APPENDICES

Appendix I. Complex Numbers (page A-10)

1. $z = -5 + 2i$, $\mathrm{Re}(z) = -5$, $\mathrm{Im}(z) = 2$

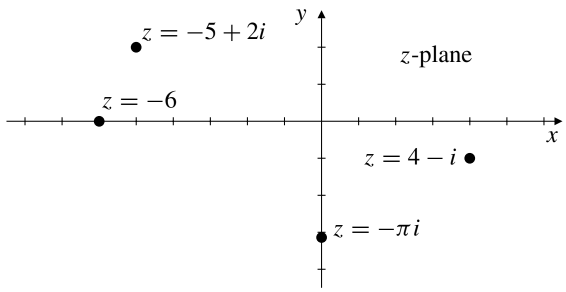

Fig. .1

2. $z = 4 - i$, $\mathrm{Re}(z) = 4$, $\mathrm{Im}(z) = -1$

3. $z = -\pi i$, $\mathrm{Re}(z) = 0$, $\mathrm{Im}(z) = -\pi$

4. $z = -6$, $\mathrm{Re}(z) = -6$, $\mathrm{Im}(z) = 0$

5. $z = -1 + i$, $|z| = \sqrt{2}$, $\mathrm{Arg}\,(z) = 3\pi/4$
$z = \sqrt{2}\,(\cos(3\pi/4) + i\sin(3\pi/4))$

6. $z = -2$, $|z| = 2$, $\mathrm{Arg}\,(z) = \pi$
$z = 2(\cos\pi + i\sin\pi)$

7. $z = 3i$, $|z| = 3$, $\mathrm{Arg}\,(z) = \pi/2$
$z = 3(\cos(\pi/2) + i\sin(\pi/2))$

8. $z = -5i$, $|z| = 5$, $\mathrm{Arg}\,(z) = -\pi/2$
$z = 5(\cos(-\pi/2) + i\sin(-\pi/2))$

9. $z = 1 + 2i$, $|z| = \sqrt{5}$, $\theta = \mathrm{Arg}\,(z) = \tan^{-1}2$
$z = \sqrt{5}(\cos\theta + i\sin\theta)$

10. $z = -2 + i$, $|z| = \sqrt{5}$, $\theta = \mathrm{Arg}\,(z) = \pi - \tan^{-1}(1/2)$
$z = \sqrt{5}(\cos\theta + i\sin\theta)$

11. $z = -3 - 4i$, $|z| = 5$, $\theta = \mathrm{Arg}\,(z) = -\pi + \tan^{-1}(4/3)$
$z = 5(\cos\theta + i\sin\theta)$

12. $z = 3 - 4i$, $|z| = 5$, $\theta = \mathrm{Arg}\,(z) = -\tan^{-1}(4/3)$
$z = 5(\cos\theta + i\sin\theta)$

13. $z = \sqrt{3} - i$, $|z| = 2$, $\mathrm{Arg}\,(z) = -\pi/6$
$z = 2(\cos(-\pi/6) + i\sin(-\pi/6))$

14. $z = -\sqrt{3} - 3i$, $|z| = 2\sqrt{3}$, $\mathrm{Arg}\,(z) = -2\pi/3$
$z = 2\sqrt{3}(\cos(-2\pi/3) + i\sin(-2\pi/3))$

15. $z = 3\cos\dfrac{4\pi}{5} + 3i\sin\dfrac{4\pi}{5}$
$|z| = 3$, $\mathrm{Arg}\,(z) = \dfrac{4\pi}{5}$

16. If $\mathrm{Arg}\,(z) = \dfrac{3\pi}{4}$ and $\mathrm{Arg}\,(w) = \dfrac{\pi}{2}$, then
$\arg(zw) = \dfrac{3\pi}{4} + \dfrac{\pi}{2} = \dfrac{5\pi}{4}$, so
$\mathrm{Arg}\,(zw) = \dfrac{5\pi}{4} - 2\pi = \dfrac{-3\pi}{4}$.

17. If $\mathrm{Arg}\,(z) = -\dfrac{5\pi}{6}$ and $\mathrm{Arg}\,(w) = \dfrac{\pi}{4}$, then
$\arg(z/w) = -\dfrac{5\pi}{6} - \dfrac{\pi}{4} = -\dfrac{13\pi}{12}$, so
$\mathrm{Arg}\,(z/w) = -\dfrac{13\pi}{12} + 2\pi = \dfrac{11\pi}{12}$.

18. $|z| = 2$, $\arg(z) = \pi \;\Rightarrow\; z = 2(\cos\pi + i\sin\pi) = -2$

19. $|z| = 5$, $\theta = \arg(z) = \pi \;\Rightarrow\; \sin\theta = 3/5$, $\cos\theta = 4/5$
$z = 4 + 3i$

20. $|z| = 1$, $\arg(z) = \dfrac{3\pi}{4} \;\Rightarrow\; z = \left(\cos\dfrac{3\pi}{4} + i\sin\dfrac{3\pi}{4}\right)$
$\Rightarrow\; z = -\dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{2}}i$

21. $|z| = \pi$, $\arg(z) = \dfrac{\pi}{6} \;\Rightarrow\; z = \pi\left(\cos\dfrac{\pi}{6} + i\sin\dfrac{\pi}{6}\right)$
$\Rightarrow\; z = \dfrac{\pi\sqrt{3}}{2} + \dfrac{\pi}{2}i$

22. $|z| = 0 \Rightarrow\; z = 0$ for any value of $\arg(z)$

23. $|z| = \dfrac{1}{2}$, $\arg(z) = -\dfrac{\pi}{3} \;\Rightarrow\; z = \dfrac{1}{2}\left(\cos\dfrac{\pi}{3} - i\sin\dfrac{\pi}{3}\right)$
$\Rightarrow\; z = \dfrac{1}{4} - \dfrac{\sqrt{3}}{4}i$

24. $\overline{5 + 3i} = 5 - 3i$

25. $\overline{-3 - 5i} = -3 + 5i$

26. $\overline{4i} = -4i$

27. $\overline{2 - i} = 2 + i$

28. $|z| = 2$ represents all points on the circle of radius 2 centred at the origin.

29. $|z| \le 2$ represents all points in the closed disk of radius 2 centred at the origin.

30. $|z - 2i| \le 3$ represents all points in the closed disk of radius 3 centred at the point $2i$.

31. $|z - 3 + 4i| \le 5$ represents all points in the closed disk of radius 5 centred at the point $3 - 4i$.

32. $\arg(z) = \pi/3$ represents all points on the ray from the origin in the first quadrant, making angle 60° with the positive direction of the real axis.

