## CHAPTER 4. SOME APPLICATIONS OF DERIVATIVES

### Section 4.1 Related Rates (page 214)

**1.** If the side and area of the square at time $t$ are $x$ and $A$, respectively, then $A = x^2$, so

$$\frac{dA}{dt} = 2x\,\frac{dx}{dt}.$$

If $x = 8$ cm and $dx/dt = 2$ cm/min, then the area is increasing at rate $dA/dt = 32$ cm$^2$/min.

**2.** As in Exercise 1, $dA/dt = 2x\,dx/dt$. If $dA/dt = -2$ ft$^2$/s and $x = 8$ ft, then $dx/dt = -2/(16)$. The side length is decreasing at 1/8 ft/s.

**3.** Let the radius and area of the ripple $t$ seconds after impact be $r$ and $A$ respectively. Then $A = \pi r^2$. We have

$$\frac{dA}{dt} = 2\pi r\frac{dr}{dt}.$$

If $r = 20$ cm and $\dfrac{dr}{dt} = 4$ cm/s, then
$\dfrac{dA}{dt} = 40\pi(4) = 160\pi.$
The area is increasing at $160\pi$ cm$^2$/s.

**4.** Let $A$ and $r$ denote the area and radius of the circle. Then

$$A = \pi r^2 \Rightarrow r = \sqrt{\frac{A}{\pi}}$$
$$\Rightarrow \frac{dr}{dt} = \left(\frac{1}{2\sqrt{A\pi}}\right)\frac{dA}{dt}.$$

When $\dfrac{dA}{dt} = -2$, and $A = 100$, $\dfrac{dr}{dt} = -\dfrac{1}{10\sqrt{\pi}}$. The radius is decreasing at the rate $\dfrac{1}{10\sqrt{\pi}}$ cm/min when the area is $100\,\text{cm}^2$.

**5.** For $A = \pi r^2$, we have $dA/dt = 2\pi r\,dr/dt$. If $dA/dt = 1/3$ km$^2$/h, then (a) $dr/dt = 1/(6\pi r)$ km/h, or (b) $dr/dt = 1/(6\pi\sqrt{A/\pi}) = 1/(6\sqrt{\pi A})$ km/h

**6.** Let the length, width, and area be $l$, $w$, and $A$ at time $t$. Thus $A = lw$.

$$\frac{dA}{dt} = l\frac{dw}{dt} + w\frac{dl}{dt}$$

When $l = 16$, $w = 12$, $\dfrac{dw}{dt} = 3$, $\dfrac{dA}{dt} = 0$, we have

$$0 = 16 \times 3 + 12\frac{dl}{dt} \Rightarrow \frac{dl}{dt} = -\frac{48}{12} = -4$$

The length is decreasing at 4 m/s.

**7.** $V = \dfrac{4}{3}\pi r^3$, so $\dfrac{dV}{dt} = 4\pi r^2\dfrac{dr}{dt}$.
When $r = 30$ cm and $dV/dt = 20$ cm$^3$/s, we have

$$20 = 4\pi(30)^2\frac{dr}{dt}$$
$$\frac{dr}{dt} = \frac{20}{3600\pi} = \frac{1}{180\pi}.$$

The radius is increasing at $1/(180\pi)$ cm/s.

**8.** The volume $V$ of the ball is given by

$$V = \frac{4}{3}\pi r^3 = \frac{4\pi}{3}\left(\frac{D}{2}\right)^3 = \frac{\pi}{6}D^3,$$

where $D = 2r$ is the diameter of the ball. We have

$$\frac{dV}{dt} = \frac{\pi}{2}D^2\,\frac{dD}{dt}.$$

When $D = 6$ cm, $dD/dt = -.5$ cm/h. At that time

$$\frac{dV}{dt} = \frac{\pi}{2}(36)(-0.5) = -9\pi \approx -28.3.$$

The volume is decreasing at about 28.3 cm$^3$/h.

**9.** The volume $V$, surface area $S$, and edge length $x$ of a cube are related by $V = x^3$ and $S = 6x^2$, so that

$$\frac{dV}{dt} = 3x^2\,\frac{dx}{dt}, \qquad \frac{dS}{dt} = 12x\,\frac{dx}{dt}.$$

If $V = 64$ cm$^3$ and $dV/dt = 2$ cm$^3$/s, then $x = 4$ cm and $dx/dt = 2/(3 \times 16) = 1/24$ cm/s. Therefore, $dS/dt = 12(4)(1/24) = 2$. The surface area is increasing at 2 cm$^2$/s.

**10.** Let $V$, $r$ and $h$ denote the volume, radius and height of the cylinder at time $t$. Thus, $V = \pi r^2 h$ and

$$\frac{dV}{dt} = 2\pi rh\frac{dr}{dt} + \pi r^2\frac{dh}{dt}.$$

If $V = 60$, $\dfrac{dV}{dt} = 2$, $r = 5$, $\dfrac{dr}{dt} = 1$, then

$$h = \frac{V}{\pi r^2} = \frac{60}{25\pi} = \frac{12}{5\pi}$$
$$\frac{dh}{dt} = \frac{1}{\pi r^2}\left(\frac{dV}{dt} - 2\pi rh\frac{dr}{dt}\right)$$
$$= \frac{1}{25\pi}\left(2 - 10\pi\frac{12}{5\pi}\right) = -\frac{22}{25\pi}.$$

The height is decreasing at the rate $\dfrac{22}{25\pi}$ cm/min.

**11.** Let the length, width, depth, and volume at time $t$ be $l$, $w$, $h$ and $V$ respectively. Thus $V = lwh$, and

$$\frac{dV}{dt} = \frac{dl}{dt}wh + lh\frac{dw}{dt} + lw\frac{dh}{dt}.$$

If $l = 6$ cm, $w$ =5cm, $h$= 4cm, $\frac{dl}{dt} = \frac{dh}{dt} = 1$m/s, and $\frac{dw}{dt} = -2$cm/s, then

$$\frac{dV}{dt} = 20 - 48 + 30 = 2.$$

The volume is increasing at a rate of 2 cm$^3$/s.

**12.** Let the length, width and area at time $t$ be $x$, $y$ and $A$ respectively. Thus $A = xy$ and

$$\frac{dA}{dt} = x\frac{dy}{dt} + y\frac{dx}{dt}.$$

If $\frac{dA}{dt} = 5$, $\frac{dx}{dt} = 10$, $x = 20$, $y = 16$, then

$$5 = 20\frac{dy}{dt} + 16(10) \Rightarrow \frac{dy}{dt} = -\frac{31}{4}.$$

Thus, the width is decreasing at $\frac{31}{4}$ m/s.

**13.** $y = x^2$. Thus $\frac{dy}{dt} = 2x\frac{dx}{dt}$. If $x = -2$ and $\frac{dx}{dt} = -3$, then $\frac{dy}{dt} = -4(-3) = 12$. $y$ is increasing at rate 12.

**14.** Since $x^2y^3 = 72$, then

$$2xy^3\frac{dx}{dt} + 3x^2y^2\frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = -\frac{2y}{3x}\frac{dx}{dt}.$$

If $x = 3$, $y = 2$, $\frac{dx}{dt} = 2$, then $\frac{dy}{dt} = -\frac{8}{9}$. Hence, the vertical velocity is $-\frac{8}{9}$ units/s.

**15.** We have

$$xy = t \Rightarrow x\frac{dy}{dt} + y\frac{dx}{dt} = 1$$
$$y = tx^2 \Rightarrow \frac{dy}{dt} = x^2 + 2xt\frac{dx}{dt}$$

At $t = 2$ we have $xy = 2$, $y = 2x^2 \Rightarrow 2x^3 = 2 \Rightarrow x = 1$, $y = 2$.
Thus $\frac{dy}{dt} + 2\frac{dx}{dt} = 1$, and $1 + 4\frac{dx}{dt} = \frac{dy}{dt}$.
So $1 + 6\frac{dx}{dt} = 1 \Rightarrow \frac{dx}{dt} = 0 \Rightarrow \frac{dy}{dt} = 1 \Rightarrow$.
Distance $D$ from origin satisfies $D = \sqrt{x^2 + y^2}$. So

$$\frac{dD}{dt} = \frac{1}{2\sqrt{x^2+y^2}}\left(2x\frac{dx}{dt} + 2y\frac{dy}{dt}\right)$$
$$= \frac{1}{\sqrt{5}}\Big(1(0) + 2(1)\Big) = \frac{2}{\sqrt{5}}.$$

The distance from the origin is increasing at a rate of $2/\sqrt{5}$.

**16.** From the figure, $x^2 + k^2 = s^2$. Thus

$$x\,\frac{dx}{dt} = s\,\frac{ds}{dt}.$$

When angle $PCA = 45°$, $x = k$ and $s = \sqrt{2}k$. The radar gun indicates that $ds/dt = 100$ km/h. Thus $dx/dt = 100\sqrt{2}k/k \approx 141$. The car is travelling at about 141 km/h.

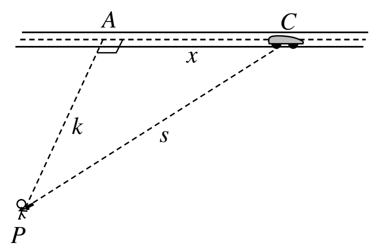



Fig. 4.1.16

**17.** We continue the notation of Exercise 16. If $dx/dt = 90$ km/h, and angle $PCA = 30°$, then $s = 2k$, $x = \sqrt{3}k$, and $ds/dt = (\sqrt{3}k/2k)(90) = 45\sqrt{3} = 77.94$. The radar gun will read about 78 km/h.

**18.** Let the distances $x$ and $y$ be as shown at time $t$. Thus $x^2 + y^2 = 25$ and $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$.
If $\frac{dx}{dt} = \frac{1}{3}$ and $y = 3$, then $x = 4$ and $\frac{4}{3} + 3\frac{dy}{dt} = 0$ so $\frac{dy}{dt} = -\frac{4}{9}$.
The top of the ladder is slipping down at a rate of $\frac{4}{9}$ m/s.

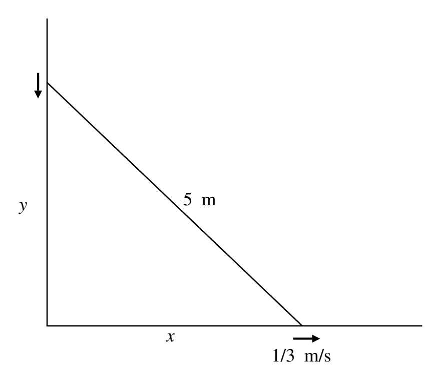



Fig. 4.1.18

**19.** Let $x$ and $y$ be the distances shown in the following figure. From similar triangles:

$$\frac{x}{2} = \frac{x+y}{5} \Rightarrow x = \frac{2y}{3} \Rightarrow \frac{dx}{dt} = \frac{2}{3}\frac{dy}{dt}.$$

Since $\dfrac{dy}{dt} = -\dfrac{1}{2}$, then

$$\frac{dx}{dt} = -\frac{1}{3} \text{ and } \frac{d}{dt}(x+y) = -\frac{1}{2} - \frac{1}{3} = -\frac{5}{6}.$$

Hence, the man's shadow is decreasing at $\frac{1}{3}$ m/s and the shadow of his head is moving towards the lamppost at a rate of $\frac{5}{6}$ m/s.

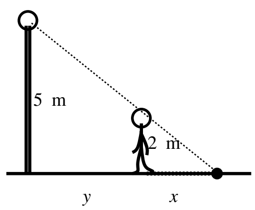



Fig. 4.1.19

**20.**

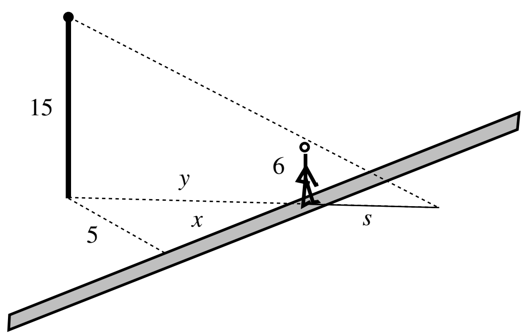



Fig. 4.1.20

Refer to the figure. $s$, $y$, and $x$ are, respectively, the length of the woman's shadow, the distances from the woman to the lamppost, and the distances from the woman to the point on the path nearest the lamppost. From one of triangles in the figure we have

$$y^2 = x^2 + 25.$$

If $x = 12$, then $y = 13$. Moreover,

$$2y\frac{dy}{dt} = 2x\frac{dx}{dt}.$$

We are given that $dx/dt = 2$ ft/s, so $dy/dt = 24/13$ ft/s when $x = 12$ ft. Now the similar triangles in the figure show that

$$\frac{s}{6} = \frac{s+y}{15},$$

so that $s = 2y/3$. Hence $ds/dt = 48/39$. The woman's shadow is changing at rate 48/39 ft/s when she is 12 ft from the point on the path nearest the lamppost.

**21.** $$C = 10,000 + 3x + \frac{x^2}{8,000}$$

$$\frac{dC}{dt} = \left(3 + \frac{x}{4,000}\right)\frac{dx}{dt}.$$

If $dC/dt = 600$ when $x = 12,000$, then $dx/dt = 100$. The production is increasing at a rate of 100 tons per day.

**22.** Let $x$, $y$ be distances travelled by $A$ and $B$ from their positions at 1:00 pm in $t$ hours.
Thus $\dfrac{dx}{dt} = 16$ km/h, $\dfrac{dy}{dt} = 20$ km/h.
Let $s$ be the distance between $A$ and $B$ at time $t$.
Thus $s^2 = x^2 + (25+y)^2$

$$2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2(25+y)\frac{dy}{dt}$$

At 1:30 $\left(t = \frac{1}{2}\right)$ we have $x = 8$, $y = 10$,
$s = \sqrt{8^2 + 35^2} = \sqrt{1289}$ so

$$\sqrt{1289}\frac{ds}{dt} = 8 \times 16 + 35 \times 20 = 828$$

and $\dfrac{ds}{dt} = \dfrac{828}{\sqrt{1289}} \approx 23.06$. At 1:30, the ships are separating at about 23.06 km/h.

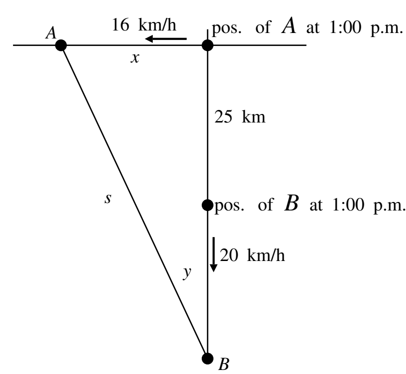



Fig. 4.1.22

**23.** Let $\theta$ and $\omega$ be the angles that the minute hand and hour hand made with the vertical $t$ minutes after 3 o'clock. Then

$$\frac{d\theta}{dt} = \frac{\pi}{30} \text{ rad/min}$$
$$\frac{d\omega}{dt} = \frac{\pi}{360} \text{ rad/min.}$$

Since $\theta = 0$ and $\omega = \dfrac{\pi}{2}$ at $t = 0$, therefore

$$\theta = \frac{\pi}{30}t \quad \text{and} \quad \omega = \frac{\pi}{360}t + \frac{\pi}{2}.$$

At the first time after 3 o'clock when the hands of the clock are together, i.e., $\theta = \omega$,

$$\Rightarrow \frac{\pi}{30}t = \frac{\pi}{360}t + \frac{\pi}{2} \Rightarrow t = \frac{180}{11}.$$

Thus, the hands will be together at $16\frac{4}{11}$ minutes after 3 o'clock.

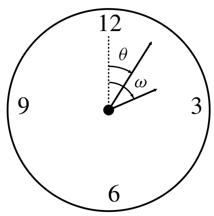



Fig. 4.1.23

**24.** Let $y$ be the height of balloon $t$ seconds after release. Then $y = 5t$ m.
Let $\theta$ be angle of elevation at $B$ of balloon at time $t$. Then $\tan\theta = y/100$. Thus

$$\sec^2\theta \frac{d\theta}{dt} = \frac{1}{100}\frac{dy}{dt} = \frac{5}{100} = \frac{1}{20}$$
$$\left(1 + \tan^2\theta\right)\frac{d\theta}{dt} = \frac{1}{20}$$
$$\left[1 + \left(\frac{y}{100}\right)^2\right]\frac{d\theta}{dt} = \frac{1}{20}.$$

When $y = 200$ we have $5\dfrac{d\theta}{dt} = \dfrac{1}{20}$ so $\dfrac{d\theta}{dt} = \dfrac{1}{100}$.

The angle of elevation of balloon at $B$ is increasing at a rate of $\dfrac{1}{100}$ rad/s.

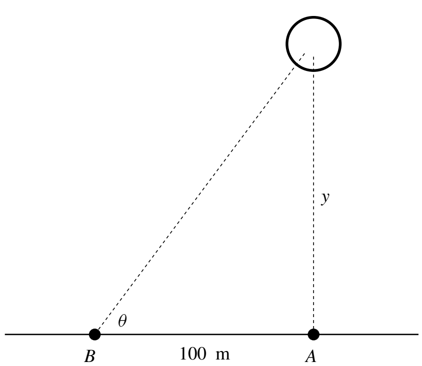



Fig. 4.1.24

**25.** Let $V$, $r$ and $h$ be the volume, radius and height of the cone. Since $h = r$, therefore

$$V = \tfrac{1}{3}\pi r^2 h = \tfrac{1}{3}\pi h^3$$
$$\frac{dV}{dt} = \pi h^2 \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{1}{\pi h^2}\frac{dV}{dt}.$$

If $\dfrac{dV}{dt} = \dfrac{1}{2}$ and $h = 3$, then $\dfrac{dV}{dt} = \dfrac{1}{18\pi}$. Hence, the height of the pile is increasing at $\dfrac{1}{18\pi}$ m/min.

**26.** Let $r$, $h$, and $V$ be the top radius, depth, and volume of the water in the tank at time $t$. Then $\dfrac{r}{h} = \dfrac{10}{8}$ and $V = \dfrac{1}{3}\pi r^2 h = \dfrac{\pi}{3}\dfrac{25}{16}h^3$. We have

$$\frac{1}{10} = \frac{\pi}{3}\frac{25}{16}3h^2\frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{16}{250\pi h^2}.$$

When $h = 4$ m, we have $\dfrac{dh}{dt} = \dfrac{1}{250\pi}$.

The water level is rising at a rate of $\dfrac{1}{250\pi}$ m/min when depth is 4 m.

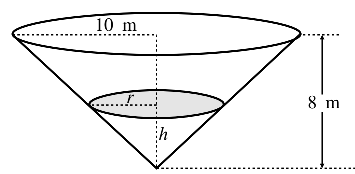



Fig. 4.1.26

**27.** Let $r$ and $h$ be the radius and height of the water in the tank at time $t$. By similar triangles,

$$\frac{r}{h} = \frac{10}{8} \Rightarrow r = \frac{5}{4}h.$$

The volume of water in the tank at time $t$ is

$$V = \frac{1}{3}\pi r^2 h = \frac{25\pi}{48}h^3.$$

Thus,

$$\frac{dV}{dt} = \frac{25\pi}{16}h^2\frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{16}{25\pi h^2}\frac{dV}{dt}.$$

If $\dfrac{dV}{dt} = \dfrac{1}{10} - \dfrac{h^3}{1000}$ and $h = 4$, then

$$\frac{dh}{dt} = \frac{16}{(25\pi)(4)^2}\left(\frac{1}{10} - \frac{4^3}{1000}\right) = \frac{9}{6250\pi}.$$

Hence, the depth of water is increasing at $\frac{9}{6250\pi}$ m/min when the water is 4 m deep. The maximum depth occurs when $\frac{dh}{dt} = 0$, i.e.,

$$\frac{16}{25\pi h^2}\left(\frac{1}{10} - \frac{h^3}{1000}\right) = 0 \Rightarrow \frac{1}{10} - \frac{h^3}{1000} = 0$$
$$\Rightarrow h = \sqrt[3]{100}.$$

Thus, the maximum depth the water in the tank can get is $\sqrt[3]{100} \approx 4.64$ m.

**28.** Let $r$, $h$, and $V$ be the top radius, depth, and volume of the water in the tank at time $t$. Then

$$\frac{r}{h} = \frac{3}{9} = \frac{1}{3}$$
$$V = \frac{1}{3}\pi r^2 h = \frac{\pi}{27}h^3$$
$$\frac{dV}{dt} = \frac{\pi}{9}h^2\frac{dh}{dt}.$$

If $\frac{dh}{dt} = 20$ cm/h $= \frac{2}{10}$ m/h when $h = 6$ m, then

$$\frac{dV}{dt} = \frac{\pi}{9} \times 36 \times \frac{2}{10} = \frac{4\pi}{5} \approx 2.51 \text{ m}^3\text{/h}.$$

Since water is coming in at a rate of 10 m$^3$/h, it must be leaking out at a rate of $10 - 2.51 \approx 7.49$ m$^3$/h.

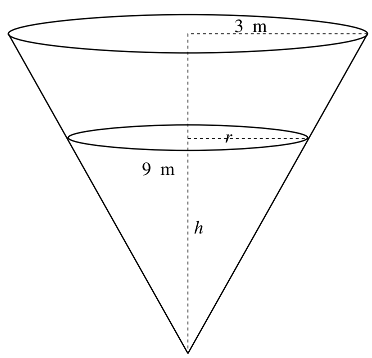



Fig. 4.1.28

**29.** Let $x$ and $s$ be the distance as shown. Then $s^2 = x^2 + 30^2$ and

$$2s\frac{ds}{dt} = 2x\frac{dx}{dt} \Rightarrow \frac{ds}{dt} = \frac{x}{s}\frac{dx}{dt}.$$

When $x = 40$, $\frac{dx}{dt} = 10$, $s = \sqrt{40^2 + 30^2} = 50$, then $\frac{ds}{dt} = \frac{40}{50}(10) = 8$. Hence, one must let out line at 8 m/min.

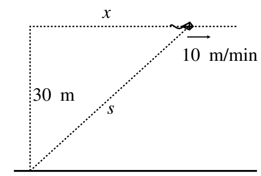



Fig. 4.1.29

**30.** Let $P$, $x$, and $y$ be your position, height above centre, and horizontal distance from centre at time $t$. Let $\theta$ be the angle shown. Then $y = 10\sin\theta$, and $x = 10\cos\theta$. We have

$$\frac{dy}{dt} = 10\cos\theta\frac{d\theta}{dt}, \qquad \frac{d\theta}{dt} = 1 \text{ rpm} = 2\pi \text{ rad/min}.$$

When $x = 6$, then $\cos\theta = \frac{6}{10}$, so $\frac{dy}{dt} = 10 \times \frac{6}{10} \times 12\pi$. You are rising or falling at a rate of $12\pi$ m/min at the time in question.

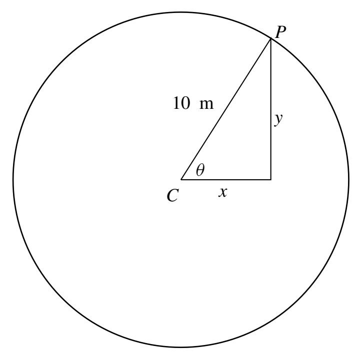



Fig. 4.1.30

**31.** Let $x$ and $y$ denote the distances of the two aircraft east and north of the airport respectively at time $t$ as shown in the following diagram. Also let the distance between the two aircraft be $s$, then $s^2 = x^2 + y^2$. Thus,

$$2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}.$$

Since $\frac{dx}{dt} = -200$ and $\frac{dy}{dt} = 150$ when $x = 144$ and $y = 60$, we have $s = \sqrt{144^2 + 60^2} = 156$, and

$$\frac{ds}{dt} = \frac{1}{156}[144(-200) + 60(150)] \approx -126.9.$$

Thus, the distance between the aircraft is decreasing at about 126.9 km/h.

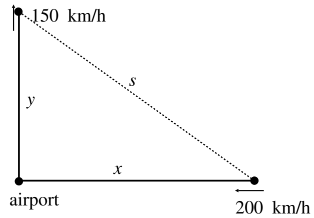



Fig. 4.1.31

**32.** $P = \frac{1}{3}x^{0.6}y^{0.4}$

$\frac{dP}{dt} = \frac{0.6}{3}x^{-0.4}y^{0.4}\frac{dx}{dt} + \frac{0.4}{3}x^{0.6}y^{-0.6}\frac{dy}{dt}.$

If $dP/dt = 0$, $x = 40$, $dx/dt = 1$, and $y = 10,000$, then

$$\frac{dy}{dt} = -\frac{6y^{0.4}}{x^{0.4}}\frac{y^{0.6}}{4x^{0.6}}\frac{dx}{dt} = -\frac{6y}{4x}\frac{dx}{dt} = -375.$$

The daily expenses are decreasing at \$375 per day.

**33.** Let the position of the ant be $(x, y)$ and the position of its shadow be $(0, s)$. By similar triangles,

$$\frac{s-y}{x} = \frac{y}{3-x} \Rightarrow s = \frac{3y}{3-x}.$$

Then,

$$\frac{ds}{dt} = \frac{3(3-x)\frac{dy}{dt} + 3y\frac{dx}{dt}}{(3-x)^2}.$$

If the ant is at $(1, 2)$ and $\frac{dx}{dt} = \frac{1}{3}$, $\frac{dy}{dt} = -\frac{1}{4}$, then

$$\frac{ds}{dt} = \frac{3(2)(-\frac{1}{4}) + 3(2)(\frac{1}{3})}{4} = \frac{1}{8}.$$

Hence, the ant's shadow is moving at $\frac{1}{8}$ units/s upwards along the $y$-axis.

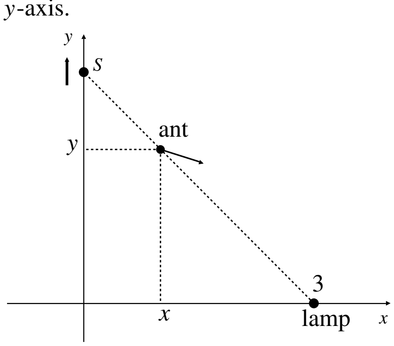



Fig. 4.1.33

**34.** Let $x$ and $y$ be the distances travelled from the intersection point by the boat and car respectively in $t$ minutes. Then

$$\frac{dx}{dt} = 20 \times \frac{1000}{60} = \frac{1000}{3}\text{m/min}$$
$$\frac{dy}{dt} = 80 \times \frac{1000}{60} = \frac{4000}{3}\text{m/min}$$

The distance $s$ between the boat and car satisfy

$$s^2 = x^2 + y^2 + 20^2, \qquad s\frac{ds}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}.$$

After one minute, $x = \frac{1000}{3}$, $y = \frac{4000}{3}$ so $s \approx 1374$. m. Thus

$$1374.5\frac{ds}{dt} = \frac{1000}{3}\frac{1000}{3} + \frac{4000}{3}\frac{4000}{3} \approx 1,888,889.$$

Hence $\frac{ds}{dt} \approx 1374.2$ m/min $\approx 82.45$ km/h after 1 minute.

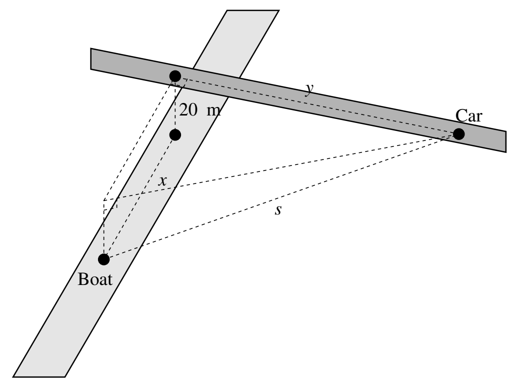



Fig. 4.1.34

**35.** Let $h$ and $b$ (measured in metres) be the depth and the surface width of the water in the trough at time $t$. We have

$$\frac{h}{(\frac{1}{2}b)} = \tan 60° = \sqrt{3} \Rightarrow b = \frac{2}{\sqrt{3}}h.$$

Thus, the volume of the water is

$$V = \left(\frac{1}{2}hb\right)(10) = \frac{10}{\sqrt{3}}h^2,$$

and

$$\frac{dV}{dt} = \frac{20}{\sqrt{3}}h\frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{\sqrt{3}}{20h}\frac{dV}{dt}.$$

If $\frac{dV}{dt} = \frac{1}{4}$ and $h = 0.2$ metres, then

$$\frac{dh}{dt} = \frac{\sqrt{3}}{20(0.2)}\left(\frac{1}{4}\right) = \frac{\sqrt{3}}{16}.$$

Hence, the water level is rising at $\frac{\sqrt{3}}{16}$ m/min.

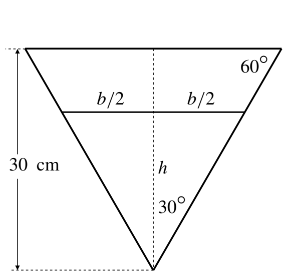



Fig. 4.1.35

**36.** Let $V$ and $h$ be the volume and depth of water in the pool at time $t$. If $h \le 2$, then

$$\frac{x}{h} = \frac{20}{2} = 10, \quad \text{so } V = \frac{1}{2}xh8 = 40h^2.$$

If $2 \le h \le 3$, then $V = 160 + 160(h-2)$.

a) If $h = 2.5\text{m}$, then $-1 = \dfrac{dV}{dt} = 160\dfrac{dh}{dt}$.
So surface of water is dropping at a rate of $\dfrac{1}{160}$ m/min.

b) If $h = 1\text{m}$, then $-1 = \dfrac{dV}{dt} = 80h\dfrac{dh}{dt} = 80\dfrac{dh}{dt}$.
So surface of water is dropping at a rate of $\dfrac{1}{80}$ m/min.

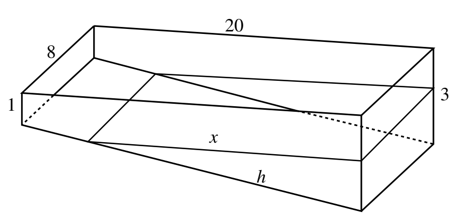



Fig. 4.1.36

**37.** Let the various distances be as shown in the figure.

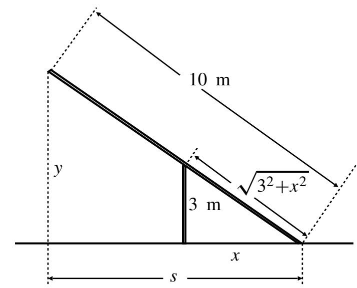



Fig. 4.1.37

a) By similar triangles,

$$\frac{y}{10} = \frac{3}{\sqrt{3^2+x^2}} \Rightarrow y = \frac{30}{\sqrt{9+x^2}}.$$

Thus,

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{-30x}{(9+x^2)^{3/2}}\frac{dx}{dt}.$$

If $x = 4$ and $\dfrac{dx}{dt} = \dfrac{1}{5}$, then

$$\frac{dy}{dt} = \frac{-30(4)}{(9+16)^{3/2}}\left(\frac{1}{5}\right) = -\frac{24}{125}.$$

Hence, the free top end of the ladder is moving vertically downward at 24/125 m/s.

b) By similar triangles,

$$\frac{x}{\sqrt{3^2+x^2}} = \frac{s}{10} \Rightarrow s = \frac{10x}{\sqrt{9+x^2}}.$$

Then,

$$\begin{aligned}\frac{ds}{dt} &= \frac{ds}{dx}\frac{dx}{dt}\\ &= \frac{(\sqrt{9+x^2})(10) - (10x)\left(\dfrac{2x}{2\sqrt{9+x^2}}\right)}{(9+x^2)}\frac{dx}{dt}\\ &= \frac{90}{(9+x^2)^{3/2}}\frac{dx}{dt}.\end{aligned}$$

If $x = 4$ and $\dfrac{dx}{dt} = \dfrac{1}{5}$, then

$$\frac{ds}{dt} = \frac{90}{(9+16)^{3/2}}\left(\frac{1}{5}\right) = \frac{18}{125}.$$

This is the rate of change of the length of the horizontal projection of the ladder. The free top end of the ladder is moving horizontally to the right at rate

$$\frac{dx}{dt} - \frac{ds}{dt} = \frac{1}{5} - \frac{18}{125} = \frac{7}{125} \text{ m/s.}$$

**38.** Let $x$, $y$, and $s$ be distances shown at time $t$. Then

$$s^2 = x^2 + 16, \qquad (15-s)^2 = y^2 + 16$$
$$s\frac{ds}{dt} = x\frac{dx}{dt}, \qquad -(15-s)\frac{ds}{dt} = y\frac{dy}{dt}.$$

When $x = 3$ and $\dfrac{dx}{dt} = \dfrac{1}{2}$, then $s = 5$ and $y = \sqrt{10^2 - 4^2} = \sqrt{84}$.
Also $\dfrac{ds}{dt} = \dfrac{3}{5}\left(\dfrac{1}{2}\right) = \dfrac{3}{10}$ so

$$\frac{dy}{dt} = -\frac{10}{\sqrt{84}}\frac{3}{10} = -\frac{3}{\sqrt{84}} \approx 0.327.$$

Crate $B$ is moving toward $Q$ at a rate of 0.327 m/s.

