heat flowing into R across $\mathcal{S}$.
The rate of flow of heat into R across surface element dS with outward normal $\hat{\mathbf{N}}$ is

$$-k\nabla T\bullet\hat{\mathbf{N}}\,dS.$$

Therefore, the rate at which heat enters R through $\mathcal{S}$ is

$$k\oiint_{\mathcal{S}}\nabla T\bullet\hat{\mathbf{N}}\,dS.$$

By conservation of energy and the Divergence Theorem we have

$$\begin{aligned}\delta c\iiint_R \frac{\partial T}{\partial t}\,dV &= k\oiint_{\mathcal{S}}\nabla T\bullet\hat{\mathbf{N}}\,dS\\ &= k\iiint_R \nabla\bullet\nabla T\,dV\\ &= k\iiint_R \nabla^2 T\,dV.\end{aligned}$$

Thus, $\displaystyle\iiint_R\left(\frac{\partial T}{\partial t} - \frac{k}{\delta c}\nabla^2 T\right)dV = 0.$

Since R is arbitrary, and the temperature T is assumed to be smooth, the integrand must vanish everywhere. Thus

$$\frac{\partial T}{\partial t} = \frac{k}{\delta c}\nabla^2 T = \frac{k}{\delta c}\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right].$$

Section 16.7 Orthogonal Curvilinear Coordinates (page 896)

1. $f(r,\theta,z) = r\theta z$ (cylindrical coordinates). By Example 9,

$$\begin{aligned}\nabla f &= \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{\partial f}{\partial z}\mathbf{k}\\ &= \theta z\,\hat{\mathbf{r}} + z\,\hat{\boldsymbol{\theta}} + r\theta\,\mathbf{k}.\end{aligned}$$

2. $f(\rho,\phi,\theta) = \rho\phi\theta$ (spherical coordinates). By Example 10,

$$\begin{aligned}\nabla f &= \frac{\partial f}{\partial\rho}\,\hat{\boldsymbol{\rho}} + \frac{1}{\rho}\frac{\partial f}{\partial\phi}\,\hat{\boldsymbol{\phi}} + \frac{1}{\rho\sin\phi}\frac{\partial f}{\partial\theta}\,\hat{\boldsymbol{\theta}}\\ &= \phi\theta\,\hat{\boldsymbol{\rho}} + \theta\,\hat{\boldsymbol{\phi}} + \frac{\phi}{\sin\phi}\,\hat{\boldsymbol{\theta}}.\end{aligned}$$

3. $\mathbf{F}(r,\theta,z) = r\hat{\mathbf{r}}$

$$\textbf{div}\,\mathbf{F} = \frac{1}{r}\left[\frac{\partial}{\partial r}(r^2)\right] = 2$$

$$\textbf{curl}\,\mathbf{F} = \frac{1}{r}\begin{vmatrix} \hat{\mathbf{r}} & r\hat{\boldsymbol{\theta}} & \mathbf{k} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ r & 0 & 0 \end{vmatrix} = \mathbf{0}.$$

4. $\mathbf{F}(r,\theta,z) = r\hat{\boldsymbol{\theta}}$

$$\textbf{div}\,\mathbf{F} = \frac{1}{r}\left[\frac{\partial}{\partial \theta}(r)\right] = 0$$

$$\textbf{curl}\,\mathbf{F} = \frac{1}{r}\begin{vmatrix} \hat{\mathbf{r}} & r\hat{\boldsymbol{\theta}} & \mathbf{k} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ 0 & r^2 & 0 \end{vmatrix} = 2\mathbf{k}.$$

5. $\mathbf{F}(\rho,\phi,\theta) = \sin\phi\,\hat{\boldsymbol{\rho}}$

$$\textbf{div}\,\mathbf{F} = \frac{1}{\rho^2\sin\phi}\left[\frac{\partial}{\partial \rho}\left(\rho^2\sin^2\phi\right)\right] = \frac{2\sin\phi}{\rho}$$

$$\textbf{curl}\,\mathbf{F} = \frac{1}{\rho^2\sin\phi}\begin{vmatrix} \hat{\boldsymbol{\rho}} & \rho\hat{\boldsymbol{\phi}} & \rho\sin\phi\,\hat{\boldsymbol{\theta}} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial \theta} \\ \sin\phi & 0 & 0 \end{vmatrix}$$
$$= -\frac{\cos\phi}{\rho}\,\hat{\boldsymbol{\theta}}.$$

6. $\mathbf{F}(\rho,\phi,\theta) = \rho\,\hat{\boldsymbol{\phi}}$

$$\textbf{div}\,\mathbf{F} = \frac{1}{\rho^2\sin\phi}\left[\frac{\partial}{\partial \phi}\left(\rho^2\sin\phi\right)\right] = \cot\phi$$

$$\textbf{curl}\,\mathbf{F} = \frac{1}{\rho^2\sin\phi}\begin{vmatrix} \hat{\boldsymbol{\rho}} & \rho\hat{\boldsymbol{\phi}} & \rho\sin\phi\,\hat{\boldsymbol{\theta}} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial \theta} \\ 0 & \rho^2 & 0 \end{vmatrix} = 2\,\hat{\boldsymbol{\theta}}.$$

7. $\mathbf{F}(\rho,\phi,\theta) = \rho\,\hat{\boldsymbol{\theta}}$

$$\textbf{div}\,\mathbf{F} = \frac{1}{\rho^2\sin\phi}\left[\frac{\partial}{\partial \theta}\left(\rho^2\right)\right] = 0$$

$$\textbf{curl}\,\mathbf{F} = \frac{1}{\rho^2\sin\phi}\begin{vmatrix} \hat{\boldsymbol{\rho}} & \rho\hat{\boldsymbol{\phi}} & \rho\sin\phi\,\hat{\boldsymbol{\theta}} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial \theta} \\ 0 & 0 & \rho^2\sin\phi \end{vmatrix}$$
$$= \cot\phi\,\hat{\boldsymbol{\rho}} - 2\,\hat{\boldsymbol{\phi}}.$$

