CHAPTER 17. ORDINARY DIFFERENTIAL EQUATIONS

NOTE: SECTIONS 17.2 AND 17.5 AND THE REVIEW EXERCISES FOR CHAPTER 17 IN CALCULUS OF SEVERAL VARIABLES HAVE MORE EXERCISES THAN THE CORRESPONDING VERSIONS IN CALCULUS: A COMPLETE COURSE AND SINGLE-VARIABLE CALCULUS. ONLY THE SOLUTIONS FOR THOSE UNITS ARE GIVEN HERE; FOR THE OTHERS SEE CHAPTER 17.

Section 17.2 Solving First-Order Equations (page 913)

1. $\frac{dy}{dx} = \frac{y}{2x}$

$2\frac{dy}{y} = \frac{dx}{x}$

$2\ln y = \ln x + C_1 \qquad \Rightarrow \qquad y^2 = Cx$

2. $\frac{dy}{dx} = \frac{3y-1}{x}$

$\int \frac{dy}{3y-1} = \int \frac{dx}{x}$

$\frac{1}{3}\ln|3y-1| = \ln|x| + \frac{1}{3}\ln C$

$\frac{3y-1}{x^3} = C$

$\Rightarrow \quad y = \frac{1}{3}(1 + Cx^3).$

3. $\frac{dy}{dx} = \frac{x^2}{y^2} \qquad \Rightarrow \qquad y^2\,dy = x^2\,dx$

$\frac{y^3}{3} = \frac{x^3}{3} + C_1, \qquad \text{or} \quad x^3 - y^3 = C$

4. $\frac{dy}{dx} = x^2y^2$

$\int \frac{dy}{y^2} = \int x^2\,dx$

$-\frac{1}{y} = \frac{1}{3}x^3 + \frac{1}{3}C$

$\Rightarrow \quad y = -\frac{3}{x^3 + C}.$

5. $\frac{dY}{dt} = tY \qquad \Rightarrow \qquad \frac{dY}{Y} = t\,dt$

$\ln Y = \frac{t^2}{2} + C_1, \qquad \text{or} \quad Y = Ce^{t^2/2}$

6. $\frac{dx}{dt} = e^x \sin t$

$\int e^{-x}\,dx = \int \sin t\,dt$

$-e^{-x} = -\cos t - C$

$\Rightarrow \quad x = -\ln(\cos t + C).$

7. $\frac{dy}{dx} = 1 - y^2 \qquad \Rightarrow \qquad \frac{dy}{1-y^2} = dx$

$\frac{1}{2}\left(\frac{1}{1+y} + \frac{1}{1-y}\right)dy = dx$

$\frac{1}{2}\ln\left|\frac{1+y}{1-y}\right| = x + C_1$

$\frac{1+y}{1-y} = Ce^{2x} \qquad \text{or} \quad y = \frac{Ce^{2x}-1}{Ce^{2x}+1}$

8. $\frac{dy}{dx} = 1 + y^2$

$\int \frac{dy}{1+y^2} = \int dx$

$\tan^{-1} y = x + C$

$\Rightarrow \quad y = \tan(x + C).$

9. $\frac{dy}{dt} = 2 + e^y \qquad \Rightarrow \qquad \frac{dy}{2+e^y} = dt$

$\int \frac{e^{-y}\,dy}{2e^{-y}+1} = \int dt$

$-\frac{1}{2}\ln(2e^{-y}+1) = t + C_1$

$2e^{-y} + 1 = C_2e^{-2t}, \qquad \text{or} \quad y = -\ln\left(Ce^{-2t} - \frac{1}{2}\right)$

10. We have

$$\frac{dy}{dx} = y^2(1-y)$$

$$\int \frac{dy}{y^2(1-y)} = \int dx = x + K.$$

Expand the left side in partial fractions:

$$\frac{1}{y^2(1-y)} = \frac{A}{y} + \frac{B}{y^2} + \frac{C}{1-y} = \frac{A(y-y^2) + B(1-y) + Cy^2}{y^2(1-y)}$$

$$\Rightarrow \begin{cases} -A + C = 0; \\ A - B = 0; \\ B = 1. \end{cases} \Rightarrow A = B = C = 1.$$

Hence,

$$\int \frac{dy}{y^2(1-y)} = \int \left(\frac{1}{y} + \frac{1}{y^2} + \frac{1}{1-y}\right)dy = \ln|y| - \frac{1}{y} - \ln|1-y|.$$

Therefore,

$$\ln\left|\frac{y}{1-y}\right| - \frac{1}{y} = x + K.$$

11. $\dfrac{dy}{dx} = \dfrac{x+y}{x-y}$ Let $y = vx$

$$v + x\frac{dv}{dx} = \frac{x(1+v)}{x(1-v)}$$
$$x\frac{dv}{dx} = \frac{1+v}{1-v} - v = \frac{1+v^2}{1-v}$$
$$\int \frac{1-v}{1+v^2}\,dv = \int \frac{dx}{x}$$
$$\tan^{-1} v - \frac{1}{2}\ln(1+v^2) = \ln|x| + C_1$$
$$\tan^{-1}(y/x) - \frac{1}{2}\ln\frac{x^2+y^2}{x^2} = \ln|x| + C_1$$
$$2\tan^{-1}(y/x) - \ln(x^2+y^2) = C.$$

12. $\dfrac{dy}{dx} = \dfrac{xy}{x^2+2y^2}$ Let $y = vx$

$$v + x\frac{dv}{dx} = \frac{vx^2}{(1+2v^2)x^2}$$
$$x\frac{dv}{dx} = \frac{v}{1+2v^2} - v = -\frac{2v^3}{1+2v^2}$$
$$\int \frac{1+2v^2}{v^3}\,dv = -2\int \frac{dx}{x}$$
$$-\frac{1}{2v^2} + 2\ln|v| = -2\ln|x| + C_1$$
$$-\frac{x^2}{2y^2} + 2\ln|y| = C_1$$
$$x^2 - 4y^2\ln|y| = Cy^2.$$

13. $\dfrac{dy}{dx} = \dfrac{x^2+xy+y^2}{x^2}$ Let $y = vx$

$$v + x\frac{dv}{dx} = \frac{x^2(1+v+v^2)}{x^2}$$
$$\int \frac{dv}{1+v^2} = \int \frac{dx}{x}$$
$$\tan^{-1} v = \ln|x| + C$$
$$\frac{y}{x} = \tan\Big(\ln|x| + C\Big)$$
$$y = x\tan\Big(\ln|x| + C\Big).$$

14. $\dfrac{dy}{dx} = \dfrac{x^3+3xy^2}{3x^2y+y^3}$ Let $y = vx$

$$v + x\frac{dv}{dx} = \frac{x^3(1+3v^2)}{x^3(3v+v^3)}$$
$$x\frac{dv}{dx} = \frac{1+3v^2}{3v+v^3} - v = \frac{1-v^4}{v(3+v^2)}$$
$$\int \frac{(3+v^2)v\,dv}{1-v^4} = \int \frac{dx}{x} \qquad \text{Let } u = v^2, \ du = 2v\,dv$$
$$\frac{1}{2}\int \frac{3+u}{1-u^2}\,du = \ln|x| + C_1$$
$$\frac{3}{4}\ln\left|\frac{u+1}{u-1}\right| - \frac{1}{4}\ln|1-u^2| = \ln|x| + C_1$$
$$3\ln\left|\frac{y^2+x^2}{y^2-x^2}\right| - \ln\left|\frac{x^4-y^4}{x^4}\right| = 4\ln|x| + C_2$$
$$\ln\left|\left(\frac{x^2+y^2}{x^2-y^2}\right)^3 \frac{1}{x^4-y^4}\right| = C_2$$
$$\ln\left|\frac{(x^2+y^2)^2}{(x^2-y^2)^4}\right| = C_2$$
$$x^2+y^2 = C(x^2-y^2)^2.$$

15. $x\dfrac{dy}{dx} = y + x\cos^2\left(\dfrac{y}{x}\right)$ (let $y = vx$)

$$xv + x^2\frac{dv}{dx} = vx + x\cos^2 v$$
$$x\frac{dv}{dx} = \cos^2 v$$
$$\sec^2 v\,dv = \frac{dx}{x}$$
$$\tan v = \ln|x| + \ln|C|$$
$$\tan\left(\frac{y}{x}\right) = \ln|Cx|$$
$$y = x\tan^{-1}(\ln|Cx|).$$

16. $\dfrac{dy}{dx} = \dfrac{y}{x} - e^{-y/x}$ (let y=vx)

$$v + x\frac{dv}{dx} = v - e^{-v}$$
$$e^v\,dv = -\frac{dx}{x}$$
$$e^v = -\ln|x| + \ln|C|$$
$$e^{y/x} = \ln\left|\frac{C}{x}\right|$$
$$y = x\ln\ln\left|\frac{C}{x}\right|.$$