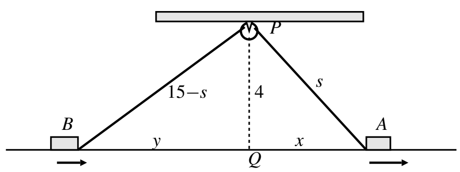



Fig. 4.1.38

**39.** Let $\theta$ be the angle of elevation, and $x$ and $y$ the horizontal and vertical distances from the launch site. We have

$$\tan\theta = \frac{y}{x} \qquad \Rightarrow \qquad \sec^2\theta\frac{d\theta}{dt} = \frac{x\dfrac{dy}{dt} - y\dfrac{dx}{dt}}{x^2}.$$

At the instant in question

$$\frac{dx}{dt} = 4\cos 30^\circ = 2\sqrt{3}, \qquad \frac{dy}{dt} = 4\sin 30^\circ = 2,$$
$$x = 50\,\text{km}, \qquad y = 100\,\text{km}.$$

Thus $\tan\theta = \dfrac{100}{50} = 2$, $\sec^2\theta = 1 + \tan^2\theta = 5$, and

$$\frac{d\theta}{dt} = \frac{1}{5}\frac{50(2) - 100(2\sqrt{3})}{(50)^2} = \frac{1 - 2\sqrt{3}}{125} \approx -0.0197.$$

Therefore, the angle of elevation is decreasing at about 0.0197 rad/s.

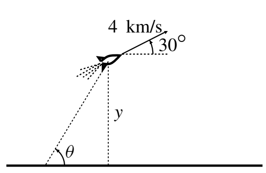



Fig. 4.1.39

**40.** Let $y$ be height of ball $t$ seconds after it drops.
Thus $\dfrac{d^2y}{dt^2} = -9.8$, $\dfrac{dy}{dt}\Big|_{t=0} = 0$, $y|_{t=0} = 20$, and

$$y = -4.9t^2 + 20, \qquad \frac{dy}{dt} = -9.8t.$$

Let $s$ be distance of shadow of ball from base of pole.
By similar triangles, $\dfrac{s - 10}{y} = \dfrac{s}{20}$.
$20s - 200 = sy$, $s = \dfrac{200}{20 - y}$
$20\dfrac{ds}{dt} = y\dfrac{ds}{dt} + s\dfrac{dy}{dt}$.

a) At $t = 1$, we have $\dfrac{dy}{dt} = -9.8$, $y = 15.1$,
$4.9\dfrac{ds}{dt} = \dfrac{200}{4.9}(-9.8)$.
The shadow is moving at a rate of 81.63 m/s after one second.

b) As the ball hits the ground, $y = 0$, $s = 10$, $t = \sqrt{\dfrac{20}{4.9}}$, and $\dfrac{dy}{dt} = -9.8\sqrt{\dfrac{20}{4.9}}$, so
$20\dfrac{ds}{dt} = 0 + 10\dfrac{dy}{dt}$.
Now $y = 0$ implies that $t = \sqrt{\dfrac{20}{4.9}}$. Thus

$$\frac{ds}{dt} = -\frac{1}{2}(9.8)\sqrt{\frac{20}{4.9}} \approx -9.90.$$

The shadow is moving at about 9.90 m/s when the ball hits the ground.

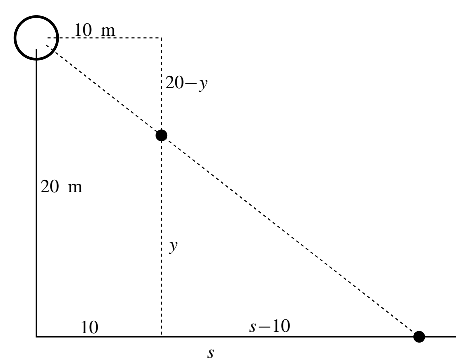



Fig. 4.1.40

**41.** Let $y(t)$ be the height of the rocket $t$ seconds after it blasts off. We have

$$\frac{d^2y}{dt^2} = 10, \qquad \frac{dy}{dt} = y = 0$$

at $t = 0$. Hence $y = 5t^2$, ($y$ in metres, $t$ in seconds). Now
$\tan\theta = \dfrac{y}{2000}$, so $\sec^2\theta\dfrac{d\theta}{dt} = \dfrac{dy/dt}{2000}$, and

$$\left(1+\left(\frac{y}{2000}\right)^2\right)\frac{d\theta}{dt} = \frac{10t}{2000} = \frac{t}{200}$$
$$\frac{d\theta}{dt} = \frac{t}{200}\cdot\frac{1}{1+\dfrac{25t^4}{2000^2}}$$
$$= \frac{t}{200}\cdot\frac{1}{1+\dfrac{t^4}{400^2}} = \frac{800t}{400^2+t^4}.$$

At $t = 10$, we have $\dfrac{d\theta}{dt} = \dfrac{8000}{400^2+100^2} \approx 0.047$ rad/s.

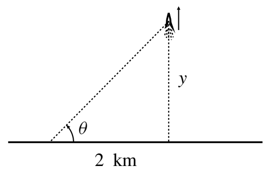


Fig. 4.1.41

## Section 4.2 Extreme Values (page 222)

1. $f(x) = x+2$ on $[-1, 1]$
$f'(x) = 1$ so $f$ is increasing.
$f$ has absolute minimum 1 at $x = -1$ and absolute maximum 3 at $x = 1$.

2. $f(x) = x+2$ on $(-\infty, 0]$
abs max 2 at $x = 0$, no min.

3. $f(x) = x+2$ on $[-1, 1)$
$f$ has absolute minimum 1 at $x = -1$ and has no absolute maximum.

4. $f(x) = x^2 - 1$
no max, abs min $-1$ at $x = 0$.

5. $f(x) = x^2 - 1$ on $[-2, 3]$
$f$ has abs min $-1$ at $x = 0$, abs max 8 at $x = 3$, and local max 3 at $x = -2$.

6. $f(x) = x^2 - 1$ on $(2, 3)$
no max or min values.

7. $f(x) = x^3 + x - 4$ on $[a, b]$
$f'(x) = 3x^2 + 1 > 0$ for all $x$.
Therefore $f$ has abs min $a^3 + a - 4$ at $x = a$ and abs max $b^3 + b - 4$ at $x = b$.

8. $f(x) = x^3 + x - 4$ on $(a, b)$
Since $f'(x) = 3x^2 + 1 > 0$ for all $x$, therefore $f$ is increasing. Since $(a, b)$ is open, $f$ has no max or min values.

9. $f(x) = x^5 + x^3 + 2x$ on $(a, b]$
$f'(x) = 5x^4 + 3x^2 + 2 > 0$ for all $x$.
$f$ has no min value, but has abs max value $b^5 + b^3 + 2b$ at $x = b$.

10. $f(x) = \dfrac{1}{x-1}$. Since $f'(x) = \dfrac{-1}{(x-1)^2} < 0$ for all $x$ in the domain of $f$, therefore $f$ has no max or min values.

11. $f(x) = \dfrac{1}{x-1}$ on $(0, 1)$
$f'(x) = -\dfrac{1}{(x-1)^2} < 0$ on $(0, 1)$
$f$ has no max or min values.

12. $f(x) = \dfrac{1}{x-1}$ on $[2, 3]$
abs min $\frac{1}{2}$ at $x = 3$, abs max 1 at $x = 2$.

13. Let $f(x) = |x-1|$ on $[-2, 2]$: $f(-2) = 3$, $f(2) = 1$.
$f'(x) = \operatorname{sgn}(x-1)$. No CP; SP $x = 1$, $f(1) = 0$.
Max value of $f$ is 3 at $x = -2$; min value is 0 at $x = 1$.

14. Let $f(x) = |x^2 - x - 2| = |(x-2)(x+1)|$ on $[-3, 3]$:
$f(-3) = 10$, $f(3) = 4$.
$f'(x) = (2x-1)\operatorname{sgn}(x^2 - x - 2)$.
CP $x = 1/2$; SP $x = -1$, and $x = 2$. $f(1/2) = 9/4$, $f(-1) = 0$, $f(2) = 0$.
Max value of $f$ is 10 at $x = -3$; min value is 0 at $x = -1$ or $x = 2$.

15. $f(x) = \dfrac{1}{x^2+1}$, $f'(x) = -\dfrac{2x}{(x^2+1)^2}$
$f$ has abs max value 1 at $x = 0$; $f$ has no min values.

16. $f(x) = (x+2)^{(2/3)}$
no max, abs min 0 at $x = -2$.

17. $f(x) = (x-2)^{1/3}$, $f'(x) = \dfrac{1}{3}(x-2)^{-2/3} > 0$
$f$ has no max or min values.

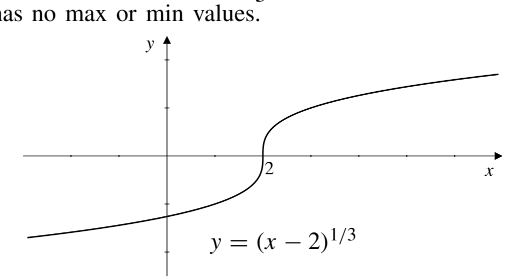


Fig. 4.2.17

18. $f(x) = x^2 + 2x$, $f'(x) = 2x + 2 = 2(x+1)$
Critical point: $x = -1$.
$f(x) \to \infty$ as $x \to \pm\infty$.

| | | CP | | |
|---|---|---|---|---|
| $f'$ | $-$ | $-1$ | $+$ | $x$ |
| $f$ | $\searrow$ | abs min | $\nearrow$ | |

Hence, $f(x)$ has no max value, and the abs min is $-1$ at $x = -1$.

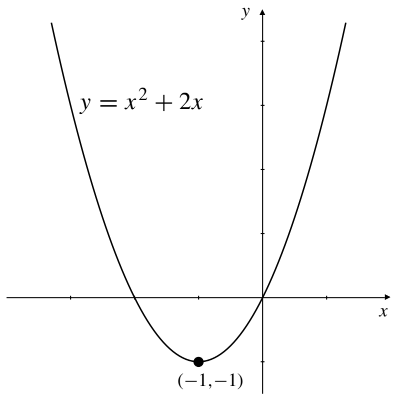



Fig. 4.2.18

**19.** $f(x) = x^3 - 3x - 2$
$f'(x) = 3x^2 - 3 = 3(x-1)(x+1)$

| | | CP | | CP | |
|---|---|---|---|---|---|
| $f'$ | $+$ | $-1$ | $-$ | $1$ | $+$ |
| $f$ | $\nearrow$ | loc max | $\searrow$ | loc min | $\nearrow$ |

$f$ has no absolute extrema.

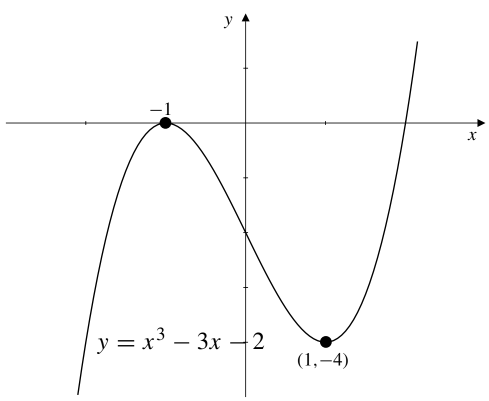



Fig. 4.2.19

**20.** $f(x) = (x^2-4)^2$, $f'(x) = 4x(x^2-4) = 4x(x+2)(x-2)$
Critical points: $x = 0$, $\pm 2$.
$f(x) \to \infty$ as $x \to \pm\infty$.

| | | CP | | CP | | CP | |
|---|---|---|---|---|---|---|---|
| $f'$ | $-$ | $-2$ | $+$ | $0$ | $-$ | $+2$ | $+$ |
| $f$ | $\searrow$ | abs min | $\nearrow$ | loc max | $\searrow$ | abs min | $\nearrow$ |

Hence, $f(x)$ has abs min 0 at $x = \pm 2$ and loc max 16 at $x = 0$.

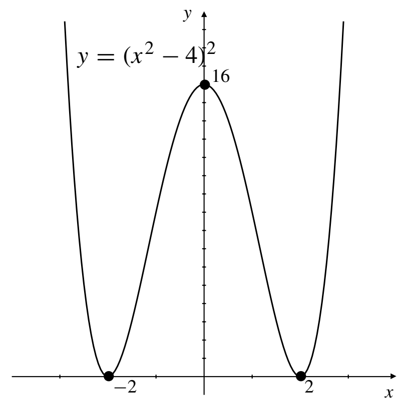



Fig. 4.2.20

**21.** $f(x) = x^3(x-1)^2$
$f'(x) = 3x^2(x-1)^2 + 2x^3(x-1)$
$= x^2(x-1)(5x-3)$
CP $x = 0, \dfrac{3}{5}, 1$

| | | CP | | CP | | CP | |
|---|---|---|---|---|---|---|---|
| $f'$ | $+$ | $0$ | $+$ | $\frac{3}{5}$ | $-$ | $1$ | $+$ |
| $f$ | $\nearrow$ | | $\nearrow$ | loc max | $\searrow$ | loc min | $\nearrow$ |

$f$ has no absolute extrema.

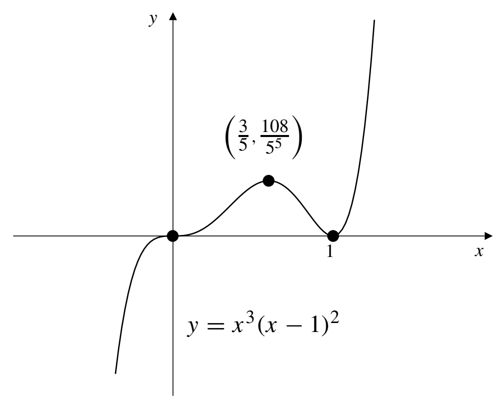



Fig. 4.2.21

**22.** $f(x) = x^2(x-1)^2$,
$f'(x) = 2x(x-1)^2 + 2x^2(x-1) = 2x(2x-1)(x-1)$
Critical points: $x = 0$, $\frac{1}{2}$ and 1.
$f(x) \to \infty$ as $x \to \pm\infty$.

| | | CP | | CP | | CP | |
|---|---|---|---|---|---|---|---|
| $f'$ | $-$ | $0$ | $+$ | $\frac{1}{2}$ | $-$ | $1$ | $+$ |
| $f$ | $\searrow$ | abs min | $\nearrow$ | loc max | $\searrow$ | abs min | $\nearrow$ |

Hence, $f(x)$ has loc max $\frac{1}{16}$ at $x = \frac{1}{2}$ and abs min 0 at $x = 0$ and $x = 1$.

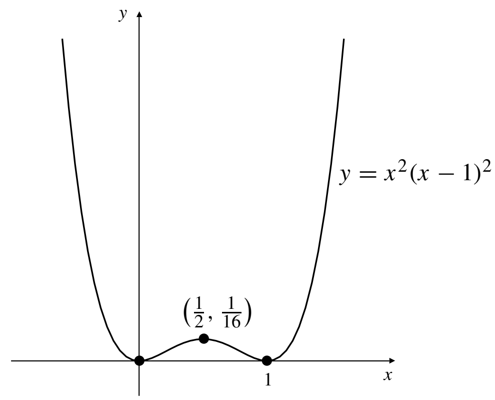



Fig. 4.2.22

**23.** $f(x) = x(x^2-1)^2$

$$\begin{aligned}
f'(x) &= (x^2-1)^2 + 2x(x^2-1)2x \\
&= (x^2-1)(x^2-1+4x^2) \\
&= (x^2-1)(5x^2-1) \\
&= (x-1)(x+1)(\sqrt{5}x-1)(\sqrt{5}x+1)
\end{aligned}$$

| | | CP | | CP | | CP | | CP | |
|---|---|---|---|---|---|---|---|---|---|
| $f'$ | $+$ | $-1$ | $-$ | $-\frac{1}{\sqrt{5}}$ | $+$ | $\frac{1}{\sqrt{5}}$ | $-$ | $1$ | $+$ |
| $f$ | $\nearrow$ | loc max | $\searrow$ | loc min | $\nearrow$ | loc max | $\searrow$ | loc min | $\nearrow$ |

$f(\pm 1) = 0$, $f(\pm 1/\sqrt{5}) = \pm 16/25\sqrt{5}$

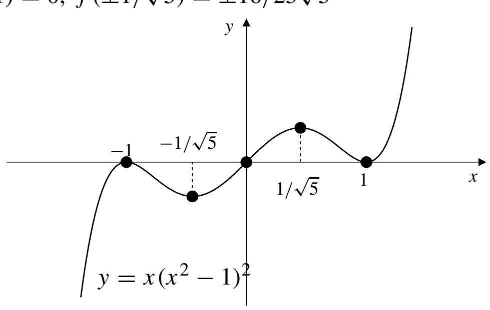



Fig. 4.2.23

**24.** $f(x) = \dfrac{x}{x^2+1}$, $f'(x) = \dfrac{1-x^2}{(x^2+1)^2}$
Critical point: $x = \pm 1$.
$f(x) \to 0$ as $x \to \pm\infty$.

| | | CP | | CP | |
|---|---|---|---|---|---|
| $f'$ | $-$ | $-1$ | $+$ | $+1$ | $-$ |
| $f$ | $\searrow$ | abs min | $\nearrow$ | abs max | $\searrow$ |

Hence, $f$ has abs max $\frac{1}{2}$ at $x = 1$ and abs min $-\frac{1}{2}$ at $x = -1$.

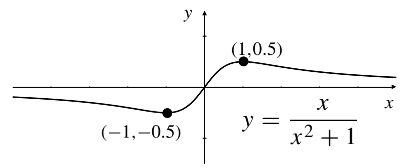



Fig. 4.2.24

**25.** $f(x) = \dfrac{x^2}{x^2+1} = 1 - \dfrac{1}{x^2+1} < 1$
$f'(x) = \dfrac{2x}{(x^2+1)^2}$

| | | CP | |
|---|---|---|---|
| $f'$ | $-$ | $0$ | $+$ |
| $f$ | $\searrow$ | abs min | $\nearrow$ |

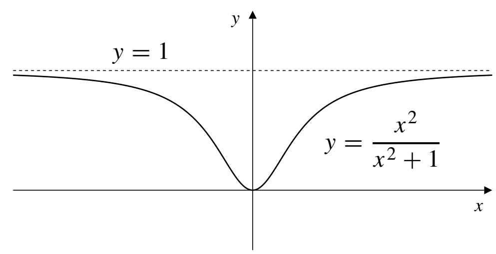



Fig. 4.2.25

**26.** $f(x) = \dfrac{x}{\sqrt{x^4+1}}$, $f'(x) = \dfrac{1-x^4}{(x^4+1)^{3/2}}$
Critical points: $x = \pm 1$.
$f(x) \to 0$ as $x \to \pm\infty$.

| | | CP | | CP | |
|---|---|---|---|---|---|
| $f'$ | $-$ | $-1$ | $+$ | $+1$ | $-$ |
| $f$ | $\searrow$ | abs min | $\nearrow$ | abs max | $\searrow$ |

Hence, $f$ has abs max $\frac{1}{\sqrt{2}}$ at $x = 1$ and abs min $-\frac{1}{\sqrt{2}}$ at $x = -1$.

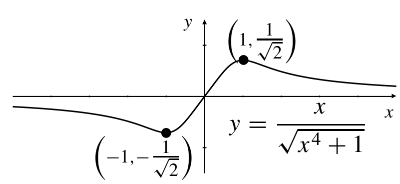



Fig. 4.2.26

**27.** $f(x) = x\sqrt{2-x^2} \quad (|x| \le \sqrt{2})$
$f'(x) = \sqrt{2-x^2} - \dfrac{x^2}{\sqrt{2-x^2}} = \dfrac{2(1-x^2)}{\sqrt{2-x^2}}$

| | SP | | CP | | CP | | SP |
|---|---|---|---|---|---|---|---|
| $f'$ | $-\sqrt{2}$ | $-$ | $-1$ | $+$ | $1$ | $-$ | $\sqrt{2}$ |
| $f$ | loc max | $\searrow$ | abs min | $\nearrow$ | abs max | $\searrow$ | loc min |

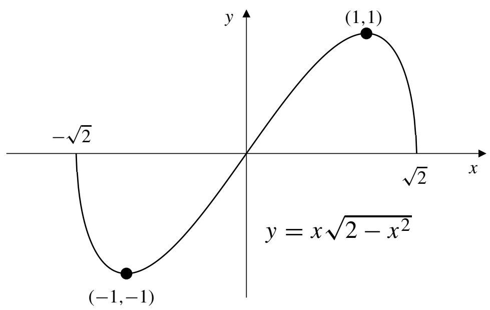


Fig. 4.2.27

**28.** $f(x) = x + \sin x$, $f'(x) = 1 + \cos x \ge 0$
$f'(x) = 0$ at $x = \pm\pi,\ \pm 3\pi, \ldots$
$f(x) \to \pm\infty$ as $x \to \pm\infty$.
Hence, $f$ has no max or min values.

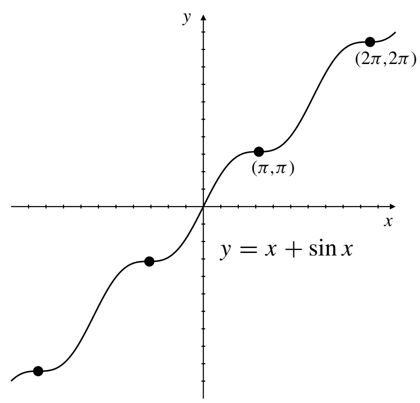


Fig. 4.2.28

**29.** $f(x) = x - 2\sin x$
$f'(x) = 1 - 2\cos x$
CP: $x = \pm\dfrac{\pi}{3} + 2n\pi$
$n = 0, \pm 1, \pm 2, \cdots$
alternating local maxima and minima

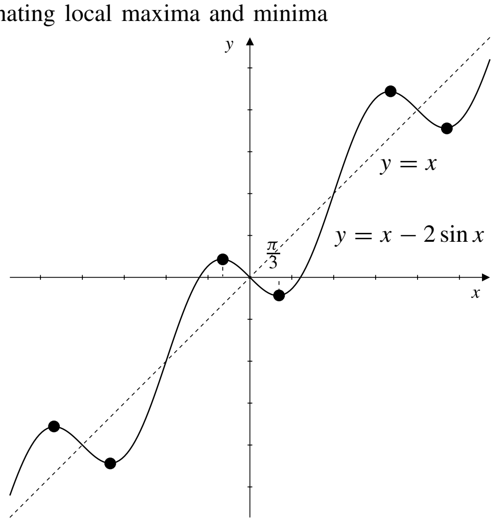


Fig. 4.2.29

**30.** $f(x) = x - 2\tan^{-1} x$, $f'(x) = 1 - \dfrac{2}{1+x^2} = \dfrac{x^2-1}{x^2+1}$
Critical points: $x = \pm 1$.
$f(x) \to \pm\infty$ as $x \to \pm\infty$.

| | | CP | | CP | |
|---|---|---|---|---|---|
| $f'$ | + | $-1$ | − | $+1$ | + |
| $f$ | ↗ | loc max | ↘ | loc min | ↗ |

Hence, $f$ has loc max $-1 + \dfrac{\pi}{2}$ at $x = -1$ and loc min $1 - \dfrac{\pi}{2}$ at $x = 1$.

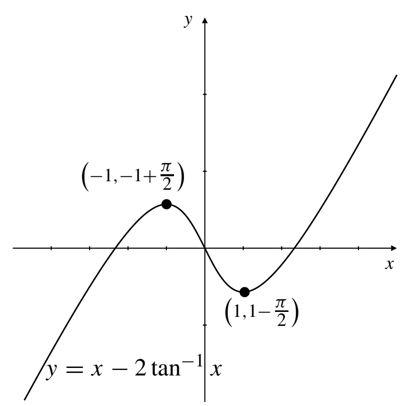


Fig. 4.2.30

**31.**
$$f(x) = 2x - \sin^{-1} x \qquad (-1 \le x \le 1)$$
$$\begin{aligned} f'(x) &= 2 - \frac{1}{\sqrt{1-x^2}} \\ &= \frac{2\sqrt{1-x^2} - 1}{\sqrt{1-x^2}} \\ &= \frac{3 - 4x^2}{\sqrt{1-x^2}(2\sqrt{1-x^2} + 1)} \end{aligned}$$
CP: $x = \pm\dfrac{\sqrt{3}}{2}$, SP: (EP:) $x = \pm 1$
$$f\left(\pm\frac{\sqrt{3}}{2}\right) = \pm\left(\sqrt{3} - \frac{\pi}{3}\right)$$

| | SP | | CP | | CP | | SP |
|---|---|---|---|---|---|---|---|
| $f'$ | $-1$ | − | $-\frac{\sqrt{3}}{2}$ | + | $\frac{\sqrt{3}}{2}$ | − | 1 |
| $f$ | loc max | ↘ | abs min | ↗ | abs max | ↘ | loc min |

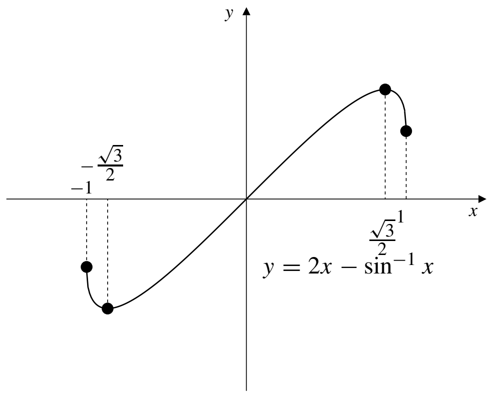



Fig. 4.2.31

**32.** $f(x) = e^{-x^2/2}$, $f'(x) = -xe^{-x^2/2}$
Critical point: $x = 0$.
$f(x) \to 0$ as $x \to \pm\infty$.

| | | CP | | |
|---|---|---|---|---|
| $f'$ | $+$ | $0$ | $-$ | $\to x$ |
| $f$ | $\nearrow$ | abs max | $\searrow$ | |

Hence, $f$ has abs max 1 at $x = 0$ and no min value.

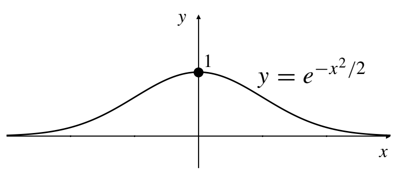



Fig. 4.2.32

**33.** $f(x) = x2^{-x}$
$$f'(x) = 2^{-x} + x(-2^{-x} \ln 2)$$
$$= 2^{-x}(1 - x \ln 2)$$

| | | CP | | |
|---|---|---|---|---|
| $f'$ | $+$ | $1/\ln 2$ | $-$ | $\to x$ |
| $f$ | $\nearrow$ | abs max | $\searrow$ | |

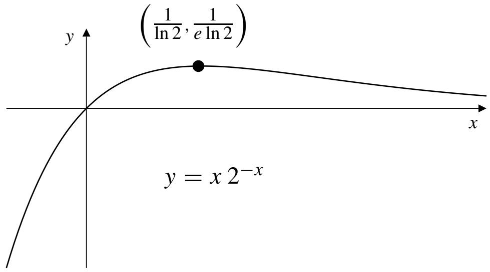



Fig. 4.2.33

**34.** $f(x) = x^2 e^{-x^2}$, $f'(x) = 2xe^{-x^2}(1 - x^2)$
Critical points: $x = 0, \pm 1$.
$f(x) \to 0$ as $x \to \pm\infty$.

| | | CP | | CP | | CP | | |
|---|---|---|---|---|---|---|---|---|
| $f'$ | $+$ | $-1$ | $-$ | $0$ | $+$ | $1$ | $-$ | $\to x$ |
| $f$ | $\nearrow$ | abs max | $\searrow$ | abs min | $\nearrow$ | abs max | $\searrow$ | |

Hence, $f$ has abs max $1/e$ at $x = \pm 1$ and abs min 0 at $x = 0$.

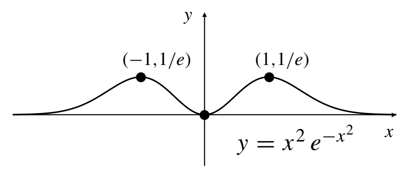



Fig. 4.2.34

**35.** $f(x) = \dfrac{\ln x}{x} \qquad (x > 0)$
$$f'(x) = \frac{\frac{x}{x} - \ln x}{x^2} = \frac{1 - \ln x}{x^2}$$
$f(x) \to -\infty$ as $x \to 0+$ (vertical asymptote), $f(x) \to 0$ as $x \to \infty$ (horizontal asymptote).

| | ASY | | CP | | |
|---|---|---|---|---|---|
| $f'$ | $0$ | $+$ | $e$ | $-$ | $\to x$ |
| $f$ | | $\nearrow$ | abs max | $\searrow$ | |

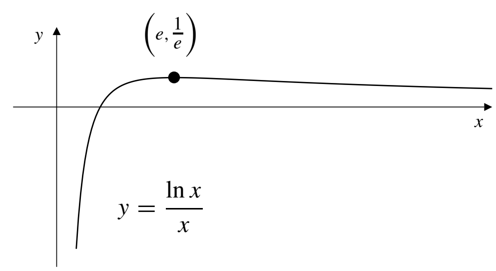



Fig. 4.2.35

**36.** Since $f(x) = |x + 1|$,
$$f'(x) = \operatorname{sgn}(x+1) = \begin{cases} 1, & \text{if } x > -1; \\ -1, & \text{if } x < -1. \end{cases}$$
$-1$ is a singular point; $f$ has no max but has abs min 0 at $x = -1$.
$f(x) \to \infty$ as $x \to \pm\infty$.