8. $\mathbf{F}(\rho,\phi,\theta) = \rho^2\,\hat{\boldsymbol{\rho}}$

$$\textbf{div}\,\mathbf{F} = \frac{1}{\rho^2\sin\phi}\left[\frac{\partial}{\partial \rho}\left(\rho^4\sin\phi\right)\right] = 4\rho$$

$$\textbf{curl}\,\mathbf{F} = \frac{1}{\rho^2\sin\phi}\begin{vmatrix} \hat{\boldsymbol{\rho}} & \rho\hat{\boldsymbol{\phi}} & \rho\sin\phi\,\hat{\boldsymbol{\theta}} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial \theta} \\ \rho^2 & 0 & 0 \end{vmatrix} = \mathbf{0}.$$

9. Let $\mathbf{r} = x(u,v)\,\mathbf{i} + y(u,v)\,\mathbf{j}$. The scale factors are

$$h_u = \left|\frac{\partial \mathbf{r}}{\partial u}\right| \quad\text{and}\quad h_v = \left|\frac{\partial \mathbf{r}}{\partial v}\right|.$$

The local basis consists of the vectors

$$\hat{\mathbf{u}} = \frac{1}{h_u}\frac{\partial \mathbf{r}}{\partial u} \quad\text{and}\quad \hat{\mathbf{v}} = \frac{1}{h_v}\frac{\partial \mathbf{r}}{\partial v}.$$

The area element is $dA = h_u h_v\,du\,dv$.

10. Since (u,v,z) constitute orthogonal curvilinear coordinates in $\mathbb{R}^3$, with scale factors h_u, h_v and $h_z = 1$, we have, for a function $f(u,v)$ independent of z,

$$\begin{aligned}\nabla f(u,v) &= \frac{1}{h_u}\frac{\partial f}{\partial u}\,\hat{\mathbf{u}} + \frac{1}{h_v}\frac{\partial f}{\partial v}\,\hat{\mathbf{v}} + \frac{1}{1}\frac{\partial f}{\partial z}\,\mathbf{k} \\ &= \frac{1}{h_u}\frac{\partial f}{\partial u}\,\hat{\mathbf{u}} + \frac{1}{h_v}\frac{\partial f}{\partial v}\,\hat{\mathbf{v}}.\end{aligned}$$

For $\mathbf{F}(u,v) = F_u(u,v)\,\hat{\mathbf{u}} + F_v(u,v)\,\hat{\mathbf{v}}$ (independent of z and having no $\mathbf{k}$ component), we have

$$\textbf{div}\,\mathbf{F}(u,v) = \frac{1}{h_u h_v}\left[\frac{\partial}{\partial u}(h_u F_u) + \frac{\partial}{\partial v}(h_v F_v)\right]$$

$$\begin{aligned}\textbf{curl}\,\mathbf{F}(u,v) &= \frac{1}{h_u h_v}\begin{vmatrix} h_u\hat{\mathbf{u}} & h_v\hat{\mathbf{v}} & \mathbf{k} \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial z} \\ h_u F_u & h_v F_v & 0 \end{vmatrix} \\ &= \frac{1}{h_u h_v}\left[\frac{\partial}{\partial u}(h_v F_v) - \frac{\partial}{\partial v}(h_u F_u)\right]\mathbf{k}.\end{aligned}$$

11. We can use the expressions calculated in the text for cylindrical coordinates, applied to functions independent of z and having no $\mathbf{k}$ components:

$$\nabla f(r,\theta) = \frac{\partial f}{\partial r}\,\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\,\hat{\boldsymbol{\theta}}$$
$$\textbf{div}\,\mathbf{F}(r,\theta) = \frac{\partial F_r}{\partial r} + \frac{F_r}{r} + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta}$$
$$\textbf{curl}\,\mathbf{F}(r,\theta) = \left[\frac{\partial F_\theta}{\partial r} + \frac{F_\theta}{r} - \frac{1}{r}\frac{\partial F_r}{\partial \theta}\right]\mathbf{k}.$$

12. $x = a\cosh u\cos v, \quad y = a\sinh u\sin v.$

a) u-curves: If $A = a\cosh u$ and $B = a\sinh u$, then

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} = \cos^2 v + \sin^2 v = 1.$$

Since $A^2 - B^2 = a^2(\cosh^2 u - \sinh^2 u) = a^2$, the u-curves are ellipses with foci at $(\pm a, 0)$.

b) v-curves: If $A = a\cos v$ and $B = a\sin v$, then

$$\frac{x^2}{A^2} - \frac{y^2}{B^2} = \cosh^2 u - \sinh^2 u = 1.$$

Since $A^2 + B^2 = a^2(\cos^2 v + \sin^2 v) = a^2$, the v-curves are hyperbolas with foci at $(\pm a, 0)$.

c) The u-curve $u = u_0$ has parametric equations

$$x = a\cosh u_0 \cos v, \qquad y = a \sinh u_0 \sin v,$$

and therefore has slope at (u_0, v_0) given by

$$m_u = \frac{dy}{dx} = \frac{dy}{dv}\Big/\frac{dx}{dv}\Big|_{(u_0,v_0)} = \frac{a\sinh u_0 \cos v_0}{-a\cosh u_0 \sin v_0}.$$

The v-curve $v = v_0$ has parametric equations

$$x = a\cosh u \cos v_0, \qquad y = a\sinh u \sin v_0,$$

and therefore has slope at (u_0, v_0) given by

$$m_v = \frac{dy}{dx} = \frac{dy}{du}\Big/\frac{dx}{du}\Big|_{(u_0,v_0)} = \frac{a\cosh u_0 \sin v_0}{a\sinh u_0 \cos v_0}.$$

Since the product of these slopes is $m_u m_v = -1$, the curves $u = u_0$ and $v = v_0$ intersect at right angles.