17. $\frac{dy}{dx} - \frac{2}{x}y = x^2$ (linear)

$$\mu = \exp\left(\int -\frac{2}{x}\,dx\right) = \frac{1}{x^2}$$

$$\frac{1}{x^2}\frac{dy}{dx} - \frac{2}{x^3}y = 1$$

$$\frac{d}{dx}\frac{y}{x^2} = 1$$

$$\frac{y}{x^2} = x + C, \qquad \text{so} \quad y = x^3 + Cx^2$$

18. We have $\frac{dy}{dx} + \frac{2y}{x} = \frac{1}{x^2}$. Let $\mu = \int \frac{2}{x}\,dx = 2\ln x = \ln x^2$, then $e^\mu = x^2$, and

$$\frac{d}{dx}(x^2y) = x^2\frac{dy}{dx} + 2xy$$
$$= x^2\left(\frac{dy}{dx} + \frac{2y}{x}\right) = x^2\left(\frac{1}{x^2}\right) = 1$$
$$\Rightarrow \quad x^2y = \int dx = x + C$$
$$\Rightarrow \quad y = \frac{1}{x} + \frac{C}{x^2}.$$

19. $\frac{dy}{dx} + 2y = 3 \qquad \mu = \exp\left(\int 2\,dx\right) = e^{2x}$

$$\frac{d}{dx}(e^{2x}y) = e^{2x}(y' + 2y) = 3e^{2x}$$

$$e^{2x}y = \frac{3}{2}e^{2x} + C \quad \Rightarrow \quad y = \frac{3}{2} + Ce^{-2x}$$

20. We have $\frac{dy}{dx} + y = e^x$. Let $\mu = \int dx = x$, then $e^\mu = e^x$, and

$$\frac{d}{dx}(e^xy) = e^x\frac{dy}{dx} + e^xy = e^x\left(\frac{dy}{dx} + y\right) = e^{2x}$$
$$\Rightarrow \quad e^xy = \int e^{2x}\,dx = \frac{1}{2}e^{2x} + C.$$

Hence, $y = \frac{1}{2}e^x + Ce^{-x}$.

21. $\frac{dy}{dx} + y = x \qquad \mu = \exp\left(\int 1\,dx\right) = e^x$

$$\frac{d}{dx}(e^xy) = e^x(y' + y) = xe^x$$

$$e^xy = \int xe^x\,dx = xe^x - e^x + C$$

$$y = x - 1 + Ce^{-x}$$

22. We have $\frac{dy}{dx} + 2e^xy = e^x$. Let $\mu = \int 2e^x\,dx = 2e^x$, then

$$\frac{d}{dx}\left(e^{2e^x}y\right) = e^{2e^x}\frac{dy}{dx} + 2e^xe^{2e^x}y$$
$$= e^{2e^x}\left(\frac{dy}{dx} + 2e^xy\right) = e^{2e^x}e^x.$$

Therefore,

$$e^{2e^x}y = \int e^{2e^x}e^x\,dx \quad \text{Let } u = 2e^x, \; du = 2e^x\,dx$$
$$= \frac{1}{2}\int e^u\,du = \frac{1}{2}e^{2e^x} + C.$$

Hence, $y = \frac{1}{2} + Ce^{-2e^x}$.

23. $\frac{dy}{dt} + 10y = 1, \qquad y\left(\frac{1}{10}\right) = \frac{2}{10}$

$$\mu = \int 10\,dt = 10t$$

$$\frac{d}{dt}(e^{10t}y) = e^{10t}\frac{dy}{dt} + 10e^{10t}y = e^{10t}$$

$$e^{10t}y(t) = \frac{1}{10}e^{10t} + C$$

$$y\left(\frac{1}{10}\right) = \frac{2}{10} \;\Rightarrow\; \frac{2e}{10} = \frac{e}{10} + C \;\Rightarrow\; C = \frac{e}{10}$$

$$y = \frac{1}{10} + \frac{1}{10}e^{1-10t}.$$

24. $\frac{dy}{dx} + 3x^2y = x^2, \qquad y(0) = 1$

$$\mu = \int 3x^2\,dx = x^3$$

$$\frac{d}{dx}(e^{x^3}y) = e^{x^3}\frac{dy}{dx} + 3x^2e^{x^3}y = x^2e^{x^3}$$

$$e^{x^3}y = \int x^2e^{x^3}\,dx = \frac{1}{3}e^{x^3} + C$$

$$y(0) = 1 \;\Rightarrow\; 1 = \frac{1}{3} + C \;\Rightarrow\; C = \frac{2}{3}$$

$$y = \frac{1}{3} + \frac{2}{3}e^{-x^3}.$$

25. $x^2y' + y = x^2e^{1/x}, \qquad y(1) = 3e$

$$y' + \frac{1}{x^2}y = e^{1/x}$$

$$\mu = \int \frac{1}{x^2}\,dx = -\frac{1}{x}$$

$$\frac{d}{dx}\left(e^{-1/x}y\right) = e^{-1/x}\left(y' + \frac{1}{x^2}y\right) = 1$$

$$e^{-1/x}y = \int 1\,dx = x + C$$

$$y(1) = 3e \;\Rightarrow\; 3 = 1 + C \;\Rightarrow\; C = 2$$

$$y = (x + 2)e^{1/x}.$$

26. $y' + (\cos x)y = 2xe^{-\sin x}, \qquad y(\pi) = 0$

$$\mu = \int \cos x\,dx = \sin x$$
$$\frac{d}{dx}(e^{\sin x}y) = e^{\sin x}(y' + (\cos x)y) = 2x$$
$$e^{\sin x}y = \int 2x\,dx = x^2 + C$$
$$y(\pi) = 0 \ \Rightarrow\ 0 = \pi^2 + C \ \Rightarrow\ C = -\pi^2$$
$$y = (x^2 - \pi^2)e^{-\sin x}.$$

27. $y(x) = 2 + \int_0^x \frac{t}{y(t)}\,dt \quad \Longrightarrow \quad y(0) = 2$

$$\frac{dy}{dx} = \frac{x}{y}, \qquad \text{i.e. } y\,dy = x\,dx$$
$$y^2 = x^2 + C$$
$$2^2 = 0^2 + C \quad \Longrightarrow \quad C = 4$$
$$y = \sqrt{4 + x^2}.$$

28. $y(x) = 1 + \int_0^x \frac{(y(t))^2}{1+t^2}\,dt \quad \Longrightarrow \quad y(0) = 1$

$$\frac{dy}{dx} = \frac{y^2}{1+x^2}, \qquad \text{i.e. } dy/y^2 = dx/(1+x^2)$$
$$-\frac{1}{y} = \tan^{-1} x + C$$
$$-1 = 0 + C \quad \Longrightarrow \quad C = -1$$
$$y = 1/(1 - \tan^{-1} x).$$

29. $y(x) = 1 + \int_1^x \frac{y(t)}{t(t+1)}\,dt \quad \Longrightarrow \quad y(1) = 1$

$$\frac{dy}{dx} = \frac{y}{x(x+1)}, \qquad \text{for } x > 0$$
$$\frac{dy}{y} = \frac{dx}{x(x+1)} = \frac{dx}{x} - \frac{dx}{x+1}$$
$$\ln y = \ln \frac{x}{x+1} + \ln C$$
$$y = \frac{Cx}{x+1}, \quad \Longrightarrow \quad 1 = C/2$$
$$y = \frac{2x}{x+1}.$$

30. $y(x) = 3 + \int_0^x e^{-y}\,dt \quad \Longrightarrow \quad y(0) = 3$

$$\frac{dy}{dx} = e^{-y}, \qquad \text{i.e. } e^y\,dy = dx$$
$$e^y = x + C \quad \Longrightarrow \quad y = \ln(x + C)$$
$$3 = y(0) = \ln C \quad \Longrightarrow \quad C = e^3$$
$$y = \ln(x + e^3).$$

31. We require $\frac{dy}{dx} = \frac{2x}{1+y^2}$. Thus

$$\int (1+y^2)\,dy = \int 2x\,dx$$
$$y + \frac{1}{3}y^3 = x^2 + C.$$

Since $(2, 3)$ lies on the curve, $12 = 4 + C$. Thus $C = 8$ and $y + \frac{1}{3}y^3 - x^2 = 8$, or $3y + y^3 - 3x^2 = 24$.

32. $\frac{dy}{dx} = 1 + \frac{2y}{x} \qquad$ Let $y = vx$

$$v + x\frac{dv}{dx} = 1 + 2v$$
$$x\frac{dv}{dx} = 1 + v$$
$$\int \frac{dv}{1+v} = \int \frac{dx}{x}$$
$$\ln|1+v| = \ln|x| + C_1$$
$$1 + \frac{y}{x} = Cx \quad \Rightarrow \quad x + y = Cx^2.$$

Since $(1, 3)$ lies on the curve, $4 = C$. Thus the curve has equation $x + y = 4x^2$.