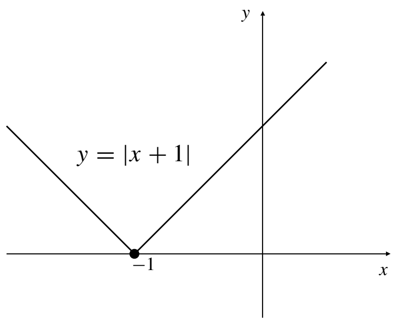


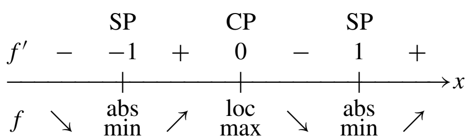

Fig. 4.2.36

**37.** $f(x) = |x^2 - 1|$

$f'(x) = 2x\,\mathrm{sgn}\,(x^2 - 1)$

CP: $x = 0$

SP: $x = \pm 1$



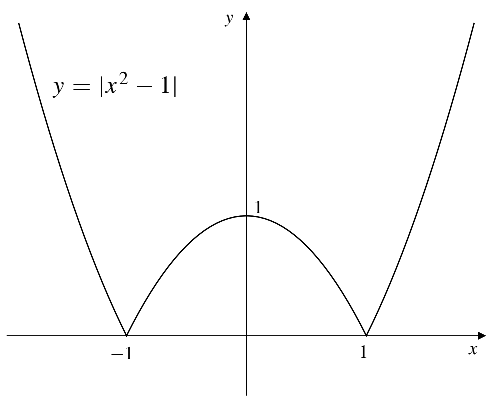
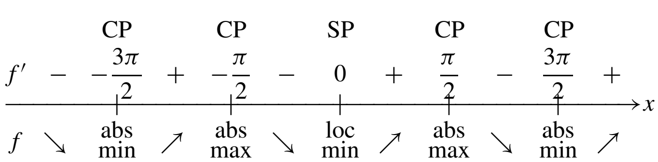




Fig. 4.2.37

**38.** $f(x) = \sin|x|$

$f'(x) = \mathrm{sgn}\,(x)\cos|x| = 0$ at $x = \pm\dfrac{\pi}{2},\ \pm\dfrac{3\pi}{2},\ \pm\dfrac{5\pi}{2}, \ldots$

0 is a singular point. Since $f(x)$ is an even function, its graph is symmetric about the origin.




Hence, $f$ has abs max 1 at $x = \pm(4k+1)\dfrac{\pi}{2}$ and abs min $-1$ at $x = \pm(4k+3)\dfrac{\pi}{2}$ where $k = 0, 1, 2, \ldots$ and loc min 0 at $x = 0$.

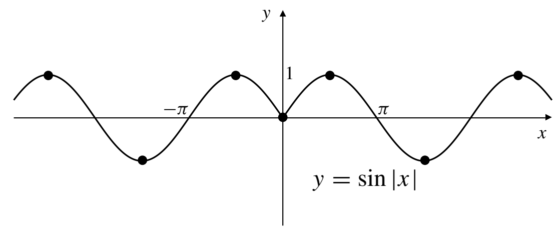



Fig. 4.2.38

**39.** $f(x) = |\sin x|$

CP: $x = \pm\dfrac{(2n+1)\pi}{2}$, SP $= \pm n\pi$

$f$ has abs max 1 at all CP.

$f$ has abs min 0 at all SP.

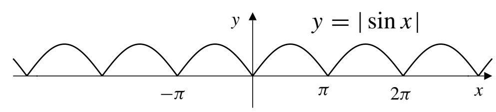


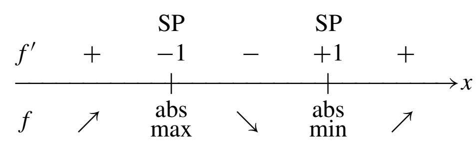

Fig. 4.2.39

**40.** $f(x) = (x-1)^{2/3} - (x+1)^{2/3}$

$f'(x) = \frac{2}{3}(x-1)^{-1/3} - \frac{2}{3}(x+1)^{-1/3}$

Singular point at $x = \pm 1$. For critical points:

$(x-1)^{-1/3} = (x+1)^{-1/3} \Rightarrow x - 1 = x + 1 \Rightarrow 2 = 0$, so there are no critical points.




Hence, $f$ has abs max $2^{2/3}$ at $x = -1$ and abs min $-2^{2/3}$ at $x = 1$.

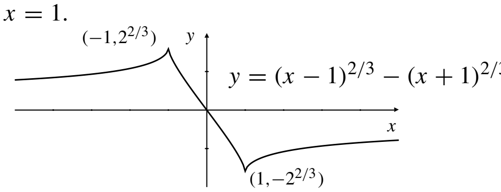



Fig. 4.2.40

**41.** $f(x) = x/\sqrt{x^2+1}$. Since

$$f'(x) = \frac{\sqrt{x^2+1} - x\,\dfrac{2x}{2\sqrt{x^2+1}}}{x^2+1} = \frac{1}{(x^2+1)^{3/2}} > 0,$$

for all $x$, $f$ cannot have any maximum or minimum value.

**42.** $f(x) = x/\sqrt{x^4+1}$. $f$ is continuous on $\mathbb{R}$, and $\lim_{x\to\pm\infty} f(x) = 0$. Since $f(1) > 0$ and $f(-1) < 0$, $f$ must have both maximum and minimum values.

$$f'(x) = \frac{\sqrt{x^4+1} - x\,\dfrac{4x^3}{2\sqrt{x^4+1}}}{x^4+1} = \frac{1 - x^4}{(x^4+1)^{3/2}}.$$

CP $x = \pm 1$. $f(\pm 1) = \pm 1/\sqrt{2}$. $f$ has max value $1/\sqrt{2}$ and min value $-1/\sqrt{2}$.

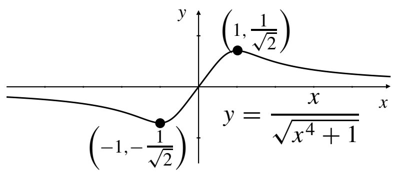



Fig. 4.2.42

**43.** $f(x) = x\sqrt{4-x^2}$ is continuous on $[-2, 2]$, and $f(\pm 2) = 0$.

$$f'(x) = \sqrt{4-x^2} + x\,\frac{-2x}{2\sqrt{4-x^2}} = \frac{2(2-x^2)}{\sqrt{4-x^2}}.$$

CP $x = \pm\sqrt{2}$. $f(\pm\sqrt{2}) = \pm 2$. $f$ has maximum value 2 at $x = \sqrt{2}$ and min value $-2$ at $x = -\sqrt{2}$.

**44.** $f(x) = x^2/\sqrt{4-x^2}$ is continuous on $(-2, 2)$, and $\lim_{x\to -2+} f(x) = \lim_{x\to 2-} f(x) = \infty$. Thus $f$ can have no maximum value, but will have a minimum value.

$$f'(x) = \frac{2x\sqrt{4-x^2} - x^2\,\dfrac{-2x}{2\sqrt{4-x^2}}}{4-x^2} = \frac{8x - x^3}{(4-x^2)^{3/2}}.$$

CP $x = 0$, $x = \pm\sqrt{8}$. $f(0) = 0$, and $\pm\sqrt{8}$ is not in the domain of $f$. $f$ has minimum value 0 at $x = 0$.

**45.** $f(x) = 1/[x\sin x]$ is continuous on $(0, \pi)$, and $\lim_{x\to 0+} f(x) = \infty = \lim_{x\to\pi-} f(x)$. Thus $f$ can have no maximum value, but will have a minimum value. Since $f$ is differentiable on $(0, \pi)$, the minimum value must occur at a CP in that interval.

**46.** $f(x) = (\sin x)/x$ is continuous and differentiable on $\mathbb{R}$ except at $x = 0$ where it is undefined.
Since $\lim_{x\to 0} f(x) = 1$ (Theorem 8 of Section 2.5), and $|f(x)| < 1$ for all $x \neq 0$ (because $|\sin x| < |x|$), $f$ cannot have a maximum value.
Since $\lim_{x\to\pm\infty} f(x) = 0$ and since $f(x) < 0$ at some points, $f$ must have a minimum value occurring at a critical point. In fact, since $|f(x)| \le 1/|x|$ for $x \neq 0$ and $f$ is even, the minimum value will occur at the two critical points closest to $x = 0$. (See Figure 2.20 In Section 2.5 of the text.)

**47.** If it exists, an absolute max value is the maximum of the set of all the local max values. Hence, if a function has an absolute max value, it must have one or more local max values. On the other hand, if a function has a local max value, it may or may not have an absolute max value. Since a local max value, say $f(x_0)$ at the point $x_0$, is defined such that it is the max within some interval $|x - x_0| < h$ where $h > 0$, the function may have greater values, and may even approach $\infty$ outside this interval. There is no absolute max value in this latter case.

**48.** No. $f(x) = -x^2$ has abs max value 0, but $g(x) = |f(x)| = x^2$ has no abs max value.

**49.** $f(x) = \begin{cases} x\sin\dfrac{1}{x} & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases}$
$|f(x)| \le |x|$ if $x > 0$ so $\lim_{x\to 0+} f(x) = 0 = f(0)$.
Therefore $f$ is continuous at $x = 0$. Clearly $x\sin\dfrac{1}{x}$ is continuous at $x > 0$. Therefore $f$ is continuous on $[0, \infty)$.
Given any $h > 0$ there exists $x_1$ in $(0, h)$ and $x_2$ in $(0, h)$ such that $f(x_1) > 0 = f(0)$ and $f(x_2) < 0 = f(0)$. Therefore $f$ cannot be a local max or min value at 0.
Specifically, let positive integer $n$ satisfy $2n\pi > \dfrac{1}{h}$ and let $x_1 = \dfrac{1}{2n\pi + \frac{\pi}{2}}$, $x_2 = \dfrac{1}{2n\pi + \frac{3\pi}{2}}$.
Then $f(x_1) = x_1 > 0$ and $f(x_2) < 0$.

## Section 4.3 Concavity and Inflections (page 227)

**1.** $f(x) = \sqrt{x}$, $f'(x) = \dfrac{1}{2\sqrt{x}}$, $f''(x) = -\dfrac{1}{4}x^{-3/2}$
$f''(x) < 0$ for all $x > 0$. $f$ is concave down on $(0, \infty)$.

**2.** $f(x) = 2x - x^2$, $f'(x) = 2 - 2x$, $f''(x) = -2 < 0$.
Thus, $f$ is concave down on $(-\infty, \infty)$.

**3.** $f(x) = x^2 + 2x + 3$, $f'(x) = 2x + 2$, $f''(x) = 2 > 0$.
$f$ is concave up on $(-\infty, \infty)$.

**4.** $f(x) = x - x^3$, $f'(x) = 1 - 3x^2$,
$f''(x) = -6x$.

| | | | | |
|---|---|---|---|---|
| $f''$ | + | 0 | − | |
| | | | | $x$ |
| $f$ | ⌣ | infl | ⌢ | |

**5.** $f(x) = 10x^3 - 3x^5$,
$f'(x) = 30x^2 - 15x^4$,
$f''(x) = 60(x - x^3) = 60x(1-x)(1+x)$.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $f''$ | + | −1 | − | 0 | + | 1 | − | |
| | | | | | | | | $x$ |
| $f$ | ⌣ | infl | ⌢ | infl | ⌣ | infl | ⌢ | |

**6.** $f(x) = 10x^3 + 3x^5$, $f'(x) = 30x^2 + 15x^4$,
$f''(x) = 60x + 60x^3 = 60x(1 + x^2)$.

| | | | | |
|---|---|---|---|---|
| $f''$ | − | 0 | + | |
| | | | | $x$ |
| $f$ | ⌢ | infl | ⌣ | |

**7.** $f(x) = (3 - x^2)^2$,
$f'(x) = -4x(3 - x^2) = -12x + 4x^3$,
$f''(x) = -12 + 12x^2 = 12(x - 1)(x + 1)$.

| | | | | | | |
|---|---|---|---|---|---|---|
| $f''$ | $+$ | $-1$ | $-$ | $1$ | $+$ | $\to x$ |
| $f$ | $\smile$ | infl | $\frown$ | infl | $\smile$ | |

**8.** $f(x) = (2 + 2x - x^2)^2$, $f'(x) = 2(2 + 2x - x^2)(2 - 2x)$,
$f''(x) = 2(2 - 2x)^2 + 2(2 + 2x - x^2)(-2)$
$= 12x(x - 2)$.

| | | | | | | |
|---|---|---|---|---|---|---|
| $f''$ | $+$ | $0$ | $-$ | $2$ | $+$ | $\to x$ |
| $f$ | $\smile$ | infl | $\frown$ | infl | $\smile$ | |

**9.** $f(x) = (x^2 - 4)^3$,
$f'(x) = 6x(x^2 - 4)^2$,
$f''(x) = 6(x^2 - 4)^2 + 24x^2(x^2 - 4)$
$= 6(x^2 - 4)(5x^2 - 4)$.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $f''$ | $+$ | $-2$ | $-$ | $-\frac{2}{\sqrt{5}}$ | $+$ | $\frac{2}{\sqrt{5}}$ | $-$ | $2$ | $+$ | $\to x$ |
| $f$ | $\smile$ | infl | $\frown$ | infl | $\smile$ | infl | $\frown$ | infl | $\smile$ | |

**10.** $f(x) = \dfrac{x}{x^2 + 3}$, $f'(x) = \dfrac{3 - x^2}{(x^2 + 3)^2}$,
$f''(x) = \dfrac{2x(x^2 - 9)}{(x^2 + 3)^3}$.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $f''$ | $-$ | $-3$ | $+$ | $0$ | $-$ | $3$ | $+$ | $\to x$ |
| $f$ | $\frown$ | infl | $\smile$ | infl | $\frown$ | infl | $\smile$ | |

**11.** $f(x) = \sin x$, $f'(x) = \cos x$, $f''(x) = -\sin x$.
$f$ is concave down on intervals $(2n\pi, (2n + 1)\pi)$ and concave up on intervals $((2n - 1)\pi, 2n\pi)$, where $n$ ranges over the integers. Points $x = n\pi$ are inflection points.

**12.** $f(x) = \cos 3x$, $f'(x) = -3\sin 3x$, $f''(x) = -9\cos 3x$.
Inflection points: $x = \left(n + \frac{1}{2}\right)\dfrac{\pi}{3}$ for $n = 0, \pm1, \pm2, \ldots$.
$f$ is concave up on $\left(\dfrac{4n + 1}{6}\pi, \dfrac{4n + 3}{6}\pi\right)$ and concave down on $\left(\dfrac{4n + 3}{6}\pi, \dfrac{4n + 5}{6}\pi\right)$.

**13.** $f(x) = x + \sin 2x$,
$f'(x) = 1 + 2\cos 2x$,
$f''(x) = -4\sin 2x$.
$f$ is concave up on intervals $\left(\dfrac{(2n - 1)\pi}{2}, n\pi\right)$, and concave down on intervals $\left(n\pi, \dfrac{(2n + 1)\pi}{2}\right)$. Points $\dfrac{n\pi}{2}$ are inflection points.

**14.** $f(x) = x - 2\sin x$, $f'(x) = 1 - 2\cos x$, $f''(x) = 2\sin x$.
Inflection points: $x = n\pi$ for $n = 0, \pm1, \pm2, \ldots$.
$f$ is concave down on $\Big((2n{+}1)\pi, (2n{+}2)\pi\Big)$ and concave up on $\Big((2n)\pi, (2n + 1)\pi\Big)$.

**15.** $f(x) = \tan^{-1} x$, $f'(x) = \dfrac{1}{1 + x^2}$,
$f''(x) = \dfrac{-2x}{(1 + x^2)^2}$.

| | | | | |
|---|---|---|---|---|
| $f''$ | $+$ | $0$ | $-$ | $\to x$ |
| $f$ | $\smile$ | infl | $\frown$ | |

**16.** $f(x) = xe^x$, $f'(x) = e^x(1 + x)$,
$f''(x) = e^x(2 + x)$.

| | | | | |
|---|---|---|---|---|
| $f''$ | $-$ | $-2$ | $+$ | $\to x$ |
| $f$ | $\frown$ | infl | $\smile$ | |

**17.** $f(x) = e^{-x^2}$, $f'(x) = -2xe^{-x^2}$,
$f''(x) = e^{-x^2}(4x^2 - 2)$.

| | | | | | | |
|---|---|---|---|---|---|---|
| $f''$ | $+$ | $-\frac{1}{\sqrt{2}}$ | $-$ | $\frac{1}{\sqrt{2}}$ | $+$ | $\to x$ |
| $f$ | $\smile$ | infl | $\frown$ | infl | $\smile$ | |

**18.** $f(x) = \dfrac{\ln(x^2)}{x}$, $f'(x) = \dfrac{2 - \ln(x^2)}{x^2}$,
$f''(x) = \dfrac{-6 + 2\ln(x^2)}{x^3}$.
$f$ has inflection point at $x = \pm e^{3/2}$ and $f$ is undefined at $x = 0$. $f$ is concave up on $(-e^{3/2}, 0)$ and $(e^{3/2}, \infty)$; and concave down on $(-\infty, -e^{3/2})$ and $(0, e^{3/2})$.

**19.** $f(x) = \ln(1 + x^2)$, $f'(x) = \dfrac{2x}{1 + x^2}$,
$$f''(x) = \frac{(1 + x^2)(2) - 2x(2x)}{(1 + x^2)^2} = \frac{2(1 - x^2)}{(1 + x^2)^2}.$$

| | | | | | | |
|---|---|---|---|---|---|---|
| $f''$ | $-$ | $-1$ | $+$ | $1$ | $-$ | $\to x$ |
| $f$ | $\frown$ | infl | $\smile$ | infl | $\frown$ | |

**20.** $f(x) = (\ln x)^2,\ f'(x) = \dfrac{2}{x}\ln x,$
$f''(x) = \dfrac{2(1-\ln x)}{x^2}$ for all $x > 0$.

$$\begin{array}{lcccccl} f'' & & 0 & + & e & - & \\ \hline & & | & & | & & \to x \\ f & & & \smile & \text{infl} & \frown & \end{array}$$

**21.** $f(x) = \dfrac{x^3}{3} - 4x^2 + 12x - \dfrac{25}{3},$
$f'(x) = x^2 - 8x + 12,$
$f''(x) = 2x - 8 = 2(x-4).$

$$\begin{array}{lcccl} f'' & - & 4 & + & \\ \hline & & | & & \to x \\ f & \frown & \text{infl} & \smile & \end{array}$$

**22.** $f(x) = (x-1)^{1/3} + (x+1)^{1/3},$
$f'(x) = \frac{1}{3}[(x-1)^{-2/3} + (x+1)^{-2/3}],$
$f''(x) = -\frac{2}{9}[(x-1)^{-5/3} + (x+1)^{-5/3}].$
$f(x) = 0 \Leftrightarrow x - 1 = -(x+1) \Leftrightarrow x = 0.$
Thus, $f$ has inflection point at $x = 0$. $f''(x)$ is undefined at $x = \pm 1$. $f$ is defined at $\pm 1$ and $x = \pm 1$ are also inflection points. $f$ is concave up on $(-\infty, -1)$ and $(0, 1)$; and down on $(-1, 0)$ and $(1, \infty)$.

**23.** According to Definition 4.3.1 and the subsequent discussion, $f(x) = ax + b$ has no concavity and therefore no inflections.

**24.** $f(x) = 3x^3 - 36x - 3,\ f'(x) = 9(x^2 - 4),\ f''(x) = 18x.$
The critical points are
$x = 2,\quad f''(2) > 0 \Rightarrow$ local min;
$x = -2,\quad f''(-2) < 0 \Rightarrow$ local max.

**25.** $f(x) = x(x-2)^2 + 1 = x^3 - 4x^2 + 4x + 1$
$f'(x) = 3x^2 - 8x + 4 = (x-2)(3x-2)$
CP: $x = 2,\ x = \dfrac{2}{3}$
$f''(x) = 6x - 8,\quad f''(2) = 4 > 0,\quad f''\left(\dfrac{2}{3}\right) = -4 < 0.$
Therefore, $f$ has a loc min at $x = 2$ and a loc max at $x = \dfrac{2}{3}$.

**26.** $f(x) = x + \dfrac{4}{x},\ f'(x) = 1 - \dfrac{4}{x^2},\ f''(x) = 8x^{-3}.$
The critical points are
$x = 2,\quad f''(2) > 0 \Rightarrow$ local min;
$x = -2,\quad f''(-2) < 0 \Rightarrow$ local max.

**27.** $f(x) = x^3 + \dfrac{1}{x}$
$f'(x) = 3x^2 - \dfrac{1}{x^2} = \dfrac{3x^4 - 1}{x^2},\quad$ CP: $x = \pm\dfrac{1}{\sqrt[4]{3}}.$
$f''(x) = 6x + \dfrac{2}{x^3}.$
$f''\left(\dfrac{1}{\sqrt[4]{3}}\right) > 0,\quad f''\left(\dfrac{-1}{\sqrt[4]{3}}\right) < 0.$
Therefore $f$ has a loc min at $\dfrac{1}{\sqrt[4]{3}}$ and a loc max at $\dfrac{-1}{\sqrt[4]{3}}$.

**28.** $f(x) = \dfrac{x}{2^x},\ f'(x) = \dfrac{1 - x\ln 2}{2^x},$
$f''(x) = \dfrac{\ln 2(x\ln 2 - 2)}{2^x}.$
The critical point is
$x = \dfrac{1}{\ln 2},\quad f''\left(\dfrac{1}{\ln 2}\right) < 0 \Rightarrow$ local max.

**29.** $f(x) = \dfrac{x}{1+x^2}$
$f'(x) = \dfrac{(1+x^2) - x2x}{(1+x^2)^2} = \dfrac{1-x^2}{(1+x^2)^2}$
CP: $x = \pm 1$
$$f''(x) = \frac{(1+x)^2(-2x) - (1-x^2)2(1+x^2)2x}{(1+x^2)^4}$$
$$= \frac{-2x - 2x^3 - 4x + 4x^3}{(1+x^2)^3} = \frac{-6x + 2x^3}{(1+x^2)^3}$$
$f''(1) = -\dfrac{1}{2},\ f''(-1) = \dfrac{1}{2}.$
$f$ has a loc max at 1 and a loc min at $-1$.

**30.** $f(x) = xe^x,\ f'(x) = e^x(1+x),\ f''(x) = e^x(2+x).$
The critical point is $x = -1$.
$f''(-1) > 0, \Rightarrow$ local min.

**31.** $f(x) = x\ln x,$
$f'(x) = 1 + \ln x,\qquad$ CP: $x = \dfrac{1}{e}$
$f''(x) = \dfrac{1}{x},\qquad f''(\dfrac{1}{e}) = e > 0.$
$f$ has a loc min at $\dfrac{1}{e}$.

**32.** $f(x) = (x^2-4)^2,\ f'(x) = 4x^3 - 16x,\ f''(x) = 12x^2 - 16.$
The critical points are
$x = 0,\quad f''(0) < 0 \Rightarrow$ local max;
$x = 2,\quad f''(2) > 0 \Rightarrow$ local min;
$x = -2,\quad f''(-2) > 0 \Rightarrow$ local min.

**33.** $f(x) = (x^2 - 4)^3$
$f'(x) = 6x(x^2 - 4)^2$
CP: $x = 0$, $x = \pm 2$
$$\begin{aligned} f''(x) &= 6(x^2 - 4)^2 + 24x^2(x^2 - 4) \\ &= 6(x^2 - 4)(5x^2 - 4) \end{aligned}$$
$f''(0) > 0, \quad f''(\pm 2) = 0.$
$f$ has a loc min at $x = 0$. Second derivative test yields no direct information about $\pm 2$. However, since $f''$ has opposite signs on opposite sides of the points 2 and $-2$, each of these points is an inflection point of $f$, and therefore $f$ cannot have a local maximum or minimum value at either.

**34.** $f(x) = (x^2 - 3)e^x$,
$f'(x) = (x^2 + 2x - 3)e^x = (x + 3)(x - 1)e^x$,
$f''(x) = (x^2 + 4x - 1)e^x$.
The critical points are
$x = -3, \quad f''(-3) < 0 \Rightarrow$ local max;
$x = 1, \quad f''(1) > 0 \Rightarrow$ local min.

**35.** $f(x) = x^2 e^{-2x^2}$
$f'(x) = e^{-2x^2}(2x - 4x^3) = 2(x - 2x^3)e^{-2x^2}$
CP: $x = 0$, $x = \pm \dfrac{1}{\sqrt{2}}$
$f''(x) = e^{-2x^2}(2 - 20x^2 + 16x^4)$
$f''(0) > 0, \quad f''\left(\pm \dfrac{1}{\sqrt{2}}\right) = -\dfrac{4}{e} < 0.$
Therefore, $f$ has a loc (and abs) min value at 0, and loc (and abs) max values at $\pm \dfrac{1}{\sqrt{2}}$.

**36.** Since
$$f(x) = \begin{cases} x^2 & \text{if } x \ge 0 \\ -x^2 & \text{if } x < 0, \end{cases}$$
we have
$$f'(x) = \begin{cases} 2x & \text{if } x \ge 0 \\ -2x & \text{if } x < 0 \end{cases} = 2|x|$$
$$f''(x) = \begin{cases} 2 & \text{if } x > 0 \\ -2 & \text{if } x < 0 \end{cases} = 2\operatorname{sgn} x.$$
$f'(x) = 0$ if $x = 0$. Thus, $x = 0$ is a critical point of $f$. It is also an inflection point since the conditions of Definition 3 are satisfied. $f''(0)$ does not exist. If a the graph of a function has a tangent line, vertical or not, at $x_0$, and has opposite concavity on opposite sides of $x_0$, the $x_0$ is an inflection point of $f$, whether or not $f''(x_0)$ even exists.

**37.** Suppose $f$ is concave up (i.e., $f''(x) > 0$) on an open interval containing $x_0$.
Let $h(x) = f(x) - f(x_0) - f'(x_0)(x - x_0)$.
Since $h'(x) = f'(x) - f'(x_0) = 0$ at $x = x_0$, $x = x_0$ is a CP of $h$.
Now $h''(x) = f''(x)$. Since $h''(x_0) > 0$, therefore $h$ has a min value at $x_0$, so $h(x) \ge h(x_0) = 0$ for $x$ near $x_0$.
Since $h(x)$ measures the distance $y = f(x)$ lies above the tangent line $y = f(x_0) + f'(x_0)(x - x_0)$ at $x$, therefore $y = f(x)$ lies above that tangent line near $x_0$.
Note: we must have $h(x) > 0$ for $x$ near $x_0$, $x \ne x_0$, for otherwise there would exist $x_1 \ne x_0$, $x_1$ near $x_0$, such that $h(x_1) = 0 = h(x_0)$. If $x_1 > x_0$, there would therefore exist $x_2$ such that $x_0 < x_2 < x_1$ and $f'(x_2) = f'(x_0)$. Therefore there would exist $x_3$ such that $x_0 < x_3 < x_2$ and $f'(x_3) = 0$, a contradiction.
The same contradiction can be obtained if $x_1 < x_0$.

**38.** Suppose that $f$ has an inflection point at $x_0$. To be specific, suppose that $f''(x) < 0$ on $(a, x_0)$ and $f''(x) > 0$ on $(x_0, b)$ for some numbers $a$ and $b$ satisfying $a < x_0 < b$.
If the graph of $f$ has a non-vertical tangent line at $x_0$, then $f'(x_0)$ exists. Let
$$F(x) = f(x) - f(x_0) - f'(x_0)(x - x_0).$$
$F(x)$ represents the signed vertical distance between the graph of $f$ and its tangent line at $x_0$. To show that the graph of $f$ crosses its tangent line at $x_0$, it is sufficient to show that $F(x)$ has opposite signs on opposite sides of $x_0$.
Observe that $F(x_0) = 0$, and $F'(x) = f'(x) - f'(x_0)$, so that $F'(x_0) = 0$ also. Since $F''(x) = f''(x)$, the assumptions above show that $F'$ has a local minimum value at $x_0$ (by the First Derivative Test). Hence $F(x) > 0$ if $a < x < x_0$ or $x_0 < x < b$. It follows (by Theorem 6) that $F(x) < 0$ if $a < x < x_0$, and $F(x) > 0$ if $x_0 < x < b$. This completes the proof for the case of a nonvertical tangent.
If $f$ has a vertical tangent at $x_0$, then its graph necessarily crosses the tangent (the line $x = x_0$) at $x_0$, since the graph of a function must cross any vertical line through a point of its domain that is not an endpoint.

**39.** $f(x) = x^n$
$g(x) = -x^n = -f(x), \quad n = 2, 3, 4, \ldots$
$f_n'(x) = nx^{n-1} = 0$ at $x = 0$
If $n$ is even, $f_n$ has a loc min, $g_n$ has a loc max at $x = 0$.
If $n$ is odd, $f_n$ has an inflection at $x = 0$, and so does $g_n$.

**40.** Let there be a function $f$ such that
$$f'(x_0) = f''(x_0) = \ldots = f^{(k-1)}(x_0) = 0,$$
$$f^{(k)}(x_0) \ne 0 \qquad \text{for some } k \ge 2.$$

If $k$ is even, then $f$ has a local min value at $x = x_0$ when $f^{(k)}(x_0) > 0$, and $f$ has a local max value at $x = x_0$ when $f^{(k)}(x_0) < 0$.
If $k$ is odd, then $f$ has an inflection point at $x = x_0$.

**41.** $f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

a) $\lim_{x\to 0+} x^{-n} f(x) = \lim_{x\to 0+} \dfrac{e^{-1/x^2}}{x^n}$ (put $y = 1/x$)

$= \lim_{y\to\infty} y^n e^{-y^2} = 0$ by Theorem 5 of Sec. 4.4

Similarly, $\lim_{x\to 0-} x^{-n} f(x) = 0$, and $\lim_{x\to 0} x^{-n} f(x) = 0$.

b) If $P(x) = \sum_{j=0}^{n} a_j x^j$ then by (a)

$$\lim_{x\to 0} P\left(\frac{1}{x}\right) f(x) = \sum_{j=0}^{n} a_j \lim_{x\to 0} x^{-j} f(x) = 0.$$

c) If $x \neq 0$ and $P_1(t) = 2t^3$, then

$$f'(x) = \frac{2}{x^3} e^{-1/x^2} = P_1\left(\frac{1}{x}\right) f(x).$$

Assume that $f^{(k)}(x) = P_k\left(\dfrac{1}{x}\right) f(x)$ for some $k \geq 1$, where $P_k$ is a polynomial. Then

$$\begin{aligned} f^{(k+1)}(x) &= -\frac{1}{x^2} P_k'\left(\frac{1}{x}\right) f(x) + P_k\left(\frac{1}{x}\right) P_1\left(\frac{1}{x}\right) f(x) \\ &= P_{k+1}\left(\frac{1}{x}\right) f(x), \end{aligned}$$

where $P_{k+1}(t) = t^2 P_k'(t) + P_1(t) P_k(t)$ is a polynomial.