d) $\mathbf{r} = a\cosh u \cos v\,\mathbf{i} + a\sinh u \sin v\,\mathbf{j}$

$$\frac{\partial \mathbf{r}}{\partial u} = a\sinh u \cos v\,\mathbf{i} + a\cosh u \sin v\,\mathbf{j}$$
$$\frac{\partial \mathbf{r}}{\partial v} = -a\cosh u \sin v\,\mathbf{i} + a\sinh u \cos v\,\mathbf{j}.$$

The scale factors are

$$h_u = \left|\frac{\partial \mathbf{r}}{\partial u}\right| = a\sqrt{\sinh^2 u \cos^2 v + \cosh^2 u \sin^2 v}$$
$$h_v = \left|\frac{\partial \mathbf{r}}{\partial v}\right| = a\sqrt{\sinh^2 u \cos^2 v + \cosh^2 u \sin^2 v} = h_u.$$

The area element is

$$\begin{aligned} dA &= h_u h_v\,du\,dv \\ &= a^2\left(\sinh^2 u \cos^2 v + \cosh^2 u \sin^2 v\right) du\,dv. \end{aligned}$$

13. $x = a\cosh u \cos v$
$y = a\sinh u \sin v$
$z = z.$
Using the result of Exercise 12, we see that the coordinate surfaces are
$u = u_0$: vertical elliptic cylinders with focal axes $x = \pm a$, $y = 0$.
$v = v_0$: vertical hyperbolic cylinders with focal axes $x = \pm a$, $y = 0$.
$z = z_0$: horizontal planes.

The coordinate curves are
u-curves: the horizontal hyperbolas in which the $v = v_0$ cylinders intersect the $z = z_0$ planes.
v-curves: the horizontal ellipses in which the $u = u_0$ cylinders intersect the $z = z_0$ planes.
z-curves: sets of four vertical straight lines where the elliptic cylinders $u = u_0$ and hyperbolic cylinders $v = v_0$ intersect.

14. $$\nabla f(r,\theta,z) = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{\partial f}{\partial z}\mathbf{k}$$
$$\begin{aligned} \nabla^2 f(r,\theta,z) &= \mathbf{div}\left(\nabla f(r,\theta,z)\right) \\ &= \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial f}{\partial \theta}\right) + \frac{\partial}{\partial z}\left(r\frac{\partial f}{\partial z}\right)\right] \\ &= \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}. \end{aligned}$$

15. $$\nabla f(\rho,\phi,\theta) = \frac{\partial f}{\partial \rho}\hat{\boldsymbol{\rho}} + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{1}{\rho\sin\phi}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}}$$
$$\begin{aligned} \nabla^2 f(\rho,\phi,\theta) &= \mathbf{div}\left(f(\rho,\phi,\theta)\right) \\ &= \frac{1}{\rho^2\sin\phi}\left[\frac{\partial}{\partial \rho}\left(\rho^2\sin\phi\frac{\partial f}{\partial \rho}\right) + \frac{\partial}{\partial \phi}\left(\rho\sin\phi\,\frac{1}{\rho}\frac{\partial f}{\partial \phi}\right)\right. \\ &\qquad \left. + \frac{\partial}{\partial \theta}\left(\frac{\rho}{\rho\sin\phi}\frac{\partial f}{\partial \theta}\right)\right] \\ &= \frac{\partial^2 f}{\partial \rho^2} + \frac{2}{\rho}\frac{\partial f}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} \\ &\qquad + \frac{\cot\phi}{\rho^2}\frac{\partial f}{\partial \phi} + \frac{1}{\rho^2\sin^2\phi}\frac{\partial^2 f}{\partial \theta^2}. \end{aligned}$$

16. $$\nabla f(u,v,w) = \frac{1}{h_u}\frac{\partial f}{\partial u}\hat{\mathbf{u}} + \frac{1}{h_v}\frac{\partial f}{\partial v}\hat{\mathbf{v}} + \frac{1}{h_w}\frac{\partial f}{\partial w}\hat{\mathbf{w}}$$
$$\begin{aligned} \nabla^2 f(u,v,w) &= \mathbf{div}\left(\nabla f(u,v,w)\right) \\ &= \frac{1}{h_u h_v h_w}\left[\frac{\partial}{\partial u}\left(\frac{h_v h_w}{h_u}\frac{\partial f}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{h_u h_w}{h_v}\frac{\partial f}{\partial v}\right)\right. \\ &\qquad \left. + \frac{\partial}{\partial w}\left(\frac{h_u h_v}{h_w}\frac{\partial f}{\partial w}\right)\right] \\ &= \frac{1}{h_u^2}\left[\frac{\partial^2 f}{\partial u^2} + \left(\frac{1}{h_v}\frac{\partial h_v}{\partial u} + \frac{1}{h_w}\frac{\partial h_w}{\partial u} - \frac{1}{h_u}\frac{\partial h_u}{\partial u}\right)\frac{\partial f}{\partial u}\right] \\ &\quad + \frac{1}{h_v^2}\left[\frac{\partial^2 f}{\partial v^2} + \left(\frac{1}{h_u}\frac{\partial h_u}{\partial v} + \frac{1}{h_w}\frac{\partial h_w}{\partial v} - \frac{1}{h_v}\frac{\partial h_v}{\partial v}\right)\frac{\partial f}{\partial v}\right] \\ &\quad + \frac{1}{h_w^2}\left[\frac{\partial^2 f}{\partial w^2} + \left(\frac{1}{h_u}\frac{\partial h_u}{\partial w} + \frac{1}{h_v}\frac{\partial h_v}{\partial w} - \frac{1}{h_w}\frac{\partial h_w}{\partial w}\right)\frac{\partial f}{\partial w}\right]. \end{aligned}$$

Review Exercises 16 (page 896)

1. The semi-ellipsoid $\mathcal{S}$ with upward normal $\hat{\mathbf{N}}$ specified in the problem and the disk D given by $x^2+y^2 \le 16$, $z = 0$, with downward normal $-\mathbf{k}$ together bound the solid region R: $0 \le z \le \frac{1}{2}\sqrt{16 - x^2 - y^2}$. By the Divergence Theorem:

$$\iint_{\mathcal{S}} \mathbf{F}\bullet\hat{\mathbf{N}}\,dS + \iint_D \mathbf{F}\bullet(-\mathbf{k})\,dA = \iiint_R \mathbf{div}\,\mathbf{F}\,dV.$$