33. $\pi \le \arg(z) \le 7\pi/4$ represents the closed wedge-shaped region in the third and fourth quadrants bounded by the ray from the origin to $-\infty$ on the real axis and the ray from the origin making angle $-45°$ with the positive direction of the real axis.

34. $(2+5i)+(3-i)=5+4i$

35. $i-(3-2i)+(7-3i)=-3+7+i+2i-3i=4$

36. $(4+i)(4-i)=16-i^2=17$

37. $(1+i)(2-3i)=2+2i-3i-3i^2=5-i$

38. $(a+bi)(\overline{2a-bi})=(a+bi)(2a+bi)=2a^2-b^2+3abi$

39. $(2+i)^3=8+12i+6i^2+i^3=2+11i$

40. $\dfrac{2-i}{2+i}=\dfrac{(2-i)^2}{4-i^2}=\dfrac{3-4i}{5}$

41. $\dfrac{1+3i}{2-i}=\dfrac{(1+3i)(2+i)}{4-i^2}=\dfrac{-1+7i}{5}$

42. $\dfrac{1+i}{i(2+3i)}=\dfrac{1+i}{-3+2i}=\dfrac{(1+i)(-3-2i)}{9+4}=\dfrac{-1-5i}{13}$

43. $\dfrac{(1+2i)(2-3i)}{(2-i)(3+2i)}=\dfrac{8+i}{8+i}=1$

44. If $z=x+yi$ and $w=u+vi$, where x, y, u, and v are real, then

$$\begin{aligned}\overline{z+w}&=\overline{x+u+(y+v)i}\\&=x+u-(y+v)i=x-yi+u-vi=\overline{z}+\overline{w}.\end{aligned}$$

45. Using the fact that $|zw|=|z||w|$, we have

$$\overline{\left(\frac{z}{w}\right)}=\overline{\left(\frac{z\overline{w}}{|w|^2}\right)}=\frac{\overline{z}\,\overline{\overline{w}}}{|w|^2}=\frac{\overline{z}w}{\overline{w}w}=\frac{\overline{z}}{\overline{w}}.$$

46.

$$\begin{aligned}z&=3+i\sqrt{3}=2\sqrt{3}\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)\\w&=-1+i\sqrt{3}=2\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)\\zw&=4\sqrt{3}\left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right)\\\frac{z}{w}&=\sqrt{3}\left(\cos\frac{-\pi}{2}+i\sin\frac{-\pi}{2}\right)=-i\sqrt{3}\end{aligned}$$

47.

$$\begin{aligned}z&=-1+i=\sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)\\w&=3i=3\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)\\zw&=3\sqrt{2}\left(\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}\right)=-3-3i\\\frac{z}{w}&=\frac{\sqrt{2}}{3}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)=\frac{1}{3}+\frac{1}{3}i\end{aligned}$$

48. $\cos(3\theta)+i\sin(3\theta)=(\cos\theta+i\sin\theta)^3$
$=\cos^3\theta+3i\cos^2\theta\sin\theta-3\cos\theta\sin^2\theta-i\sin^3\theta$
Thus

$$\begin{aligned}\cos(3\theta)&=\cos^3\theta-3\cos\theta\sin^2\theta=4\cos^3\theta-3\cos\theta\\\sin(3\theta)&=3\cos^2\theta\sin\theta-\sin^3\theta=3\sin\theta-4\sin^3\theta.\end{aligned}$$

49.
a) $\overline{z}=2/z$ can be rewritten $|z|^2=z\overline{z}=2$, so is satisfied by all numbers z on the circle of radius $\sqrt{2}$ centred at the origin.

b) $\overline{z}=-2/z$ can be rewritten $|z|^2=z\overline{z}=-2$, which has no solutions since the square of $|z|$ is nonnegative for all complex z.

50. If $z=w=-1$, then $zw=1$, so $\sqrt{zw}=1$. But if we use $\sqrt{z}=\sqrt{-1}=i$ and the same value for $\sqrt{w}$, then $\sqrt{z}\sqrt{w}=i^2=-1\ne\sqrt{zw}$.

51. The three cube roots of $-1=\cos\pi+i\sin\pi$ are of the form $\cos\theta+i\sin\theta$ where $\theta=\pi/3$, $\theta=\pi$, and $\theta=5\pi/3$. Thus they are

$$\frac{1}{2}+i\frac{\sqrt{3}}{2},\quad -1,\quad \frac{1}{2}-i\frac{\sqrt{3}}{2}.$$

52. The three cube roots of $-8i=8\left(\cos\dfrac{3\pi}{2}+i\sin\dfrac{3\pi}{2}\right)$ are of the form $2(\cos\theta+i\sin\theta)$ where $\theta=\pi/2$, $\theta=7\pi/6$, and
$\theta=11\pi/6$. Thus they are

$$2i,\quad -\sqrt{3}-i,\quad \sqrt{3}-i.$$

53. The three cube roots of $-1+i=\sqrt{2}\left(\cos\dfrac{3\pi}{4}+i\sin\dfrac{3\pi}{4}\right)$ are of the form $2^{1/6}(\cos\theta+i\sin\theta)$ where $\theta=\pi/4$, $\theta=11\pi/12$, and $\theta=19\pi/12$.

54. The four fourth roots of $4=4(\cos 0+i\sin 0)$ are of the form $\sqrt{2}(\cos\theta+i\sin\theta)$ where $\theta=0$, $\theta=\pi/2$, π, and $\theta=3\pi/2$. Thus they are $\sqrt{2}$, $i\sqrt{2}$, $-\sqrt{2}$, and $-i\sqrt{2}$.

55. The equation $z^4+1-i\sqrt{3}=0$ has solutions that are the four fourth roots of $-1+i\sqrt{3}=2\left(\cos\dfrac{2\pi}{3}+i\sin\dfrac{2\pi}{3}\right)$. Thus they are of the form $2^{1/4}(\cos\theta+i\sin\theta)$, where $\theta=\pi/6$, $2\pi/3$, $7\pi/6$, and $5\pi/3$. They are the complex numbers

$$\pm 2^{1/4}\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right),\quad \pm 2^{1/4}\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right).$$

56. The equation $z^5 + a^5 = 0$ ($a > 0$) has solutions that are the five fifth roots of $-a^5 = a(\cos\pi + i\sin\pi)$; they are of the form $a(\cos\theta + i\sin\theta)$, where $\theta = \pi/5,\ 3\pi/5,\ \pi,\ 7\pi/5$, and $9\pi/5$.