33. If $\xi = x - x_0$, $\eta = y - y_0$, and

$$\frac{dy}{dx} = \frac{ax + by + c}{ex + fy + g},$$

then

$$\frac{d\eta}{d\xi} = \frac{dy}{dx} = \frac{a(\xi + x_0) + b(\eta + y_0) + c}{e(\xi + x_0) + f(\eta + y_0) + g}$$
$$= \frac{a\xi + b\eta + (ax_0 + by_0 + c)}{e\xi + f\eta + (ex_0 + fy_0 + g)}$$
$$= \frac{a\xi + b\eta}{e\xi + f\eta}$$

provided x_0 and y_0 are chosen such that

$$ax_0 + by_0 + c = 0, \qquad \text{and} \qquad ex_0 + fy_0 + g = 0.$$

34. The system $x_0 + 2y_0 - 4 = 0$, $2x_0 - y_0 - 3 = 0$ has solution $x_0 = 2$, $y_0 = 1$. Thus, if $\xi = x - 2$ and $\eta = y - 1$, where

$$\frac{dy}{dx} = \frac{x + 2y - 4}{2x - y - 3},$$

then

$$\frac{d\eta}{d\xi} = \frac{\xi + 2\eta}{2\xi - \eta} \qquad \text{Let } \eta = v\xi$$
$$v + \xi\frac{dv}{d\xi} = \frac{1+2v}{2-v}$$
$$\xi\frac{dv}{d\xi} = \frac{1+2v}{2-v} - v = \frac{1+v^2}{2-v}$$
$$\int \left(\frac{2-v}{1+v^2}\right) dv = \int \frac{d\xi}{\xi}$$
$$2\tan^{-1} v - \frac{1}{2}\ln(1+v^2) = \ln|\xi| + C_1$$
$$4\tan^{-1}\frac{\eta}{\xi} - \ln(\xi^2 + \eta^2) = C.$$

Hence the solution of the original equation is

$$4\tan^{-1}\frac{y-1}{x-2} - \ln\Big((x-2)^2+(y-1)^2\Big) = C.$$

35. $(xy^2+y)\,dx + (x^2y+x)\,dy = 0$
$d\left(\frac{1}{2}x^2y^2+xy\right)=0$
$x^2y^2+2xy=C.$

36. $(e^x\sin y+2x)\,dx+(e^x\cos y+2y)\,dy=0$
$d(e^x\sin y+x^2+y^2)=0$
$e^x\sin y+x^2+y^2=C.$

37. $e^{xy}(1+xy)\,dx+x^2e^{xy}\,dy=0$
$d\Big(xe^{xy}\Big)=0 \quad\Rightarrow\quad xe^{xy}=C.$

38. $\left(2x+1-\frac{y^2}{x^2}\right)dx+\frac{2y}{x}\,dy=0$
$d\left(x^2+x+\frac{y^2}{x}\right)=0$
$x^2+x+\frac{y^2}{x}=C.$

39. $(x^2+2y)\,dx-x\,dy=0$
$M=x^2+2y, \qquad N=-x$
$\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=-\frac{3}{x}$ (indep. of y)
$\frac{d\mu}{\mu}=-\frac{3}{x}\,dx \quad\Rightarrow\quad \mu=\frac{1}{x^3}$
$\left(\frac{1}{x}+\frac{2y}{x^3}\right)dx-\frac{1}{x^2}\,dy=0$
$d\left(\ln|x|-\frac{y}{x^2}\right)=0$
$\ln|x|-\frac{y}{x^2}=C_1$
$y=x^2\ln|x|+Cx^2.$

40. $(xe^x+x\ln y+y)\,dx+\left(\frac{x^2}{y}+x\ln x+x\sin y\right)dy=0$
$M=xe^x+x\ln y+y, \qquad N=\frac{x^2}{y}+x\ln x+x\sin y$
$\frac{\partial M}{\partial y}=\frac{x}{y}+1, \qquad \frac{\partial N}{\partial x}=\frac{2x}{y}+\ln x+1+\sin y$
$\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=\frac{1}{N}\left(-\frac{x}{y}-\ln x-\sin y\right)=-\frac{1}{x}$
$\frac{d\mu}{\mu}=-\frac{1}{x}\,dx \quad\Rightarrow\quad \mu=\frac{1}{x}$
$\left(e^x+\ln y+\frac{y}{x}\right)dx+\left(\frac{x}{y}+\ln x+\sin y\right)dy$
$d\left(e^x+x\ln y+y\ln x-\cos y\right)=0$
$e^x+x\ln y+y\ln x-\cos y=C.$

41. Since $a>b>0$ and $k>0$,

$$\lim_{t\to\infty}x(t)=\lim_{t\to\infty}\frac{ab\Big(e^{(b-a)kt}-1\Big)}{be^{(b-a)kt}-a}=\frac{ab(0-1)}{0-a}=b.$$

42. Since $b>a>0$ and $k>0$,

$$\lim_{t\to\infty}x(t)=\lim_{t\to\infty}\frac{ab\Big(e^{(b-a)kt}-1\Big)}{be^{(b-a)kt}-a}=\lim_{t\to\infty}\frac{ab\Big(1-e^{(a-b)kt}\Big)}{b-ae^{(a-b)kt}}=\frac{ab(1-0)}{b-0}=a.$$

43. The solution given, namely

$$x=\frac{ab\left(e^{(b-a)kt}-1\right)}{be^{(b-a)kt}-a},$$

is indeterminate (0/0) if $a=b$.
If $a=b$ the original differential equation becomes

$$\frac{dx}{dt}=k(a-x)^2,$$

which is separable and yields the solution

$$\frac{1}{a-x}=\int\frac{dx}{(a-x)^2}=k\int dt=kt+C.$$

Since $x(0)=0$, we have $C=\frac{1}{a}$, so $\frac{1}{a-x}=kt+\frac{1}{a}$.
Solving for x, we obtain

$$x=\frac{a^2kt}{1+akt}.$$

This solution also results from evaluating the limit of solution obtained for the case $a\neq b$ as b approaches a (using l'Hôpital's Rule, say).

44. Given that $m\frac{dv}{dt}=mg-kv$, then

$$\int\frac{dv}{g-\frac{k}{m}v}=\int dt$$
$$-\frac{m}{k}\ln\left|g-\frac{k}{m}v\right|=t+C.$$

Since $v(0) = 0$, therefore $C = -\frac{m}{k}\ln g$. Also, $g - \frac{k}{m}v$ remains positive for all $t > 0$, so

$$\frac{m}{k}\ln\frac{g}{g - \frac{k}{m}v} = t$$
$$\frac{g - \frac{k}{m}v}{g} = e^{-kt/m}$$
$$\Rightarrow \quad v = v(t) = \frac{mg}{k}\left(1 - e^{-kt/m}\right).$$

Note that $\lim_{t\to\infty} v(t) = \frac{mg}{k}$. This limiting velocity can be obtained directly from the differential equation by setting $\frac{dv}{dt} = 0$.

45. We proceed by separation of variables:

$$m\frac{dv}{dt} = mg - kv^2$$
$$\frac{dv}{dt} = g - \frac{k}{m}v^2$$
$$\frac{dv}{g - \frac{k}{m}v^2} = dt$$
$$\int \frac{dv}{\frac{mg}{k} - v^2} = \frac{k}{m}\int dt = \frac{kt}{m} + C.$$

Let $a^2 = mg/k$, where $a > 0$. Thus, we have

$$\int \frac{dv}{a^2 - v^2} = \frac{kt}{m} + C$$
$$\frac{1}{2a}\ln\left|\frac{a+v}{a-v}\right| = \frac{kt}{m} + C$$
$$\ln\left|\frac{a+v}{a-v}\right| = \frac{2akt}{m} + C_1 = 2\sqrt{\frac{kg}{m}}\,t + C_1$$
$$\frac{a+v}{a-v} = C_2 e^{2t\sqrt{kg/m}}.$$

Assuming $v(0) = 0$, we get $C_2 = 1$. Thus

$$a + v = e^{2t\sqrt{kg/m}}(a - v)$$
$$v\left(1 + e^{2t\sqrt{kg/m}}\right) = a\left(e^{2t\sqrt{kg/m}} - 1\right)$$
$$= \sqrt{\frac{mg}{k}}\left(e^{2t\sqrt{kg/m}} - 1\right)$$
$$v = \sqrt{\frac{mg}{k}}\,\frac{e^{2t\sqrt{kg/m}} - 1}{e^{2t\sqrt{kg/m}} + 1}$$

Clearly $v \to \sqrt{\frac{mg}{k}}$ as $t \to \infty$. This also follows from setting $\frac{dv}{dt} = 0$ in the given differential equation.