By induction, $f^{(n)} = P_n\left(\dfrac{1}{n}\right) f(x)$ for $n \neq 0$, where $P_n$ is a polynomial.

d) $f'(0) = \lim_{h\to 0} \dfrac{f(h) - f(0)}{h} = \lim_{h\to 0} h^{-1} f(h) = 0$ by (a). Suppose that $f^{(k)}(0) = 0$ for some $k \geq 1$. Then

$$\begin{aligned} f^{(k+1)}(0) &= \lim_{h\to 0} \frac{f^{(k)}(h) - f^{(k)}(0)}{h} \\ &= \lim_{h\to 0} h^{-1} f^{(k)}(h) \\ &= \lim_{h\to 0} h^{-1} P_k\left(\frac{1}{h}\right) f(h) = 0 \end{aligned}$$

by (b).
Thus $f^{(n)}(0) = 0$ for $n = 1, 2, \ldots$ by induction.

e) Since $f'(x) < 0$ if $x < 0$ and $f'(x) > 0$ if $x > 0$, therefore $f$ has a local min value at 0 and $-f$ has a loc max value there.

f) If $g(x) = xf(x)$ then $g'(x) = f(x) + xf'(x)$, $g''(x) = 2f'(x) + xf''(x)$.
In general, $g^{(n)}(x) = nf^{(n-1)}(x) + xf^{(n)}(x)$ (by induction).
Then $g^{(n)}(0) = 0$ for all $n$ (by (d)).
Since $g(x) < 0$ if $x < 0$ and $g(x) > 0$ if $x > 0$, $g$ cannot have a max or min value at 0. It must have an inflection point there.

**42.** We are given that

$$f(x) = \begin{cases} x^2 \sin \dfrac{1}{x}, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$$

If $x \neq 0$, then

$$\begin{aligned} f'(x) &= 2x \sin \frac{1}{x} - \cos \frac{1}{x} \\ f''(x) &= 2 \sin \frac{1}{x} - \frac{2}{x} \cos \frac{1}{x} - \frac{1}{x^2} \sin \frac{1}{x}. \end{aligned}$$

If $x = 0$, then

$$f'(x) = \lim_{h\to 0} \frac{h^2 \sin \frac{1}{h} - 0}{h} = 0.$$

Thus 0 is a critical point of $f$. There are points $x$ arbitrarily close to 0 where $f(x) > 0$, for example $x = \dfrac{2}{(4n+1)\pi}$, and other such points where $f(x) < 0$, for example $x = \dfrac{2}{(4n+3)\pi}$. Therefore $f$ does not have a local max or min at $x = 0$. Also, there are points arbitrarily close to 0 where $f''(x) > 0$, for example $x = \dfrac{1}{(2n+1)\pi}$, and other such points where $f''(x) < 0$, for instance $x = \dfrac{1}{2n\pi}$. Therefore $f$ does not have constant concavity on any interval $(0, a)$ where $a > 0$, so 0 is not an inflection point of $f$ either.

## Section 4.4 Sketching the Graph of a Function (page 236)

**1.** Function (d) appears to be the derivative of function (c), and function (b) appears to be the derivative of function (d). Thus graph (c) is the graph of $f$, (d) is the graph of $f'$, (b) is the graph of $f''$, and (a) must be the graph of the other function $g$.

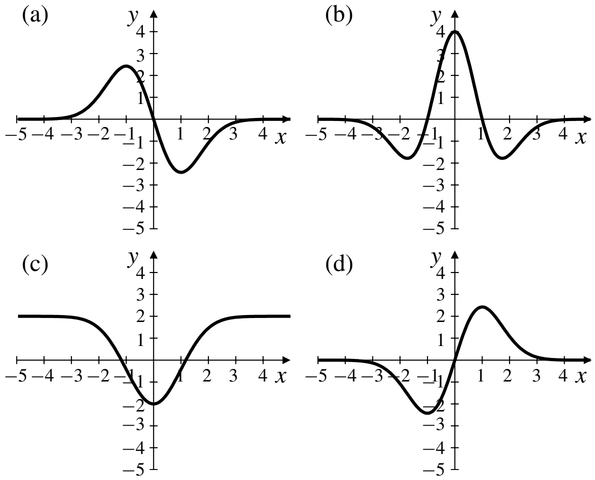



Fig. 4.4.1

**2.**

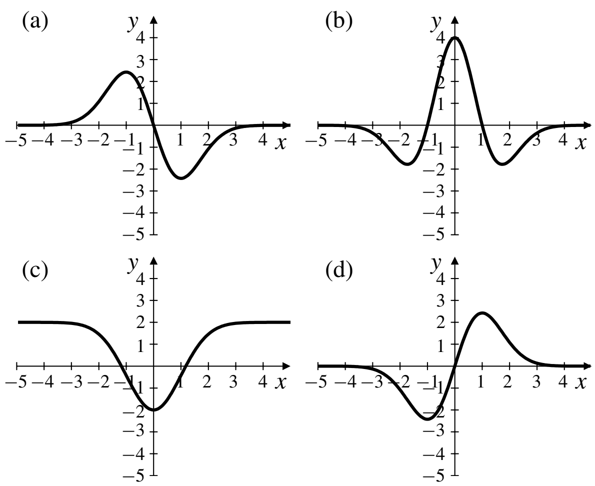



Fig. 4.4.2

The function graphed in Fig. 4.2(a):
is odd, is asymptotic to $y = 0$ at $\pm\infty$,
is increasing on $(-\infty, -1)$ and $(1, \infty)$,
is decreasing on $(-1, 1)$,
has CPs at $x = -1$ (max) and 1 (min),
is concave up on $(-\infty, -2)$ and $(0, 2)$ (approximately),
is concave down on $(-2, 0)$ and $(2, \infty)$ (approximately),
has inflections at $x = \pm 2$ (approximately).

The function graphed in Fig. 4.2(b):
is even, is asymptotic to $y = 0$ at $\pm\infty$,
is increasing on $(-1.7, 0)$ and $(1.7, \infty)$ (approximately),
is decreasing on $(-\infty, -1.7)$ and $(0, 1.7)$ (approximately),
has CPs at $x = 0$ (max) and $\pm 1.7$ (min) (approximately),
is concave up on $(-2.5, -1)$ and $(1, 2.5)$ (approximately),
is concave down on $(-\infty, -2.5)$, $(-1, 1)$, and $(2.5, \infty)$ (approximately),
has inflections at $\pm 2.5$ and $\pm 1$ (approximately).

The function graphed in Fig. 4.2(c):
is even, is asymptotic to $y = 2$ at $\pm\infty$,
is increasing on $(0, \infty)$,
is decreasing on $(-\infty, 0)$,
has a CP at $x = 0$ (min),
is concave up on $(-1, 1)$ (approximately),
is concave down on $(-\infty, -1)$ and $(1, \infty)$ (approximately),
has inflections at $x = \pm 1$ (approximately).

The function graphed in Fig. 4.2(d):
is odd, is asymptotic to $y = 0$ at $\pm\infty$,
is increasing on $(-1, 1)$,
is decreasing on $(-\infty, -1)$ and $(1, \infty)$,
has CPs at $x = -1$ (min) and 1 (max),
is concave down on $(-\infty, -1.7)$ and $(0, 1.7)$ (approximately),
is concave up on $(-1.7, 0)$ and $(1.7, \infty)$ (approximately),
has inflections at 0 and $\pm 1.7$ (approximately).

**3.** $f(x) = x/(1 - x^2)$ has slope 1 at the origin, so its graph must be (c).
$g(x) = x^3/(1 - x^4)$ has slope 0 at the origin, but has the same sign at all points as does $f(x)$, so its graph must be (b).
$h(x) = (x^3 - x)/\sqrt{1 + x^6}$ has no vertical asymptotes, so its graph must be (d).
$k(x) = x^3/\sqrt{|x^4 - 1|}$ is positive for all positive $x \neq 1$, so its graph must be (a).

**4.**

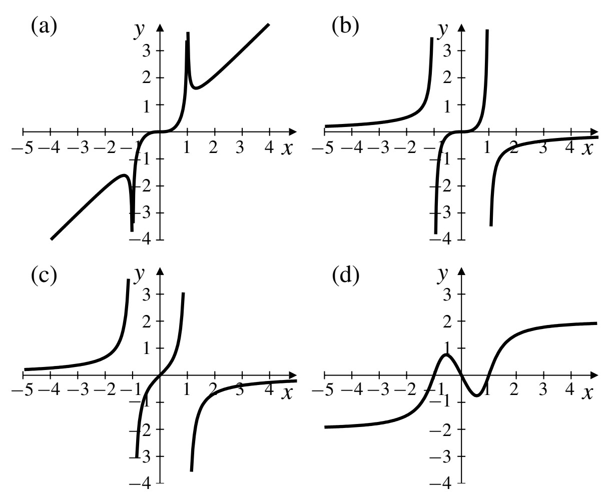



Fig. 4.4.4

The function graphed in Fig. 4.4(a):
is odd, is asymptotic to $x = \pm 1$ and $y = x$,
is increasing on $(-\infty, -1.5)$, $(-1, 1)$, and $(1.5, \infty)$ (approximately),
is decreasing on $(-1.5, -1)$ and $(1, 1.5)$ (approximately),
has CPs at $x = -1.5$, $x = 0$, and $x = 1.5$,
is concave up on $(0, 1)$ and $(1, \infty)$,
is concave down on $(-\infty, -1)$ and $(-1, 0)$,
has an inflection at $x = 0$.

The function graphed in Fig. 4.4(b):
is odd, is asymptotic to $x = \pm 1$ and $y = 0$,
is increasing on $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$,
has a CP at $x = 0$,
is concave up on $(-\infty, -1)$ and $(0, 1)$,
is concave down on $(-1, 0)$ and $(1, \infty)$,
has an inflection at $x = 0$.

The function graphed in Fig. 4.4(c):
is odd, is asymptotic to $x = \pm 1$ and $y = 0$,
is increasing on $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$,
has no CP,
is concave up on $(-\infty, -1)$ and $(0, 1)$,
is concave down on $(-1, 0)$ and $(1, \infty)$,
has an inflection at $x = 0$.

The function graphed in Fig. 4.4(d):
is odd, is asymptotic to $y = \pm 2$,
is increasing on $(-\infty, -0.7)$ and $(0.7, \infty)$ (approximately),
is decreasing on $(-0.7, 0.7)$ (approximately),
has CPs at $x = \pm 0.7$ (approximately),
is concave up on $(-\infty, -1)$ and $(0, 1)$ (approximately),
is concave down on $(-1, 0)$ and $(1, \infty)$ (approximately),
has an inflection at $x = 0$ and $x = \pm 1$ (approximately).

**5.** $f(0) = 1 \quad f(\pm 1) = 0 \quad f(2) = 1$
$\lim_{x\to\infty} f(x) = 2$, $\lim_{x\to-\infty} f(x) = -1$

| | | SP | | CP | |
|---|---|---|---|---|---|
| $f'$ | $+$ | $0$ | $-$ | $1$ | $+$ |
| $f$ | $\nearrow$ | loc max | $\searrow$ | loc min | $\nearrow$ |

| | | | | | |
|---|---|---|---|---|---|
| $f''$ | $+$ | $0$ | $+$ | $2$ | $-$ |
| $f$ | $\smile$ | | $\smile$ | infl | $\frown$ |

0 must be a SP because $f'' > 0$ on both sides and it is a loc max. 1 must be a CP because $f''$ is defined there so $f'$ must be too.

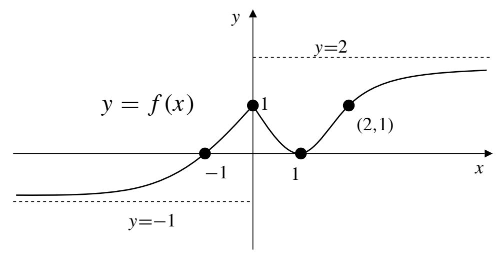


Fig. 4.4.5

**6.** According to the given properties:
Oblique asymptote: $y = x - 1$.
Critical points: $x = 0, \ 2$. Singular point: $x = -1$.
Local max 2 at $x = 0$; local min 0 at $x = 2$.

| | | SP | | CP | | CP | |
|---|---|---|---|---|---|---|---|
| $f'$ | $+$ | $-1$ | $+$ | $0$ | $-$ | $2$ | $+$ |
| $f$ | $\nearrow$ | | $\nearrow$ | loc max | $\searrow$ | loc min | $\nearrow$ |

Inflection points: $x = -1, \ 1, \ 3$.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $f'$ | $+$ | $-1$ | $-$ | $1$ | $+$ | $3$ | $-$ |
| $f$ | $\smile$ | infl | $\frown$ | infl | $\smile$ | infl | $\frown$ |

Since $\lim_{x\to\pm\infty} \big(f(x) + 1 - x\big) = 0$, the line $y = x - 1$ is an oblique asymptote.

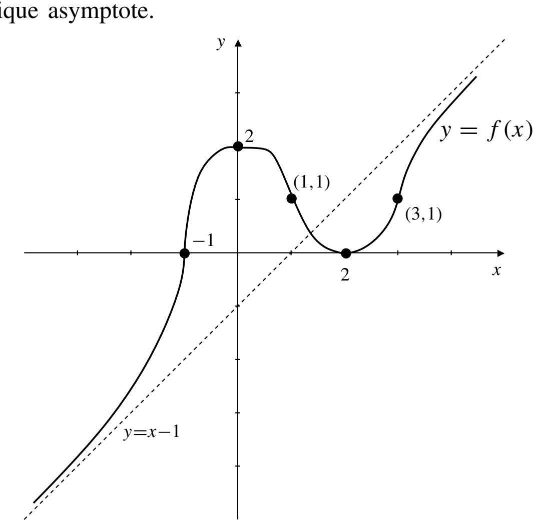


Fig. 4.4.6

**7.** $y = (x^2 - 1)^3$
$$y' = 6x(x^2-1)^2 = 6x(x-1)^2(x+1)^2$$
$$y'' = 6[(x^2-1)^2 + 4x^2(x^2-1)] = 6(x^2-1)(5x^2-1) = 6(x-1)(x+1)(\sqrt{5}x-1)(\sqrt{5}x+1)$$
From $y$: Asymptotes: none. Symmetry: even. Intercepts: $x = \pm 1$.
From $y'$: CP: $x = 0$, $x = \pm 1$. SP: none.

| | | CP | | CP | | CP | |
|---|---|---|---|---|---|---|---|
| $y'$ | $-$ | $-1$ | $-$ | $0$ | $+$ | $1$ | $+$ |
| $y$ | $\searrow$ | | $\searrow$ | abs min | $\nearrow$ | | $\nearrow$ |

From $y''$: $y'' = 0$ at $x = \pm 1$, $x = \pm\dfrac{1}{\sqrt{5}}$.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $y''$ | $+$ | $-1$ | $-$ | $-\frac{1}{\sqrt{5}}$ | $+$ | $\frac{1}{\sqrt{5}}$ | $-$ | $1$ | $+$ |
| $y$ | $\smile$ | infl | $\frown$ | infl | $\smile$ | infl | $\frown$ | infl | $\smile$ |

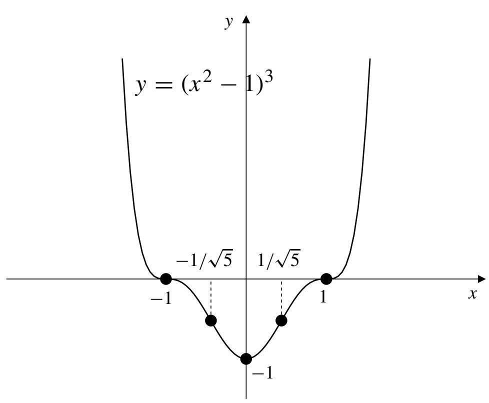


Fig. 4.4.7

**8.** $y = x(x^2-1)^2$, $y' = (x^2-1)(5x^2-1)$, $y'' = 4x(5x^2-3)$.
From $y$: Intercepts: $(0,0)$, $(1,0)$. Symmetry: odd (i.e., about the origin).
From $y'$: Critical point: $x = \pm 1$, $\pm\dfrac{1}{\sqrt{5}}$.

| | | CP | | CP | | CP | | CP | |
|---|---|---|---|---|---|---|---|---|---|
| $y'$ | $+$ | $-1$ | $-$ | $-\frac{1}{\sqrt{5}}$ | $+$ | $\frac{1}{\sqrt{5}}$ | $-$ | $1$ | $+$ |
| $y$ | $\nearrow$ | loc max | $\searrow$ | loc min | $\nearrow$ | loc max | $\searrow$ | loc min | $\nearrow$ |

From $y''$: Inflection points at $x = 0$, $\pm\sqrt{\frac{3}{5}}$.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $y''$ | $-$ | $-\sqrt{\frac{3}{5}}$ | $+$ | $0$ | $-$ | $\sqrt{\frac{3}{5}}$ | $+$ |
| $y$ | $\frown$ | infl | $\smile$ | infl | $\frown$ | infl | $\smile$ |

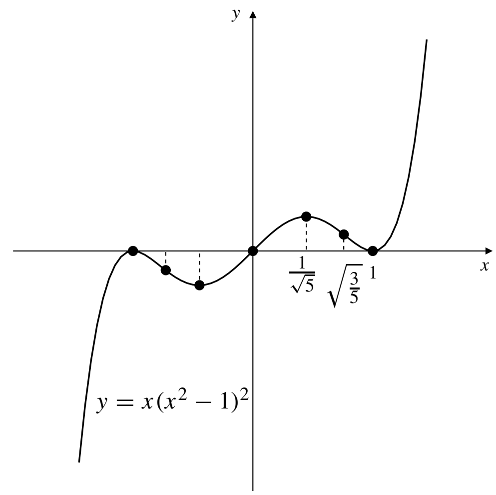


Fig. 4.4.8

**9.** $y = \dfrac{2-x}{x} = \dfrac{2}{x} - 1$, $\quad y' = -\dfrac{2}{x^2}$, $\quad y'' = \dfrac{4}{x^3}$.
From $y$: Asymptotes: $x = 0$, $y = -1$.
Symmetry: none obvious.
Intercept: $(2,0)$. Points: $(-1,-3)$.
From $y'$: CP: none. SP: none.

| | | ASY | |
|---|---|---|---|
| $y'$ | $-$ | $0$ | $-$ |
| $y$ | $\searrow$ | | $\searrow$ |

From $y''$: $y'' = 0$ nowhere.

| | | ASY | |
|---|---|---|---|
| $y''$ | $-$ | $0$ | $+$ |
| $y$ | $\frown$ | | $\smile$ |

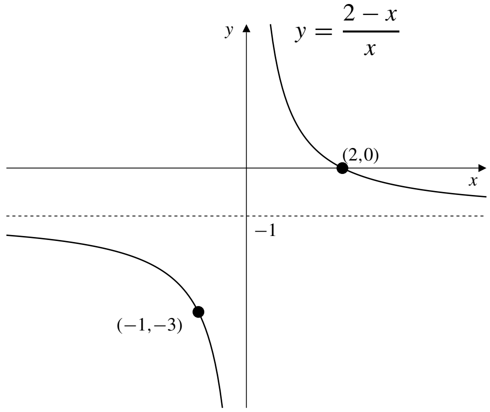



Fig. 4.4.9

**10.** $y = \dfrac{x-1}{x+1} = 1 - \dfrac{2}{x+1}$, $y' = \dfrac{2}{(x+1)^2}$, $y'' = \dfrac{-4}{(x+1)^3}$.
From $y$: Intercepts: $(0, -1)$, $(1, 0)$. Asymptotes: $y = 1$ (horizontal), $x = -1$ (vertical). No obvious symmetry.
Other points: $(-2, 3)$.
From $y'$: No critical point.

| | | ASY | | |
|---|---|---|---|---|
| $y'$ | $+$ | $-1$ | $+$ | $x$ |
| $y$ | $\nearrow$ | | $\nearrow$ | |

From $y''$: No inflection point.

| | | ASY | | |
|---|---|---|---|---|
| $y''$ | $+$ | $-1$ | $-$ | $x$ |
| $y$ | $\smile$ | | $\frown$ | |

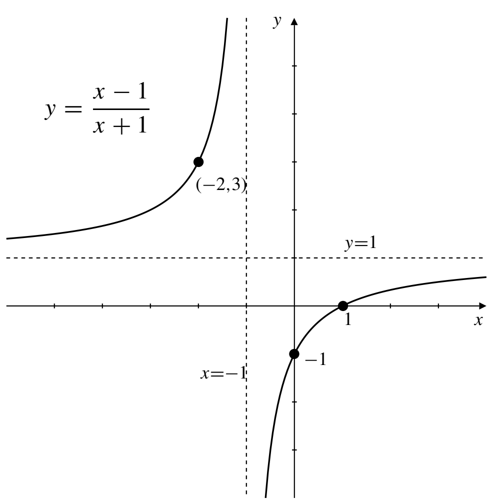



Fig. 4.4.10

**11.**
$$y = \frac{x^3}{1+x}$$
$$y' = \frac{(1+x)3x^2 - x^3}{(1+x)^2} = \frac{3x^2 + 2x^3}{(1+x)^2}$$
$$y'' = \frac{(1+x)^2(6x+6x^2) - (3x^2+2x^3)2(1+x)}{(1+x)^4}$$
$$= \frac{6x(1+x)^2 - 6x^2 - 4x^3}{(1+x)^3} = \frac{6x + 6x^2 + 2x^3}{(1+x)^3}$$
$$= \frac{2x(3+3x+x^2)}{(1+x)^3}$$
From $y$:
Asymptotes: $x = -1$. Symmetry: none.
Intercepts $(0, 0)$. Points $(-3/2, 27/4)$.
From $y'$ CP: $x = 0$, $x = -\dfrac{3}{2}$.

| | | CP | | ASY | | CP | | |
|---|---|---|---|---|---|---|---|---|
| $y'$ | $-$ | $-\frac{3}{2}$ | $+$ | $-1$ | $+$ | $0$ | $+$ | $x$ |
| $y$ | $\searrow$ | loc min | $\nearrow$ | | $\nearrow$ | | $\nearrow$ | |

From $y''$: $y'' = 0$ only at $x = 0$.

| | | ASY | | | | |
|---|---|---|---|---|---|---|
| $y''$ | $+$ | $-1$ | $-$ | $0$ | $+$ | $x$ |
| $y$ | $\smile$ | | $\frown$ | infl | $\smile$ | |

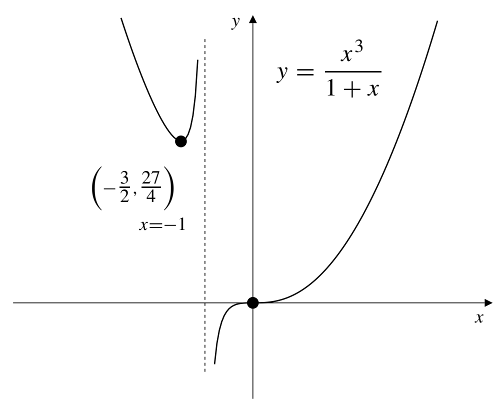



Fig. 4.4.11

**12.** $y = \dfrac{1}{4+x^2}$, $y' = \dfrac{-2x}{(4+x^2)^2}$, $y'' = \dfrac{6x^2-8}{(4+x^2)^3}$.
From $y$: Intercept: $(0, \frac{1}{4})$. Asymptotes: $y = 0$ (horizontal). Symmetry: even (about $y$-axis).
From $y'$: Critical point: $x = 0$.

| | | CP | | |
|---|---|---|---|---|
| $y'$ | $+$ | $0$ | $-$ | $x$ |
| $y$ | $\nearrow$ | abs max | $\searrow$ | |

From $y''$: $y''=0$ at $x=\pm\dfrac{2}{\sqrt{3}}$.

| $y''$ | $+$ | $-\frac{2}{\sqrt{3}}$ | $-$ | $\frac{2}{\sqrt{3}}$ | $+$ | $x$ |
|---|---|---|---|---|---|---|
| $y$ | $\smile$ | infl | $\frown$ | infl | $\smile$ | |

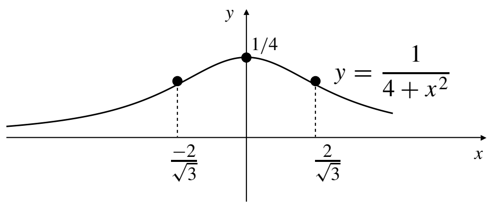



Fig. 4.4.12

**13.** $y=\dfrac{1}{2-x^2}$, $\quad y'=\dfrac{2x}{(2-x^2)^2}$

$$y''=\frac{2}{(2-x^2)^2}+\frac{8x^2}{(2-x^2)^3}=\frac{4+6x^2}{(2-x^2)^3}$$

From $y$: Asymptotes: $y=0$, $x=\pm\sqrt{2}$.
Symmetry: even.
Intercepts $(0,\frac{1}{2})$. Points $(\pm 2,-\frac{1}{2})$.
From $y'$: CP $x=0$.

| | | ASY | | CP | | ASY | | |
|---|---|---|---|---|---|---|---|---|
| $y''$ | $-$ | $-\sqrt{2}$ | $-$ | $0$ | $+$ | $\sqrt{2}$ | $+$ | $x$ |
| $y$ | $\searrow$ | | $\searrow$ | loc min | $\nearrow$ | | $\nearrow$ | |

$y''$ : $y''=0$ nowhere.

| | | ASY | | ASY | | |
|---|---|---|---|---|---|---|
| $y''$ | $-$ | $-\sqrt{2}$ | $+$ | $\sqrt{2}$ | $-$ | $x$ |
| $y$ | $\frown$ | | $\smile$ | | $\frown$ | |

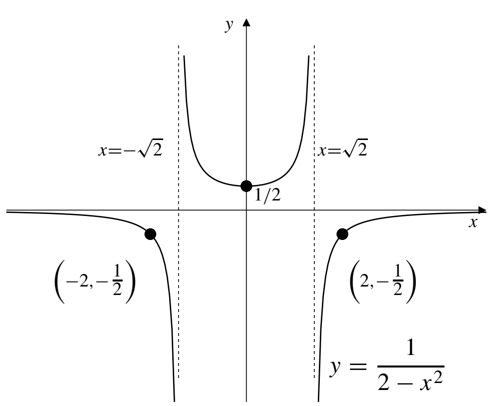



Fig. 4.4.13

**14.** $y=\dfrac{x}{x^2-1}$, $y'=-\dfrac{x^2+1}{(x^2-1)^2}$, $y''=\dfrac{2x(x^2+3)}{(x^2-1)^3}$.
From $y$: Intercept: $(0,0)$. Asymptotes: $y=0$ (horizontal), $x=\pm 1$ (vertical). Symmetry: odd. Other points: $(2,\frac{2}{3})$, $(-2,-\frac{2}{3})$.
From $y'$: No critical or singular points.

| | | ASY | | ASY | | |
|---|---|---|---|---|---|---|
| $y'$ | $-$ | $-1$ | $-$ | $1$ | $-$ | $x$ |
| $y$ | $\searrow$ | | $\searrow$ | | $\searrow$ | |

From $y''$: $y''=0$ at $x=0$.

| | | ASY | | | | ASY | | |
|---|---|---|---|---|---|---|---|---|
| $y''$ | $-$ | $-1$ | $+$ | $0$ | $-$ | $1$ | $+$ | $x$ |
| $y$ | $\frown$ | | $\smile$ | infl | $\frown$ | | $\smile$ | |

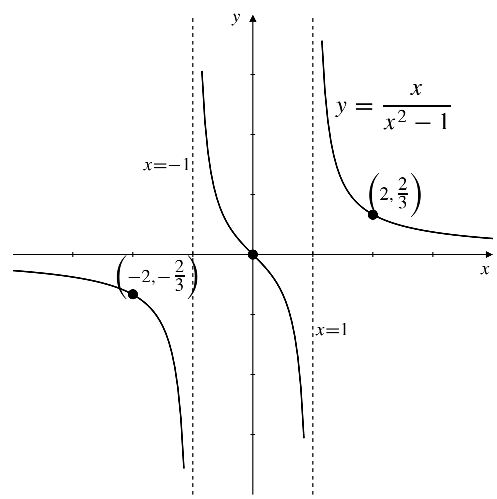



Fig. 4.4.14

**15.** $y=\dfrac{x^2}{x^2-1}=1+\dfrac{1}{x^2-1}$

$$y'=\frac{-2x}{(x^2-1)^2}$$

$$y''=-2\frac{(x^2-1)^2-x2(x^2-1)2x}{(x^2-1)^4}=\frac{2(3x^2+1)}{(x^2-1)^3}$$

From $y$: Asymptotes: $y=1$, $x=\pm 1$. Symmetry: even.
Intercepts $(0,0)$. Points $\left(\pm 2,\dfrac{4}{3}\right)$.
From $y'$: CP $x=0$.

| | | ASY | | CP | | ASY | | |
|---|---|---|---|---|---|---|---|---|
| $y'$ | $+$ | $-1$ | $+$ | $0$ | $-$ | $1$ | $-$ | $x$ |
| $y$ | $\nearrow$ | | $\nearrow$ | loc max | $\searrow$ | | $\searrow$ | |

From $y''$: $y'' = 0$ nowhere.

| | | ASY | | ASY | | |
|---|---|---|---|---|---|---|
| $y''$ | $+$ | $-1$ | $-$ | $1$ | $+$ | $x$ |
| $y$ | $\smile$ | | $\frown$ | | $\smile$ | |

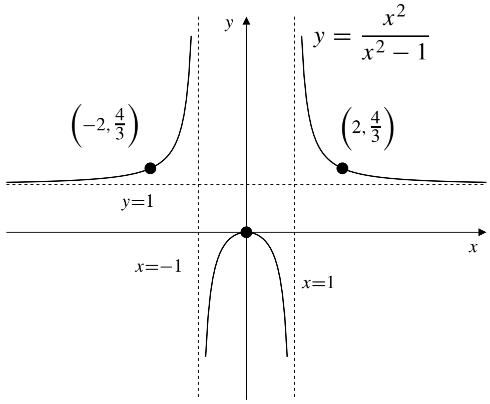



Fig. 4.4.15

**16.** $y = \dfrac{x^3}{x^2-1}$, $y' = \dfrac{x^2(x^2-3)}{(x^2-1)^2}$, $y'' = \dfrac{2x(x^2+3)}{(x^2-1)^3}$.
From $y$: Intercept: $(0, 0)$. Asymptotes: $x = \pm 1$ (vertical), $y = x$ (oblique). Symmetry: odd. Other points: $\left(\pm\sqrt{3}, \pm\dfrac{3\sqrt{3}}{2}\right)$.
From $y'$: Critical point: $x = 0$, $\pm\sqrt{3}$.

| | | CP | | ASY | | CP | | ASY | | CP | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $y'$ | $+$ | $-\sqrt{3}$ | $-$ | $-1$ | $-$ | $0$ | $-$ | $1$ | $-$ | $\sqrt{3}$ | $+$ | $x$ |
| $y$ | $\nearrow$ | loc max | $\searrow$ | | $\searrow$ | | $\searrow$ | | $\searrow$ | loc min | $\nearrow$ | |

From $y''$: $y'' = 0$ at $x = 0$.

| | | ASY | | | | ASY | | |
|---|---|---|---|---|---|---|---|---|
| $y''$ | $-$ | $-1$ | $+$ | $0$ | $-$ | $1$ | $+$ | $x$ |
| $y$ | $\frown$ | | $\smile$ | infl | $\frown$ | | $\smile$ | |