57. The n nth roots of unity are

$$\begin{aligned}
\omega_1 &= 1\\
\omega_2 &= \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\\
\omega_3 &= \cos\frac{4\pi}{n} + i\sin\frac{4\pi}{n} = \omega_2^2\\
\omega_4 &= \cos\frac{6\pi}{n} + i\sin\frac{6\pi}{n} = \omega_2^3\\
&\vdots\\
\omega_n &= \cos\frac{2(n-1)\pi}{n} + i\sin\frac{2(n-1)\pi}{n} = \omega_2^{n-1}.
\end{aligned}$$

Hence

$$\begin{aligned}
\omega_1 + \omega_2 + \omega_3 + \cdots + \omega_n &= 1 + \omega_2 + \omega_2^2 + \cdots + \omega_2^{n-1}\\
&= \frac{1 - \omega_2^n}{1 - \omega_2} = \frac{0}{1 - \omega_2} = 0.
\end{aligned}$$

Appendix II. Complex Functions (page A-19)

In Solutions 1–12, $z = x + yi$ and $w = u + vi$, where x, y, u, and v are real.

1. The function $w = \overline{z}$ transforms the closed rectangle $0 \le x \le 1$, $0 \le y \le 2$ to the closed rectangle $0 \le u \le 1$, $-2 \le v \le 0$.

2. The function $w = \overline{z}$ transforms the line $x + y = 1$ to the line $u - v = 1$.

3. The function $w = z^2$ transforms the closed annular sector $1 \le |z| \le 2$, $\pi/2 \le \arg(z) \le 3\pi/4$ to the closed annular sector $1 \le |w| \le 4$, $\pi \le \arg(w) \le 3\pi/2$.

4. The function $w = z^3$ transforms the closed quarter-circular disk $0 \le |z| \le 2$, $0 \le \arg(z) \le \pi/2$ to the closed three-quarter disk $0 \le |w| \le 8$, $0 \le \arg(w) \le 3\pi/2$.

5. The function $w = 1/z = \overline{z}/|z|^2$ transforms the closed quarter-circular disk $0 \le |z| \le 2$, $0 \le \arg(z) \le \pi/2$ to the closed region lying on or outside the circle $|w| = 1/2$ and in the fourth quadrant, that is, having $-\pi/2 \le \arg(w) \le 0$.

6. The function $w = -iz$ rotates the z-plane $-90°$, so transforms the wedge $\pi/4 \le \arg(z) \le \pi/3$ to the wedge $-\pi/4 \le \arg(z) \le -\pi/6$.

7. The function $w = \sqrt{z}$ transforms the ray $\arg(z) = -\pi/3$ (that is, $\mathrm{Arg}(z) = 5\pi/3$) to the ray $\arg(w) = 5\pi/6$.

8. The function $w = z^2 = x^2 - y^2 + 2xyi$ transforms the line $x = 1$ to $u = 1 - y^2$, $v = 2y$, which is the parabola $v^2 = 4 - 4u$ with vertex at $w = 1$, opening to the left.

9. The function $w = z^2 = x^2 - y^2 + 2xyi$ transforms the line $y = 1$ to $u = x^2 - 1$, $v = 2x$, which is the parabola $v^2 = 4u + 4$ with vertex at $w = -1$ and opening to the right.

10. The function $w = 1/z = (x - yi)/(x^2 + y^2)$ transforms the line $x = 1$ to the curve given parametrically by

$$u = \frac{1}{1 + y^2}, \qquad v = \frac{-y}{1 + y^2}.$$

This curve is, in fact, a circle,

$$u^2 + v^2 = \frac{1 + y^2}{(1 + y^2)^2} = u,$$

with centre $w = 1/2$ and radius $1/2$.

11. The function $w = e^z = e^x\cos y + ie^x\sin y$ transforms the horizontal strip $-\infty < x < \infty$, $\pi/4 \le y \le \pi/2$ to the wedge $\pi/4 \le \arg(w) \le \pi/2$, or, equivalently, $u \ge 0$, $v \ge u$.

12. The function $w = e^{iz} = e^{-y}(\cos x + i\sin x)$ transforms the vertical half-strip $0 < x < \pi/2$, $0 < y < \infty$ to the first-quadrant part of the unit open disk $|w| = e^{-y} < 1$, $0 < \arg(w) = x < \pi/2$, that is $u > 0$, $v > 0$, $u^2 + v^2 < 1$.

13.
$$\begin{aligned}
&f(z) = z^2 = (x + yi)^2 = x^2 - y^2 + 2xyi\\
&u = x^2 - y^2, \qquad v = 2xy\\
&\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x}\\
&f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 2x + 2yi = 2z.
\end{aligned}$$

14.
$$\begin{aligned}
&f(z) = z^3 = (x + yi)^3 = x^3 - 3xy^2 + (3x^2y - y^3)i\\
&u = x^3 - 3xy^2, \qquad v = 3x^2y - y^3\\
&\frac{\partial u}{\partial x} = 3(x^2 - y^2) = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -6xy = -\frac{\partial v}{\partial x}\\
&f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 3(x^2 - y^2 + 2xyi) = 3z^2.
\end{aligned}$$

15.
$$\begin{aligned}
&f(z) = \frac{1}{z} = \frac{x - yi}{x^2 + y^2}\\
&u = \frac{x}{x^2 + y^2}, \qquad v = \frac{-y}{x^2 + y^2}\\
&\frac{\partial u}{\partial x} = \frac{y^2 - x^2}{(x^2 + y^2)^2} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = \frac{-2xy}{(x^2 + y^2)^2} = -\frac{\partial v}{\partial x}\\
&f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{-(x^2 - y^2) + 2xyi}{(x^2 + y^2)^2} = \frac{-(\overline{z})^2}{(z\overline{z})^2} = -\frac{1}{z^2}.
\end{aligned}$$

16. $f(z) = e^{z^2} = e^{x^2-y^2}(\cos(2xy) + i\sin(2xy))$

$u = e^{x^2-y^2}\cos(2xy), \qquad v = e^{x^2-y^2}\sin(2xy)$

$$\frac{\partial u}{\partial x} = e^{x^2-y^2}(2x\cos(2xy) - 2y\sin(2xy)) = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -e^{x^2-y^2}(2y\cos(2xy) + 2x\sin(2xy)) = -\frac{\partial v}{\partial x}$$

$$\begin{aligned} f'(z) &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \\ &= e^{x^2-y^2}[2x\cos(2xy) - 2y\sin(2xy) \\ &\quad + i(2y\cos(2xy) + 2x\sin(2xy))] \\ &= (2x + 2yi)e^{x^2-y^2}(\cos(2xy) + i\sin(2xy)) = 2ze^{z^2}. \end{aligned}$$

17. $e^{yi} = \cos y + i\sin y$ (for real y). Replacing y by $-y$, we get $e^{-yi} = \cos y - i\sin y$ (since cos is even and sin is odd). Adding and subtracting these two formulas gives

$$e^{yi} + e^{-yi} = 2\cos y, \qquad e^{yi} - e^{-yi} = 2i\sin y.$$

Thus $\cos y = \dfrac{e^{yi} + e^{-yi}}{2}$ and $\sin y = \dfrac{e^{yi} - e^{-yi}}{2i}$.