For $\mathbf{F} = x^2z\mathbf{i} + (y^2z + 3y)\mathbf{j} + x^2\mathbf{k}$ we have

$$\iiint_R \mathbf{div}\,\mathbf{F}\,dV = \iiint_R (2xz + 2yz + 3)\,dV$$
$$= 0 + 0 + 3\iiint_R dV = 3 \times (\text{volume of } R)$$
$$= \frac{3}{2}\,\frac{4}{3}\pi 4^2 2 = 64\pi.$$

The flux of $\mathbf{F}$ across $\mathcal{S}$ is

$$\iint_{\mathcal{S}} \mathbf{F} \bullet \hat{\mathbf{N}}\,dS = 64\pi + \iint_D \mathbf{F} \bullet \mathbf{k}\,dA$$
$$= 64\pi + \iint_D x^2\,dA$$
$$= 64\pi + \int_0^{2\pi} \cos^2\theta\,d\theta \int_0^4 r^3\,dr = 128\pi.$$

2. Let R be the region inside the cylinder $\mathcal{S}$ and between the planes $z = 0$ and $z = b$. The oriented boundary of R consists of $\mathcal{S}$ and the disks D_1 with normal $\hat{\mathbf{N}}_1 = \mathbf{k}$ and D_2 with normal $\hat{\mathbf{N}}_2 = -\mathbf{k}$ as shown in the figure. For $\mathbf{F} = x\mathbf{i} + \cos(z^2)\mathbf{j} + e^z\mathbf{k}$ we have $\mathbf{div}\,\mathbf{F} = 1 + e^z$ and

$$\iiint_R \mathbf{div}\,\mathbf{F}\,dV = \iint_{D_2} dx\,dy \int_0^b (1 + e^z)\,dz$$
$$= \iint_{D_2} [b + (e^b - 1)]\,dx\,dy$$
$$= \pi a^2 b + \pi a^2(e^b - 1).$$

Also $\displaystyle\iint_{D_2} \mathbf{F} \bullet (-\mathbf{k})\,dA = -\iint_{D_2} e^0\,dA = -\pi a^2$

$$\iint_{D_1} \mathbf{F} \bullet \mathbf{k}\,dA = \iint_{D_1} e^b\,dA = \pi a^2 e^b.$$

By the Divergence Theorem

$$\iint_{\mathcal{S}} \mathbf{F} \bullet \hat{\mathbf{N}}\,dS + \iint_{D_1} \mathbf{F} \bullet \mathbf{k}\,dA + \iint_{D_2} \mathbf{F} \bullet (-\mathbf{k})\,dA$$
$$= \iiint_R \mathbf{div}\,\mathbf{F}\,dV = \pi a^2 b + \pi a^2(e^b - 1).$$

Therefore, $\displaystyle\iint_{\mathcal{S}} \mathbf{F} \bullet \hat{\mathbf{N}}\,dS = \pi a^2 b.$

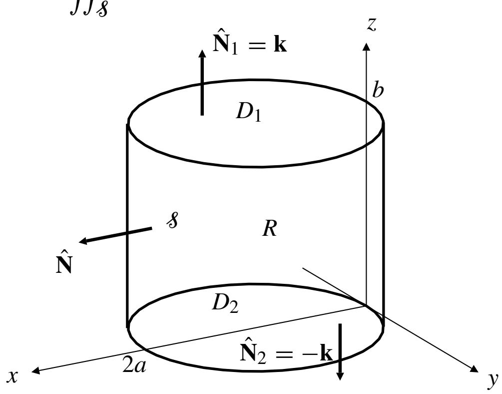

Fig. R-16.2

3. $$\oint_{\mathcal{C}} (3y^2 + 2xe^{y^2})\,dx + (2x^2ye^{y^2})\,dy$$
$$= \iint_P [4xye^{y^2} - (6y + 4xye^{y^2})]\,dA$$
$$= -6\iint_P y\,dA = -6\bar{y}A = -6,$$

since P has area $A = 2$ and its centroid has y-coordinate $\bar{y} = 1/2$.

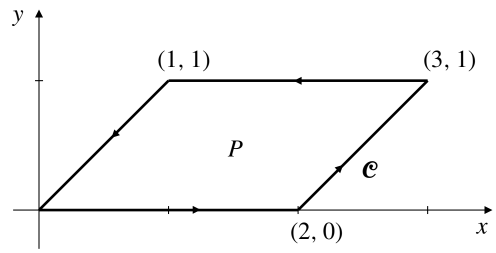

Fig. R-16.3

4. If $\mathbf{F} = -z\mathbf{i} + x\mathbf{j} + y\mathbf{k}$, then

$$\mathbf{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -z & x & y \end{vmatrix} = \mathbf{i} - \mathbf{j} + \mathbf{k}.$$

The unit normal $\hat{\mathbf{N}}$ to a region in the plane $2x + y + 2z = 7$ is

$$\hat{\mathbf{N}} = \pm\frac{2\mathbf{i} + \mathbf{j} + 2\mathbf{k}}{3}.$$

If $\mathcal{C}$ is the boundary of a disk D of radius a in that plane, then

$$\oint_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{r} = \iint_D \mathbf{curl}\,\mathbf{F} \bullet \hat{\mathbf{N}}\,dS$$
$$= \pm\iint_D \frac{2 - 1 + 2}{3}\,dS = \pm\pi a^2.$$

5. If $\mathcal{S}_a$ is the sphere of radius a centred at the origin, then

$$\mathbf{div}\,\mathbf{F}(0,0,0) = \lim_{a\to 0+} \frac{1}{\frac{4}{3}\pi a^3} \oiint_{\mathcal{S}_a} \mathbf{F} \bullet \hat{\mathbf{N}}\,dS$$
$$= \lim_{a\to 0+} \frac{3}{4\pi a^3}(\pi a^3 + 2a^4) = \frac{3}{4}.$$