46. The balance in the account after t years is $y(t)$ and $y(0) = 1000$. The balance must satisfy

$$\frac{dy}{dt} = 0.1y - \frac{y^2}{1,000,000}$$
$$\frac{dy}{dt} = \frac{10^5 y - y^2}{10^6}$$
$$\int \frac{dy}{10^5 y - y^2} = \int \frac{dt}{10^6}$$
$$\frac{1}{10^5}\int\left(\frac{1}{y} + \frac{1}{10^5 - y}\right)dy = \frac{t}{10^6} - \frac{C}{10^5}$$
$$\ln|y| - \ln|10^5 - y| = \frac{t}{10} - C$$
$$\frac{10^5 - y}{y} = e^{C-(t/10)}$$
$$y = \frac{10^5}{e^{C-(t/10)} + 1}.$$

Since $y(0) = 1000$, we have

$$1000 = y(0) = \frac{10^5}{e^C + 1} \quad \Rightarrow \quad C = \ln 99,$$

and

$$y = \frac{10^5}{99e^{-t/10} + 1}.$$

The balance after 1 year is

$$y = \frac{10^5}{99e^{-1/10} + 1} \approx \$1,104.01.$$

As $t \to \infty$, the balance can grow to

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty}\frac{10^5}{e^{(4.60-0.1t)} + 1} = \frac{10^5}{0 + 1} = \$100,000.$$

For the account to grow to \$50,000, t must satisfy

$$50,000 = y(t) = \frac{100,000}{99e^{-t/10} + 1}$$
$$\Rightarrow \quad 99e^{-t/10} + 1 = 2$$
$$\Rightarrow \quad t = 10\ln 99 \approx 46 \text{ years.}$$

47. The hyperbolas $xy = C$ satisfy the differential equation

$$y + x\frac{dy}{dx} = 0, \qquad \text{or} \quad \frac{dy}{dx} = -\frac{y}{x}.$$

Curves that intersect these hyperbolas at right angles must therefore satisfy $\frac{dy}{dx} = \frac{x}{y}$, or $x\,dx = y\,dy$, a separated equation with solutions $x^2 - y^2 = C$, which is also a family of rectangular hyperbolas. (Both families are degenerate at the origin for $C = 0$.)

48. Let $x(t)$ be the number of kg of salt in the solution in the tank after t minutes. Thus, $x(0) = 50$. Salt is coming into the tank at a rate of 10 g/L $\times$ 12 L/min $=$ 0.12 kg/min. Since the contents flow out at a rate of 10 L/min, the volume of the solution is increasing at 2 L/min and thus, at any time t, the volume of the solution is $1000 + 2t$ L. Therefore the concentration of salt is $\dfrac{x(t)}{1000+2t}$ L. Hence, salt is being removed at a rate

$$\frac{x(t)}{1000+2t}\ \text{kg/L} \times 10\ \text{L/min} = \frac{5x(t)}{500+t}\ \text{kg/min}.$$

Therefore,

$$\begin{aligned}\frac{dx}{dt} &= 0.12 - \frac{5x}{500+t}\\ \frac{dx}{dt} &+ \frac{5}{500+t}x = 0.12.\end{aligned}$$

Let $\mu = \displaystyle\int \frac{5}{500+t}\,dt = 5\ln|500+t| = \ln(500+t)^5$ for $t > 0$. Then $e^{\mu} = (500+t)^5$, and

$$\begin{aligned}\frac{d}{dt}\Big[(500+t)^5 x\Big] &= (500+t)^5\frac{dx}{dy} + 5(500+t)^4 x\\ &= (500+t)^5\left(\frac{dx}{dy} + \frac{5x}{500+t}\right)\\ &= 0.12(500+t)^5.\end{aligned}$$

Hence,

$$\begin{aligned}(500+t)^5 x &= 0.12\int (500+t)^5\,dt = 0.02(500+t)^6 + C\\ &\Rightarrow x = 0.02(500+t) + C(500+t)^{-5}.\end{aligned}$$

Since $x(0) = 50$, we have $C = 1.25 \times 10^{15}$ and

$$x = 0.02(500+t) + (1.25\times 10^{15})(500+t)^{-5}.$$

After 40 min, there will be

$$x = 0.02(540) + (1.25\times 10^{15})(540)^{-5} = 38.023\ \text{kg}$$

of salt in the tank.

49. If $\mu(y)M(x,y)\,dx + \mu(y)N(x,y)\,dy$ is exact, then

$$\begin{aligned}&\frac{\partial}{\partial y}\Big(\mu(y)M(x,y)\Big) = \frac{\partial}{\partial x}\Big(\mu(y)N(x,y)\Big)\\ &\mu'(y)M + \mu\frac{\partial M}{\partial y} = \mu\frac{\partial N}{\partial x}\\ &\frac{\mu'}{\mu} = \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right).\end{aligned}$$

Thus M and N must be such that

$$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

depends only on y.

50.
$$\begin{aligned}&2y^2(x+y^2)\,dx + xy(x+6y^2)\,dy = 0\\ &(2xy^2+2y^4)\mu(y)\,dx + (x^2y+6xy^3)\mu(y)\,dy = 0\\ &\frac{\partial M}{\partial y} = (4xy+8y^3)\mu(y) + (2xy^2+2y^4)\mu'(y)\\ &\frac{\partial N}{\partial x} = (2xy+6y^3)\mu(y).\end{aligned}$$
For exactness we require
$$\begin{aligned}&(2xy^2+2y^4)\mu'(y) = [(2xy+6y^3) - (4xy+8y^3)]\mu(y)\\ &y(2xy+2y^3)\mu'(y) = -(2xy+2y^3)\mu(y)\\ &y\mu'(y) = -\mu(y) \quad\Rightarrow\quad \mu(y) = \frac{1}{y}\\ &(2xy+2y^3)\,dx + (x^2+6xy^2)\,dy = 0\\ &d(x^2y+2xy^3) = 0 \quad\Rightarrow\quad x^2y+2xy^3 = C.\end{aligned}$$

51. Consider $y\,dx - (2x + y^3e^y)\,dy = 0$.
Here $M = y$, $N = -2x - y^3e^y$, $\dfrac{\partial M}{\partial y} = 1$, and $\dfrac{\partial N}{\partial x} = -2$.
Thus

$$\begin{aligned}&\frac{\mu'}{\mu} = -\frac{3}{y} \quad\Rightarrow\quad \mu = \frac{1}{y^3}\\ &\frac{1}{y^2}\,dx - \left(\frac{2x}{y^3} + e^y\right)dy = 0\\ &d\left(\frac{x}{y^2} - e^y\right) = 0\\ &\frac{x}{y^2} - e^y = C, \qquad\text{or}\qquad x - y^2e^y = Cy^2.\end{aligned}$$

52. If $\mu(xy)$ is an integrating factor for $M\,dx + N\,dy = 0$, then

$$\begin{aligned}&\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N), \qquad\text{or}\\ &x\mu'(xy)M + \mu(xy)\frac{\partial M}{\partial y} = y\mu'(xy)N + \mu(xy)\frac{\partial N}{\partial x}.\end{aligned}$$

Thus M and N will have to be such that the right-hand side of the equation

$$\frac{\mu'(xy)}{\mu(xy)} = \frac{1}{xM - yN}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

depends only on the product xy.

53. For $\left(x\cos x + \dfrac{y^2}{x}\right)dx - \left(\dfrac{x\sin x}{y} + y\right)dy$ we have

$$\begin{aligned}
&M = x\cos x + \frac{y^2}{x}, \qquad N = -\frac{x\sin x}{y} - y\\
&\frac{\partial M}{\partial y} = \frac{2y}{x}, \qquad \frac{\partial N}{\partial x} = -\frac{\sin x}{y} - \frac{x\cos x}{y}\\
&\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = -\left(\frac{\sin x}{y} + \frac{x\cos x}{y} + \frac{2y}{x}\right)\\
&xM - yN = x^2\cos x + y^2 + x\sin x + y^2\\
&\frac{1}{xM - yN}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = -\frac{1}{xy}.
\end{aligned}$$

Thus, an integrating factor is given by

$$\frac{\mu'(t)}{\mu(t)} = -\frac{1}{t} \qquad \Rightarrow \qquad \mu(t) = \frac{1}{t}.$$

We multiply the original equation by $1/(xy)$ to make it exact:

$$\begin{aligned}
&\left(\frac{\cos x}{y} + \frac{y}{x^2}\right)dx - \left(\frac{\sin x}{y^2} + \frac{1}{x}\right)dy = 0\\
&d\left(\frac{\sin x}{y} - \frac{y}{x}\right) = 0\\
&\frac{\sin x}{y} - \frac{y}{x} = C.
\end{aligned}$$

The solution is $x\sin x - y^2 = Cxy$.