18.
$$\begin{aligned} e^{z+2\pi i} &= e^x(\cos(y + 2\pi) + i\sin(y + 2\pi)) \\ &= e^x(\cos y + i\sin y) = e^z. \end{aligned}$$

Thus e^z is periodic with period $2\pi i$. So is $e^{-z} = 1/e^z$. Since $e^{i(z+2\pi)} = e^{zi+2\pi i} = e^{zi}$, therefore e^{zi} and also e^{-zi} are periodic with period 2π. Hence

$$\cos z = \frac{e^{zi} + e^{-zi}}{2} \text{ and } \sin z = \frac{e^{zi} - e^{-zi}}{2i}$$

are periodic with period 2π, and

$$\cosh z = \frac{e^z + e^{-z}}{2} \text{ and } \sinh z = \frac{e^z - e^{-z}}{2}$$

are periodic with period $2\pi i$.

19.
$$\begin{aligned} \frac{d}{dz}\cos z &= \frac{d}{dz}\frac{e^{zi} + e^{-zi}}{2} = \frac{ie^{zi} - e^{-zi}}{2} = -\sin z \\ \frac{d}{dz}\sin z &= \frac{d}{dz}\frac{e^{zi} - e^{-zi}}{2i} = \frac{ie^{zi} + e^{-zi}}{2i} = \cos z \\ \frac{d}{dz}\cosh z &= \frac{d}{dz}\frac{e^z + e^{-z}}{2} = \frac{e^z - e^{-z}}{2} = \sinh z \\ \frac{d}{dz}\sinh z &= \frac{d}{dz}\frac{e^z - e^{-z}}{2} = \frac{e^z + e^{-z}}{2} = \cosh z \end{aligned}$$

20.
$$\begin{aligned} \cosh(iz) &= \frac{e^{iz} + e^{-iz}}{2} = \cosh z \\ -i\sinh(iz) &= \frac{1}{i}\frac{e^{iz} - e^{-iz}}{2} = \sin z \\ \cos(iz) &= \frac{e^{-z} + e^z}{2} = \cosh z \\ \sin(iz) &= \frac{e^{-z} - e^z}{2i} = i\frac{-e^{-z} + e^z}{2} = i\sinh z \end{aligned}$$

21.
$$\begin{aligned} \cos z = 0 &\Leftrightarrow e^{zi} = -e^{-zi} \Leftrightarrow e^{2zi} = -1 \\ &\Leftrightarrow e^{-2y}[\cos(2x) + i\sin(2x)] = -1 \\ &\Leftrightarrow \sin(2x) = 0, \quad e^{-2y}\cos(2x) = -1 \\ &\Leftrightarrow y = 0,\ \cos(2x) = -1 \\ &\Longleftrightarrow y = 0,\ x = \pm\frac{\pi}{2},\ \pm\frac{3\pi}{2}, \ldots \end{aligned}$$

Thus the only complex zeros of $\cos z$ are its real zeros at $z = (2n + 1)\pi/2$ for integers n.

22.
$$\begin{aligned} \sin z = 0 &\Leftrightarrow e^{zi} = e^{-zi} \Leftrightarrow e^{2zi} = 1 \\ &\Leftrightarrow e^{-2y}[\cos(2x) + i\sin(2x)] = 1 \\ &\Leftrightarrow \sin(2x) = 0, \quad e^{-2y}\cos(2x) = 1 \\ &\Leftrightarrow y = 0,\ \cos(2x) = 1 \\ &\Longleftrightarrow y = 0,\ x = 0, \pm\pi,\ \pm 2\pi, \ldots \end{aligned}$$

Thus the only complex zeros of $\sin z$ are its real zeros at $z = n\pi$ for integers n.

23. By Exercises 20 and 21, $\cosh z = 0$ if and only if $\cos(iz) = 0$, that is, if and only if $z = (2n + 1)\pi i/2$ for integer n.
Similarly, $\sinh z = 0$ if and only if $\sin(iz) = 0$, that is, if and only if $z = n\pi i$ for integer n.

24.
$$\begin{aligned} e^z &= e^{x+yi} = e^x\cos y + ie^x\sin y \\ e^{-z} &= e^{-x-yi} = e^{-x}\cos y - e^{-x}\sin y \\ \cosh z &= \frac{e^z + e^{-z}}{2} = \frac{e^x + e^{-x}}{2}\cos y + i\frac{e^x - e^{-x}}{2}\sin y \\ &= \cosh x\cos y + i\sinh x\sin y \end{aligned}$$

$$\text{Re}(\cosh z) = \cosh x\cos y, \quad \text{Im}(\cosh z) = \sinh x\sin y.$$

25.
$$\begin{aligned} \sinh z &= \frac{e^z - e^{-z}}{2} = \frac{e^x - e^{-x}}{2}\cos y + i\frac{e^x + e^{-x}}{2}\sin y \\ &= \sinh x\cos y + i\cosh x\sin y \end{aligned}$$

$$\text{Re}(\sinh z) = \sinh x\cos y, \quad \text{Im}(\cosh z) = \cosh x\sin y.$$

26.
$$\begin{aligned} e^{iz} &= e^{-y+xi} = e^{-y}\cos x + ie^{-y}\sin x \\ e^{-iz} &= e^{y-xi} = e^y\cos x - ie^y\sin x \\ \cos z &= \frac{e^{iz} + e^{-iz}}{2} = \frac{e^{-y} + e^y}{2}\cos x + i\frac{e^{-y} - e^y}{2}\sin x \\ &= \cos x\cosh y - i\sin x\sinh y \end{aligned}$$

$$\text{Re}(\cos z) = \cos x\cosh y, \quad \text{Im}(\cos z) = -\sin x\sinh y$$

$$\begin{aligned} \sin z &= \frac{e^{iz} - e^{-iz}}{2i} = \frac{e^{-y} - e^y}{2i}\cos x + i\frac{e^{-y} + e^y}{2i}\sin x \\ &= \sin x\cosh y + i\cos x\sinh y \end{aligned}$$

$$\text{Re}(\sin z) = \sin x\cosh y, \quad \text{Im}(\sin z) = \cos x\sinh y.$$

27. $z^2 + 2iz = 0 \Rightarrow z = 0$ or $z = -2i$

28.
$$\begin{aligned} z^2 - 2z + i = 0 \Rightarrow (z - 1)^2 &= 1 - i \\ &= \sqrt{2}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right) \\ \Rightarrow z &= 1 \pm 2^{1/4}\left(\cos\frac{7\pi}{8} + i\sin\frac{7\pi}{8}\right) \end{aligned}$$