6. If $\mathcal{S}$ is any surface with upward normal $\hat{\mathbf{N}}$ and boundary the curve $\mathcal{C}$: $x^2 + y^2 = 1$, $z = 2$, then $\mathcal{C}$ is oriented counterclockwise as seen from above, and it has parametrization

$$\mathbf{r} = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + 2\mathbf{k} \quad (0 \le 2 \le 2\pi).$$

Thus $d\mathbf{r} = (-\sin t\mathbf{i} + \cos t\mathbf{j})\,dt$, and if $\mathbf{F} = -y\mathbf{i} + x\cos(1 - x^2 - y^2)\mathbf{j} + yz\mathbf{k}$, then the flux of $\mathbf{curl\,F}$ upward through $\mathcal{S}$ is

$$\begin{aligned}\iint_{\mathcal{S}} \mathbf{curl\,F} \bullet \hat{\mathbf{N}}\,dS &= \oint_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{r}\\ &= \int_0^{2\pi} (\sin^2 t + \cos^2 t + 0)\,dt = 2\pi.\end{aligned}$$

7. $\mathbf{F}(\mathbf{r}) = r^\lambda \mathbf{r}$ where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $r = |\mathbf{r}|$. Since $r^2 = x^2 + y^2 + z^2$, therefore $\partial r/\partial x = x/r$ and

$$\frac{\partial}{\partial x}(r^\lambda x) = \lambda r^{\lambda-1}\frac{x^2}{r} + r^\lambda = r^{\lambda-2}(\lambda x^2 + r^2).$$

Similar expressions hold for $(\partial/\partial y)(r^\lambda y)$ and $(\partial/\partial z)(r^\lambda z)$, so

$$\mathbf{div\,F}(\mathbf{r}) = r^{\lambda-2}(\lambda r^2 + 3r^2) = (\lambda + 3)r^\lambda.$$

$\mathbf{F}$ is solenoidal on any set in $\mathbb{R}^3$ that excludes the origin if an only if $\lambda = -3$. In this case $\mathbf{F}$ is not defined at $\mathbf{r} = \mathbf{0}$. There is no value of λ for which $\mathbf{F}$ is solenoidal on all of $\mathbb{R}^3$.

8. If $\mathbf{curl\,F} = \mu\mathbf{F}$ on $\mathbb{R}^3$, where $\mu \neq 0$ is a constant, then

$$\mathbf{div\,F} = \frac{1}{\mu}\mathbf{div\,curl\,F} = 0$$

by Theorem 3(g) of Section 7.2. By part (i) of the same theorem,

$$\begin{aligned}\nabla^2\mathbf{F} &= \nabla(\mathbf{div\,F}) - \mathbf{curl\,curl\,F}\\ &= 0 - \mu\mathbf{curl\,F} = -\mu^2\mathbf{F}.\end{aligned}$$

Thus $\nabla^2\mathbf{F} + \mu^2\mathbf{F} = \mathbf{0}$.

9. Apply the variant of the Divergence Theorem given in Theorem 7(b) of Section 7.3, namely

$$\iiint_P \mathbf{grad}\,\phi\,dV = \oiint_{\mathcal{S}} \phi\hat{\mathbf{N}}\,dS,$$

to the scalar field $\phi = 1$ over the polyhedron P. Here $\mathcal{S} = \bigcup_{i=1}^{n} F_i$ is the surface of P, oriented with outward normal field $\hat{\mathbf{N}}_i$ on the face F_i. If $\mathbf{N}_i = A_i\hat{\mathbf{N}}_i$, where A_i is the area of F_i, then, since $\mathbf{grad}\,\phi = \mathbf{0}$, we have

$$\mathbf{0} = \oiint_{\mathcal{S}} \hat{\mathbf{N}}\,dS = \sum_{i=1}^{n}\iint_{F_i}\frac{\mathbf{N}_i}{A_i}\,dS = \sum_{i=1}^{n}\frac{\mathbf{N}_i}{A_i}A_i = \sum_{i=1}^{n}\mathbf{N}_i.$$

10. Let $\mathcal{C}$ be a simple, closed curve in the xy-plane bounding a region R. If

$$\mathbf{F} = (2y^3 - 3y + xy^2)\mathbf{i} + (x - x^3 + x^2y)\mathbf{j},$$

then by Green's Theorem, the circulation of $\mathbf{F}$ around $\mathcal{C}$ is

$$\begin{aligned}&\oint_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{r}\\ &= \iint_R \left[\frac{\partial}{\partial x}(x - x^3 + x^2y) - \frac{\partial}{\partial y}(2y^3 - 3y + xy^2)\right] dA\\ &= \iint_R (1 - 3x^2 + 2xy - 6y^2 + 3 - 2xy)\,dA\\ &= \iint_R (4 - 3x^2 - 6y^2)\,dx\,dy.\end{aligned}$$

The last integral has a maximum value when the region R is bounded by the ellipse $3x^2 + 6y^2 = 4$, oriented counterclockwise; this is the largest region in the xy-plane where the integrand is nonnegative.

11. Let $\mathcal{S}$ be a closed, oriented surface in $\mathbb{R}^3$ bounding a region R, and having outward normal field $\hat{\mathbf{N}}$. If

$$\mathbf{F} = (4x + 2x^3z)\mathbf{i} - y(x^2 + z^2)\mathbf{j} - (3x^2z^2 + 4y^2z)\mathbf{k},$$

then by the Divergence Theorem, the flux of $\mathbf{F}$ through $\mathcal{S}$ is

$$\oiint_{\mathcal{S}} \mathbf{F}\bullet\hat{\mathbf{N}}\,dS = \iiint_R \mathbf{div\,F}\,dV = \iiint_R (4 - x^2 - 4y^2 - z^2)\,dV.$$

The last integral has a maximum value when the region R is bounded by the ellipsoid $x^2 + 4y^2 + z^2 = 4$ with outward normal; this is the largest region in $\mathbb{R}^3$ where the integrand is nonnegative.