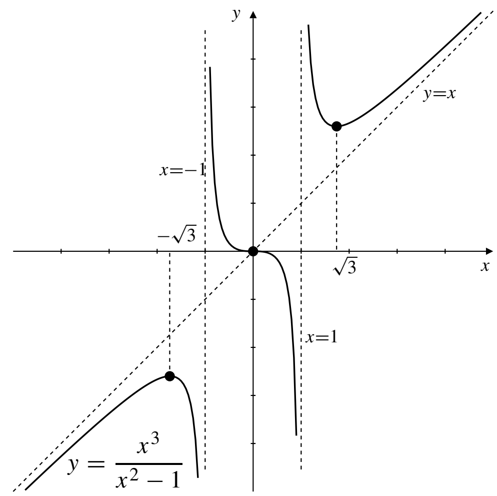



Fig. 4.4.16

**17.** $$y = \frac{x^3}{x^2+1} = \frac{x^3+x-x}{x^2+1} = x - \frac{x}{x^2+1}$$
$$y' = \frac{(x^2+1)3x^2 - x^3 2x}{(x^2+1)^2} = \frac{x^4+3x^2}{(x^2+1)^2} = \frac{x^2(x^2+3)}{(x^2+1)^2}$$
$$\begin{aligned} y'' &= \frac{(x^2+1)^2(4x^3+6x) - (x^4+3x^2)2(x^2+1)2x}{(x^2+1)^4} \\ &= \frac{4x^5+10x^3+6x-4x^5-12x^3}{(x^2+1)^3} \\ &= \frac{2x(3-x^2)}{(x^2+1)^3} \end{aligned}$$
From $y$: Asymptotes: $y = x$ (oblique). Symmetry: odd.
Intercepts $(0, 0)$.
Points $(\pm\sqrt{3}, \pm\frac{3}{4}\sqrt{3})$.
From $y'$: CP: $x = 0$.

| | | CP | | |
|---|---|---|---|---|
| $y'$ | $+$ | $0$ | $+$ | $x$ |
| $y$ | $\nearrow$ | | $\nearrow$ | |

From $y''$: $y'' = 0$ at $x = 0$, $x = \pm\sqrt{3}$.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $y''$ | $+$ | $-\sqrt{3}$ | $-$ | $0$ | $+$ | $\sqrt{3}$ | $-$ | $x$ |
| $y$ | $\smile$ | infl | $\frown$ | infl | $\smile$ | infl | $\frown$ | |

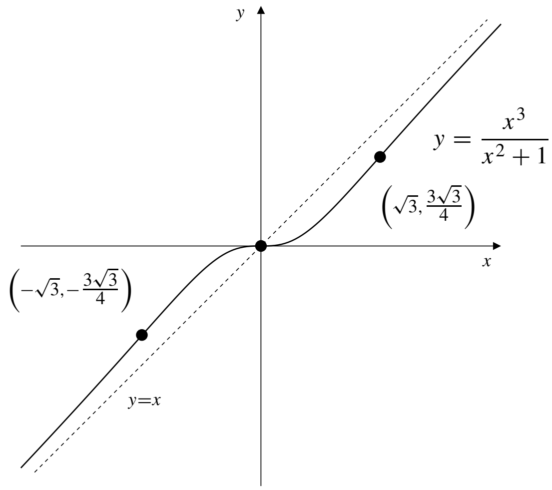



Fig. 4.4.17

**18.** $y = \dfrac{x^2}{x^2+1}$, $y' = \dfrac{2x}{(x^2+1)^2}$, $y'' = \dfrac{2(1-3x^2)}{(x^2+1)^3}$.
From $y$: Intercept: $(0, 0)$. Asymptotes: $y = 1$ (horizontal). Symmetry: even.
From $y'$: Critical point: $x = 0$.

| | | CP | | |
|---|---|---|---|---|
| $y'$ | $-$ | $0$ | $+$ | $\to x$ |
| $y$ | $\searrow$ | abs min | $\nearrow$ | |

From $y''$: $y'' = 0$ at $x = \pm\dfrac{1}{\sqrt{3}}$.

| | | | | | | |
|---|---|---|---|---|---|---|
| $y''$ | $-$ | $-\frac{1}{\sqrt{3}}$ | $+$ | $\frac{1}{\sqrt{3}}$ | $-$ | $\to x$ |
| $y$ | $\frown$ | infl | $\smile$ | infl | $\frown$ | |

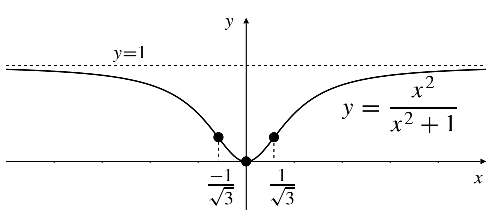



Fig. 4.4.18

**19.**
$$y = \frac{x^2-4}{x+1} = x - 1 - \frac{3}{x+1}$$
$$y' = 1 + \frac{3}{(x+1)^2} = \frac{(x+1)^2+3}{(x+1)^2}$$
$$y'' = -\frac{6}{(x+1)^3}$$
From $y$: Asymptotes: $y = x - 1$ (oblique), $x = -1$.
Symmetry: none.
Intercepts $(0, -4)$, $(\pm 2, 0)$.
From $y'$: CP: none.

| | | ASY | | |
|---|---|---|---|---|
| $y'$ | $+$ | $-1$ | $+$ | $\to x$ |
| $y$ | $\nearrow$ | | $\nearrow$ | |

From $y''$: $y'' = 0$ nowhere.

| | | ASY | | |
|---|---|---|---|---|
| $y''$ | $+$ | $-1$ | $-$ | $\to x$ |
| $y$ | $\smile$ | | $\frown$ | |

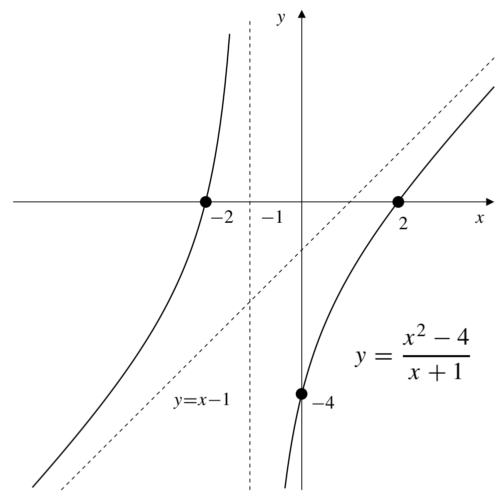



Fig. 4.4.19

**20.** $y = \dfrac{x^2-2}{x^2-1}$, $y' = \dfrac{2x}{(x^2-1)^2}$, $y'' = \dfrac{-2(3x^2+1)}{(x^2-1)^3}$.
From $y$: Intercept: $(0, 2)$, $(\pm\sqrt{2}, 0)$. Asymptotes: $y = 1$ (horizontal), $x = \pm 1$ (vertical). Symmetry: even.
From $y'$: Critical point: $x = 0$.

| | | ASY | | CP | | ASY | | |
|---|---|---|---|---|---|---|---|---|
| $f'$ | $-$ | $-1$ | $-$ | $0$ | $+$ | $1$ | $+$ | $\to x$ |
| $f$ | $\searrow$ | | $\searrow$ | loc min | $\nearrow$ | | $\nearrow$ | |

From $y''$: $y'' = 0$ nowhere.

| | | ASY | | ASY | | |
|---|---|---|---|---|---|---|
| $y'$ | $-$ | $-1$ | $+$ | $1$ | $-$ | $\to x$ |
| $y$ | $\frown$ | | $\smile$ | | $\frown$ | |

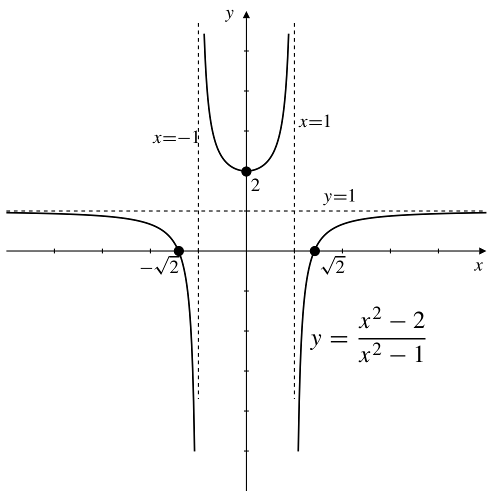



Fig. 4.4.20

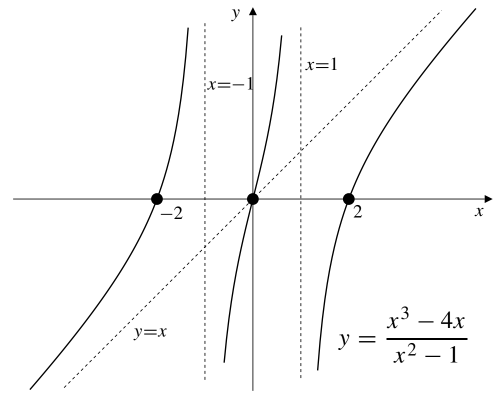



Fig. 4.4.21

**21.** $$y = \frac{x^3 - 4x}{x^2 - 1} = \frac{x(x-2)(x+2)}{x^2 - 1}$$

$$\begin{aligned} y' &= \frac{(x^2-1)(3x^2-4) - (x^3-4x)2x}{(x^2-1)^2} \\ &= \frac{3x^4 - 7x^2 + 4 - 2x^4 + 8x^2}{(x^2-1)^2} \\ &= \frac{x^4 + x^2 + 4}{(x^2-1)^2} \\ y'' &= \frac{(x^2-1)^2(4x^3+2x) - (x^4+x^2+4)2(x^2-1)2x}{(x^2-1)^4} \\ &= \frac{4x^5 - 2x^3 - 2x - 4x^5 - 4x^3 - 16x}{(x^2-1)^3} \\ &= \frac{-6x^3 - 18x}{(x^2-1)^3} = -6x\frac{x^2+3}{(x^2-1)^3} \end{aligned}$$

From $y$: Asymptotes: $y = x$ (oblique), $x = \pm 1$.
Symmetry: odd. Intercepts $(0,0)$, $(\pm 2, 0)$.
From $y'$: CP: none.

| | | ASY | | ASY | | |
|---|---|---|---|---|---|---|
| $y'$ | $+$ | $-1$ | $+$ | $1$ | $+$ | $\to x$ |
| $y$ | $\nearrow$ | | $\nearrow$ | | $\nearrow$ | |

From $y''$: $y'' = 0$ at $x = 0$.

| | | ASY | | | | ASY | | |
|---|---|---|---|---|---|---|---|---|
| $y''$ | $+$ | $-1$ | $-$ | $0$ | $+$ | $1$ | $-$ | $\to x$ |
| $y$ | $\smile$ | | $\frown$ | infl | $\smile$ | | $\frown$ | |

**22.** $y = \dfrac{x^2-1}{x^2} = 1 - \dfrac{1}{x^2}$, $y' = \dfrac{2}{x^3}$, $y'' = -\dfrac{6}{x^4}$.
From $y$: Intercepts: $(\pm 1, 0)$. Asymptotes: $y = 1$ (horizontal), $x = 0$ (vertical). Symmetry: even.
From $y'$: No critical points.

| | | ASY | | |
|---|---|---|---|---|
| $y'$ | $-$ | $0$ | $+$ | $\to x$ |
| $y$ | $\searrow$ | | $\nearrow$ | |

From $y''$: $y''$ is negative for all $x$.

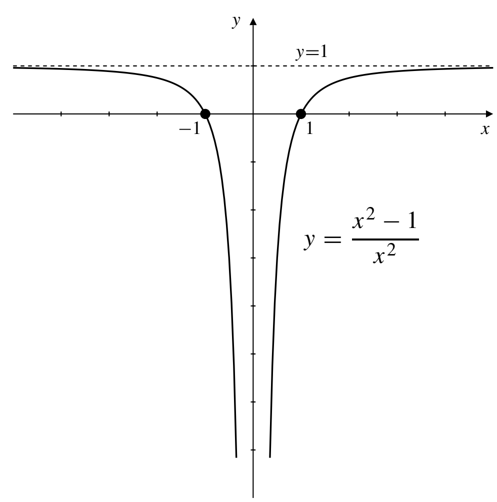



Fig. 4.4.22

**23.** $$y = \frac{x^5}{(x^2-1)^2} = x + \frac{2x^3 - x}{(x^2-1)^2}$$

$$y' = \frac{(x^2-1)^2 5x^4 - x^5 2(x^2-1)2x}{(x^2-1)^4}$$
$$= \frac{5x^6 - 5x^4 - 4x^6}{(x^2-1)^3} = \frac{x^4(x^2-5)}{(x^2-1)^3}$$
$$y'' = \frac{(x^2-1)^3(6x^5-20x^3) - (x^6-5x^4)3(x^2-1)^2 2x}{(x^2-1)^6}$$
$$= \frac{6x^7 - 26x^5 + 20x^3 - 6x^7 + 30x^5}{(x^2-1)^4}$$
$$= \frac{4x^3(x^2+5)}{(x^2-1)^4}$$

From $y$: Asymptotes: $y = x$, $x = \pm 1$. Symmetry: odd. Intercepts $(0,0)$. Points $\left(\pm\sqrt{5}, \pm\frac{25}{16}\sqrt{5}\right)$.

From $y'$: CP $x = 0$, $x = \pm\sqrt{5}$.

| | | CP | | ASY | | CP | | ASY | | CP | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $y'$ | $+$ | $-\sqrt{5}$ | $-$ | $-1$ | $+$ | $0$ | $+$ | $1$ | $-$ | $\sqrt{5}$ | $+$ |
| $y$ | $\nearrow$ | loc max | $\searrow$ | | $\nearrow$ | | $\nearrow$ | | $\searrow$ | loc min | $\nearrow$ |

From $y''$: $y'' = 0$ if $x = 0$.

| | | ASY | | | | ASY | |
|---|---|---|---|---|---|---|---|
| $y''$ | $-$ | $-1$ | $-$ | $0$ | $+$ | $1$ | $+$ |
| $y$ | $\frown$ | | $\frown$ | infl | $\smile$ | | $\smile$ |

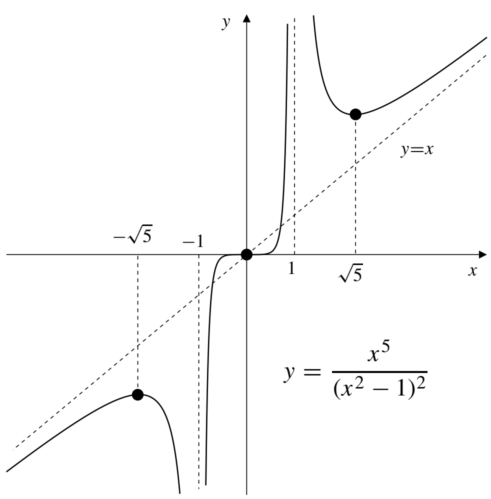



Fig. 4.4.23

**24.** $y = \dfrac{(2-x)^2}{x^3}$, $y' = -\dfrac{(x-2)(x-6)}{x^4}$,
$$y'' = \frac{2(x^2-12x+24)}{x^5} = \frac{2(x-6+2\sqrt{3})(x-6-2\sqrt{3})}{x^5}.$$

From $y$: Intercept: $(2,0)$. Asymptotes: $y = 0$ (horizontal), $x = 0$ (vertical). Symmetry: none obvious. Other points: $(-2,-2)$, $(-10,-0.144)$.

From $y'$: Critical points: $x = 2,\ 6$.

| | | ASY | | CP | | CP | |
|---|---|---|---|---|---|---|---|
| $y'$ | $-$ | $0$ | $-$ | $2$ | $+$ | $6$ | $-$ |
| $y$ | $\searrow$ | | $\searrow$ | loc min | $\nearrow$ | loc max | $\searrow$ |

From $y''$: $y'' = 0$ at $x = 6 \pm 2\sqrt{3}$.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $y''$ | $-$ | $0$ | $+$ | $6+2\sqrt{3}$ | $-$ | $6-2\sqrt{3}$ | $+$ |
| $y$ | $\frown$ | | $\smile$ | infl | $\frown$ | infl | $\smile$ |

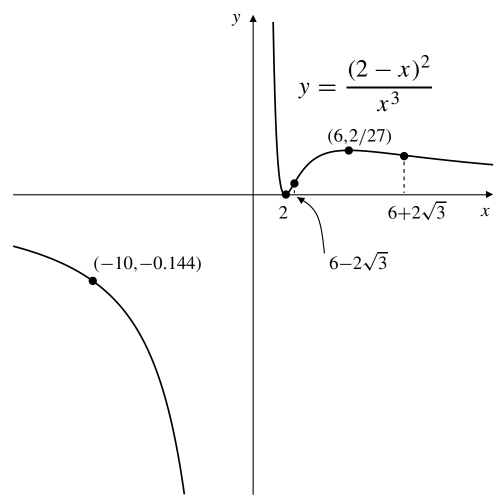



Fig. 4.4.24

**25.** $$y = \frac{1}{x^3 - 4x} = \frac{1}{x(x-2)(x+2)}$$
$$y' = -\frac{3x^2-4}{(x^3-4x)^2} = -\frac{3x^2-4}{x^2(x^2-4)^2}$$
$$y'' = -\frac{(x^3-4x)^2(6x) - (3x^2-4)2(x^3-4x)(3x^2-4)}{(x^3-4x)^4}$$
$$= -\frac{6x^4 - 24x^2 - 18x^4 + 48x^2 - 32}{(x^3-4x)^3}$$
$$= \frac{12(x^2-1)^2 + 20}{x^3(x^2-4)^3}$$

From $y$: Asymptotes: $y = 0$, $x = 0, -2, 2$.
Symmetry: odd. No intercepts.
Points: $\left(\pm\frac{2}{\sqrt{3}}, \pm\frac{16}{3\sqrt{3}}\right)$, $\left(\pm 3, \pm\frac{1}{15}\right)$

From $y'$: CP: $x = \pm\dfrac{2}{\sqrt{3}}$.

| | | ASY | | CP | | ASY | | CP | | ASY | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $y'$ | $-$ | $-2$ | $-$ | $-\frac{2}{\sqrt{3}}$ | $+$ | $0$ | $+$ | $\frac{2}{\sqrt{3}}$ | $-$ | $2$ | $-$ |
| $y$ | $\searrow$ | | $\searrow$ | loc min | $\nearrow$ | | $\nearrow$ | loc max | $\searrow$ | | $\searrow$ |

From $y''$: $y''=0$ nowhere.

| | | ASY | | ASY | | ASY | | |
|---|---|---|---|---|---|---|---|---|
| $y''$ | $-$ | $-2$ | $+$ | $0$ | $-$ | $2$ | $+$ | $x$ |
| $y$ | $\frown$ | | $\smile$ | | $\frown$ | | $\smile$ | |

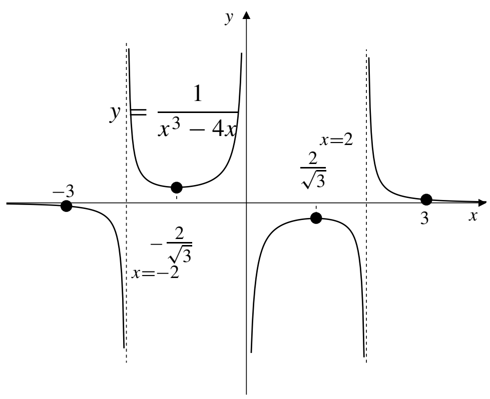



Fig. 4.4.25

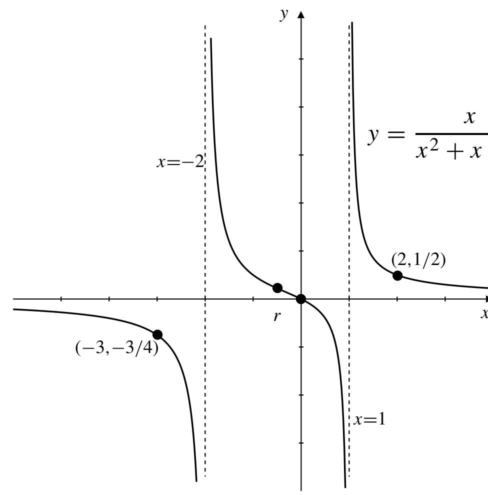



Fig. 4.4.26

**26.** $y = \dfrac{x}{x^2+x-2} = \dfrac{x}{(2+x)(x-1)}$,
$y' = \dfrac{-(x^2+2)}{(x+2)^2(x-1)^2}$, $y'' = \dfrac{2(x^3+6x+2)}{(x+2)^3(x-1)^3}$.
From $y$: Intercepts: $(0,0)$. Asymptotes: $y=0$ (horizontal), $x=1$, $x=-2$ (vertical). Other points: $(-3,-\frac{3}{4})$, $(2,\frac{1}{2})$.
From $y'$: No critical point.

| | | ASY | | ASY | | |
|---|---|---|---|---|---|---|
| $y'$ | $-$ | $-2$ | $-$ | $1$ | $-$ | $x$ |
| $y$ | $\searrow$ | | $\searrow$ | | $\searrow$ | |

From $y''$: $y''=0$ if $f(x)=x^3+6x+2=0$. Since $f'(x)=3x^2+6\ge 6$, $f$ is increasing and can only have one root. Since $f(0)=2$ and $f(-1)=-5$, that root must be between $-1$ and $0$. Let the root be $r$.

| | | ASY | | | | ASY | | |
|---|---|---|---|---|---|---|---|---|
| $y''$ | $-$ | $-2$ | $+$ | $r$ | $-$ | $1$ | $+$ | $x$ |
| $y$ | $\frown$ | | $\smile$ | infl | $\frown$ | | $\smile$ | |

**27.** $y = \dfrac{x^3-3x^2+1}{x^3} = 1 - \dfrac{3}{x} + \dfrac{1}{x^3}$
$y' = \dfrac{3}{x^2} - \dfrac{3}{x^4} = \dfrac{3(x^2-1)}{x^4}$
$y'' = -\dfrac{6}{x^3} + \dfrac{12}{x^5} = 6\dfrac{2-x^2}{x^5}$
From $y$ : Asymptotes: $y=1$, $x=0$. Symmetry: none.
Intercepts: since $\lim_{x\to 0+} y = \infty$, and $\lim_{x\to 0-} y = -\infty$, there are intercepts between $-1$ and $0$, between $0$ and $1$, and between $2$ and $3$.
Points: $(-1,3)$, $(1,-1)$, $(2,-\frac{3}{8})$, $(3,\dfrac{1}{27})$.
From $y'$: CP: $x=\pm 1$.

| | | CP | | ASY | | CP | | |
|---|---|---|---|---|---|---|---|---|
| $y'$ | $+$ | $-1$ | $-$ | $0$ | $-$ | $1$ | $+$ | $x$ |
| $y$ | $\nearrow$ | loc max | $\searrow$ | | $\searrow$ | loc min | $\nearrow$ | |

From $y''$: $y''=0$ at $x=\pm\sqrt{2}$.

| | | | | ASY | | | | |
|---|---|---|---|---|---|---|---|---|
| $y''$ | $+$ | $-\sqrt{2}$ | $-$ | $0$ | $+$ | $\sqrt{2}$ | $-$ | $x$ |
| $y$ | $\smile$ | infl | $\frown$ | | $\smile$ | infl | $\frown$ | |

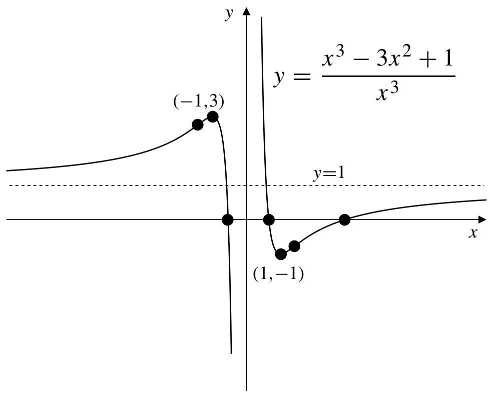



Fig. 4.4.27

**28.** $y = x + \sin x$, $y' = 1 + \cos x$, $y'' = -\sin x$.
From $y$: Intercept: $(0, 0)$. Other points: $(k\pi, k\pi)$, where $k$ is an integer. Symmetry: odd.
From $y'$: Critical point: $x = (2k + 1)\pi$, where $k$ is an integer.

| | | CP | | CP | | CP | | |
|---|---|---|---|---|---|---|---|---|
| $f'$ | + | $-\pi$ | + | $\pi$ | − | $3\pi$ | + | $x$ |
| $f$ | ↗ | | ↗ | | ↗ | | ↗ | |

From $y''$: $y'' = 0$ at $x = k\pi$, where $k$ is an integer.

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $y''$ | + | $-2\pi$ | − | $-\pi$ | + | $0$ | − | $\pi$ | + | $2\pi$ | − | $x$ |
| $y$ | ⌣ | infl | ⌢ | infl | ⌣ | infl | ⌢ | infl | ⌣ | infl | ⌢ | |

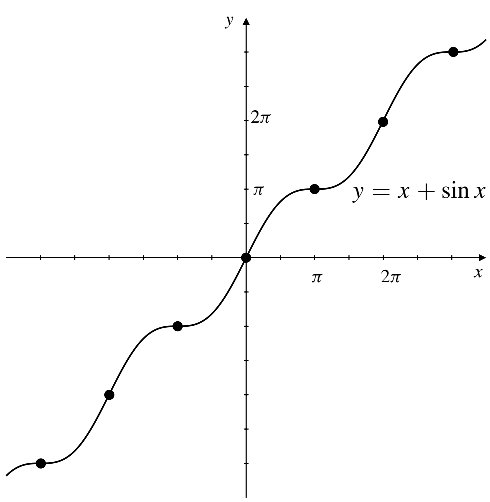



Fig. 4.4.28

**29.** $y = x + 2\sin x$, $\quad y' = 1 + 2\cos x$, $\quad y'' = -2\sin x$.
$y = 0$ if $x = 0$
$y' = 0$ if $x = -\dfrac{1}{2}$, i.e., $x = \pm\dfrac{2\pi}{3} \pm 2n\pi$
$y'' = 0$ if $x = \pm n\pi$
From $y$: Asymptotes: (none). Symmetry: odd.
Points: $\left(\pm\dfrac{2\pi}{3}, \pm\dfrac{2\pi}{3} + \sqrt{3}\right)$, $\left(\pm\dfrac{8\pi}{3}, \pm\dfrac{8\pi}{3} + \sqrt{3}\right)$, $\left(\pm\dfrac{4\pi}{3}, \pm\dfrac{4\pi}{3} - \sqrt{3}\right)$.
From $y'$: CP: $x = \pm\dfrac{2\pi}{3} \pm 2n\pi$.

| | | CP | | CP | | CP | | CP | | CP | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $y'$ | − | $-\frac{8\pi}{3}$ | + | $-\frac{4\pi}{3}$ | + | $-\frac{2\pi}{3}$ | + | $\frac{2\pi}{3}$ | − | $\frac{4\pi}{3}$ | + | $x$ |
| $y$ | ↘ | loc min | ↗ | loc max | ↘ | loc min | ↗ | loc max | ↘ | loc min | ↗ | |

From $y''$: $y'' = 0$ at $x = \pm n\pi$.

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $y''$ | + | $-2\pi$ | − | $-\pi$ | + | $0$ | − | $\pi$ | + | $2\pi$ | − | $x$ |
| $y$ | ⌣ | infl | ⌢ | infl | ⌣ | infl | ⌢ | infl | ⌣ | infl | ⌢ | |

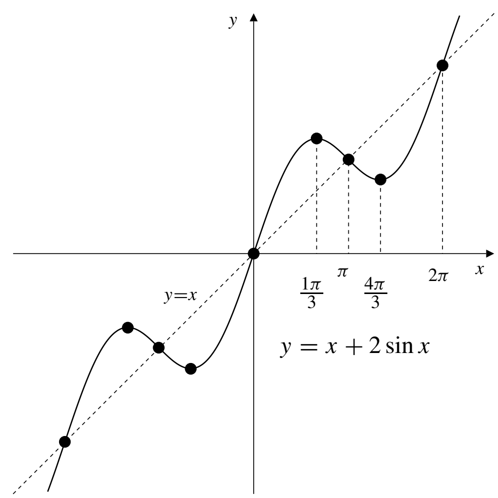



Fig. 4.4.29

**30.** $y = e^{-x^2}$, $y' = -2xe^{-x^2}$, $y'' = (4x^2 - 2)e^{-x^2}$.
From $y$: Intercept: $(0, 1)$. Asymptotes: $y = 0$ (horizontal). Symmetry: even.
From $y'$: Critical point: $x = 0$.

| | | CP | | |
|---|---|---|---|---|
| $y'$ | + | $0$ | − | $x$ |
| $y$ | ↗ | abs max | ↘ | |

From $y''$: $y'' = 0$ at $x = \pm\dfrac{1}{\sqrt{2}}$.

| $y''$ | $+$ | $-\frac{1}{\sqrt{2}}$ | $-$ | $\frac{1}{\sqrt{2}}$ | $+$ | $x$ |
|---|---|---|---|---|---|---|
| $y$ | $\smile$ | infl | $\frown$ | infl | $\smile$ | |

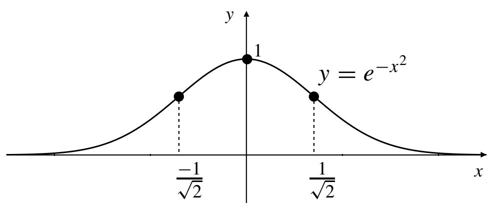



Fig. 4.4.30

**31.** $y = xe^x$, $\quad y' = e^x(1+x)$, $\quad y'' = e^x(2+x)$.
From $y$: Asymptotes: $y = 0$ (at $x = -\infty$).
Symmetry: none. Intercept $(0,0)$.
Points: $\left(-1, -\dfrac{1}{e}\right)$, $\left(-2, -\dfrac{2}{e^2}\right)$,
From $y'$: CP: $x = -1$.

| $y'$ | $-$ | CP $-1$ | $+$ | $x$ |
|---|---|---|---|---|
| $y$ | $\searrow$ | abs min | $\nearrow$ | |

From $y''$: $y'' = 0$ at $x = -2$.

| $y''$ | $-$ | $-2$ | $+$ | $x$ |
|---|---|---|---|---|
| $y$ | $\frown$ | infl | $\smile$ | |

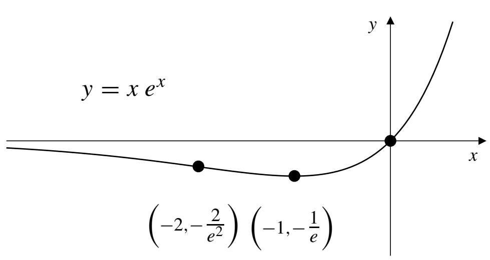



Fig. 4.4.31

**32.** $y = e^{-x}\sin x \qquad (x \ge 0)$,
$y' = e^{-x}(\cos x - \sin x)$, $y'' = -2e^{-x}\cos x$.
From $y$: Intercept: $(k\pi, 0)$, where $k$ is an integer.
Asymptotes: $y = 0$ as $x \to \infty$.