29. $z^2 + 2z + 5 = 0 \Rightarrow (z+1)^2 = -4$
$\Rightarrow z = -1 \pm 2i$

30. $z^2 - 2iz - 1 = 0 \Rightarrow (z-i)^2 = 0$
$\Rightarrow z = i$ (double root)

31.
$$\begin{aligned} &z^3 - 3iz^2 - 2z = z(z^2 - 3iz - 2) = 0 \\ &\Rightarrow z = 0 \text{ or } z^2 - 3iz - 2 = 0 \\ &\Rightarrow z = 0 \text{ or } \left(z - \frac{3}{2}i\right)^2 = -\frac{1}{4} \\ &\Rightarrow z = 0 \text{ or } z = \left(\frac{3}{2} \pm \frac{1}{2}\right)i \\ &\Rightarrow z = 0 \text{ or } z = i \text{ or } z = 2i \end{aligned}$$

32.
$$\begin{aligned} &z^4 - 2z^2 + 4 = 0 \ \Rightarrow \ (z^2-1)^2 = -3 \\ &z^2 = 1 - i\sqrt{3} \qquad \text{or} \qquad z^2 = 1 + i\sqrt{3} \\ &z^2 = 2\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right), \quad z^2 = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) \\ &z = \pm\sqrt{2}\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right), \quad \text{or} \\ &z = \pm\sqrt{2}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) \\ &z = \pm\left(\sqrt{\frac{3}{2}} - \frac{i}{\sqrt{2}}\right), \quad z = \pm\left(\sqrt{\frac{3}{2}} + \frac{i}{\sqrt{2}}\right) \end{aligned}$$

33.
$$\begin{aligned} &z^4 + 1 = 0 \ \Rightarrow \ z^2 = i \text{ or } z^2 = -i \\ &\qquad \Rightarrow \ z = \pm\frac{1+i}{\sqrt{2}}, \quad z = \pm\frac{1-i}{\sqrt{2}} \\ &z^4 + 1 = \left(z - \frac{1+i}{\sqrt{2}}\right)\left(z - \frac{1-i}{\sqrt{2}}\right) \\ &\qquad \times \left(z + \frac{1+i}{\sqrt{2}}\right)\left(z + \frac{1-i}{\sqrt{2}}\right) \\ &\quad = \left(\left[z - \frac{1}{\sqrt{2}}\right]^2 + \frac{1}{2}\right)\left(\left[z + \frac{1}{\sqrt{2}}\right]^2 + \frac{1}{2}\right) \\ &\quad = (z^2 - \sqrt{2}z + 1)(z^2 + \sqrt{2}z + 1) \end{aligned}$$

34. Since $P(z) = z^4 - 4z^3 + 12z^2 - 16z + 16$ has real coefficients, if $z_1 = 1 - \sqrt{3}i$ is a zero of $P(z)$, then so is $\overline{z_1}$. Now

$$(z - z_1)(z - \overline{z_1}) = (z-1)^2 + 3 = z^2 - 2z + 4.$$

By long division (details omitted) we discover that

$$\frac{z^4 - 4z^3 + 12z^2 - 16z + 16}{z^2 - 2z + 4} = z^2 - 2z + 4.$$

Thus z_1 and $\overline{z_1}$ are both *double zeros* of $P(z)$. These are the only zeros.

35. Since $P(z) = z^5 + 3z^4 + 4z^3 + 4z^2 + 3z + 1$ has real coefficients, if $z_1 = i$ is a zero of $P(z)$, then so is $z_2 = -i$. Now

$$(z - z_1)(z - z_2) = (z - i)(z + i) = z^2 + 1.$$

By long division (details omitted) we discover that

$$\begin{aligned} \frac{z^5 + 3z^4 + 4z^3 + 4z^2 + 3z + 1}{z^2 + 1} &= z^3 + 3z^2 + 3z + 1 \\ &= (z+1)^3. \end{aligned}$$

Thus $P(z)$ has the five zeros: i, $-i$, -1, -1, and -1.

36. Since $P(z) = z^5 - 2z^4 - 8z^3 + 8z^2 + 31z - 30$ has real coefficients, if $z_1 = -2 + i$ is a zero of $P(z)$, then so is $z_2 = -2 - i$. Now

$$(z - z_1)(z - z_2) = z^2 + 4z + 5.$$

By long division (details omitted) we discover that

$$\begin{aligned} &\frac{z^5 - 2z^4 - 8z^3 + 8z^2 + 31z - 30}{z^2 + 4z + 5} \\ &= z^3 - 6z^2 + 11z - 6. \end{aligned}$$

Observe that $z_3 = 1$ is a zero of $z^3 - 6z^2 + 11z - 6$. By long division again:

$$\frac{z^3 - 6z^2 + 11z - 6}{z - 1} = z^2 - 5z + 6 = (z-2)(z-3).$$

Hence $P(z)$ has the five zeros $-2 + i$, $-2 - i$, 1, 2, and 3.

37. If $w = z^4 + z^3 - 2iz - 3$ and $|z| = 2$, then $|z^4| = 16$ and

$$|w - z^4| = |z^3 - 2iz - 3| \le 8 + 4 + 3 = 15 < 16.$$

By the mapping principle described in the proof of Theorem 2, the image in the w-plane of the circle $|z| = 2$ is a closed curve that winds around the origin the same number of times that the image of z^4 does, namely 4 times.

Appendix III. Continuous Functions (page A-25)

1. To be proved: If $a < b < c$, $f(x) \le g(x)$ for $a \le x \le c$, $\lim_{x\to b} f(x) = L$, and $\lim_{x\to b} g(x) = M$, then $L \le M$.

Proof: Suppose, to the contrary, that $L > M$. Let $\epsilon = (L - M)/3$, so $\epsilon > 0$. There exist numbers $\delta_1 > 0$ and $\delta_2 > 0$ such that if $a \le x \le b$, then

$$|x - b| < \delta_1 \Rightarrow |f(x) - L| < \epsilon$$
$$|x - b| < \delta_2 \Rightarrow |g(x) - M| < \epsilon.$$

Thus if $|x - b| < \delta = \min\{\delta_1, \delta_2, b - a, c - b\}$, then

$$f(x) - g(x) > L - \epsilon - M - \epsilon = L - M - 2\epsilon = \frac{L - M}{3} > 0.$$

This contradicts the fact that $f(x) \le g(x)$ on $[a, b]$. Therefore $L \le M$.