12. Let $\mathcal{C}$ be a simple, closed curve on the plane $x + y + z = 1$, oriented counterclockwise as seen from above, and bounding a plane region $\mathcal{S}$ on $x + y + z = 1$. Then $\mathcal{S}$ has normal $\hat{\mathbf{N}} = (\mathbf{i} + \mathbf{j} + \mathbf{k})/\sqrt{3}$. If $\mathbf{F} = xy^2\mathbf{i} + (3z - xy^2)\mathbf{j} + (4y - x^2y)\mathbf{k}$, then

$$\begin{aligned}\mathbf{curl\,F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ xy^2 & 3z - xy^2 & 4y - x^2y \end{vmatrix}\\ &= (1 - x^2)\mathbf{i} + 2xy\mathbf{j} - (y^2 + 2xy)\mathbf{k}.\end{aligned}$$

By Stokes's Theorem we have

$$\oint_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{r} = \iint_{\mathcal{S}} \mathbf{curl\,F} \bullet \hat{\mathbf{N}}\,dS = \iint_{\mathcal{S}} \frac{1 - x^2 - y^2}{\sqrt{3}}\,dS.$$

The last integral will be maximum if the projection of $\mathcal{S}$ onto the xy-plane is the disk $x^2+y^2 \le 1$. This maximum value is

$$\iint_{x^2+y^2\le 1} \frac{1-x^2-y^2}{\sqrt{3}}\sqrt{3}\,dx\,dy$$
$$= \int_0^{2\pi} d\theta \int_0^1 (1-r^2)r\,dr = 2\pi\left(\frac{1}{2}-\frac{1}{4}\right) = \frac{\pi}{2}.$$

Challenging Problems 16 (page 897)

1. By Theorem 1 of Section 7.1, we have

$$\mathbf{div}\,\mathbf{v}(\mathbf{r}_1) = \lim_{\epsilon\to 0+} \frac{3}{4\pi\epsilon^3} \oiint_{\mathcal{S}_\epsilon} \mathbf{v}(\mathbf{r}) \bullet \hat{\mathbf{N}}(\mathbf{r})\,dS.$$

Here $\mathcal{S}_\epsilon$ is the sphere of radius ϵ centred at the point (with position vector) $\mathbf{r}_1$ and having outward normal field $\hat{\mathbf{N}}(\mathbf{r})$. If $\mathbf{r}$ is (the position vector of) any point on $\mathcal{S}_\epsilon$, then $\mathbf{r} = \mathbf{r}_1 + \epsilon\hat{\mathbf{N}}(\mathbf{r})$, and

$$\oiint_{\mathcal{S}_\epsilon} \mathbf{v}(\mathbf{r}) \bullet \hat{\mathbf{N}}(\mathbf{r})\,dS$$
$$= \oiint_{\mathcal{S}_\epsilon} \Big[\mathbf{v}(\mathbf{r}_1) + \Big(\mathbf{v}(\mathbf{r}) - \mathbf{v}(\mathbf{r}_1)\Big)\Big] \bullet \hat{\mathbf{N}}(\mathbf{r})\,dS$$
$$= \mathbf{v}(\mathbf{r}_1) \bullet \oiint_{\mathcal{S}_\epsilon} \hat{\mathbf{N}}(\mathbf{r})\,dS$$
$$+ \oiint_{\mathcal{S}_\epsilon} \Big(\mathbf{v}(\mathbf{r}) - \mathbf{v}(\mathbf{r}_1)\Big) \bullet \frac{\mathbf{r}-\mathbf{r}_1}{\epsilon}\,dS.$$

But $\oiint_{\mathcal{S}_\epsilon} \hat{\mathbf{N}}(\mathbf{r})\,dS = \mathbf{0}$ by Theorem 7(b) of Section 7.3 with $\phi = 1$. Also, since $\mathbf{v}$ satisfies

$$\mathbf{v}(\mathbf{r}_2) - \mathbf{v}(\mathbf{r}_1) = C|\mathbf{r}_2 - \mathbf{r}_1|^2,$$

we have

$$\oiint_{\mathcal{S}_\epsilon} \Big(\mathbf{v}(\mathbf{r}) - \mathbf{v}(\mathbf{r}_1)\Big) \bullet \frac{\mathbf{r}-\mathbf{r}_1}{\epsilon}\,dS$$
$$= \oiint_{\mathcal{S}_\epsilon} \frac{C\epsilon^2}{\epsilon}\,dS = 4\pi C\epsilon^3.$$

Thus

$$\mathbf{div}\,\mathbf{v}(\mathbf{r}_1) = \lim_{\epsilon\to 0+} \frac{3}{4\pi\epsilon^3}(0 + 4\pi C\epsilon^3) = 3C.$$

The divergence of the large-scale velocity field of matter in the universe is three times Hubble's constant C.

2. a) The steradian measure of a half-cone of semi-vertical angle α is

$$\int_0^{2\pi} d\theta \int_0^\alpha \sin\phi\,d\phi = 2\pi(1-\cos\alpha).$$

b) If $\mathcal{S}$ is the intersection of a smooth surface with the general half-cone K, and is oriented with normal field $\hat{\mathbf{N}}$ pointing away from the vertex P of K, and if $\mathcal{S}_a$ is the intersection with K of a sphere of radius a centred at P, with a chosen so that $\mathcal{S}$ and $\mathcal{S}_a$ do not intersect in K, then $\mathcal{S}$, $\mathcal{S}_a$, and the walls of K bound a solid region R that does not contain the origin. If $\mathbf{F} = \mathbf{r}/|\mathbf{r}|^3$, then $\mathbf{div}\,\mathbf{F} = 0$ in R (see Example 3 in Section 7.1), and $\mathbf{F} \bullet \hat{\mathbf{N}} = 0$ on the walls of K. It follows from the Divergence Theorem applied to $\mathbf{F}$ over R that

$$\iint_{\mathcal{S}} \mathbf{F} \bullet \hat{\mathbf{N}}\,dS = \iint_{\mathcal{S}_a} \mathbf{F} \bullet \frac{\mathbf{r}}{|\mathbf{r}|}\,dS$$
$$= \frac{a^2}{a^4}\iint_{\mathcal{S}_a} dS = \frac{1}{a^2}(\text{area of } \mathcal{S}_a)$$
$$= \text{area of } \mathcal{S}_1.$$