From $y'$: Critical points: $x = \dfrac{\pi}{4} + k\pi$, where $k$ is an integer.

| $y'$ | $0$ | $+$ | CP $\frac{\pi}{4}$ | $-$ | CP $\frac{5\pi}{4}$ | $+$ | CP $\frac{9\pi}{4}$ | $-$ | $x$ |
|---|---|---|---|---|---|---|---|---|---|
| $y$ | | $\nearrow$ | abs max | $\searrow$ | abs min | $\nearrow$ | loc max | $\searrow$ | |

From $y''$: $y'' = 0$ at $x = (k + \frac{1}{2})\pi$, where $k$ is an integer.

| $y''$ | $0$ | $-$ | $\frac{\pi}{2}$ | $+$ | $\frac{3\pi}{2}$ | $-$ | $\frac{5\pi}{2}$ | $+$ | $x$ |
|---|---|---|---|---|---|---|---|---|---|
| $y$ | | $\frown$ | infl | $\smile$ | infl | $\frown$ | infl | $\smile$ | |

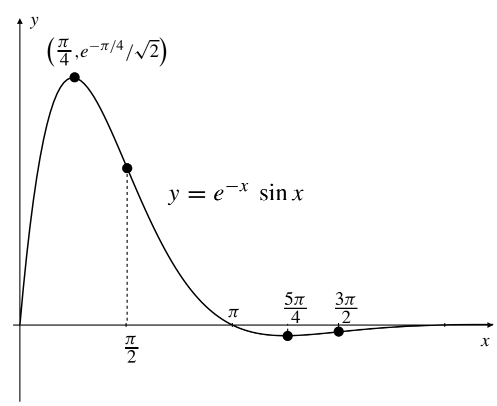



Fig. 4.4.32

**33.** $y = x^2 e^{-x^2}$

$$y' = e^{-x^2}(2x - 2x^3) = 2x(1-x^2)e^{-x^2}$$
$$y'' = e^{-x^2}(2 - 6x^2 - 2x(2x - 2x^3))$$
$$= (2 - 10x^2 + 4x^4)e^{-x^2}$$

From $y$: Asymptotes: $y = 0$.
Intercept: $(0,0)$. Symmetry: even.
Points $\left(\pm 1, \dfrac{1}{e}\right)$
From $y'$: CP $x = 0$, $x = \pm 1$.

| $y'$ | $+$ | CP $-1$ | $-$ | CP $0$ | $+$ | CP $1$ | $-$ | $x$ |
|---|---|---|---|---|---|---|---|---|
| $y$ | $\nearrow$ | abs max | $\searrow$ | abs min | $\nearrow$ | abs max | $\searrow$ | |

From $y''$: $y''=0$ if
$2x^4-5x^2+1=0$

$$x^2=\frac{5\pm\sqrt{25-8}}{4}=\frac{5\pm\sqrt{17}}{4}.$$

so $x=\pm a=\pm\sqrt{\dfrac{5+\sqrt{17}}{4}}$, $x=\pm b=\pm\sqrt{\dfrac{5-\sqrt{17}}{4}}$.

| $y''$ | $+$ | $-a$ | $-$ | $-b$ | $+$ | $b$ | $-$ | $a$ | $+$ | $\to x$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $y$ | $\smile$ | infl | $\frown$ | infl | $\smile$ | infl | $\frown$ | infl | $\smile$ | |

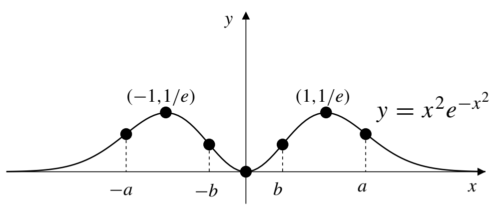


Fig. 4.4.33

**34.** $y=x^2e^x$, $y'=(2x+x^2)e^x=x(2+x)e^x$,
$y''=(x^2+4x+2)e^x=(x+2-\sqrt{2})(x+2+\sqrt{2})e^x$.
From $y$: Intercept: $(0,0)$.
Asymptotes: $y=0$ as $x\to-\infty$.
From $y'$: Critical point: $x=0$, $x=-2$.

| | | CP | | CP | | |
|---|---|---|---|---|---|---|
| $y'$ | $+$ | $-2$ | $-$ | $0$ | $+$ | $\to x$ |
| $y$ | $\nearrow$ | loc max | $\searrow$ | abs min | $\nearrow$ | |

From $y''$: $y''=0$ at $x=-2\pm\sqrt{2}$.

| $y''$ | $+$ | $-2-\sqrt{2}$ | $-$ | $-2+\sqrt{2}$ | $+$ | $\to x$ |
|---|---|---|---|---|---|---|
| $y$ | $\smile$ | infl | $\frown$ | infl | $\smile$ | |

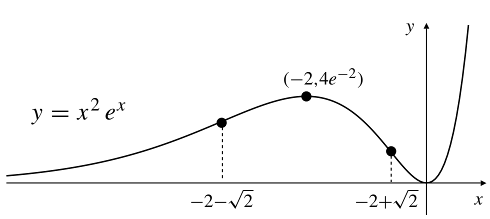


Fig. 4.4.34

**35.** $y=\dfrac{\ln x}{x}$, $\quad y'=\dfrac{1-\ln x}{x^2}$

$$y''=\frac{x^2\left(-\frac{1}{x}\right)-(1-\ln x)2x}{x^4}=\frac{2\ln x-3}{x^3}$$

From $y$: Asymptotes: $x=0$, $y=0$.
Symmetry: none. Intercept: $(1,0)$.
Points: $\left(e,\dfrac{1}{e}\right)$, $\left(e^{3/2},\dfrac{3}{2e^{3/2}}\right)$.
From $y'$: CP: $x=e$.

| | ASY | | CP | | |
|---|---|---|---|---|---|
| $y'$ | $0$ | $+$ | $e$ | $-$ | $\to x$ |
| $y$ | | $\nearrow$ | abs max | $\searrow$ | |

From $y''$: $y''=0$ at $x=e^{3/2}$.

| | ASY | | | | |
|---|---|---|---|---|---|
| $y''$ | $0$ | $-$ | $e^{3/2}$ | $+$ | $\to x$ |
| $y$ | | $\frown$ | infl | $\smile$ | |

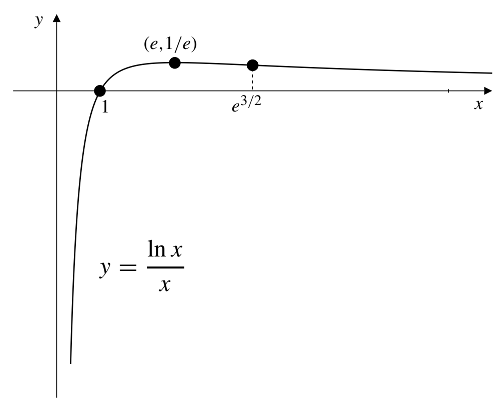


Fig. 4.4.35

**36.** $y=\dfrac{\ln x}{x^2}\quad(x>0)$,
$y'=\dfrac{1-2\ln x}{x^3}$, $y''=\dfrac{6\ln x-5}{x^4}$.
From $y$: Intercepts: $(1,0)$. Asymptotes: $y=0$, since $\lim_{x\to\infty}\dfrac{\ln x}{x^2}=0$, and $x=0$, since $\lim_{x\to0+}\dfrac{\ln x}{x^2}=-\infty$.
From $y'$: Critical point: $x=e^{1/2}$.

| | | | CP | | |
|---|---|---|---|---|---|
| $y'$ | $0$ | $+$ | $\sqrt{e}$ | $-$ | $\to x$ |
| $y$ | | $\nearrow$ | abs max | $\searrow$ | |

From $y''$: $y''=0$ at $x=e^{5/6}$.

| $y''$ | $0$ | $-$ | $e^{5/6}$ | $+$ | $\to x$ |
|---|---|---|---|---|---|
| $y$ | | $\frown$ | infl | $\smile$ | |

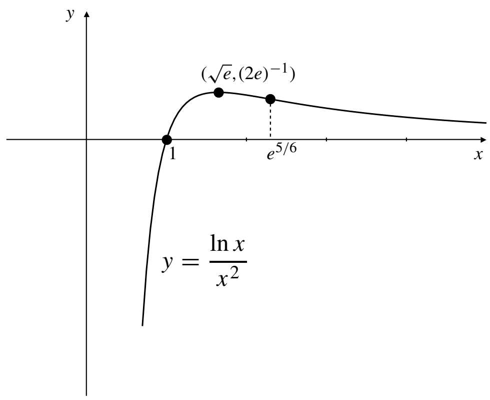



Fig. 4.4.36

**37.** $y = \dfrac{1}{\sqrt{4-x^2}} = (4-x^2)^{-1/2}$

$$y' = -\frac{1}{2}(4-x^2)^{-3/2}(-2x) = \frac{x}{(4-x^2)^{3/2}}$$

$$y'' = \frac{(4-x^2)^{3/2} - x\frac{3}{2}(4-x^2)^{1/2}(-2x)}{(4-x^2)^3}$$

$$= \frac{4+2x^2}{(4-x^2)^{5/2}}$$

From $y$: Asymptotes: $x = \pm 2$. Domain $-2 < x < 2$. Symmetry: even. Intercept: $(0, \frac{1}{2})$.
From $y'$: CP: $x = 0$.

| | ASY | | CP | | ASY |
|---|---|---|---|---|---|
| $y'$ | $-2$ | $-$ | $0$ | $+$ | $2$ |
| $y$ | | ↘ | abs min | ↗ | |

From $y''$: $y'' = 0$ nowhere, $y'' > 0$ on $(-2, 2)$. Therefore, $y$ is concave up.

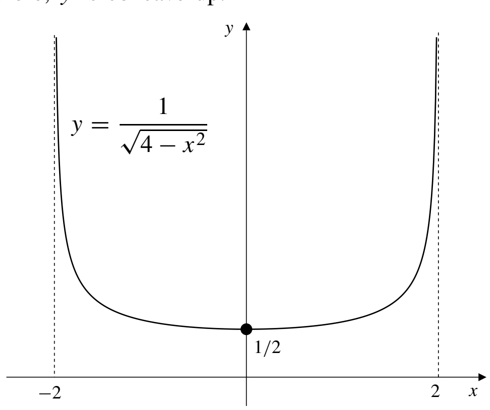



Fig. 4.4.37

**38.** $y = \dfrac{x}{\sqrt{x^2+1}}$, $y' = (x^2+1)^{-3/2}$, $y'' = -3x(x^2+1)^{-5/2}$.
From $y$: Intercept: $(0,0)$. Asymptotes: $y = 1$ as $x \to \infty$, and $y = -1$ as $x \to -\infty$. Symmetry: odd.
From $y'$: No critical point. $y' > 0$ and $y$ is increasing for all $x$.
From $y''$: $y'' = 0$ at $x = 0$.

| | | | |
|---|---|---|---|
| $y''$ | $+$ | $0$ | $-$ |
| $y$ | $\smile$ | infl | $\frown$ |

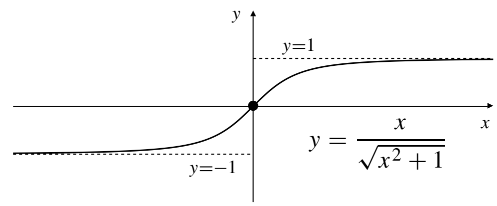



Fig. 4.4.38

**39.** $y = (x^2-1)^{1/3}$

$$y' = \frac{2}{3}x(x^2-1)^{-2/3}$$

$$y'' = \frac{2}{3}[(x^2-1)^{-2/3} - \frac{2}{3}x(x^2-1)^{-5/3}2x]$$

$$= -\frac{2}{3}(x^2-1)^{-5/3}\left(1 + \frac{x^2}{3}\right)$$

From $y$: Asymptotes: none.
Symmetry: even. Intercepts: $(\pm 1, 0)$, $(0, -1)$.
From $y'$: CP: $x = 0$. SP: $x = \pm 1$.

| | | SP | | CP | | SP | |
|---|---|---|---|---|---|---|---|
| $y'$ | $-$ | $-1$ | $-$ | $0$ | $+$ | $1$ | $+$ |
| $y$ | ↘ | | ↘ | abs min | ↗ | | ↗ |

From $y''$: $y'' = 0$ nowhere.

| | | | | | |
|---|---|---|---|---|---|
| $y''$ | $-$ | $-1$ | $+$ | $1$ | $-$ |
| $y$ | $\frown$ | infl | $\smile$ | infl | $\frown$ |

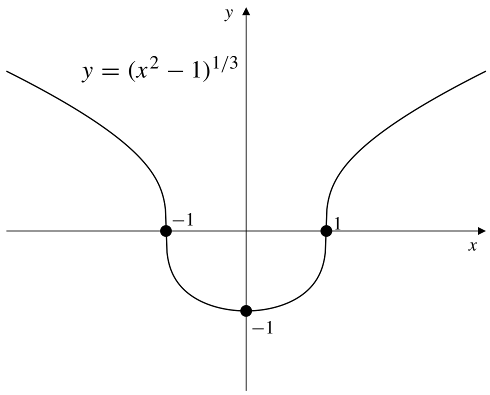



Fig. 4.4.39

**40.** According to Theorem 5 of Section 4.4,

$$\lim_{x\to 0+} x\ln x = 0.$$

Thus,

$$\lim_{x\to 0} x\ln|x| = \lim_{x\to 0+} x\ln x = 0.$$

If $f(x) = x\ln|x|$ for $x \neq 0$, we may define $f(0)$ such that $f(0) = \lim_{x\to 0} x\ln|x| = 0$. Then $f$ is continuous on the whole real line and

$$f'(x) = \ln|x| + 1, \qquad f''(x) = \frac{1}{|x|}\operatorname{sgn}(x).$$

From $f$: Intercept: $(0,0)$, $(\pm 1, 0)$. Asymptotes: none. Symmetry: odd.
From $f'$: CP: $x = \pm\frac{1}{e}$. SP: $x = 0$.

| | | CP | | SP | | CP | |
|---|---|---|---|---|---|---|---|
| $f'$ | $+$ | $-\frac{1}{e}$ | $-$ | $0$ | $-$ | $\frac{1}{e}$ | $+$ |
| $f$ | $\nearrow$ | loc max | $\searrow$ | | $\searrow$ | loc min | $\nearrow$ |

From $f''$: $f''$ is undefined at $x = 0$.

| | | | |
|---|---|---|---|
| $f''$ | $-$ | $0$ | $+$ |
| $f$ | $\frown$ | infl | $\smile$ |

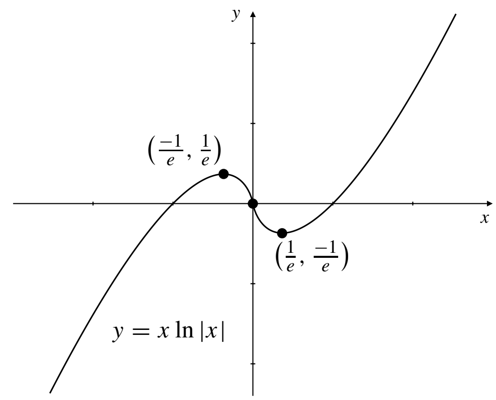



Fig. 4.4.40

**41.** $y = 0$ is an asymptote of $y = \dfrac{\sin x}{1+x^2}$.
Curve crosses asymptote at infinitely many points: $x = n\pi$ $(n = 0, \pm 1, \pm 2, \ldots)$.

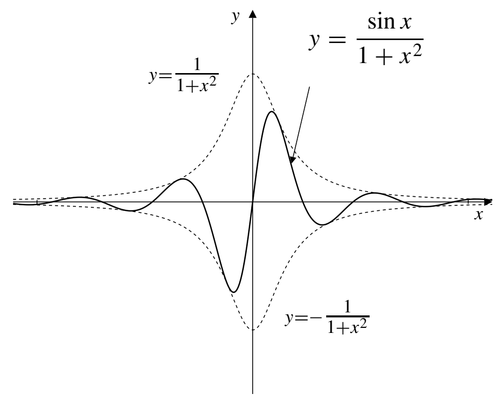



Fig. 4.4.41

## Section 4.5 Extreme-Value Problems (page 242)

**1.** Let the numbers be $x$ and $7 - x$. Then $0 \le x \le 7$. The product is $P(x) = x(7-x) = 7x - x^2$.
$P(0) = P(7) = 0$ and $P(x) > 0$ if $0 < x < 7$. Thus maximum $P$ occurs at a CP:

$$0 = \frac{dP}{dx} = 7 - 2x \Rightarrow x = \frac{7}{2}.$$

The maximum product is $P(7/2) = 49/4$.

**2.** Let the numbers be $x$ and $\frac{8}{x}$ where $x > 0$. Their sum is $S = x + \frac{8}{x}$. Since $S \to \infty$ as $x \to \infty$ or $x \to 0+$, the minimum sum must occur at a critical point:

$$0 = \frac{dS}{dx} = 1 - \frac{8}{x^2} \Rightarrow x = 2\sqrt{2}.$$

Thus, the smallest possible sum is $2\sqrt{2}+\dfrac{8}{2\sqrt{2}}=4\sqrt{2}$.

**3.** Let the numbers be $x$ and $60-x$. Then $0\le x\le 60$.
Let $P(x)=x^2(60-x)=60x^2-x^3$.
Clearly, $P(0)=P(60)=0$ amd $P(x)>0$ if $0<x<60$.
Thus maximum $P$ occurs at a CP:

$$0=\frac{dP}{dx}=120x-3x^2=3x(40-x).$$

Therefore, $x=0$ or 40.
Max must correspond to $x=40$. The numbers are 40 and 20.

**4.** Let the numbers be $x$ and $16-x$. Let $P(x)=x^3(16-x)^5$. Since $P(x)\to-\infty$ as $x\to\pm\infty$, so the maximum must occur at a critical point:

$$\begin{aligned}0=P'(x)&=3x^2(16-x)^5-5x^3(16-x)^4\\&=x^2(16-x)^4(48-8x).\end{aligned}$$

The critical points are 0, 6 and 16. Clearly, $P(0)=P(16)=0$, and $P(6)=216\times10^5$. Thus, $P(x)$ is maximum if the numbers are 6 and 10.

**5.** Let the numbers be $x$ and $10-x$. We want to minimize

$$S(x)=x^3+(10-x)^2,\quad 0\le x\le 10.$$

$S(0)=100$ and $S(10)=1,000$. For CP:

$$0=S'(x)=3x^2-2(10-x)=3x^2+2x-20.$$

The only positive CP is $x=(-2+\sqrt{4+240})/6\approx 2.270$. Since $S(2.270)\approx 71.450$, the minimum value of $S$ is about 71.45.

**6.** If the numbers are $x$ and $n-x$, then $0\le x\le n$ and the sum of their squares is

$$S(x)=x^2+(n-x)^2.$$

Observe that $S(0)=S(n)=n^2$. For critical points:

$$0=S'(x)=2x-2(n-x)=2(2x-n)\Rightarrow x=n/2.$$

Since $S(n/2)=n^2/2$, this is the smallest value of the sum of squares.

**7.** Let the dimensions of a rectangle be $x$ and $y$. Then the area is $A=xy$ and the perimeter is $P=2x+2y$.
Given $A$ we can express

$$P=P(x)=2x+\frac{2A}{x},\qquad (0<x<\infty).$$

Evidently, minimum $P$ occurs at a CP. For CP:

$$0=\frac{dP}{dx}=2-\frac{2A}{x^2}\quad\Rightarrow\quad x^2=A=xy\Rightarrow x=y.$$

Thus min $P$ occurs for $x=y$, i.e., for a square.

**8.** Let the width and the length of a rectangle of given perimeter $2P$ be $x$ and $P-x$. Then the area of the rectangle is

$$A(x)=x(P-x)=Px-x^2.$$

Since $A(x)\to-\infty$ as $x\to\pm\infty$ the maximum must occur at a critical point:

$$0=\frac{dA}{dx}=P-2x\Rightarrow x=\frac{P}{2}$$

Hence, the width and the length are $\dfrac{P}{2}$ and $(P-\dfrac{P}{2})=\dfrac{P}{2}$. Since the width equals the length, it is a square.

**9.** Let the dimensions of the isosceles triangle be as shown. Then $2x+2y=P$ (given constant). The area is

$$A=xh=x\sqrt{y^2-x^2}=x\sqrt{\left(\frac{P}{2}-x\right)^2-x^2}.$$

Evidently, $y\ge x$ so $0\le x\le P/4$. If $x=0$ or $x=P/4$, then $A=0$. Thus the maximum of $A$ must occur at a CP. For max $A$:

$$0=\frac{dA}{dx}=\sqrt{\frac{P^2}{4}-Px}-\frac{Px}{2\sqrt{\dfrac{P^2}{4}-Px}},$$

i.e., $\dfrac{P^2}{2}-2Px-Px=0$, or $x=\dfrac{P}{6}$. Thus $y=P/3$ and the triangle is equilateral since all three sides are $P/3$.

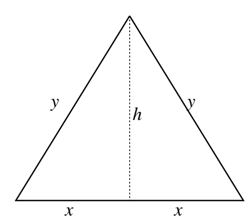



Fig. 4.5.9

**10.** Let the various dimensions be as shown in the figure. Since $h=10\sin\theta$ and $b=20\cos\theta$, the area of the triangle is

$$\begin{aligned}A(\theta)&=\tfrac12 bh=100\sin\theta\cos\theta\\&=50\sin2\theta\qquad\text{for }0<\theta<\frac{\pi}{2}.\end{aligned}$$

Since $A(\theta) \to 0$ as $\theta \to 0$ and $\theta \to \frac{\pi}{2}$, the maximum must be at a critial point:

$$0 = A'(\theta) = 100\cos 2\theta \Rightarrow 2\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{4}.$$

Hence, the largest possible area is

$$A(\pi/4) = 50\sin\left[2\left(\frac{\pi}{4}\right)\right] = 50\,\mathrm{m}^2.$$

(Remark: alternatively, we may simply observe that the largest value of $\sin 2\theta$ is 1; therefore the largest possible area is $50(1) = 50$ m$^2$.)

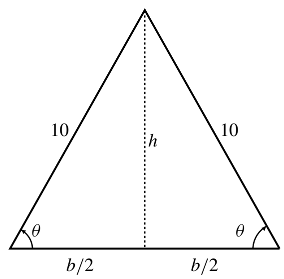



Fig. 4.5.10

**11.** Let the corners of the rectangle be as shown.
The area of the rectangle is $A = 2xy = 2x\sqrt{R^2 - x^2}$ (for $0 \le x \le R$).
If $x = 0$ or $x = R$ then $A = 0$; otherwise $A > 0$.
Thus maximum $A$ must occur at a critical point:

$$0 = \frac{dA}{dx} = 2\left[\sqrt{R^2 - x^2} - \frac{x^2}{\sqrt{R^2 - x^2}}\right] \Rightarrow R^2 - 2x^2 = 0.$$

Thus $x = \frac{R}{\sqrt{2}}$ and the maximum area is

$$2\frac{R}{\sqrt{2}}\sqrt{R^2 - \frac{R^2}{2}} = R^2 \text{ square units.}$$

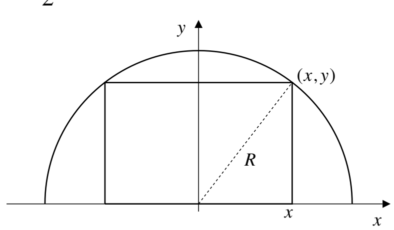



Fig. 4.5.11

**12.** Let $x$ be as shown in the figure. The perimeter of the rectangle is

$$P(x) = 4x + 2\sqrt{R^2 - x^2} \qquad (0 \le x \le R).$$

For critical points:

$$0 = \frac{dP}{dx} = 4 + \frac{-2x}{\sqrt{R^2 - x^2}}$$
$$\Rightarrow 2\sqrt{R^2 - x^2} = x \Rightarrow x = \frac{2R}{\sqrt{5}}.$$

Since

$$\frac{d^2P}{dx^2} = \frac{-2R^2}{(R^2 - x^2)^{3/2}} < 0$$

therefore $P(x)$ is concave down on $[0, R]$, so it must have an absolute maximum value at $x = \frac{2R}{\sqrt{5}}$. The largest perimeter is therefore

$$P\left(\frac{2R}{\sqrt{5}}\right) = 4\left(\frac{2R}{\sqrt{5}}\right) + \sqrt{R^2 - \frac{4R^2}{5}} = \frac{10R}{\sqrt{5}} \text{ units.}$$

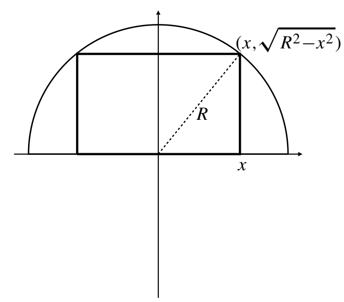



Fig. 4.5.12

**13.** Let the upper right corner be $(x, y)$ as shown. Then $x \ge 0$ and $y = b\sqrt{1 - \frac{x^2}{a^2}}$, so $x \le a$.
The area of the rectangle is

$$A(x) = 4xy = 4bx\sqrt{1 - \frac{x^2}{a^2}}, \qquad (0 \le x \le a).$$

Clearly, $A = 0$ if $x = 0$ or $x = a$, so maximum $A$ must occur at a critical point:

$$0 = \frac{dA}{dx} = 4b\left(\sqrt{1 - \frac{x^2}{a^2}} - \frac{\frac{2x^2}{a^2}}{2\sqrt{1 - \frac{x^2}{a^2}}}\right)$$

Thus $1 - \frac{x^2}{a^2} - \frac{x^2}{a^2} = 0$ and $x = \frac{a}{\sqrt{2}}$. Thus $y = \frac{b}{\sqrt{2}}$.

The largest area is $4\frac{a}{\sqrt{2}}\frac{b}{\sqrt{2}} = 2ab$ square units.

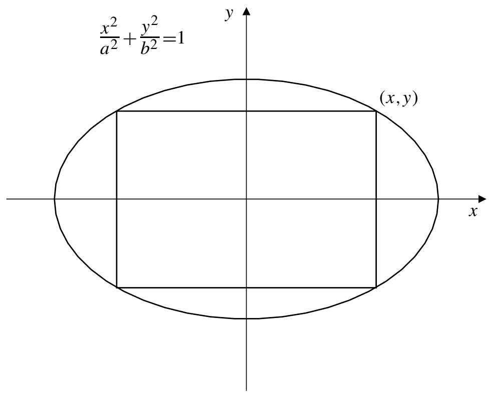



Fig. 4.5.13

**14.** See the diagrams below.

a) The area of the rectangle is $A = xy$. Since

$$\frac{y}{a-x} = \frac{b}{a} \Rightarrow y = \frac{b(a-x)}{a}.$$

Thus, the area is

$$A = A(x) = \frac{bx}{a}(a-x) \qquad (0 \le x \le a).$$

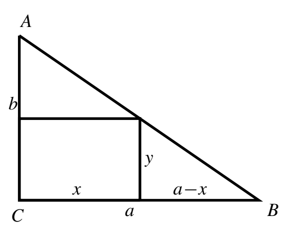
For critical points:

$$0 = A'(x) = \frac{b}{a}(a-2x) \Rightarrow x = \frac{a}{2}.$$

Since $A''(x) = -\dfrac{2b}{a} < 0$, $A$ must have a maximum value at $x = \dfrac{a}{2}$. Thus, the largest area for the rectangle is

$$\frac{b}{a}\left(\frac{a}{2}\right)\left(a - \frac{a}{2}\right) = \frac{ab}{4} \text{ square units,}$$

that is, half the area of the triangle $ABC$.



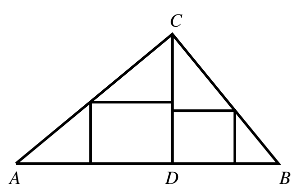




Fig. 4.5.14(a) Fig. 4.5.14(b)

(b) This part has the same answer as part (a). To see this, let $CD \perp AB$, and solve separate problems for the largest rectangles in triangles $ACD$ and $BCD$ as shown. By part (a), both maximizing rectangles have the same height, namely half the length of $CD$. Thus, their union is a rectangle of area half of that of triangle $ABC$.

**15.** NEED FIGURE If the sides of the triangle are 10 cm, 10 cm, and $2x$ cm, then the area of the triangle is $A(x) = x\sqrt{100 - x^2}$ cm$^2$, where $0 \le x \le 10$. Evidently $A(0) = A(10) = 0$ and $A(x) > 0$ for $0 < x < 10$. Thus $A$ will be maximum at a critical point. For a critical point

$$\begin{aligned} 0 = A'(x) &= \sqrt{100 - x^2} - x\left(\frac{1}{2\sqrt{100 - x^2}}(-2x)\right) \\ &= \frac{100 - x^2 - x^2}{\sqrt{100 - x^2}}. \end{aligned}$$

Thus the critical point is given by $2x^2 = 100$, so $x = \sqrt{50}$. The maximum area of the triangle is $A(\sqrt{50}) = 50$ cm$^2$.

**16.** NEED FIGURE If the equal sides of the isosceles triangle are 10 cm long and the angles opposite these sides are $\theta$, then the area of the triangle is

$$A(\theta) = \frac{1}{2}(10)(10\sin\theta) = 50\sin\theta \text{ cm}^2,$$

which is evidently has maximum value 50 cm$^2$ when $\theta = \pi/2$, that is, when the triangle is right-angled. This solution requires no calculus, and so is easier than the one given for the previous problem.

**17.** Let the width and the height of the billboard be $w$ and $h$ m respectively. The area of the board is $A = wh$. The printed area is $(w-8)(h-4) = 100$.
Thus $h = 4 + \dfrac{100}{w-8}$ and $A = 4w + \dfrac{100w}{w-8}$, $(w > 8)$.
Clearly, $A \to \infty$ if $w \to \infty$ or $w \to 8+$. Thus minimum $A$ occurs at a critical point:

$$\begin{aligned} 0 &= \frac{dA}{dw} = 4 + \frac{100}{w-8} - \frac{100w}{(w-8)^2} \\ 100w &= 4(w^2 - 16w + 64) + 100w - 800 \\ w^2 &- 16w - 136 = 0 \\ w &= \frac{16 \pm \sqrt{800}}{2} = 8 \pm 10\sqrt{2}. \end{aligned}$$

Since $w > 0$ we must have $w = 8 + 10\sqrt{2}$.
Thus $h = 4 + \dfrac{100}{10\sqrt{2}} = 4 + 5\sqrt{2}$.
The billboard should be $8 + 10\sqrt{2}$ m wide and $4 + 5\sqrt{2}$ m high.

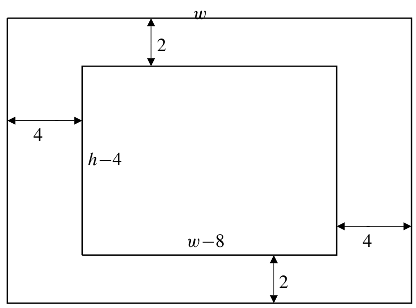



Fig. 4.5.17

**18.** Let $x$ be the side of the cut-out squares. Then the volume of the box is

$$V(x) = x(70 - 2x)(150 - 2x) \qquad (0 \le x \le 35).$$

Since $V(0) = V(35) = 0$, the maximum value will occur at a critical point:

$$\begin{aligned} 0 = V'(x) &= 4(2625 - 220x + 3x^2) \\ &= 4(3x - 175)(x - 15) \\ \Rightarrow x &= 15 \text{ or } \frac{175}{3}. \end{aligned}$$

The only critical point in $[0, 35]$ is $x = 15$. Thus, the largest possible volume for the box is

$$V(15) = 15(70 - 30)(150 - 30) = 72,000\,\text{cm}^3.$$

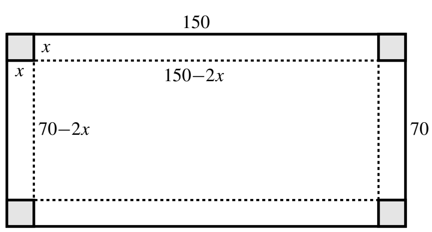



Fig. 4.5.18

**19.** Let the rebate be $\$x$. Then number of cars sold per month is

$$2000 + 200\left(\frac{x}{50}\right) = 2000 + 4x.$$

The profit per car is $1000 - x$, so the total monthly profit is

$$\begin{aligned} P &= (2000 + 4x)(1000 - x) = 4(500 + x)(1000 - x) \\ &= 4(500,000 + 500x - x^2). \end{aligned}$$

For maximum profit:

$$0 = \frac{dP}{dx} = 4(500 - 2x) \Rightarrow x = 250.$$

(Since $\dfrac{d^2P}{dx^2} = -8 < 0$ any critical point gives a local max.) The manufacturer should offer a rebate of \$250 to maximize profit.