2. To be proved: If $f(x) \le K$ on $[a, b)$ and $(b, c]$, and if $\lim_{x\to b} f(x) = L$, then $L \le K$.

Proof: If $L > K$, then let $\epsilon = (L - K)/2$; thus $\epsilon > 0$. There exists $\delta > 0$ such that $\delta < b - a$ and $\delta < c - b$, and such that if $0 < |x - b| < \delta$, then $|f(x) - L| < \epsilon$. In this case

$$f(x) > L - \epsilon = L - \frac{L - K}{2} > K,$$

which contradicts the fact that $f(x) \le K$ on $[a, b)$ and $(b, c]$. Therefore $L \le K$.

3. Let $\epsilon > 0$ be given. Let $\delta = \epsilon^{1/r}$, $(r > 0)$. Then

$$0 < x < \delta \quad \Rightarrow \quad 0 < x^r < \delta^r = \epsilon.$$

Thus $\lim_{x\to 0+} x^r = 0$.

4. a) Let $f(x) = C$, $g(x) = x$. Let $\epsilon > 0$ be given and let $\delta = \epsilon$. For any real number x, if $|x - a| < \delta$, then

$$|f(x) - f(a)| = |C - C| = 0 < \epsilon,$$
$$|g(x) - g(a)| = |x - a| < \delta = \epsilon.$$

Thus $\lim_{x\to a} f(x) = f(a)$ and $\lim_{x\to a} g(x) = g(a)$, and f and g are both continuous at every real number a.

5. A polynomial is constructed by adding and multiplying finite numbers of functions of the type of f and g in Exercise 4. By Theorem 1(a), such sums and products are continuous everywhere, since their components have been shown to be continuous everywhere.

6. If P and Q are polynomials, they are continuous everywhere by Exercise 5. If $Q(a) \ne 0$, then $\lim_{x\to a} \dfrac{P(x)}{Q(x)} = \dfrac{P(a)}{Q(a)}$ by Theorem 1(a). Hence P/Q is continuous everywhere except at the zeros of Q.

7. Suppose n is a positive integer and $a > 0$. Let $\epsilon > 0$ be given. Let $b = a^{1/n}$, and let $\delta = \min\{a(1 - 2^{-n}), b^{n-1}\epsilon\}$. If $|x - a| < \delta$, then $x > a/2^n$, and if $y = x^{1/n}$, then $y > b/2$. Thus

$$\begin{aligned}\left|x^{1/n} - a^{1/n}\right| &= |y - b| \\ &= \frac{|y^n - b^n|}{y^{n-1} + y^{n-2}b + \cdots + b^{n-1}} \\ &< \frac{|x - a|}{b^{n-1}} < \frac{b^{n-1}\epsilon}{b^{n-1}} = \epsilon.\end{aligned}$$

Thus $\lim_{x\to a} x^{1/n} = a^{1/n}$, and $x^{1/n}$ is continuous at $x = a$.

8. By Exercise 5, x^m is continuous everywhere. By Exercise 7, $x^{1/n}$ is continuous at each $a > 0$. Thus for $a > 0$ we have

$$\begin{aligned}\lim_{x\to a} x^{m/n} = \lim_{x\to a} \left(x^{1/n}\right)^m &= \left(\lim_{x\to a} x^{1/n}\right)^m \\ &= (a^{1/n})^m = a^{m/n},\end{aligned}$$

and $x^{m/n}$ is continuous at each positive number.

9. If m and n are integers and n is odd, then $(-x)^{m/n} = cx^{m/n}$, where $c = (-1)^{m/n}$ is either -1 or 1 depending on the parity of m. Since $x^{m/n}$ is continuous at each positive number a, so is $cx^{m/n}$. Thus $(-x)^{m/n}$ is continuous at each positive number, and $x^{m/n}$ is continuous at each negative number.

If $r = m/n > 0$, then $\lim_{x\to 0+} x^r = 0$ by Exercise 3. Hence $\lim_{x\to 0-} x^r = (-1)^r \lim_{x\to 0+} x^r = 0$, also. Therefore $\lim_{x\to 0} x^r = 0$, and x^r is continuous at $x = 0$.

10. Let $\epsilon > 0$ be given. Let $\delta = \epsilon$. If a is any real number then

$$\left||x| - |a|\right| \le |x - a| < \epsilon \quad \text{if} \quad |x - a| < \delta.$$

Thus $\lim_{x\to a} |x| = |a|$, and the absolute value function is continuous at every real number.

11. By the definition of sin, $P_t = (\cos t, \sin t)$, and $P_a = (\cos a, \sin a)$ are two points on the unit circle $x^2 + y^2 = 1$. Therefore

$$\begin{aligned}|t - a| &= \text{length of the arc from } P_t \text{ to } P_a \\ &> \text{length of the chord from } P_t \text{ to } P_a \\ &= \sqrt{(\cos t - \cos a)^2 + (\sin t - \sin a)^2}.\end{aligned}$$

If $\epsilon > 0$ is given, and $|t - a| < \delta = \epsilon$, then the above inequality implies that

$$|\cos t - \cos a| \le |t - a| < \epsilon,$$
$$|\sin t - \sin a| \le |t - a| < \epsilon.$$

Thus sin is continuous everywhere.

12. The proof that cos is continuous everywhere is almost identical to that for sin in Exercise 11.

13. Let $a > 0$ and $\epsilon > 0$. Let $\delta = \min\left\{\frac{a}{2}, \frac{\epsilon a}{2}\right\}$.
If $|x-a| < \delta$, then $x > \frac{a}{2}$, so $\frac{1}{t} < \frac{2}{a}$ whenever t is between a and x. Thus

$$\begin{aligned}&|\ln x - \ln a|\\ &= \text{area under } y = \frac{1}{t} \text{ between } t = a \text{ and } t = x\\ &< \frac{2}{a}|x-a| < \frac{2}{a}\frac{\epsilon a}{2} = \epsilon.\end{aligned}$$

Thus $\lim_{x\to a} \ln x = \ln a$, and ln is continuous at each point a in its domain $(0, \infty)$.

14. Let a be any real number, and let $\epsilon > 0$ be given. Assume (making ϵ smaller if necessary) that $\epsilon < e^a$. Since

$$\ln\left(1 - \frac{\epsilon}{e^a}\right) + \ln\left(1 + \frac{\epsilon}{e^a}\right) = \ln\left(1 - \frac{\epsilon^2}{e^{2a}}\right) < 0,$$

we have $\ln\left(1 + \frac{\epsilon}{e^a}\right) < -\ln\left(1 - \frac{\epsilon}{e^a}\right)$.
Let $\delta = \ln\left(1 + \frac{\epsilon}{e^a}\right)$. If $|x-a| < \delta$, then

$$\begin{aligned}&\ln\left(1 - \frac{\epsilon}{e^a}\right) < x - a < \ln\left(1 + \frac{\epsilon}{e^a}\right)\\ &1 - \frac{\epsilon}{e^a} < e^{x-a} < 1 + \frac{\epsilon}{e^a}\\ &\left|e^{x-a} - 1\right| < \frac{\epsilon}{e^a}\\ &|e^x - e^a| = e^a|e^{x-a} - 1| < \epsilon.\end{aligned}$$

Thus $\lim_{x\to a} e^x = e^a$ and e^x is continuous at every point a in its domain.