Section 17.5 Linear Differential Equations with Constant Coefficients (page 934)

1.
$$\begin{aligned}
&y'' + 7y' + 10y = 0\\
\text{auxiliary eqn}\quad &r^2 + 7r + 10 = 0\\
&(r+5)(r+2) = 0 \quad \Rightarrow \quad r = -5, -2\\
&y = Ae^{-5t} + Be^{-2t}
\end{aligned}$$

2.
$$\begin{aligned}
&y'' - 2y' - 3y = 0\\
\text{auxiliary eqn}\quad &r^2 - 2r - 3 = 0 \quad \Rightarrow \quad r = -1,\ r = 3\\
&y = Ae^{-t} + Be^{3t}
\end{aligned}$$

3.
$$\begin{aligned}
&y'' + 2y' = 0\\
\text{auxiliary eqn}\quad &r^2 + 2r = 0 \quad \Rightarrow \quad r = 0,\ -2\\
&y = A + Be^{-2t}
\end{aligned}$$

4. $4y'' - 4y' - 3y = 0$
$4r^2 - 4r - 3 = 0 \Rightarrow (2r+1)(2r-3) = 0$
Thus, $r_1 = -\frac{1}{2}$, $r_2 = \frac{3}{2}$, and $y = Ae^{-(1/2)t} + Be^{(3/2)t}$.

5.
$$\begin{aligned}
&y'' + 8y' + 16y = 0\\
\text{auxiliary eqn}\quad &r^2 + 8r + 16 = 0 \quad \Rightarrow \quad r = -4,\ -4\\
&y = Ae^{-4t} + Bte^{-4t}
\end{aligned}$$

6. $y'' - 2y' + y = 0$
$r^2 - 2r + 1 = 0 \Rightarrow (r-1)^2 = 0$
Thus, $r = 1,\ 1$, and $y = Ae^t + Bte^t$.

7.
$$\begin{aligned}
&y'' - 6y' + 10y = 0\\
\text{auxiliary eqn}\quad &r^2 - 6r + 10 = 0 \quad \Rightarrow \quad r = 3 \pm i\\
&y = Ae^{3t}\cos t + Be^{3t}\sin t
\end{aligned}$$

8. $9y'' + 6y' + y = 0$
$9r^2 + 6r + 1 = 0 \Rightarrow (3r+1)^2 = 0$
Thus, $r = -\frac{1}{3},\ -\frac{1}{3}$, and $y = Ae^{-(1/3)t} + Bte^{-(1/3)t}$.

9.
$$\begin{aligned}
&y'' + 2y' + 5y = 0\\
\text{auxiliary eqn}\quad &r^2 + 2r + 5 = 0 \quad \Rightarrow \quad r = -1 \pm 2i\\
&y = Ae^{-t}\cos 2t + Be^{-t}\sin 2t
\end{aligned}$$

10. For $y'' - 4y' + 5y = 0$ the auxiliary equation is $r^2 - 4r + 5 = 0$, which has roots $r = 2 \pm i$. Thus, the general solution of the DE is $y = Ae^{2t}\cos t + Be^{2t}\sin t$.

11. For $y'' + 2y' + 3y = 0$ the auxiliary equation is $r^2 + 2r + 3 = 0$, which has solutions $r = -1 \pm \sqrt{2}i$. Thus the general solution of the given equation is
$y = Ae^{-t}\cos(\sqrt{2}t) + Be^{-t}\sin(\sqrt{2}t)$.

12. Given that $y'' + y' + y = 0$, hence $r^2 + r + 1 = 0$. Since $a = 1$, $b = 1$ and $c = 1$, the discriminant is $D = b^2 - 4ac = -3 < 0$ and $-(b/2a) = -\frac{1}{2}$ and $\omega = \sqrt{3}/2$. Thus, the general solution is
$y = Ae^{-(1/2)t}\cos\left(\frac{\sqrt{3}}{2}t\right) + Be^{-(1/2)t}\sin\left(\frac{\sqrt{3}}{2}t\right)$.

13. $\begin{cases} 2y'' + 5y' - 3y = 0\\ y(0) = 1\\ y'(0) = 0 \end{cases}$

The DE has auxiliary equation $2r^2 + 5y - 3 = 0$, with roots $r = \frac{1}{2}$ and $r = -3$. Thus $y = Ae^{t/2} + Be^{-3t}$.
Now $1 = y(0) = A + B$, and $0 = y'(0) = \dfrac{A}{2} - 3B$.
Thus $B = 1/7$ and $A = 6/7$. The solution is
$y = \dfrac{6}{7}e^{t/2} + \dfrac{1}{7}e^{-3t}$.

14. Given that $y'' + 10y' + 25y = 0$, hence $r^2 + 10r + 25 = 0 \Rightarrow (r+5)^2 = 0 \Rightarrow r = -5$. Thus,

$$\begin{aligned}
y &= Ae^{-5t} + Bte^{-5t}\\
y' &= -5e^{-5t}(A + Bt) + Be^{-5t}.
\end{aligned}$$

Since

$$\begin{aligned}
0 &= y(1) = Ae^{-5} + Be^{-5}\\
2 &= y'(1) = -5e^{-5}(A+B) + Be^{-5},
\end{aligned}$$

we have $A = -2e^5$ and $B = 2e^5$.
Thus, $y = -2e^5e^{-5t} + 2te^5e^{-5t} = 2(t-1)e^{-5(t-1)}$.

15. $\begin{cases} y'' + 4y' + 5y = 0 \\ y(0) = 2 \\ y'(0) = 0 \end{cases}$

The auxiliary equation for the DE is $r^2 + 4r + 5 = 0$, which has roots $r = -2 \pm i$. Thus

$$y = Ae^{-2t}\cos t + Be^{-2t}\sin t$$
$$y' = (-2Ae^{-2t} + Be^{-2t})\cos t - (Ae^{-2t} + 2Be^{-2t})\sin t.$$

Now $2 = y(0) = A \Rightarrow A = 2$, and
$2 = y'(0) = -2A + B \Rightarrow B = 6$.
Therefore $y = e^{-2t}(2\cos t + 6\sin t)$.

16. The auxiliary equation $r^2 - (2+\epsilon)r + (1+\epsilon)$ factors to $(r - 1 - \epsilon)(r - 1) = 0$ and so has roots $r = 1 + \epsilon$ and $r = 1$. Thus the DE $y'' - (2+\epsilon)y' + (1+\epsilon)y = 0$ has general solution $y = Ae^{(1+\epsilon)t} + Be^t$. The function $y_\epsilon(t) = \dfrac{e^{(1+\epsilon)t} - e^t}{\epsilon}$ is of this form with $A = -B = 1/\epsilon$. We have, substituting $\epsilon = h/t$,

$$\begin{aligned} \lim_{\epsilon\to 0} y_\epsilon(t) &= \lim_{\epsilon\to 0} \frac{e^{(1+\epsilon)t} - e^t}{\epsilon} \\ &= t \lim_{h\to 0} \frac{e^{t+h} - e^t}{h} \\ &= t\left(\frac{d}{dt}e^t\right) = t\,e^t \end{aligned}$$

which is, along with e^t, a solution of the CASE II DE $y'' - 2y' + y = 0$.

17. Given that $a > 0$, $b > 0$ and $c > 0$:
Case 1: If $D = b^2 - 4ac > 0$ then the two roots are

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Since

$$\begin{aligned} b^2 - 4ac &< b^2 \\ \pm\sqrt{b^2 - 4ac} &< b \\ -b \pm \sqrt{b^2 - 4ac} &< 0 \end{aligned}$$

therefore r_1 and r_2 are negative. The general solution is

$$y(t) = Ae^{r_1 t} + Be^{r_2 t}.$$

If $t \to \infty$, then $e^{r_1 t} \to 0$ and $e^{r_2 t} \to 0$.
Thus, $\lim_{t\to\infty} y(t) = 0$.
Case 2: If $D = b^2 - 4ac = 0$ then the two equal roots $r_1 = r_2 = -b/(2a)$ are negative. The general solution is

$$y(t) = Ae^{r_1 t} + Bte^{r_2 t}.$$

If $t \to \infty$, then $e^{r_1 t} \to 0$ and $e^{r_2 t} \to 0$ at a faster rate than $Bt \to \infty$. Thus, $\lim_{t\to\infty} y(t) = 0$.
Case 3: If $D = b^2 - 4ac < 0$ then the general solution is

$$y = Ae^{-(b/2a)t}\cos(\omega t) + Be^{-(b/2a)t}\sin(\omega t)$$

where $\omega = \dfrac{\sqrt{4ac - b^2}}{2a}$. If $t \to \infty$, then the amplitude of both terms $Ae^{-(b/2a)t} \to 0$ and $Be^{-(b/2a)t} \to 0$. Thus, $\lim_{t\to\infty} y(t) = 0$.