The area of $\mathcal{S}_1$ (the part of the sphere of radius 1 in K) is the measure (in steradians) of the solid angle subtended by K at its vertex P. Hence this measure is given by

$$\iint_{\mathcal{S}} \frac{\mathbf{r}}{|\mathbf{r}|^3} \bullet \hat{\mathbf{N}}\,dS.$$

3. a) Verification of the identity

$$\frac{\partial}{\partial t}\left(\mathbf{G} \bullet \frac{\partial \mathbf{r}}{\partial s}\right) - \frac{\partial}{\partial s}\left(\mathbf{G} \bullet \frac{\partial \mathbf{r}}{\partial t}\right)$$
$$= \frac{\partial \mathbf{F}}{\partial t} \bullet \frac{\partial \mathbf{r}}{\partial s} + \left((\nabla \times \mathbf{F}) \times \frac{\partial \mathbf{r}}{\partial t}\right) \bullet \frac{\partial \mathbf{r}}{\partial s}.$$

can be carried out using the following MapleV commands:

```
>  with(linalg):
>  F:=(x,y,z,t)-> [F1(x,y,z,t),
>    F2(x,y,z,t),F3(x,y,z,t)];
>  r:=(s,t)->[x(s,t),y(s,t),z(s,t)];
>
G:=(s,t)->F(x(s,t),y(s,t),z(s,t),t);
>  g:=(s,t)-> dotprod(G(s,t),
>    map(diff,r(s,t),s));
>  h:=(s,t)-> dotprod(G(s,t),
>    map(diff,r(s,t),t));
>  LH1:=diff(g(s,t),t);
>  LH2:=diff(h(s,t),s);
>  LHS:=simplify(LH1-LH2);
>
RH1:=dotprod(subs(x=x(s,t),y=y(s,t),
>    z=z(s,t),diff(F(x,y,z,t),t)),
>    diff(r(s,t),s));
>
RH2:=dotprod(crossprod(subs(x=x(s,t),
>    y=y(s,t),z=z(s,t),
```

```
>    curl(F(x,y,z,t),[x,y,z])),
>    diff(r(s,t),t)),diff(r(s,t),s));
>  RHS:=RH1+RH2; LHS-RHS; simplify(%);
```

We omit the output here; some of the commands produce screenfulls of output. The output of the final command is 0, indicating that the identity is valid.

b) As suggested by the hint,

$$\begin{aligned}
\frac{d}{dt}\int_{C_t} \mathbf{F} \bullet d\mathbf{r} &= \int_a^b \frac{\partial}{\partial t}\left(\mathbf{G} \bullet \frac{\partial \mathbf{r}}{\partial s}\right) ds \\
&= \int_a^b \left[\frac{\partial}{\partial s}\left(\mathbf{G} \bullet \frac{\partial \mathbf{r}}{\partial t}\right)\right. \\
&\quad \left. + \left(\frac{\partial}{\partial t}\left(\mathbf{G} \bullet \frac{\partial \mathbf{r}}{\partial s}\right) - \frac{\partial}{\partial s}\left(\mathbf{G} \bullet \frac{\partial \mathbf{r}}{\partial t}\right)\right)\right] ds \\
&= \mathbf{G} \bullet \frac{\partial \mathbf{r}}{\partial t}\bigg|_{s=a}^{s=b} \\
&\quad + \int_a^b \left[\frac{\partial \mathbf{F}}{\partial t} + \left((\nabla \times \mathbf{F}) \times \frac{\partial \mathbf{r}}{\partial t}\right)\right] \bullet \frac{\partial \mathbf{r}}{\partial s}\, ds \\
&= \mathbf{F}\Big(\mathbf{r}(b,t), t\Big) \bullet \mathbf{v}_C(b,t) - \mathbf{F}\Big(\mathbf{r}(a,t), t\Big) \bullet \mathbf{v}_C(a,t) \\
&\quad + \int_{C_t} \frac{\partial \mathbf{F}}{\partial t} \bullet d\mathbf{r} + \int_{C_t} \Big((\nabla \times \mathbf{F}) \times \mathbf{v}_C\Big) \bullet d\mathbf{r}.
\end{aligned}$$

4. a) Verification of the identity

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\mathbf{G} \bullet \left[\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right]\right) - \frac{\partial}{\partial u}\left(\mathbf{G} \bullet \left[\frac{\partial \mathbf{r}}{\partial t} \times \frac{\partial \mathbf{r}}{\partial v}\right]\right) \\
&\quad - \frac{\partial}{\partial v}\left(\mathbf{G} \bullet \left[\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial t}\right]\right) \\
&= \frac{\partial \mathbf{F}}{\partial t} \bullet \left[\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right] + (\nabla \bullet \mathbf{F})\frac{\partial \mathbf{r}}{\partial t} \bullet \left[\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right].
\end{aligned}$$

can be carried out using the following MapleV commands:

```
>  with(linalg):
>  F:=(x,y,z,t)->[F1(x,y,z,t),
>    F2(x,y,z,t),F3(x,y,z,t)];
>  r:=(u,v,t)->[x(u,v,t),y(u,v,t),
>    z(u,v,t)];
>  ru:=(u,v,t)->diff(r(u,v,t),u);
>  rv:=(u,v,t)->diff(r(u,v,t),v);
>  rt:=(u,v,t)->diff(r(u,v,t),t);
>  G:=(u,v,t)->F(x(u,v,t),
>    y(u,v,t),z(u,v,t),t);
>
ruxv:=(u,v,t)->crossprod(ru(u,v,t),
>    rv(u,v,t));
>
rtxv:=(u,v,t)->crossprod(rt(u,v,t),
>    rv(u,v,t));
>
ruxt:=(u,v,t)->crossprod(ru(u,v,t),
>    rt(u,v,t));
>  LH1:=diff(dotprod(G(u,v,t),
>    ruxv(u,v,t)),t);
>  LH2:=diff(dotprod(G(u,v,t),
>    rtxv(u,v,t)),u);
>  LH3:=diff(dotprod(G(u,v,t),
>    ruxt(u,v,t)),v);
>  LHS:=simplify(LH1-LH2-LH3);
>  RH1:=dotprod(subs(x=x(u,v,t),
>    y=y(u,v,t),z=z(u,v,t),
>
diff(F(x,y,z,t),t)),ruxv(u,v,t));
>  RH2:=(divf(u,v,t))*
>
(dotprod(rt(u,v,t),ruxv(u,v,t)));
>  RHS:=simplify(RH1+RH2);
>  simplify(LHS-RHS);
```