15. Suppose $a \le x_n \le b$ for each n, and $\lim x_n = L$. Then $a \le L \le b$ by Theorem 3. Let $\epsilon > 0$ be given. Since f is continuous on $[a, b]$, there exists $\delta > 0$ such that if $a \le x \le b$ and $|x - L| < \delta$ then $|f(x) - f(L)| < \epsilon$. Since $\lim x_n = L$, there exists an integer N such that if $n \ge N$ then $|x_n - L| < \delta$. Hence $|f(x_n) - f(L)| < \epsilon$ for such n. Therefore $\lim(f(x_n) = f(L)$.

16. Let $g(t) = \dfrac{t}{1+|t|}$. For $t \ne 0$ we have

$$g'(t) = \frac{1 + |t| - t \operatorname{sgn} t}{(1+|t|)^2} = \frac{1 + |t| - |t|}{(1+|t|)^2} = \frac{1}{(1+|t|)^2} > 0.$$

If $t = 0$, g is also differentiable, and has derivative 1:

$$g'(0) = \lim_{h\to 0}\frac{g(h) - g(0)}{h} = \lim_{h\to 0}\frac{1}{1+|h|} = 1.$$

Thus g is continuous and increasing on $\mathbb{R}$.
If f is continuous on $[a, b]$, then

$$h(x) = g\Big(f(x)\Big) = \frac{f(x)}{1 + |f(x)|}$$

is also continuous there, being the composition of continuous functions. Also, $h(x)$ is bounded on $[a, b]$, since

$$\left|g\Big(f(x)\Big)\right| \le \frac{|f(x)|}{1 + |f(x)|} \le 1.$$

By assumption in this problem, $h(x)$ must assume maximum and minimum values; there exist c and d in $[a, b]$ such that

$$g\Big(f(c)\Big) \le g\Big(f(x)\Big) \le g\Big(f(d)\Big)$$

for all x in $[a, b]$. Since g is increasing, so is its inverse g^{-1}. Therefore

$$f(c) \le f(x) \le f(d)$$

for all x in $[a, b]$, and f is bounded on that interval.

Appendix IV. The Riemann Integral (page A-30)

1. $f(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1\\ 0 & \text{if } 1 < x \le 2\end{cases}$
Let $0 < \epsilon < 1$. Let $P = \{0, 1 - \frac{\epsilon}{3}, 1 + \frac{\epsilon}{3}, 2\}$. Then

$$\begin{aligned}L(f, P) &= 1\left(1 - \frac{\epsilon}{3}\right) + 0 + 0 = 1 - \frac{\epsilon}{3}\\ U(f, P) &= 1\left(1 - \frac{\epsilon}{3}\right) + 1\left(\frac{2\epsilon}{3}\right) + 0 = 1 + \frac{\epsilon}{3}.\end{aligned}$$

Since $U(f, P) - L(f, P) < \epsilon$, f is integrable on $[0, 2]$. Since $L(f, P) < 1 < U(f, P)$ for every ϵ, therefore $\displaystyle\int_0^2 f(x)\,dx = 1$.

2. $f(x) = \begin{cases} 1 & \text{if } x = 1/n \quad (n = 1, 2, 3, \ldots)\\ 0 & \text{otherwise}\end{cases}$
If P is any partition of $[0, 1]$ then $L(f, P) = 0$. Let $0 < \epsilon \le 2$. Let N be an integer such that $N + 1 > \frac{2}{\epsilon} \ge N$. A partition P of $[0, 1]$ can be constructed so that the first two points of P are 0 and $\frac{\epsilon}{2}$, and such that each of the N points $\frac{1}{n}$ $(n = 1, 2, 3, \ldots, n)$ lies in a subinterval of P having length at most $\frac{\epsilon}{2N}$. Since every number $\frac{1}{n}$ with n a positive integer lies either in $\left[0, \frac{\epsilon}{2}\right]$ or one of these other N subintervals of P, and since $\max f(x) = 1$ for these subintervals and $\max f(x) = 0$ for all other subintervals of P, therefore $U(f, P) \le \frac{\epsilon}{2} + N\frac{\epsilon}{2N} = \epsilon$. By Theorem 3, f is integrable on $[0, 1]$. Evidently

$$\int_0^1 f(x)\,dx = \text{least upper bound } L(f, P) = 0.$$

3. $f(x) = \begin{cases} 1/n & \text{if } x = m/n \text{ in lowest terms} \\ 0 & \text{otherwise} \end{cases}$

Clearly $L(f, P) = 0$ for every partition P of $[0, 1]$. Let $\epsilon > 0$ be given. To show that f is integrable we must exhibit a partition P for which $U(f, P) < \epsilon$. We can assume $\epsilon < 1$. Choose a positive integer N such that $2/N < \epsilon$. There are only finitely many integers n such that $1 \le n \le N$. For each such n, there are only finitely many integers m such that $0 \le m/n \le 1$. Therefore there are only finitely many points x in $[0, 1]$ where $f(x) > \epsilon/2$. Let P be a partition of $[0, 1]$ such that all these points are contained in subintervals of the partition having total length less than $\epsilon/2$. Since $f(x) \le 1$ on these subintervals, and $f(x) < \epsilon/2$ on all other subintervals P, therefore $U(f, P) \le 1 \times (\epsilon/2) + (\epsilon/2) \times 1 = \epsilon$, and f is integrable on $[0, 1]$. Evidently $\int_0^1 f(x)\,dx = 0$, since all lower sums are 0.

4. Suppose, to the contrary, that $I_* > I^*$. Let $\epsilon = \dfrac{I_* - I^*}{3}$, so $\epsilon > 0$. By the definition of I_* and I^*, there exist partitions P_1 and P_2 of $[a, b]$, such that $L(f, P_1) \ge I_* - \epsilon$ and $U(f, P_2) \le I^* + \epsilon$. By Theorem 2, $L(f, P_1) \le U(f, P_2)$, so

$$3\epsilon = I_* - I^* \le L(f, P_1) + \epsilon - U(f, P_2) + \epsilon \le 2\epsilon.$$

Since $\epsilon > 0$, it follows that $3 \le 2$. This contradiction shows that we must have $I_* \le I^*$.