18. The auxiliary equation $ar^2 + br + c = 0$ has roots

$$r_1 = \frac{-b - \sqrt{D}}{2a}, \quad r_2 = \frac{-b + \sqrt{D}}{2a},$$

where $D = b^2 - 4ac$. Note that $a(r_2 - r_1) = \sqrt{D} = -(2ar_1 + b)$. If $y = e^{r_1 t}u$, then $y' = e^{r_1 t}(u' + r_1 u)$, and $y'' = e^{r_1 t}(u'' + 2r_1 u' + r_1^2 u)$. Substituting these expressions into the DE $ay'' + by' + cy = 0$, and simplifying, we obtain

$$e^{r_1 t}(au'' + 2ar_1 u' + bu') = 0,$$

or, more simply, $u'' - (r_2 - r_1)u' = 0$. Putting $v = u'$ reduces this equation to first order:

$$v' = (r_2 - r_1)v,$$

which has general solution $v = Ce^{(r_2 - r_1)t}$. Hence

$$u = \int Ce^{(r_2 - r_1)t}\,dt = Be^{(r_2 - r_1)t} + A,$$

and $y = e^{r_1 t}u = Ae^{r_1 t} + Be^{r_2 t}$.

19. $y''' - 4y'' + 3y' = 0$
Auxiliary: $r^3 - 4r^2 + 3r = 0$
$r(r-1)(r-3) = 0 \Rightarrow r = 0, 1, 3$
General solution: $y = C_1 + C_2 e^t + C_3 e^{3t}$.

20. $y^{(4)} - 2y'' + y = 0$
Auxiliary: $r^4 - 2r^2 + 1 = 0$
$(r^2 - 1)^2 = 0 \Rightarrow r = -1, -1, 1, 1$
General solution: $y = C_1 e^{-t} + C_2 te^{-t} + C_3 e^t + C_4 te^t$.

21. $y^{(4)} + 2y'' + y = 0$
Auxiliary: $r^4 + 2r^2 + 1 = 0$
$(r^2 + 1)^2 = 0 \Rightarrow r = -i, -i, i, i$
General solution:
$y = C_1\cos t + C_2\sin t + C_3 t\cos t + C_4 t\sin t$.

22. $y^{(4)} + 4y^{(3)} + 6y'' + 4y' + y = 0$
Auxiliary: $r^4 + 4r^3 + 6r^2 + 4r + 1 = 0$
$(r+1)^4 = 0 \Rightarrow r = -1, -1, -1, -1$
General solution: $y = e^{-t}(C_1 + C_2 t + C_3 t^2 + C_4 t^3)$.

23. If $y = e^{2t}$, then $y''' - 2y' - 4y = e^{2t}(8 - 4 - 4) = 0$. The auxiliary equation for the DE is $r^3 - 2r - 4 = 0$, for which we already know that $r = 2$ is a root. Dividing the left side by $r - 2$, we obtain the quotient $r^2 + 2r + 2$. Hence the other two auxiliary roots are $-1 \pm i$.
General solution: $y = C_1 e^{2t} + C_2 e^{-t}\cos t + C_3 e^{-t}\sin t$.

24. Aux. eqn: $(r^2 - r - 2)^2(r^2 - 4)^2 = 0$

$$(r+1)^2(r-2)^2(r-2)^2(r+2)^2 = 0$$
$$r = 2, 2, 2, 2, -1, -1, -2, -2.$$

The general solution is

$$y = e^{2t}(C_1 + C_2 t + C_3 t^2 + C_4 t^3) + e^{-t}(C_5 + C_6 t) + e^{-2t}(C_7 + C_8 t).$$

25.
$$x^2 y'' - xy' + y = 0$$
aux: $r(r-1) - r + 1 = 0$
$$r^2 - 2r + 1 = 0$$
$$(r-1)^2 = 0, \quad r = 1, 1.$$
Thus $y = Ax + Bx\ln x$.

26.
$$x^2 y'' - xy' - 3y = 0$$
$$r(r-1) - r - 3 = 0 \Rightarrow r^2 - 2r - 3 = 0$$
$$\Rightarrow (r-3)(r+1) = 0 \Rightarrow r_1 = -1 \text{ and } r_2 = 3$$
Thus, $y = Ax^{-1} + Bx^3$.

27.
$$x^2 y'' + xy' - y = 0$$
aux: $r(r-1) + r - 1 = 0 \quad \Rightarrow \quad r = \pm 1$
$$y = Ax + \frac{B}{x}.$$

28. Consider $x^2 y'' - xy' + 5y = 0$. Since $a = 1$, $b = -1$, and $c = 5$, therefore $(b-a)^2 < 4ac$. Then $k = (a-b)/2a = 1$ and $\omega^2 = 4$. Thus, the general solution is $y = Ax\cos(2\ln x) + Bx\sin(2\ln x)$.

29.
$$x^2 y'' + xy' = 0$$
aux: $r(r-1) + r = 0 \quad \Rightarrow \quad r = 0,\ 0.$
Thus $y = A + B\ln x$.

30. Given that $x^2 y'' + xy' + y = 0$. Since $a = 1$, $b = 1$, $c = 1$ therefore $(b-a)^2 < 4ac$. Then $k = (a-b)/2a = 0$ and $\omega^2 = 1$. Thus, the general solution is $y = A\cos(\ln x) + B\sin(\ln x)$.

31. $x^3 y''' + xy' - y = 0$.
Trying $y = x^r$ leads to the auxiliary equation

$$r(r-1)(r-2) + r - 1 = 0$$
$$r^3 - 3r^2 + 3r - 1 = 0$$
$$(r-1)^3 = 0 \ \Rightarrow \ r = 1, 1, 1.$$

Thus $y = x$ is a solution. To find the general solution, try $y = xv(x)$. Then

$$y' = xv' + v, \quad y'' = xv'' + 2v', \quad y''' = xv''' + 3v''.$$

Now
$$x^3 y''' + xy' - y = x^4 v''' + 3x^3 v'' + x^2 v' + xv - xv = x^2(x^2 v''' + 3xv'' + v'),$$
and y is a solution of the given equation if $v' = w$ is a solution of $x^2 w'' + 3xw' + w = 0$. This equation has auxiliary equation $r(r-1) + 3r + 1 = 0$, that is $(r+1)^2 = 0$, so its solutions are

$$v' = w = \frac{C_2}{x} + \frac{2C_3 \ln x}{x}$$
$$v = C_1 + C_2 \ln x + C_3(\ln x)^2.$$

The general solution of the given equation is, therefore,

$$y = C_1 x + C_2 x\ln x + C_3 x(\ln x)^2.$$

32. Because $y'' + 4y = 0$, therefore $y = A\cos 2t + B\sin 2t$. Now

$$y(0) = 2 \Rightarrow A = 2,$$
$$y'(0) = -5 \Rightarrow B = -\tfrac{5}{2}.$$

Thus, $y = 2\cos 2t - \frac{5}{2}\sin 2t$.
circular frequency = $\omega = 2$, frequency = $\dfrac{\omega}{2\pi} = \dfrac{1}{\pi} \approx 0.318$
period = $\dfrac{2\pi}{\omega} = \pi \approx 3.14$
amplitude = $\sqrt{(2)^2 + (-\frac{5}{2})^2} \simeq 3.20$

33.
$$\begin{cases} y'' + 100y = 0 \\ y(0) = 0 \\ y'(0) = 3 \end{cases}$$
$$y = A\cos(10t) + B\sin(10t)$$
$$A = y(0) = 0, \quad 10B = y'(0) = 3$$
$$y = \frac{3}{10}\sin(10t)$$

34. For $y'' + y = 0$, we have $y = A\sin t + B\cos t$. Since,

$$y(2) = 3 = A\sin 2 + B\cos 2$$
$$y'(2) = -4 = A\cos 2 - B\sin 2,$$

therefore
$$A = 3\sin 2 - 4\cos 2$$
$$B = 4\sin 2 + 3\cos 2.$$

Thus,

$$y = (3\sin 2 - 4\cos 2)\sin t + (4\sin 2 + 3\cos 2)\cos t = 3\cos(t-2) - 4\sin(t-2).$$

35. $\begin{cases} y'' + \omega^2 y = 0 \\ y(a) = A \\ y'(a) = B \end{cases}$

$y = A\cos\Big(\omega(t-a)\Big) + \frac{B}{\omega}\sin\Big(\omega(t-a)\Big)$

36. $y = \mathcal{A}\cos\Big(\omega(t-c)\Big) + \mathcal{B}\sin\Big(\omega(t-c)\Big)$
(easy to calculate $y'' + \omega^2 y = 0$)

$$\begin{aligned} y &= \mathcal{A}\Big(\cos(\omega t)\cos(\omega c) + \sin(\omega t)\sin(\omega c)\Big) \\ &\quad + \mathcal{B}\Big(\sin(\omega t)\cos(\omega c) - \cos(\omega t)\sin(\omega c)\Big) \\ &= \Big(\mathcal{A}\cos(\omega c) - \mathcal{B}\sin(\omega c)\Big)\cos\omega t \\ &\quad + \Big(\mathcal{A}\sin(\omega c) + \mathcal{B}\cos(\omega c)\Big)\sin\omega t \\ &= A\cos\omega t + B\sin\omega t \end{aligned}$$

where $A = \mathcal{A}\cos(\omega c) - \mathcal{B}\sin(\omega c)$ and
$B = \mathcal{A}\sin(\omega c) + \mathcal{B}\cos(\omega c)$

37. If $y = A\cos\omega t + B\sin\omega t$ then

$$\begin{aligned} y'' + \omega^2 y &= -A\omega^2\cos\omega t - B\omega^2\sin\omega t \\ &\qquad + \omega^2(A\cos\omega t + B\sin\omega t) = 0 \end{aligned}$$

for all t. So y is a solution of (†).