Again the final output is 0, indicating that the identity is valid.

b) If $\mathcal{C}_t$ is the oriented boundary of $\mathcal{S}_t$ and L_t is the corresponding counterclockwise boundary of the parameter region R in the uv-plane, then

$$\begin{aligned}
&\oint_{\mathcal{C}_t}\left(\mathbf{F} \times \frac{\partial \mathbf{r}}{\partial t}\right) \bullet d\mathbf{r} \\
&= \oint_{L_t}\left(\mathbf{G} \times \frac{\partial \mathbf{r}}{\partial t}\right) \bullet \left(\frac{\partial \mathbf{r}}{\partial u}\, du + \frac{\partial \mathbf{r}}{\partial v}\, dv\right) \\
&= \oint_{L_t}\left[-\mathbf{G} \bullet \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial t}\right) + \mathbf{G} \bullet \left(\frac{\partial \mathbf{r}}{\partial t} \times \frac{\partial \mathbf{r}}{\partial v}\right)\right] dt \\
&= \iint_R \left[\frac{\partial}{\partial u}\left(\mathbf{G} \bullet \left(\frac{\partial \mathbf{r}}{\partial t} \times \frac{\partial \mathbf{r}}{\partial v}\right)\right)\right. \\
&\qquad \left. + \frac{\partial}{\partial v}\left(\mathbf{G} \bullet \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial t}\right)\right)\right] du\, dv,
\end{aligned}$$

by Green's Theorem.

c) Using the results of (a) and (b), we calculate

$$\begin{aligned}
\frac{d}{dt}\iint_{\mathcal{S}_t} \mathbf{F} \bullet \hat{\mathbf{N}}\, dS &= \iint_R \frac{\partial}{\partial t}\left[\mathbf{G} \bullet \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right)\right] du\, dv \\
&= \iint_R \frac{\partial \mathbf{F}}{\partial t} \bullet \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right) du\, dv \\
&\quad + \iint_R (\mathbf{div}\,\mathbf{F})\frac{\partial \mathbf{r}}{\partial t} \bullet \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right) du\, dv \\
&\quad + \iint_R \left[\frac{\partial}{\partial u}\left(\mathbf{G} \bullet \left(\frac{\partial \mathbf{r}}{\partial t} \times \frac{\partial \mathbf{r}}{\partial v}\right)\right)\right. \\
&\qquad \left. + \frac{\partial}{\partial v}\left(\mathbf{G} \bullet \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial t}\right)\right)\right] du\, dv \\
&= \iint_{\mathcal{S}_t} \frac{\partial \mathbf{F}}{\partial t} \bullet \hat{\mathbf{N}}\, dS + \iint_{\mathcal{S}_t} (\mathbf{div}\,\mathbf{F})\mathbf{v}_S \bullet \hat{\mathbf{N}}\, dS
\end{aligned}$$

$$+ \oint\!\!\!\oint \mathcal{C}_t (\mathbf{F} \times \mathbf{v}_C) \bullet d\mathbf{r}.$$

5. We have

$$\frac{1}{\Delta t}\left[\iiint_{D_{t+\Delta t}} f(\mathbf{r}, t+\Delta t)\, dV - \iiint_{D_t} f(\mathbf{r}, t)\, dV\right]$$
$$= \iiint_{D_t} \frac{f(\mathbf{r}, t+\Delta t) - f(\mathbf{r}, t)}{\Delta t}\, dV$$
$$+ \frac{1}{\Delta t}\iiint_{D_{t+\Delta t}-D_t} f(\mathbf{r}, t+\Delta t)\, dV$$
$$- \frac{1}{\Delta t}\iiint_{D_t-D_{t+\Delta t}} f(\mathbf{r}, t+\Delta t)\, dV$$
$$= I_1 + I_2 - I_3.$$

Evidently $I_1 \to \iiint_{D_t} \frac{\partial f}{\partial t}\, dV$ as $\Delta t \to 0$.

I_2 and I_3 are integrals over the parts of ΔD_t where the surface $\S_t$ is moving outwards and inwards, respectively, that is, where $\mathbf{v}_S \bullet \hat{\mathbf{N}}$ is, respectively, positive and negative. Since $dV = |\mathbf{v}_S \bullet \hat{\mathbf{N}}|\, dS\, \Delta T$, we have

$$I_2 - I_3 = \iint_{S_t} f(\mathbf{r}, t+\Delta t)\mathbf{v}_S \bullet \hat{\mathbf{N}}\, dS$$
$$= \iint_{S_t} f(\mathbf{r}, t)\mathbf{v}_S \bullet \hat{\mathbf{N}}\, dS$$
$$+ \iint_{S_t} \Big(f(\mathbf{r}, t+\Delta t) - f(\mathbf{r}, t)\Big)\mathbf{v}_S \bullet \hat{\mathbf{N}}\, dS.$$

The latter integral approaches 0 as $\Delta t \to 0$ because

$$\left|\iint_{S_t} \Big(f(\mathbf{r}, t+\Delta t) - f(\mathbf{r}, t)\Big)\mathbf{v}_S \bullet \hat{\mathbf{N}}\, dS\right|$$
$$\le \max|\mathbf{v}_S| \left|\frac{\partial f}{\partial t}\right| (\text{area of } S_t)\Delta t.$$