**20.** If the manager charges $\$(40+x)$ per room, then $(80-2x)$ rooms will be rented.
The total income will be $\$(80 - 2x)(40 + x)$ and the total cost will be $\$(80 - 2x)(10) + (2x)(2)$. Therefore, the profit is

$$\begin{aligned} P(x) &= (80 - 2x)(40 + x) - [(80 - 2x)(10) + (2x)(2)] \\ &= 2400 + 16x - 2x^2 \qquad \text{for } x > 0. \end{aligned}$$

If $P'(x) = 16 - 4x = 0$, then $x = 4$. Since $P''(x) = -4 < 0$, $P$ must have a maximum value at $x = 4$. Therefore, the manager should charge \$44 per room.

**21.** Head for point $C$ on road $x$ km east of $A$. Travel time is

$$T = \frac{\sqrt{12^2 + x^2}}{15} + \frac{10 - x}{39}.$$

We have $\quad T(0) = \dfrac{12}{15} + \dfrac{10}{39} = 1.0564$ hrs

$$T(10) = \frac{\sqrt{244}}{15} = 1.0414 \text{ hrs}$$

For critical points:

$$\begin{aligned} 0 = \frac{dT}{dx} &= \frac{1}{15}\frac{x}{\sqrt{12^2 + x^2}} - \frac{1}{39} \\ &\Rightarrow 13x = 5\sqrt{12^2 + x^2} \\ &\Rightarrow (13^2 - 5^2)x^2 = 5^2 \times 12^2 \Rightarrow x = 5 \end{aligned}$$

$$T(5) = \frac{13}{15} + \frac{5}{39} = 0.9949 < \begin{cases} T(0) \\ T(10). \end{cases}$$

(Or note that

$$\begin{aligned} \frac{d^2T}{dt^2} &= \frac{1}{15}\,\frac{\sqrt{12^2 + x^2} - \dfrac{x^2}{\sqrt{12^2 + x^2}}}{12^2 + x^2} \\ &= \frac{12^2}{15(12^2 + x^2)^{3/2}} > 0 \end{aligned}$$

so any critical point is a local minimum.)
To minimize travel time, head for point 5 km east of $A$.

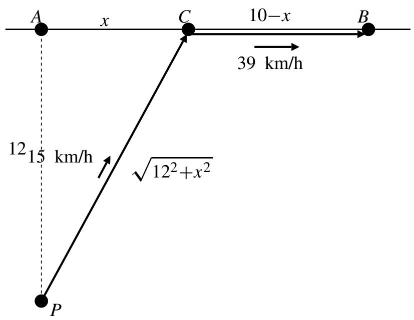



Fig. 4.5.21

**22.** This problem is similar to the previous one except that the 10 in the numerator of the second fraction in the expression for $T$ is replaced with a 4. This has no effect on the critical point of $T$, namely $x = 5$, which now lies outside the appropriate interval $0 \le x \le 4$. Minimum $T$ must occur at an endpoint. Note that

$$T(0) = \frac{12}{15} + \frac{4}{39} = 0.9026$$
$$T(4) = \frac{1}{15}\sqrt{12^2 + 4^2} = 0.8433.$$

The minimum travel time corresponds to $x = 4$, that is, to driving in a straight line to $B$.

**23.** Use $x$ m for the circle and $1 - x$ m for square. The sum of areas is

$$A = \pi r^2 + s^2 = \frac{\pi x^2}{4\pi^2} + \left(\frac{1-x}{4}\right)^2$$
$$= \frac{x^2}{4\pi} + \frac{(1-x)^2}{4^2} \qquad (0 \le x \le 1)$$

Now $A(0) = \dfrac{1}{16}$, $A(1) = \dfrac{1}{4\pi} > A(0)$. For CP:

$$0 = \frac{dA}{dx} = \frac{x}{2\pi} - \frac{1-x}{8} \Rightarrow x\left(\frac{1}{2\pi} + \frac{1}{8}\right) = \frac{1}{8} \Rightarrow x = \frac{\pi}{4+\pi}.$$

Since $\dfrac{d^2A}{dx^2} = \dfrac{1}{2\pi} + \dfrac{1}{8} > 0$, the CP gives local minimum for $A$.

a) For max total area use none of wire for the square, i.e., $x = 1$.

b) For minimum total area use $1 - \dfrac{\pi}{4+\pi} = \dfrac{4}{4+\pi}$ m for square.

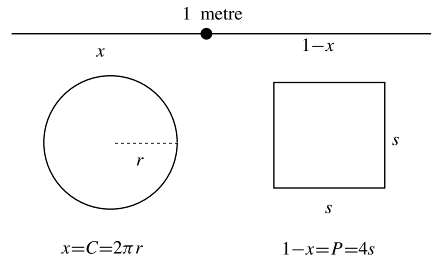



Fig. 4.5.23

**24.** Let the dimensions of the rectangle be as shown in the figure. Clearly,

$$x = a\sin\theta + b\cos\theta,$$
$$y = a\cos\theta + b\sin\theta.$$

Therefore, the area is

$$\begin{aligned} A(\theta) &= xy \\ &= (a\sin\theta + b\cos\theta)(a\cos\theta + b\sin\theta) \\ &= ab + (a^2+b^2)\sin\theta\cos\theta \\ &= ab + \frac{1}{2}(a^2+b^2)\sin 2\theta \qquad \text{for } 0 \le \theta \le \frac{\pi}{2}. \end{aligned}$$

If $A'(\theta) = (a^2+b^2)\cos 2\theta = 0$, then $\theta = \dfrac{\pi}{4}$. Since $A''(\theta) = -2(a^2+b^2)\sin 2\theta < 0$ when $0 \le \theta \le \dfrac{\pi}{2}$, therefore $A(\theta)$ must have a maximum value at $\theta = \dfrac{\pi}{4}$. Hence, the area of the largest rectangle is

$$\begin{aligned} A\left(\frac{\pi}{4}\right) &= ab + \frac{1}{2}(a^2+b^2)\sin\left(\frac{\pi}{2}\right) \\ &= ab + \frac{1}{2}(a^2+b^2) = \frac{1}{2}(a+b)^2 \quad \text{sq. units.} \end{aligned}$$

(Note: $x = y = \dfrac{a}{\sqrt{2}} + \dfrac{b}{\sqrt{2}}$ indicates that the rectangle containing the given rectangle with sides $a$ and $b$, has largest area when it is a square.)

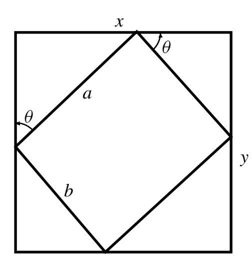



Fig. 4.5.24

**25.** Let the line have intercepts $x$, $y$ as shown. Let $\theta$ be angle shown. The length of line is

$$L = \frac{9}{\cos\theta} + \frac{\sqrt{3}}{\sin\theta} \qquad (0 < \theta < \frac{\pi}{2}).$$

Clearly, $L \to \infty$ if $\theta \to 0+$ or $\theta \to \frac{\pi}{2}-$.
Thus the minimum length occurs at a critical point.
For CP:

$$0 = \frac{dL}{d\theta} = \frac{9\sin\theta}{\cos^2\theta} - \frac{\sqrt{3}\cos\theta}{\sin^2\theta} \Rightarrow \tan^3\theta = \left(\frac{1}{\sqrt{3}}\right)^3$$
$$\Rightarrow \theta = \frac{\pi}{6}$$

Shortest line segment has length

$$L = \frac{9}{\sqrt{3}/2} + \frac{\sqrt{3}}{1/2} = 8\sqrt{3} \text{ units.}$$

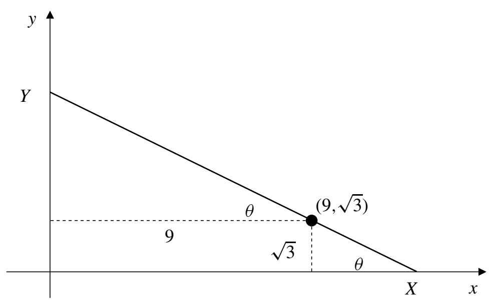



Fig. 4.5.25

**26.** The longest beam will have length equal to the minimum of $L = x + y$, where $x$ and $y$ are as shown in the figure below:

$$x = \frac{a}{\cos\theta}, \quad y = \frac{b}{\sin\theta}.$$

Thus,

$$L = L(\theta) = \frac{a}{\cos\theta} + \frac{b}{\sin\theta} \qquad \left(0 < \theta < \frac{\pi}{2}\right).$$

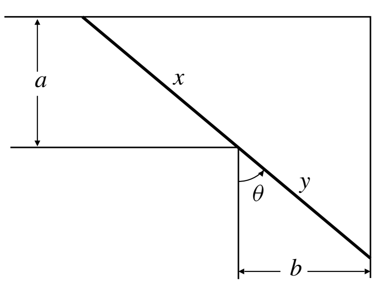



Fig. 4.5.26

If $L'(\theta) = 0$, then

$$\begin{aligned} & \frac{a\sin\theta}{\cos^2\theta} - \frac{b\cos\theta}{\sin^2\theta} = 0 \\ \Leftrightarrow \quad & \frac{a\sin^3\theta - b\cos^3\theta}{\cos^2\theta\sin^2\theta} = 0 \\ \Leftrightarrow \quad & a\sin^3\theta - b\cos^3\theta = 0 \\ \Leftrightarrow \quad & \tan^3\theta = \frac{b}{a} \\ \Leftrightarrow \quad & \tan\theta = \frac{b^{1/3}}{a^{1/3}}. \end{aligned}$$

Clearly, $L(\theta) \to \infty$ as $\theta \to 0+$ or $\theta \to \frac{\pi}{2}-$. Thus, the minimum must occur at $\theta = \tan^{-1}\left(\frac{b^{1/3}}{a^{1/3}}\right)$. Using the triangle above for $\tan\theta = \frac{b^{1/3}}{a^{1/3}}$, it follows that

$$\cos\theta = \frac{a^{1/3}}{\sqrt{a^{2/3} + b^{2/3}}}, \quad \sin\theta = \frac{b^{1/3}}{\sqrt{a^{2/3} + b^{2/3}}}.$$

Hence, the minimum is

$$\begin{aligned} L(\theta) &= \frac{a}{\left(\frac{a^{1/3}}{\sqrt{a^{2/3}+b^{2/3}}}\right)} + \frac{b}{\left(\frac{b^{1/3}}{\sqrt{a^{2/3}+b^{2/3}}}\right)} \\ &= \left(a^{2/3} + b^{2/3}\right)^{3/2} \text{ units.} \end{aligned}$$

**27.** If the largest beam that can be carried horizontally around the corner is $l$ m long (by Exercise 26, $l = (a^{2/3} + b^{2/3})^{2/3}$ m), then at the point of maximum clearance, one end of the beam will be on the floor at the outer wall of one hall, and the other will be on the ceiling at the outer wall of the second hall. Thus the horizontal projection of the beam will be $l$. So the beam will have length

$$\sqrt{l^2 + c^2} = [(a^{2/3} + b^{2/3})^3 + c^2]^{1/2} \text{ units.}$$

**28.** Let $\theta$ be the angle of inclination of the ladder. The height of the fence is

$$h(\theta) = 6\sin\theta - 2\tan\theta \qquad \left(0 < \theta < \frac{\pi}{2}\right).$$

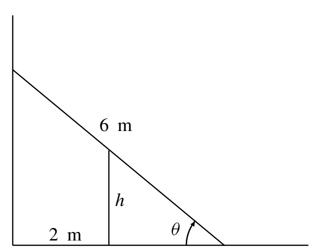



Fig. 4.5.28

For critical points:

$$\begin{aligned}0 &= h'(\theta) = 6\cos\theta - 2\sec^2\theta \\ &\Rightarrow 3\cos\theta = \sec^2\theta \Rightarrow 3\cos^3\theta = 1 \\ &\Rightarrow \cos\theta = \left(\tfrac{1}{3}\right)^{1/3}.\end{aligned}$$

Since $h''(\theta) = -6\sin\theta - 4\sec^2\theta\tan\theta < 0$ for $0 < \theta < \dfrac{\pi}{2}$, therefore $h(\theta)$ must be maximum at $\theta = \cos^{-1}\left(\frac{1}{3}\right)^{1/3}$. Then

$$\sin\theta = \frac{\sqrt{3^{2/3}-1}}{3^{1/3}}, \quad \tan\theta = \sqrt{3^{2/3}-1}.$$

Thus, the maximum height of the fence is

$$\begin{aligned}h(\theta) &= 6\left(\frac{\sqrt{3^{2/3}-1}}{3^{1/3}}\right) - 2\sqrt{3^{2/3}-1} \\ &= 2(3^{2/3}-1)^{3/2} \approx 2.24\,\text{m}.\end{aligned}$$

**29.** Let $(x, y)$ be a point on $x^2y^4 = 1$. Then $x^2y^4 = 1$ and the square of distance from $(x, y)$ to $(0, 0)$ is
$S = x^2 + y^2 = \dfrac{1}{y^4} + y^2, \quad (y \neq 0)$
Clearly, $S \to \infty$ as $y \to 0$ or $y \to \pm\infty$, so minimum $S$ must occur at a critical point. For CP:

$$\begin{aligned}0 = \frac{dS}{dy} = \frac{-4}{y^5} + 2y \Rightarrow y^6 = 2 \Rightarrow y = \pm 2^{1/6} \\ \Rightarrow x = \pm\frac{1}{2^{1/3}}\end{aligned}$$

Thus the shortest distance from origin to curve is

$$S = \sqrt{\frac{1}{2^{2/3}} + 2^{1/3}} = \sqrt{\frac{3}{2^{2/3}}} = \frac{3^{1/2}}{2^{1/3}}\ \text{units.}$$

**30.** The square of the distance from $(8, 1)$ to the curve $y = 1 + x^{3/2}$ is

$$\begin{aligned}S &= (x-8)^2 + (y-1)^2 \\ &= (x-8)^2 + (1 + x^{3/2} - 1)^2 \\ &= x^3 + x^2 - 16x + 64.\end{aligned}$$

Note that $y$, and therefore also $S$, is only defined for $x \geq 0$. If $x = 0$ then $S = 64$. Also, $S \to \infty$ if $x \to \infty$. For critical points:

$$\begin{aligned}0 &= \frac{dS}{dx} = 3x^2 + 2x - 16 = (3x+8)(x-2) \\ &\Rightarrow x = -\tfrac{8}{3} \text{ or } 2.\end{aligned}$$

Only $x = 2$ is feasible. At $x = 2$ we have $S = 44 < 64$. Therefore the minimum distance is $\sqrt{44} = 2\sqrt{11}$ units.

**31.** Let the cylinder have radius $r$ and height $h$. By symmetry, the centre of the cylinder is at the centre of the sphere. Thus

$$r^2 + \frac{h^2}{4} = R^2.$$

The volume of cylinder is

$$V = \pi r^2 h = \pi h\left(R^2 - \frac{h^2}{4}\right), \quad (0 \leq h \leq 2R).$$

Clearly, $V = 0$ if $h = 0$ or $h = 2R$, so maximum $V$ occurs at a critical point. For CP:

$$\begin{aligned}0 &= \frac{dV}{dh} = \pi\left[R^2 - \frac{h^2}{4} - \frac{2h^2}{4}\right] \\ &\Rightarrow h^2 = \frac{4}{3}R^2 \qquad \Rightarrow h = \frac{2R}{\sqrt{3}} \\ &\Rightarrow r = \sqrt{\frac{2}{3}}R.\end{aligned}$$

The largest cylinder has height $\dfrac{2R}{\sqrt{3}}$ units and radius $\sqrt{\dfrac{2}{3}}R$ units.

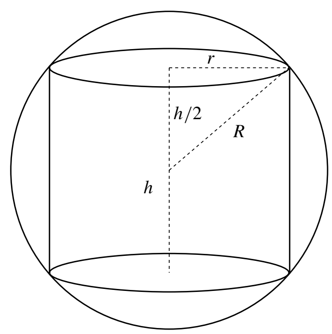



Fig. 4.5.31

**32.** Let the radius and the height of the circular cylinder be $r$ and $h$. By similar triangles,

$$\frac{h}{R-r} = \frac{H}{R} \Rightarrow h = \frac{H(R-r)}{R}.$$

Hence, the volume of the circular cylinder is

$$\begin{aligned}V(r) &= \pi r^2 h = \frac{\pi r^2 H(R-r)}{R} \\ &= \pi H\left(r^2 - \frac{r^3}{R}\right) \qquad \text{for } 0 \leq r \leq R.\end{aligned}$$

Since $V(0) = V(R) = 0$, the maximum value of $V$ must be at a critical point. If $\dfrac{dV}{dr} = \pi H\left(2r - \dfrac{3r^2}{R}\right) = 0$, then $r = \dfrac{2R}{3}$. Therefore the cylinder has maximum volume if its radius is $r = \dfrac{2R}{3}$ units, and its height is

$$h = \frac{H\left(R - \frac{2R}{3}\right)}{R} = \frac{H}{3} \text{ units.}$$

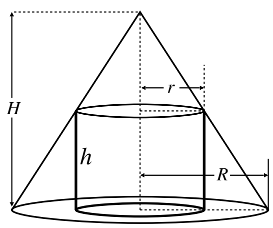



Fig. 4.5.32

**33.** Let the box have base dimensions $x$ m and height $y$ m. Then $x^2y = \text{volume} = 4$.
Most economical box has minimum surface area (bottom and sides). This area is

$$\begin{aligned} S &= x^2 + 4xy = x^2 + 4x\left(\frac{4}{x^2}\right) \\ &= x^2 + \frac{16}{x}, \quad (0 < x < \infty). \end{aligned}$$

Clearly, $S \to \infty$ if $x \to \infty$ or $x \to 0+$. Thus minimum $S$ occurs at a critical point. For CP:

$$0 = \frac{dS}{dx} = 2x - \frac{16}{x^2} \Rightarrow x^3 = 8 \Rightarrow x = 2 \Rightarrow y = 1.$$

Most economical box has base 2 m $\times$ 2 m and height 1 m.

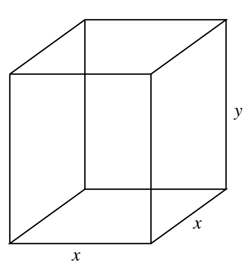



Fig. 4.5.33

**34.**

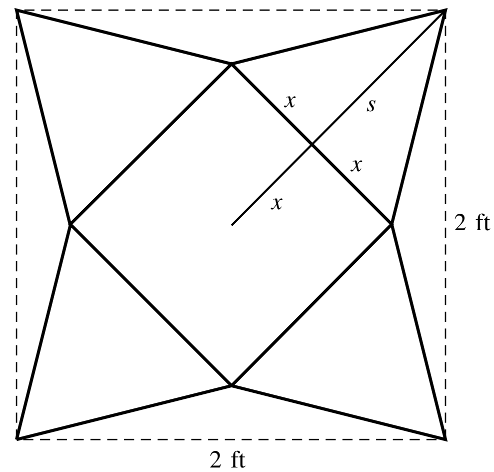



Fig. 4.5.34

From the figure, if the side of the square base of the pyramid is $2x$, then the slant height of triangular walls of the pyramid is $s = \sqrt{2} - x$. The vertical height of the pyramid is

$$h = \sqrt{s^2 - x^2} = \sqrt{2 - 2\sqrt{2}x + x^2 - x^2} = \sqrt{2}\sqrt{1 - \sqrt{2}x}.$$

Thus the volume of the pyramid is

$$V = \frac{4\sqrt{2}}{3}x^2\sqrt{1 - \sqrt{2}x},$$

for $0 \le x \le 1/\sqrt{2}$. $V = 0$ at both endpoints, so the maximum will occur at an interior critical point. For CP:

$$\begin{aligned} 0 &= \frac{dV}{dx} = \frac{4\sqrt{2}}{3}\left[2x\sqrt{1 - \sqrt{2}x} - \frac{\sqrt{2}x^2}{2\sqrt{1 - \sqrt{2}x}}\right] \\ 4x(1 - \sqrt{2}x) &= \sqrt{2}x^2 \\ 4x &= 5\sqrt{2}x^2 \quad , x = 4/(5\sqrt{2}). \end{aligned}$$

$V(4/(5\sqrt{2})) = 32\sqrt{2}/(75\sqrt{5})$. The largest volume of such a pyramid is $32\sqrt{2}/(75\sqrt{5})$ ft$^3$.

**35.** Let the dimensions be as shown. The perimeter is $\pi\dfrac{x}{2} + x + 2y = 10$. Therefore,

$$\left(1 + \frac{\pi}{2}\right)x + 2y = 10, \quad \text{or} \quad (2 + \pi)x + 4y = 20.$$

The area of the window is

$$A = xy + \frac{1}{2}\pi\left(\frac{x}{2}\right)^2 = \pi\frac{x^2}{8} + x\left(5 - \frac{(2 + \pi)x}{4}\right).$$

To maximize light admitted, maximize the area $A$. For CP:

$$0 = \frac{dA}{dx} = \frac{\pi x}{4} + 5 - \frac{2+\pi}{4}x - \frac{2+\pi}{4}x \Rightarrow x = \frac{20}{4+\pi}$$
$$\Rightarrow y = \frac{10}{4+\pi}.$$

To admit greatest amount of light, let width $= \dfrac{20}{4+\pi}$ m and height (of the rectangular part) be $\dfrac{10}{4+\pi}$ m.

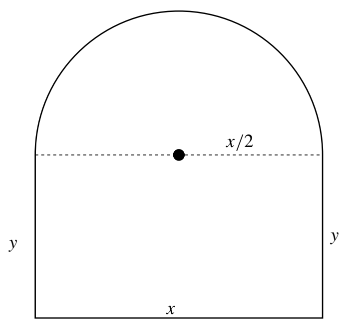



Fig. 4.5.35

**36.** Let $h$ and $r$ be the length and radius of the cylindrical part of the tank. The volume of the tank is

$$V = \pi r^2 h + \tfrac{4}{3}\pi r^3.$$

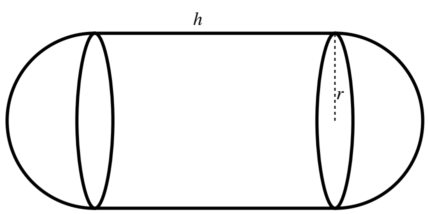



Fig. 4.5.36

If the cylindrical wall costs \$$k$ per unit area and the hemispherical wall \$$2k$ per unit area, then the total cost of the tank wall is

$$\begin{aligned} C &= 2\pi rhk + 8\pi r^2 k \\ &= 2\pi rk \frac{V - \frac{4}{3}\pi r^3}{\pi r^2} + 8\pi r^2 k \\ &= \frac{2Vk}{r} + \frac{16}{3}\pi r^2 k \qquad (0 < r < \infty). \end{aligned}$$

Since $C \to \infty$ as $r \to 0+$ or $r \to \infty$, the minimum cost must occur at a critical point. For critical points,

$$0 = \frac{dC}{dr} = -2Vkr^{-2} + \frac{32}{3}\pi rk \qquad \Leftrightarrow \qquad r = \left(\frac{3V}{16\pi}\right)^{1/3}.$$

Since $V = \pi r^2 h + \frac{4}{3}\pi r^3$,

$$r^3 = \frac{3}{16\pi}\left(\pi r^2 h + \frac{4}{3}\pi r^3\right) \Rightarrow r = \frac{1}{4}h$$
$$\Rightarrow h = 4r = 4\left(\frac{3V}{16\pi}\right)^{1/3}.$$

Hence, in order to minimize the cost, the radius and length of the cylindrical part of the tank should be $\left(\dfrac{3V}{16\pi}\right)^{1/3}$ and $4\left(\dfrac{3V}{16\pi}\right)^{1/3}$ units respectively.

**37.** Let $D'$ be chosen so that mirror $AB$ is the right bisector of $DD'$. Let $CD'$ meet $AB$ at $X$. Therefore, the travel time along $CXD$ is

$$T_X = \frac{CX + XD}{\text{speed}} = \frac{CX + XD'}{\text{speed}} = \frac{CD'}{\text{speed}}.$$

If $Y$ is any other point on $AB$, travel time along $CYD$ is

$$T_Y = \frac{CY + YD}{\text{speed}} = \frac{CY + YD'}{\text{speed}} > \frac{CD'}{\text{speed}}.$$

(The sum of two sides of a triangle is greater than the third side.) Therefore, $X$ minimizes travel time. Clearly, $XN$ bisects $\angle CXD$.

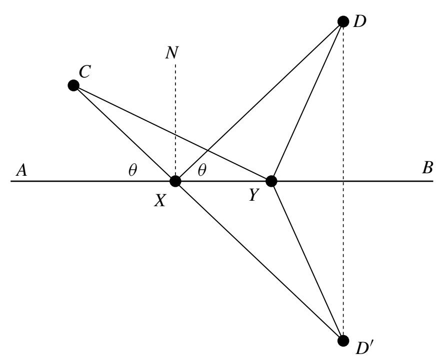



Fig. 4.5.37

**38.** If the path of the light ray is as shown in the figure then the time of travel from $A$ to $B$ is

$$T = T(x) = \frac{\sqrt{a^2 + x^2}}{v_1} + \frac{\sqrt{b^2 + (c-x)^2}}{v_2}.$$

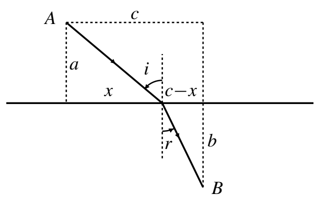



Fig. 4.5.38

To minimize $T$, we look for a critical point:

$$\begin{aligned}0 = \frac{dT}{dx} &= \frac{1}{v_1}\frac{x}{\sqrt{a^2+x^2}} - \frac{1}{v_2}\frac{c-x}{\sqrt{b^2+(c-x)^2}}\\ &= \frac{1}{v_1}\sin i - \frac{1}{v_2}\sin r.\end{aligned}$$

Thus,

$$\frac{\sin i}{\sin r} = \frac{v_1}{v_2}.$$

**39.** Let the width be $w$, and the depth be $h$. Therefore

$$\left(\frac{h}{2}\right)^2 + \left(\frac{w}{2}\right)^2 = R^2.$$

The stiffness is $S = wh^3 = h^3\sqrt{4R^2-h^2}$ for $(0 \le h \le 2R)$. We have $S = 0$ if $h = 0$ or $h = 2R$. For maximum stiffness:

$$0 = \frac{dS}{dh} = 3h^2\sqrt{4R^2-h^2} - \frac{h^4}{\sqrt{4R^2-h^2}}.$$

Thus $3(4R^2 - h^2) = h^2$ so $h = \sqrt{3}R$, and $w = R$. The stiffest beam has width $R$ and depth $\sqrt{3}R$.

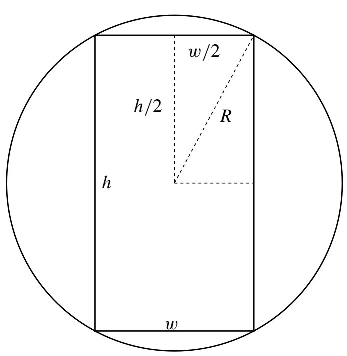



Fig. 4.5.39

**40.** The curve $y = 1 + 2x - x^3$ has slope $m = y' = 2 - 3x^2$. Evidently $m$ is greatest for $x = 0$, in which case $y = 1$ and $m = 2$. Thus the tangent line with maximal slope has equation $y = 1 + 2x$.

**41.** $\dfrac{dQ}{dt} = kQ^3(L-Q)^5 \qquad (k, L > 0)$

$Q$ grows at the greatest rate when $f(Q) = Q^3(L-Q)^5$ is maximum, i.e., when

$$\begin{aligned}0 = f'(Q) &= 3Q^2(L-Q)^5 - 5Q^3(L-Q)^4\\ &= Q^2(L-Q)^4(3L-8Q) \quad\Rightarrow\quad Q = 0, L, \frac{3L}{8}.\end{aligned}$$

Since $f(0) = f(L) = 0$ and $f\left(\dfrac{3L}{8}\right) > 0$, $Q$ is growing most rapidly when $Q = \dfrac{3L}{8}$.

**42.** Let $h$ and $r$ be the height and base radius of the cone and $R$ be the radius of the sphere. From similar triangles,

$$\begin{aligned}&\frac{r}{\sqrt{h^2+r^2}} = \frac{R}{h-R}\\ \Rightarrow\qquad &h = \frac{2r^2R}{r^2-R^2} \qquad (r > R).\end{aligned}$$

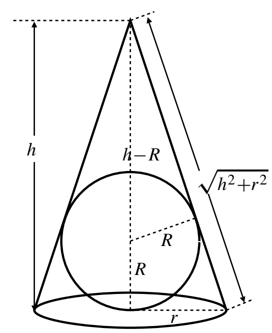



Fig. 4.5.42

Then the volume of the cone is

$$V = \frac{1}{3}\pi r^2 h = \frac{2}{3}\pi R\frac{r^4}{r^2-R^2} \qquad (R < r < \infty).$$

Clearly $V \to \infty$ if $r \to \infty$ or $r \to R+$. Therefore to minimize $V$, we look for a critical point:

$$\begin{aligned}&0 = \frac{dV}{dr} = \frac{2}{3}\pi R\left[\frac{(r^2-R^2)(4r^3) - r^4(2r)}{(r^2-R^2)^2}\right]\\ \Leftrightarrow\quad &4r^5 - 4r^3R^2 - 2r^5 = 0\\ \Leftrightarrow\quad &r = \sqrt{2}R.\end{aligned}$$

Hence, the smallest possible volume of a right circular cone which can contain sphere of radius $R$ is

$$V = \frac{2}{3}\pi R\left(\frac{4R^4}{2R^2-R^2}\right) = \frac{8}{3}\pi R^3 \text{ cubic units.}$$

**43.** If $x$ cars are loaded, the total time for the trip is

$$T = t + 1 + \frac{x}{1{,}000} \qquad \text{where } x = f(t) = \frac{1{,}000\,t}{e^{-t}+t}.$$

We can minimize the average time per car (or, equivalently, maximize the number of cars per hour). The average time (in hours) per car is

$$\begin{aligned} A = \frac{T}{x} &= \frac{e^{-t}+t}{1{,}000} + \frac{e^{-t}+t}{1{,}000t} + \frac{1}{1{,}000} \\ &= \frac{1}{1{,}000}\left[\left(e^{-t}+t\right)\left(1+\frac{1}{t}\right)+1\right]. \end{aligned}$$

This expression approaches $\infty$ as $t \to 0+$ or $t \to \infty$. For a minimum we should look for a positive critical point. Thus we want

$$0 = \frac{1}{1{,}000}\left[\left(-e^{-t}+1\right)\left(1+\frac{1}{t}\right) - \left(e^{-t}+t\right)\frac{1}{t^2}\right],$$

which simplifies to

$$t^2 + t + 1 = t^2 e^t.$$

Both sides of this equation are increasing functions but the left side has smaller slope than the right side for $t > 0$. Since the left side is 1 while the right side is 0 at $t = 0$, there will exist a unique solution in $t > 0$. Using a graphing calculator or computer program we determine that the critical point is approximately $t = 1.05032$. For this value of $t$ we have $x \approx 750.15$, so the movement of cars will be optimized by loading 750 cars for each sailing.