38. If $f(t)$ is any solution of (†) then $f''(t) = -\omega^2 f(t)$ for all t. Thus,

$$\begin{aligned} &\frac{d}{dt}\Big[\omega^2\Big(f(t)\Big)^2 + \Big(f'(t)\Big)^2\Big] \\ &= 2\omega^2 f(t)f'(t) + 2f'(t)f''(t) \\ &= 2\omega^2 f(t)f'(t) - 2\omega^2 f(t)f'(t) = 0 \end{aligned}$$

for all t. Thus, $\omega^2\Big(f(t)\Big)^2 + \Big(f'(t)\Big)^2$ is constant. (This can be interpreted as a conservation of energy statement.)

39. If $g(t)$ satisfies (†) and also $g(0) = g'(0) = 0$, then by Exercise 20,

$$\begin{aligned} &\omega^2\Big(g(t)\Big)^2 + \Big(g'(t)\Big)^2 \\ &= \omega^2\Big(g(0)\Big)^2 + \Big(g'(0)\Big)^2 = 0. \end{aligned}$$

Since a sum of squares cannot vanish unless each term vanishes, $g(t) = 0$ for all t.

40. If $f(t)$ is any solution of (†), let $g(t) = f(t) - A\cos\omega t - B\sin\omega t$ where $A = f(0)$ and $B\omega = f'(0)$. Then g is also solution of (†). Also $g(0) = f(0) - A = 0$ and $g'(0) = f'(0) - B\omega = 0$. Thus, $g(t) = 0$ for all t by Exercise 24, and therefore $f(x) = A\cos\omega t + B\sin\omega t$. Thus, it is proved that every solution of (†) is of this form.

41. We are given that $k = -\frac{b}{2a}$ and $\omega^2 = \frac{4ac - b^2}{4a^2}$ which is positive for Case III. If $y = e^{kt}u$, then

$$\begin{aligned} y' &= e^{kt}\Big(u' + ku\Big) \\ y'' &= e^{kt}\Big(u'' + 2ku' + k^2 u\Big). \end{aligned}$$

Substituting into $ay'' + by' + cy = 0$ leads to

$$\begin{aligned} 0 &= e^{kt}\Big(au'' + (2ka + b)u' + (ak^2 + bk + c)u\Big) \\ &= e^{kt}\Big(au'' + 0 + ((b^2/(4a) - (b^2/(2a) + c)u\Big) \\ &= a\,e^{kt}\Big(u'' + \omega^2 u\Big). \end{aligned}$$

Thus u satisfies $u'' + \omega^2 u = 0$, which has general solution

$$u = A\cos(\omega t) + B\sin(\omega t)$$

by the previous problem. Therefore $ay'' + by' + cy = 0$ has general solution

$$y = Ae^{kt}\cos(\omega t) + Be^{kt}\sin(\omega t).$$

42. From Example 9, the spring constant is $k = 9 \times 10^4$ gm/sec^2. For a frequency of 10 Hz (i.e., a circular frequency $\omega = 20\pi$ rad/sec.), a mass m satisfying $\sqrt{k/m} = 20\pi$ should be used. So,

$$m = \frac{k}{400\pi^2} = \frac{9 \times 10^4}{400\pi^2} = 22.8 \text{ gm}.$$

The motion is determined by

$$\begin{cases} y'' + 400\pi^2 y = 0 \\ y(0) = -1 \\ y'(0) = 2 \end{cases}$$

therefore, $y = A\cos 20\pi t + B\sin 20\pi t$ and

$$\begin{aligned} y(0) &= -1 \Rightarrow A = -1 \\ y'(0) &= 2 \Rightarrow B = \frac{2}{20\pi} = \frac{1}{10\pi}. \end{aligned}$$

Thus, $y = -\cos 20\pi t + \frac{1}{10\pi}\sin 20\pi t$, with y in cm and t in second, gives the displacement at time t. The amplitude is $\sqrt{(-1)^2 + (\frac{1}{10\pi})^2} \approx 1.0005$ cm.

43. Frequency $= \dfrac{\omega}{2\pi}$, $\omega^2 = \dfrac{k}{m}$ (k = spring const, m = mass)
Since the spring does not change, $\omega^2 m = k$ (constant)
For $m = 400$ gm, $\omega = 2\pi(24)$ (frequency = 24 Hz)
If $m = 900$ gm, then $\omega^2 = \dfrac{4\pi^2(24)^2(400)}{900}$
so $\omega = \dfrac{2\pi \times 24 \times 2}{3} = 32\pi$.
Thus frequency $= \dfrac{32\pi}{2\pi} = 16$ Hz
For $m = 100$ gm, $\omega = \dfrac{4\pi^2(24)^2 400}{100}$
so $\omega = 96\pi$ and frequency $= \dfrac{\omega}{2\pi} = 48$ Hz.

44. Using the addition identities for cosine and sine,

$$\begin{aligned} y &= e^{kt}[A\cos\omega(t-t_0)B\sin\omega(t-t_0)] \\ &= e^{kt}[A\cos\omega t\cos\omega t_0 + A\sin\omega t\sin\omega t_0 \\ &\quad + B\sin\omega t\cos\omega t_0 - B\cos\omega t\sin\omega t_0] \\ &= e^{kt}[A_1\cos\omega t + B_1\sin\omega t], \end{aligned}$$

where $A_1 = A\cos\omega t_0 - B\sin\omega t_0$ and $B_1 = A\sin\omega t_0 + B\cos\omega t_0$. Under the conditions of this problem we know that $e^{kt}\cos\omega t$ and $e^{kt}\sin\omega t$ are independent solutions of $ay'' + by' + cy = 0$, so our function y must also be a solution, and, since it involves two arbitrary constants, it is a general solution.

45. Expanding the hyperbolic functions in terms of exponentials,

$$\begin{aligned} y &= e^{kt}[A\cosh\omega(t-t_0)B\sinh\omega(t-t_0)] \\ &= e^{kt}\left[\frac{A}{2}e^{\omega(t-t_0)} + \frac{A}{2}e^{-\omega(t-t_0)}\right. \\ &\quad \left. + \frac{B}{2}e^{\omega(t-t_0)} - \frac{B}{2}e^{-\omega(t-t_0)}\right] \\ &= A_1 e^{(k+\omega)t} + B_1 e^{(k-\omega)t} \end{aligned}$$

where $A_1 = (A/2)e^{-\omega t_0} + (B/2)e^{-\omega t_0}$ and $B_1 = (A/2)e^{\omega t_0} - (B/2)e^{\omega t_0}$. Under the conditions of this problem we know that $Rr = k \pm \omega$ are the two real roots of the auxiliary equation $ar^2+br+c = 0$, so $e^{(k\pm\omega)t}$ are independent solutions of $ay'' + by' + cy = 0$, and our function y must also be a solution. Since it involves two arbitrary constants, it is a general solution.