5. Theorem 3 of Section 6.4: Proofs of parts (c)–(h).

c) Multiplying a function by a constant multiplies all its Riemann sums by the same constant. If the constant is positive, upper and lower sums remain upper and lower; if the constant is negative upper sums become lower and vice versa. Therefore

$$\int_a^b Af(x)\,dx = A\int_a^b f(x)\,dx.$$

It therefore remains to be proved only that the integral of a sum of functions is the sum of the integrals. Suppose that

$$\int_a^b f(x)\,dx = I, \quad \text{and} \quad \int_a^b g(x)\,dx = J.$$

If $\epsilon > 0$, then there exist partitions P_1 and P_2 of $[a, b]$ such that

$$U(f, P_1) - \frac{\epsilon}{2} \le I < L(f, P_1) + \frac{\epsilon}{2}$$
$$U(g, P_2) - \frac{\epsilon}{2} \le J < L(g, P_2) + \frac{\epsilon}{2}.$$

Let P be the common refinement of P_1 and P_2. Then the above inequalities hold with P replacing P_1 and P_2. If $m_1 \le f(x) \le M_1$ and $m_2 \le g(x) \le M_2$ on any interval, then $m_1 + m_2 \le f(x) + g(x) \le M_1 + M_2$ there. It follows that

$$U(f + g, P) \le U(f, P) + U(g, P),$$
$$L(f, P) + L(g, P) \le L(f + g, P).$$

Therefore

$$U(f + g, P) - \epsilon \le I + J \le L(f + g, P) + \epsilon.$$

Hence $\displaystyle\int_a^b \Big(f(x) + g(x)\Big)\,dx = I + J.$

d) Assume $a < b < c$; the other cases are similar. Let $\epsilon > 0$. If

$$\int_a^b f(x)\,dx = I, \quad \text{and} \quad \int_b^c f(x)\,dx = J,$$

then there exist partitions P_1 of $[a, b]$, and P_2 of $[b, c]$ such that

$$L(f, P_1) \le I < L(f, P_1) + \frac{\epsilon}{2}$$
$$L(f, P_2) \le J < L(f, P_2) + \frac{\epsilon}{2}$$

(with similar inequalities for upper sums). Let P be the partition of $[a, c]$ formed by combining all the subdivision points of P_1 and P_2. Then

$$L(f, P) = L(f, P_1) + L(f, P_2) \le I + J < L(f, P) + \epsilon.$$

Similarly, $U(f, P) - \epsilon < I + J \le U(f, P)$. Therefore

$$\int_a^c f(x)\,dx = I + J.$$

e) Let

$$\int_a^b f(x)\,dx = I, \quad \text{and} \quad \int_a^b g(x)\,dx = J,$$

where $f(x) \le g(x)$ on $[a, b]$. We want to show that $I \le J$. Suppose, to the contrary, that $I > J$. Then there would exist a partition P of $[a, b]$ for which

$$I < L(f, P) + \frac{I - J}{2}, \quad \text{and} \quad U(g, P) - \frac{I - J}{2} < J.$$

Thus $L(f, P) > \dfrac{I + J}{2} > U(g, P) \ge L(g, P)$. However, $f(x) \le g(x)$ on $[a, b]$ implies that $L(f, P) \le L(g, P)$ for any partition. Thus we have a contradiction, and so $I \le J$.

f) Since $-|f(x)| \le f(x) \le |f(x)|$ for any x, we have by part (e), if $a \le b$,

$$-\int_a^b |f(x)|\,dx \le \int_a^b f(x)\,dx \le \int_a^b |f(x)|\,dx.$$

Therefore $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx.$

g) By parts (b), (c) and (d),

$$\begin{aligned}\int_{-a}^a f(x)\,dx &= \int_{-a}^0 f(x)\,dx + \int_0^a f(x)\,dx\\ &= \int_0^a f(-x)\,dx + \int_0^a f(x)\,dx\\ &= \int_0^a [f(-x)+f(x)]\,dx.\end{aligned}$$

If f is odd, the last integral is 0. If f is even, the last integral is $\int_0^a 2f(x)\,dx$. Thus both (g) and (h) are proved.

6. Let $\epsilon > 0$ be given. Let $\delta = \epsilon^2/2$. Let $0 \le x \le 1$ and $0 \le y \le 1$. If $x < \epsilon^2/4$ and $y < \epsilon^2/4$ then $|\sqrt{x} - \sqrt{y}| \le \sqrt{x} + \sqrt{y} < \epsilon$.
If $|x-y| < \delta$ and either $x \ge \epsilon^2/4$ or $y \ge \epsilon^2/4$ then

$$|\sqrt{x} - \sqrt{y}| = \frac{|x-y|}{\sqrt{x}+\sqrt{y}} < \frac{2}{\epsilon} \times \frac{\epsilon^2}{2} = \epsilon.$$

Thus $f(x) = \sqrt{x}$ is uniformly continuous on [0, 1].

7. Suppose f is uniformly continuous on $[a, b]$. Taking $\epsilon = 1$ in the definition of uniform continuity, we can find a positive number δ such that $|f(x) - f(y)| < 1$ whenever x and y are in $[a, b]$ and $|x-y| < \delta$. Let N be a positive integer such that $h = (b-a)/N$ satisfies $h < \delta$.
If $x_k = a + kh$, $(0 \le k \le N)$, then each of the subintervals of the partition $P = \{x_0, x_1, \dots, x_N\}$ has length less than δ. Thus

$$|f(x_k) - f(x_{k-1})| < 1 \qquad \text{for} \quad 1 \le k \le N.$$

By repeated applications of the triangle inequality,

$$|f(x_{k-1}) - f(a)| = |f(x_{k-1}) - f(x_0)| < k - 1.$$

If x is any point in $[a, b]$, then x belongs to one of the intervals $[x_{k-1}, x_k]$, so, by the triangle inequality again,

$$|f(x) - f(a)| \le |f(x) - f(x_{k-1})| + |f(x_{k-1}) - f(a)| < k \le N.$$

Thus $|f(x)| < |f(a)| + N$, and f is bounded on $[a, b]$.

8. Suppose that $|f(x)| \le K$ on $[a, b]$ (where $K > 0$), and that f is integrable on $[a, b]$. Let $\epsilon > 0$ be given, and let $\delta = \epsilon/K$. If x and y belong to $[a, b]$ and $|x - y| < \delta$, then

$$\begin{aligned}|F(x) - F(y)| &= \left|\int_a^x f(t)\,dt - \int_a^y f(t)\,dt\right|\\ &= \left|\int_y^x f(t)\,dt\right| \le K|x-y| < K\frac{\epsilon}{K} = \epsilon.\end{aligned}$$

(See Theorem 3(f) of Section 6.4.) Thus F is uniformly continuous on $[a, b]$.