46. $\begin{cases} y'' + 2y' + 5y = 0 \\ y(3) = 2 \\ y'(3) = 0 \end{cases}$

The DE has auxiliary equation $r^2 + 2r + 5 = 0$ with roots $r = -1 \pm 2i$. By the second previous problem, a general solution can be expressed in the form $y = e^{-t}[A\cos 2(t-3) + B\sin 2(t-3)]$ for which

$$\begin{aligned} y' &= -e^{-t}[A\cos 2(t-3) + B\sin 2(t-3)] \\ &\quad + e^{-t}[-2A\sin 2(t-3) + 2B\cos 2(t-3)]. \end{aligned}$$

The initial conditions give

$$\begin{aligned} 2 &= y(3) = e^{-3}A \\ 0 &= y'(3) = -e^{-3}(A + 2B) \end{aligned}$$

Thus $A = 2e^3$ and $B = -A/2 = -e^3$. The IVP has solution

$$y = e^{3-t}[2\cos 2(t-3) - \sin 2(t-3)].$$

47. $\begin{cases} y'' + 4y' + 3y = 0 \\ y(3) = 1 \\ y'(3) = 0 \end{cases}$

The DE has auxiliary equation $r^2 + 4r + 3 = 0$ with roots $r = -2 + 1 = -1$ and $r = -2 - 1 = -3$ (i.e. $k \pm \omega$, where $k = -2$ and $\omega = 1$). By the second previous problem, a general solution can be expressed in the form $y = e^{-2t}[A\cosh(t-3) + B\sinh(t-3)]$ for which

$$\begin{aligned} y' &= -2e^{-2t}[A\cosh(t-3) + B\sinh(t-3)] \\ &\quad + e^{-2t}[A\sinh(t-3) + B\cosh(t-3)]. \end{aligned}$$

The initial conditions give

$$\begin{aligned} 1 &= y(3) = e^{-6}A \\ 0 &= y'(3) = -e^{-6}(-2A + B) \end{aligned}$$

Thus $A = e^6$ and $B = 2A = 2e^6$. The IVP has solution

$$y = e^{6-2t}[\cosh(t-3) + 2\sinh(t-3)].$$

48. Let $u(x) = c - k^2 y(x)$. Then $u(0) = c - k^2 a$.
Also $u'(x) = -k^2 y'(x)$, so $u'(0) = -k^2 b$. We have

$$u''(x) = -k^2 y''(x) = -k^2\Big(c - k^2 y(x)\Big) = -k^2 u(x)$$

This IVP for the equation of simple harmonic motion has solution

$$u(x) = (c - k^2 a)\cos(kx) - kb\sin(kx)$$

so that

$$\begin{aligned} y(x) &= \frac{1}{k^2}\Big(c - u(x)\Big) \\ &= \frac{c}{k^2}\Big(c - (c - k^2 a)\cos(kx) + kb\sin(kx)\Big) \\ &= \frac{c}{k^2}(1 - \cos(kx) + a\cos(kx) + \frac{b}{k}\sin(kx). \end{aligned}$$

49. Since $x'(0) = 0$ and $x(0) = 1 > 1/5$, the motion will be governed by $x'' = -x + (1/5)$ until such time $t > 0$ when $x'(t) = 0$ again.

Let $u = x - (1/5)$. Then $u'' = x'' = -(x - 1/5) = -u$, $u(0) = 4/5$, and $u'(0) = x'(0) = 0$. This simple harmonic motion initial-value problem has solution $u(t) = (4/5)\cos t$. Thus $x(t) = (4/5)\cos t + (1/4)$ and $x'(t) = u'(t) = -(4/5)\sin t$. These formulas remain valid until $t = \pi$ when $x'(t)$ becomes 0 again. Note that $x(\pi) = -(4/5) + (1/5) = -(3/5)$.

Since $x(\pi) < -(1/5)$, the motion for $t > \pi$ will be governed by $x'' = -x - (1/5)$ until such time $t > \pi$ when $x'(t) = 0$ again.

Let $v = x + (1/5)$. Then $v'' = x'' = -(x + 1/5) = -v$, $v(\pi) = -(3/5) + (1/5) = -(2/5)$, and $v'(\pi) = x'(\pi) = 0$. Thius initial-value problem has solution $v(t) = -(2/5)\cos(t - \pi) = (2/5)\cos t$, so that $x(t) = (2/5)\cos t - (1/5)$ and $x'(t) = -(2/5)\sin t$. These formulas remain valid for $t \ge \pi$ until $t = 2\pi$ when x' becomes 0 again. We have $x(2\pi) = (2/5) - (1/5) = 1/5$ and $x'(2\pi) = 0$.

The conditions for stopping the motion are met at $t = 2\pi$; the mass remains at rest thereafter. Thus

$$x(t) = \begin{cases} \frac{4}{5}\cos t + \frac{1}{5} & \text{if } 0 \le t \le \pi \\ \frac{2}{5}\cos t - \frac{1}{5} & \text{if } \pi < t \le 2\pi \\ \frac{1}{5} & \text{if } t > 2\pi \end{cases}$$

Review Exercises 17 (page 945)

SOLUTIONS FOR EXERCISES 1–26 ARE IN CHAPTER 17

27. $$\frac{dy}{dx} = \frac{3y}{x-1} \Rightarrow \int \frac{dy}{y} = 3\frac{dx}{x-1}$$
$$\Rightarrow \ln|y| = \ln|x-1|^3 + \ln|C|$$
$$\Rightarrow y = C(x-1)^3.$$
Since $y = 4$ when $x = 2$, we have $4 = C(2-1)^3 = C$, so the equation of the curve is $y = 4(x-1)^3$.

28. The ellipses $3x^2 + 4y^2 = C$ all satisfy the differential equation

$$6x + 8y\frac{dy}{dx} = 0, \quad \text{or} \quad \frac{dy}{dx} = -\frac{3x}{4y}.$$

A family of curves that intersect these ellipses at right angles must therefore have slopes given by $\dfrac{dy}{dx} = \dfrac{4y}{3x}$. Thus

$$3\int \frac{dy}{y} = 4\int \frac{dx}{x}$$
$$3\ln|y| = 4\ln|x| + \ln|C|.$$

The family is given by $y^3 = Cx^4$.

29. $[(x + A)e^x \sin y + \cos y]\,dx + x[e^x \cos y + B\sin y]\,dy = 0$ is $M\,dx + N\,dy$. We have

$$\frac{\partial M}{\partial y} = (x + A)e^x \cos y - \sin y$$
$$\frac{\partial N}{\partial x} = e^x \cos y + B\sin y + xe^x \cos y.$$

These expressions are equal (and the DE is exact) if $A = 1$ and $B = -1$. If so, the left side of the DE is $d\phi(x, y)$, where

$$\phi(x, y) = xe^x \sin y + x\cos y.$$

The general solution is $xe^x \sin y + x\cos y = C$.

30. $(x^2 + 3y^2)\,dx + xy\,dy = 0$. Multiply by x^n:

$$x^n(x^2 + 3y^2)\,dx + x^{n+1}y\,dy = 0$$

is exact provided $6x^n y = (n+1)x^n y$, that is, provided $n = 5$. In this case the left side is $d\phi$, where

$$\phi(x, y) = \frac{1}{2}x^6y^2 + \frac{1}{8}x^8.$$

The general solution of the given DE is

$$4x^6y^2 + x^8 = C.$$

31. $x^2y'' - x(2 + x\cot x)y' + (2 + x\cot x)y = 0$
If $y = x$, then $y' = 1$ and $y'' = 0$, so the DE is clearly satisfied by y. To find a second, independent solution, try $y = xv(x)$. Then $y' = v + xv'$, and $y'' = 2v' + xv''$. Substituting these expressions into the given DE, we obtain

$$2x^2v' + x^3v'' - (xv + x^2v')(2 + x\cot x)$$
$$+ xv(2 + x\cot x) = 0$$
$$x^3v'' - x^3v'\cot x = 0,$$

or, putting $w = v'$, $w' = (\cot x)w$, that is,

$$\frac{dw}{w} = \frac{\cos x\,dx}{\sin x}$$
$$\ln w = \ln\sin x + \ln C_2$$
$$v' = w = C_2\sin x \;\Rightarrow\; v = C_1 - C_2\cos x.$$

A second solution of the DE is $x\cos x$, and the general solution is
$$y = C_1x + C_2x\cos x.$$

32. $x^2y'' - x(2 + x\cot x)y' + (2 + x\cot x)y = x^3 \sin x$
Look for a particular solution of the form
$y = xu_1(x) + x\cos x u_2(x)$, where

$$\begin{aligned} xu_1' + x\cos x u_2' &= 0 \\ u_1' + (\cos x - x\sin x)u_2' &= x\sin x. \end{aligned}$$

Divide the first equation by x and subtract from the second equation to get

$$-x\sin x u_2' = x\sin x.$$

Thus $u_2' = -1$ and $u_2 = -x$. The first equation now gives $u_1' = \cos x$, so that $u_1 = \sin x$. The general solution of the DE is

$$y = x\sin x - x^2\cos x + C_1x + C_2x\cos x.$$

33. Suppose $y' = f(x, y)$ and $y(x_0) = y_0$, where $f(x, y)$ is continuous on the whole xy-plane and satisfies $|f(x, y)| \le K$ there. By the Fundamental Theorem of Calculus, we have

$$\begin{aligned} y(x) - y_0 &= y(x) - y(x_0) \\ &= \int_{x_0}^{x} y'(t)\,dt = \int_{x_0}^{x} f\Big(t, y(t)\Big)\,dt. \end{aligned}$$

Therefore,

$$|y(x) - y_0| \le K|x - x_0|.$$

Thus $y(x)$ is bounded above and below by the lines $y = y_0 \pm K(x - x_0)$, and cannot have a vertical asymptote anywhere.

Remark: we don't seem to have needed the continuity of $\partial f/\partial y$, only the continuity of f (to enable the use of the Fundamental Theorem).