**44.** Let distances and angles be as shown. Then $\tan\alpha = \dfrac{2}{x}$, $\tan(\theta+\alpha) = \dfrac{12}{x}$

$$\begin{aligned} \frac{12}{x} &= \frac{\tan\theta + \tan\alpha}{1-\tan\theta\tan\alpha} = \frac{\tan\theta + \frac{2}{x}}{1 - \frac{2}{x}\tan\theta} \\ \frac{12}{x} - \frac{24}{x^2}\tan\theta &= \tan\theta + \frac{2}{x} \\ \tan\theta\left(1+\frac{24}{x^2}\right) &= \frac{10}{x}, \quad \text{so } \tan\theta = \frac{10x}{x^2+24} = f(x). \end{aligned}$$

To maximize $\theta$ (i.e., to get the best view of the mural), we can maximize $\tan\theta = f(x)$.
Since $f(0) = 0$ and $f(x) \to 0$ as $x \to \infty$, we look for a critical point.

$$\begin{aligned} 0 = f'(x) = 10\left[\frac{x^2+24-2x^2}{(x^2+24)^2}\right] &\Rightarrow x^2 = 24 \\ &\Rightarrow x = 2\sqrt{6} \end{aligned}$$

Stand back $2\sqrt{6}$ ft ($\approx 4.9$ ft) to see the mural best.

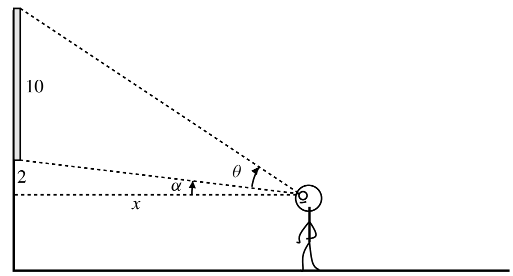
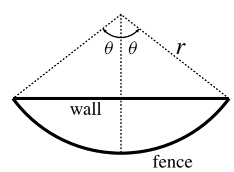



Fig. 4.5.44

**45.** Let $r$ be the radius of the circular arc and $\theta$ be the angle shown in the left diagram below. Thus,

$$2r\theta = 100 \qquad \Rightarrow \qquad r = \frac{50}{\theta}.$$



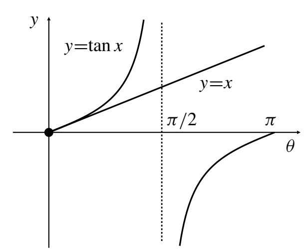




Fig. 4.5.45(a) Fig. 4.5.45(b)

The area of the enclosure is

$$\begin{aligned} A &= \frac{2\theta}{2\pi}\pi r^2 - (r\cos\theta)(r\sin\theta) \\ &= \frac{50^2}{\theta} - \frac{50^2}{\theta^2}\frac{\sin 2\theta}{2} \\ &= 50^2\left(\frac{1}{\theta} - \frac{\sin 2\theta}{2\theta^2}\right) \end{aligned}$$

for $0 < \theta \le \pi$. Note that $A \to \infty$ as $\theta \to 0+$, and for $\theta = \pi$ we are surrounding the entire enclosure with fence (a circle) and not using the wall at all. Evidently this would not produce the greatest enclosure area, so the maximum area must correspond to a critical point of $A$:

$$\begin{aligned} 0 = \frac{dA}{d\theta} &= 50^2\left(-\frac{1}{\theta^2} - \frac{2\theta^2(2\cos 2\theta) - \sin 2\theta(4\theta)}{4\theta^4}\right) \\ \Leftrightarrow \qquad & \frac{1}{\theta^2} + \frac{\cos 2\theta}{\theta^2} = \frac{\sin 2\theta}{\theta^3} \\ \Leftrightarrow \qquad & 2\theta\cos^2\theta = 2\sin\theta\cos\theta \\ \Leftrightarrow \qquad & \cos\theta = 0 \quad \text{or} \quad \tan\theta = \theta. \end{aligned}$$

Observe that $\tan\theta = \theta$ has no solutions in $(0, \pi]$. (The graphs of $y = \tan\theta$ and $y = \theta$ cross at $\theta = 0$ but nowhere else between 0 and $\pi$.) Thus, the greatest enclosure area must correspond to $\cos\theta = 0$, that is, to $\theta = \frac{\pi}{2}$. The largest enclosure is thus semicircular, and has area $\frac{2}{\pi}(50)^2 = \frac{5000}{\pi}$ m$^2$.

**46.** Let the cone have radius $r$ and height $h$.
Let sector of angle $\theta$ from disk be used.
Then $2\pi r = R\theta$ so $r = \frac{R}{2\pi}\theta$.
Also $h = \sqrt{R^2 - r^2} = \sqrt{R^2 - \frac{R^2\theta^2}{4\pi^2}} = \frac{R}{2\pi}\sqrt{4\pi^2 - \theta^2}$
The cone has volume

$$V = \frac{\pi r^2 h}{3} = \frac{\pi}{3}\frac{R^2}{4\pi^2}\theta^2\frac{R}{2\pi}\sqrt{4\pi^2 - \theta}$$
$$= \frac{R^3}{24\pi^2}f(\theta) \quad \text{where } f(\theta) = \theta^2\sqrt{4\pi^2 - \theta^2} \quad (0 \le \theta \le 2\pi)$$

$V(0) = V(2\pi) = 0$ so maximum $V$ must occur at a critical point. For CP:

$$0 = f'(\theta) = 2\theta\sqrt{4\pi^2 - \theta^2} - \frac{\theta^3}{\sqrt{4\pi^2 - \theta^2}}$$
$$\Rightarrow 2(4\pi^2 - \theta^2) = \theta^2 \qquad \Rightarrow \theta^2 = \frac{8}{3}\pi^2.$$

The largest cone has volume $V\left(\pi\sqrt{\frac{8}{3}}\right) = \frac{2\pi R^3}{9\sqrt{3}}$ cu. units.

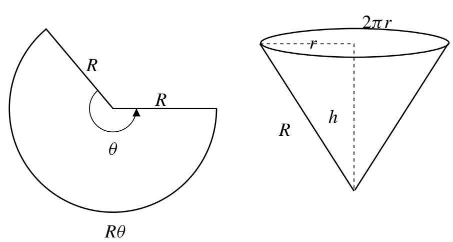



Fig. 4.5.46

**47.** Let the various distances be as labelled in the diagram.

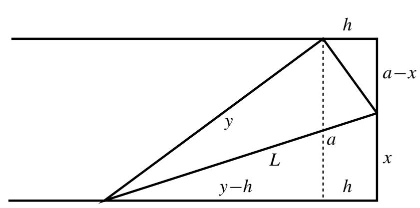



Fig. 4.5.47

From the geometry of the various triangles in the diagram we have

$$x^2 = h^2 + (a - x)^2 \Rightarrow h^2 = 2ax - a^2$$
$$y^2 = a^2 + (y - h)^2 \Rightarrow h^2 = 2hy - a^2$$

hence $hy = ax$. Then

$$L^2 = x^2 + y^2 = x^2 + \frac{a^2x^2}{h^2}$$
$$= x^2 + \frac{a^2x^2}{2ax - a^2} = \frac{2ax^3}{2ax - a^2}$$

for $\frac{a}{2} < x \le a$. Clearly, $L \to \infty$ as $x \to \frac{a}{2}+$, and $L(a) = \sqrt{2}a$. For critical points of $L^2$:

$$0 = \frac{d(L^2)}{dx} = \frac{(2ax - a^2)(6ax^2) - (2ax^3)(2a)}{(2ax - a^2)^2}$$
$$= \frac{2a^2x^2(4x - 3a)}{(2ax - a^2)^2}.$$

The only critical point in $\left(\frac{a}{2}, a\right]$ is $x = \frac{3a}{4}$. Since $L\left(\frac{3a}{4}\right) = \frac{3\sqrt{3}a}{4} < L(a)$, therefore the least possible length for the fold is $\frac{3\sqrt{3}a}{4}$ cm.

## Section 4.6 Finding Roots of Equations (page 251)

**1.** $f(x) = x^2 - 2$, $f'(x) = 2x$.
Newton's formula $x_{n+1} = g(x_n)$, where

$$g(x) = x - \frac{x^2 - 2}{2x} = \frac{x^2 + 2}{2x}.$$

Starting with $x_0 = 1.5$, get $x_3 = x_4 = 1.41421356237$.

**2.** $f(x) = x^2 - 3$, $f'(x) = 2x$.
Newton's formula $x_{n+1} = g(x_n)$, where

$$g(x) = x - \frac{x^2 - 3}{2x} = \frac{x^2 + 3}{2x}.$$

Starting with $x_0 = 1.5$, get $x_4 = x_5 = 1.73205080757$.

**3.** $f(x) = x^3 + 2x - 1$, $f'(x) = 3x^2 + 2$.
Newton's formula $x_{n+1} = g(x_n)$, where

$$g(x) = x - \frac{x^3 + 2x - 1}{3x^2 + 2} = \frac{2x^3 + 1}{3x^2 + 2}.$$

Starting with $x_0 = 0.5$, get $x_3 = x_4 = 0.45339765152$.

**4.** $f(x) = x^3 + 2x^2 - 2$, $f'(x) = 3x^2 + 4x$.
Newton's formula $x_{n+1} = g(x_n)$, where

$$g(x) = x - \frac{x^3 + 2x^2 - 2}{3x^2 + 4x} = \frac{2x^3 + 2x^2 + 2}{3x^2 + 4x}.$$

Starting with $x_0 = 1.5$, get $x_5 = x_6 = 0.839286755214$.

**5.** $f(x) = x^4 - 8x^2 - x + 16$, $f'(x) = 4x^3 - 16x - 1$.
Newton's formula $x_{n+1} = g(x_n)$, where

$$g(x) = x - \frac{x^4 - 8x^2 - x + 16}{4x^3 - 16x - 1} = \frac{3x^4 - 8x^2 - 16}{4x^3 - 16x - 1}.$$

Starting with $x_0 = 1.5$, get $x_4 = x_5 = 1.64809536561$.
Starting with $x_0 = 2.5$, get $x_5 = x_6 = 2.35239264766$.

**6.** $f(x) = x^3 + 3x^2 - 1$, $f'(x) = 3x^2 + 6x$.
Newton's formula $x_{n+1} = g(x_n)$, where

$$g(x) = x - \frac{x^3 + 3x^2 - 1}{3x^2 + 6x} = \frac{2x^3 + 3x^2 + 1}{3x^2 + 6x}.$$

Because $f(-3) = -1$, $f(-2) = 3$, $f(-1) = 1$, $f(0) = -1$, $f(1) = 3$, there are roots between $-3$ and $-2$, between $-1$ and 0, and between 0 and 1.
Starting with $x_0 = -2.5$, get $x_5 = x_6 = -2.87938524157$.
Starting with $x_0 = -0.5$, get $x_4 = x_5 = -0.652703644666$.
Starting with $x_0 = 0.5$, get $x_4 = x_5 = 0.532088886328$.

**7.** $f(x) = \sin x - 1 + x$, $f'(x) = \cos x + 1$.
Newton's formula is $x_{n+1} = g(x_n)$, where

$$g(x) = x - \frac{\sin x - 1 + x}{\cos x + 1}.$$

The graphs of $\sin x$ and $1-x$ suggest a root near $x = 0.5$. Starting with $x_0 = 0.5$, get
$x_3 = x_4 = 0.510973429389$.

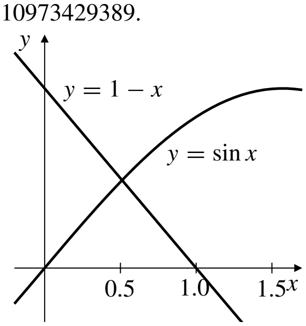



Fig. 4.6.7

**8.** $f(x) = x^2 - \cos x$, $f'(x) = 2x + \sin x$.
Newton's formula is $x_{n+1} = g(x_n)$, where

$$g(x) = x - \frac{x^2 - \cos x}{2x + \sin x}.$$

The graphs of $\cos x$ and $x^2$, suggest a root near $x = \pm 0.8$. Starting with $x_0 = 0.8$, get $x_3 = x_4 = 0.824132312303$. The other root is the negative of this one, because $\cos x$ and $x^2$ are both even functions.

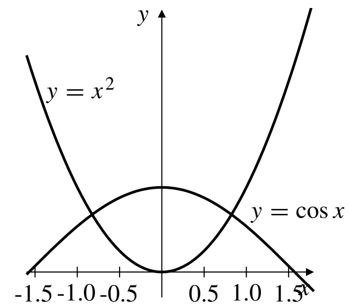



Fig. 4.6.8

**9.** Since $\tan x$ takes all real values between any two consecutive odd multiples of $\pi/2$, its graph intersects $y = x$ infinitely often. Thus, $\tan x = x$ has infinitely many solutions. The one between $\pi/2$ and $3\pi/2$ is close to $3\pi/2$, so start with $x_0 = 4.5$. Newton's formula here is

$$x_{n+1} = x_n - \frac{\tan x_n - x_n}{\sec^2 x_n - 1}.$$

We get $x_3 = x_4 = 4.49340945791$.

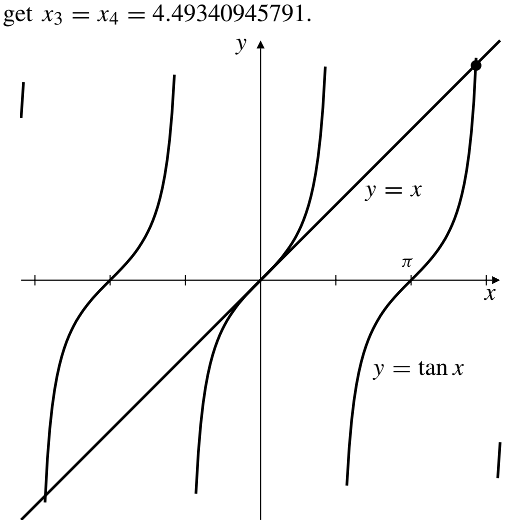



Fig. 4.6.9

**10.** A graphing calculator shows that the equation

$$(1 + x^2)\sqrt{x} - 1 = 0$$

has a root near $x = 0.6$. Use of a solve routine or Newton's Method gives $x = 0.56984029099806$.

**12.** Let $f(x) = \dfrac{\sin x}{1+x^2}$. Since $|f(x)| \le 1/(1+x^2) \to 0$ as $x \to \pm\infty$ and $f(0) = 0$, the maximum and minimum values of $f$ will occur at the two critical points of $f$ that are closest to the origin on the right and left, respectively. For CP:

$$0 = f'(x) = \frac{(1+x^2)\cos x - 2x\sin x}{(1+x^2)^2}$$
$$0 = (1+x^2)\cos x - 2x\sin x$$

with $0 < x < \pi$ for the maximum and $-\pi < x < 0$ for the minimum. Solving this equation using a solve routine or Newton's Method starting, say, with $x_0 = 1.5$, we get $x = \pm 0.79801699184239$. The corresponding max and min values of $f$ are $\pm 0.437414158279$.

**13.** Let $f(x) = \dfrac{\cos x}{1+x^2}$. Note that $f$ is an even function, and that $f$ has maximum value 1 at $x = 0$. (Clearly $f(0) = 1$ and $|f(x)| < 1$ if $x \ne 0$.) The minimum value will occur at the critical points closest to but not equal to 0. For CP:

$$0 = f'(x) = \frac{(1+x^2)(-\sin x) - 2x\cos x}{(1+x^2)^2}$$
$$0 = (1+x^2)\sin x + 2x\cos x.$$

The first CP to the right of zero is between $\pi/2$ and $3\pi/2$, so start with $x = 2.5$, say, and get $x = 2.5437321475261$. The minimum value is $f(x) = -0.110639672192$.

**14.** For $x^2 = 0$ we have $x_{n+1} = x_n - (x_n^2/(2x_n)) = x_n/2$. If $x_0 = 1$, then $x_1 = 1/2$, $x_2 = 1/4$, $x_3 = 1/8$.

a) $x_n = 1/2^n$, by induction.

b) $x_n$ approximates the root $x = 0$ to within 0.0001 provided $2^n > 10,000$. We need $n \ge 14$ to ensure this.

c) To ensure that $x_n^2$ is within 0.0001 of 0 we need $(1/2^n)^2 < 0.0001$, that is, $2^{2n} > 10,000$. We need $n \ge 7$.

d) Convergence of Newton approximations to the root $x = 0$ of $x^2 = 0$ is slower than usual because the derivative $2x$ of $x^2$ is zero at the root.

**15.** $f(x) = \begin{cases} \sqrt{x} & \text{if } x \ge 0 \\ \sqrt{-x} & \text{if } x < 0 \end{cases}$,

$f'(x) = \begin{cases} 1/(2\sqrt{x}) & \text{if } x > 0 \\ -1/(2\sqrt{-x}) & \text{if } x < 0 \end{cases}$.

The Newton's Method formula says that

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - 2x_n = -x_n.$$

If $x_0 = a$, then $x_1 = -a$, $x_2 = a$, and, in general, $x_n = (-1)^n a$. The approximations oscillate back and forth between two numbers.

If one observed that successive approximations were oscillating back and forth between two values $a$ and $b$, one should try their average, $(a+b)/2$, as a new starting guess. It may even turn out to be the root!

**16.** Newton's Method formula for $f(x) = x^{1/3}$ is

$$x_{n+1} = x_n - \frac{x_n^{1/3}}{(1/3)x_n^{-2/3}} = x_n - 3x_n = -2x_n.$$

If $x_0 = 1$, then $x_1 = -2$, $x_2 = 4$, $x_3 = -8$, $x_4 = 16$, and, in general, $x_n = (-2)^n$. The successive "approximations" oscillate ever more widely, diverging from the root at $x = 0$.

**17.** Newton's Method formula for $f(x) = x^{2/3}$ is

$$x_{n+1} = x_n - \frac{x_n^{2/3}}{(2/3)x_n^{-1/3}} = x_n - \tfrac{3}{2}x_n = -\tfrac{1}{2}x_n.$$

If $x_0 = 1$, then $x_1 = -1/2$, $x_2 = 1/4$, $x_3 = -1/8$, $x_4 = 1/16$, and, in general, $x_n = (-1/2)^n$. The successive approximations oscillate around the root $x = 0$, but still converge to it (though more slowly than is usual for Newton's Method).

**18.** To solve $1 + \frac{1}{4}\sin x = x$, start with $x_0 = 1$ and iterate $x_{n+1} = 1 + \frac{1}{4}\sin x_n$. $x_5$ and $x_6$ round to 1.23613.

**19.** To solve $\cos(x/3) = x$, start with $x_0 = 0.9$ and iterate $x_{n+1} = \cos(x_n/3)$. $x_4$ and $x_5$ round to 0.95025.

**20.** To solve $(x+9)^{1/3} = x$, start with $x_0 = 2$ and iterate $x_{n+1} = (x_n + 9)^{1/3}$. $x_4$ and $x_5$ round to 2.24004.

**21.** To solve $1/(2+x^2) = x$, start with $x_0 = 0.5$ and iterate $x_{n+1} = 1/(2+x_n^2)$. $x_6$ and $x_7$ round to 0.45340.

**22.** To solve $x^3 + 10x - 10 = 0$, start with $x_0 = 1$ and iterate $x_{n+1} = 1 - \frac{1}{10}x_n^3$. $x_7$ and $x_8$ round to 0.92170.

**23.** $r$ is a fixed point of $N(x)$

$$\begin{aligned} &\iff \quad r = N(r) = r - \frac{f(r)}{f'(r)} \\ &\iff \quad 0 = -f(r)/f'(r) \\ &\iff \quad f(r) = 0 \end{aligned}$$

i.e., if and only if $r$ is a root of $f(x) = 0$. In this case, $x_{n+1} = N(x_n)$ is the $n$th Newton's Method approximation to the root, starting from the initial guess $x_0$.

**24.** Let $g(x) = f(x) - x$ for $a \le x \le b$. $g$ is continuous (because $f$ is), and since $a \le f(x) \le b$ whenever $a \le x \le b$ (by condition (i)), we know that $g(a) \ge 0$ and $g(b) \le 0$. By the Intermediate-Value Theorem there exists $r$ in $[a, b]$ such that $g(r) = 0$, that is, such that $f(r) = r$.

**25.** We are given that there is a constant $K$ satisfying $0 < K < 1$, such that

$$|f(u) - f(v)| \le K|u - v|$$

holds whenever $u$ and $v$ are in $[a, b]$. Pick any $x_0$ in $[a, b]$, and let $x_1 = f(x_0)$, $x_2 = f(x_1)$, and, in general, $x_{n+1} = f(x_n)$. Let $r$ be the fixed point of $f$ in $[a, b]$ found in Exercise 24. Thus $f(r) = r$. We have

$$|x_1 - r| = |f(x_0) - f(r)| \le K|x_0 - r|$$
$$|x_2 - r| = |f(x_1) - f(r)| \le K|x_1 - r| \le K^2|x_0 - r|,$$

and, in general, by induction

$$|x_n - r| \le K^n|x_0 - r|.$$

Since $K < 1$, $\lim_{n\to\infty} K^n = 0$, so $\lim_{n\to\infty} x_n = r$. The iterates converge to the fixed point as claimed in Theorem 6.

## Section 4.7 Linear Approximations (page 256)

**1.** $f(x) = x^2$, $f'(x) = 2x$, $f(3) = 9$, $f'(3) = 6$.
Linearization at $x = 3$: $L(x) = 9 + 6(x - 3)$.

**2.** $f(x) = x^{-3}$, $f'(x) = -3x^{-4}$, $f(2) = 1/8$, $f'(2) = -3/16$.
Linearization at $x = 2$: $L(x) = \frac{1}{8} - \frac{3}{16}(x - 2)$.

**3.** $f(x) = \sqrt{4 - x}$, $f'(x) = -1/(2\sqrt{4 - x})$, $f(0) = 2$, $f'(0) = -1/4$.
Linearization at $x = 0$: $L(x) = 2 - \frac{1}{4}x$.

**4.** $f(x) = \sqrt{3 + x^2}$, $f'(x) = x/\sqrt{3 + x^2}$, $f(1) = 2$, $f'(1) = 1/2$.
Linearization at $x = 1$: $L(x) = 2 + \frac{1}{2}(x - 1)$.

**5.** $f(x) = (1 + x)^{-2}$, $f'(x) = -2(1 + x)^{-3}$, $f(2) = 1/9$, $f'(2) = -2/27$.
Linearization at $x = 2$: $L(x) = \frac{1}{9} - \frac{2}{27}(x - 2)$.

**6.** $f(x) = x^{-1/2}$, $f'(x) = (-1/2)x^{-3/2}$, $f(4) = 1/2$, $f'(4) = -1/16$.
Linearization at $x = 4$: $L(x) = \frac{1}{2} - \frac{1}{16}(x - 4)$.

**7.** $f(x) = \sin x$, $f'(x) = \cos x$, $f(\pi) = 0$, $f'(\pi) = -1$.
Linearization at $x = \pi$: $L(x) = -(x - \pi)$.

**8.** $f(x) = \cos(2x)$, $f'(x) = -2\sin(2x)$, $f(\pi/3) = -1/2$, $f'(\pi/3) = -\sqrt{3}$.
Linearization at $x = \pi/3$: $L(x) = -\frac{1}{2} - \sqrt{3}\left(x - \frac{\pi}{3}\right)$.

**9.** $f(x) = \sin^2 x$, $f'(x) = 2\sin x\cos x$, $f(\pi/6) = 1/4$, $f'(\pi/6) = \sqrt{3}/2$.
Linearization at $x = \pi/6$: $L(x) = \frac{1}{4} + (\sqrt{3}/2)\left(x - \frac{\pi}{6}\right)$.

**10.** $f(x) = \tan x$, $f'(x) = \sec^2 x$, $f(\pi/4) = 1$, $f'(\pi/4) = 2$.
Linearization at $x = \pi/4$: $L(x) = 1 + 2\left(x - \frac{\pi}{4}\right)$.

**11.** If $A$ and $x$ are the area and side length of the square, then $A = x^2$. If $x = 10$ cm and $\Delta x = 0.4$ cm, then

$$\Delta A \approx \frac{dA}{dx}\,\Delta x = 2x\,\Delta x = 20(0.4) = 8.$$

The area increases by about 8 cm$^2$.

**12.** If $V$ and $x$ are the volume and side length of the cube, then $V = x^3$. If $x = 20$ cm and $\Delta V = -12$ cm$^3$, then

$$-12 = \Delta V \approx \frac{dV}{dx}\,\Delta x = 3x^2\,\Delta x = 1,200\,\Delta x,$$

so that $\Delta x = -1/100$. The edge length must decrease by about 0.01 cm in to decrease the volume by 12 cm$^3$.

**13.** The circumference $C$ and radius $r$ of the orbit are linked by $C = 2\pi r$. Thus $\Delta C = 2\pi\,\Delta r$. If $\Delta r = -10$ mi then $\Delta C \approx 2\pi\,\Delta r = 20\pi$. The circumference of the orbit will decrease by about $20\pi \approx 62.8$ mi if the radius decreases by 10 mi. Note that the answer does not depend on the actual radius of the orbit.

**14.** $a = g[R/(R + h)]^2$ implies that

$$\Delta a \approx \frac{da}{dh}\,\Delta h = gR^2\,\frac{-2}{(R + h)^3}\,\Delta h.$$

If $h = 0$ and $\Delta h = 10$ mi, then

$$\Delta a \approx -\frac{20g}{R} = -\frac{20 \times 32}{3960} \approx 0.16 \text{ ft/s}^2.$$

**15.** $f(x) = x^{1/2} \quad f'(x) = \frac{1}{2}x^{-1/2} \quad f''(x) = -\frac{1}{4}x^{-3/2}$

$$\sqrt{50} = f(50) \approx f(49) + f'(49)(50 - 49)$$
$$= 7 + \frac{1}{14} = \frac{99}{14} \approx 7.071.$$

$f''(x) < 0$ on $[49, 50]$, so error is negative: $\sqrt{50} < \dfrac{99}{14}$

$|f''(x)| < \dfrac{1}{4 \times 49^{3/2}} = \dfrac{1}{4 \times 7^3} = \dfrac{1}{1372} \approx 0.00073 = k$ on $(49, 50)$.

Thus $|\text{error}| \le \dfrac{k}{2}(50 - 49)^2 = \dfrac{1}{2744} = 0.00036$. We have

$$\frac{99}{14} - \frac{1}{2744} \le \sqrt{50} \le \frac{99}{14},$$

i.e., $7.071064 \le \sqrt{50} \le 7.071429$

**16.** Let $f(x) = \sqrt{x}$, then $f'(x) = \frac{1}{2}x^{-1/2}$ and $f''(x) = -\frac{1}{4}x^{-3/2}$. Hence,

$$\sqrt{47} = f(47) \approx f(49) + f'(49)(47 - 49)$$
$$= 7 + \left(\frac{1}{14}\right)(-2) = \frac{48}{7} \approx 6.8571429.$$

Clearly, if $x \geq 36$, then

$$|f''(x)| \leq \frac{1}{4 \times 6^3} = \frac{1}{864} = K.$$

Since $f''(x) < 0$, $f$ is concave down. Therefore, the error $E = \sqrt{47} - \dfrac{48}{7} < 0$ and

$$|E| < \frac{K}{2}(47-49)^2 = \frac{1}{432}.$$

Thus,

$$\frac{48}{7} - \frac{1}{432} < \sqrt{47} < \frac{48}{7}$$
$$6.8548 < \sqrt{47} < 6.8572.$$

**17.** $f(x) = x^{1/4}, \quad f'(x) = \frac{1}{4}x^{-3/4}, \quad f''(x) = -\frac{3}{16}x^{-7/4}$

$$\begin{aligned}\sqrt[4]{85} &= f(85) \approx f(81) + f'(81)(85-81)\\ &= 3 + \frac{4}{4 \times 27} = 3 + \frac{1}{27} = \frac{82}{27} \approx 3.037.\end{aligned}$$

$f''(x) < 0$ on $[81, 85]$ so error is negative: $\sqrt[4]{85} < \dfrac{82}{27}$.

$|f''(x)| < \dfrac{3}{16 \times 3^7} = \dfrac{1}{11,664} = k$ on $[81, 85]$.

Thus $|\text{Error}| \leq \dfrac{k}{2}(85-81)^2 = 0.00069$.

$$\frac{82}{27} - \frac{1}{1458} < \sqrt[4]{85} < \frac{82}{27},$$

or $3.036351 \leq \sqrt[4]{85} \leq 3.037037$

**18.** Let $f(x) = \dfrac{1}{x}$, then $f'(x) = -\dfrac{1}{x^2}$ and $f''(x) = \dfrac{2}{x^3}$. Hence,

$$\begin{aligned}\frac{1}{2.003} &= f(2.003) \approx f(2) + f'(2)(0.003)\\ &= \frac{1}{2} + \left(-\frac{1}{4}\right)(0.003) = 0.49925.\end{aligned}$$

If $x \geq 2$, then $|f''(x)| \leq \frac{2}{8} = \frac{1}{4}$. Since $f''(x) > 0$ for $x > 0$, $f$ is concave up. Therefore, the error

$$E = \frac{1}{2.003} - 0.49925 > 0$$

and

$$|E| < \frac{1}{8}(0.003)^2 = 0.000001125.$$

Thus,

$$0.49925 < \frac{1}{2.003} < 0.49925 + 0.000001125$$
$$0.49925 < \frac{1}{2.003} < 0.499251125.$$

**19.** $f(x) = \cos x, \quad f'(x) = -\sin x, \quad f''(x) = -\cos x$

$$\begin{aligned}\cos 46^\circ &= \cos\left(\frac{\pi}{4} + \frac{\pi}{180}\right)\\ &\approx \cos\frac{\pi}{4} - \sin\left(\frac{\pi}{4}\right)\left(\frac{\pi}{180}\right)\\ &= \frac{1}{\sqrt{2}}\left(1 - \frac{\pi}{180}\right) \approx 0.694765.\end{aligned}$$

$f''(0) < 0$ on $[45^\circ, 46^\circ]$ so

$$|\text{Error}| < \frac{1}{2\sqrt{2}}\left(\frac{\pi}{180}\right)^2 \approx 0.0001.$$

We have

$$\frac{1}{\sqrt{2}}\left(1 - \frac{\pi}{180} - \frac{\pi^2}{2 \times 180^2}\right) < \cos 46^\circ < \frac{1}{\sqrt{2}}\left(1 - \frac{\pi}{180}\right)$$

i.e., $0.694658 \leq \cos 46^\circ < 0.694765$.

**20.** Let $f(x) = \sin x$, then $f'(x) = \cos x$ and $f''(x) = -\sin x$. Hence,

$$\begin{aligned}\sin\left(\frac{\pi}{5}\right) &= f\left(\frac{\pi}{6} + \frac{\pi}{30}\right) \approx f\left(\frac{\pi}{6}\right) + f'\left(\frac{\pi}{6}\right)\left(\frac{\pi}{30}\right)\\ &= \frac{1}{2} + \frac{\sqrt{3}}{2}\left(\frac{\pi}{30}\right) \approx 0.5906900.\end{aligned}$$

If $x \leq \dfrac{\pi}{4}$, then $|f''(x)| \leq \dfrac{1}{\sqrt{2}}$. Since $f''(x) < 0$ on $0 < x \leq 90^\circ$, $f$ is concave down. Therefore, the error $E$ is negative and

$$|E| < \frac{1}{2\sqrt{2}}\left(\frac{\pi}{30}\right)^2 = 0.0038772.$$

Thus,

$$0.5906900 - 0.0038772 < \sin\left(\frac{\pi}{5}\right) < 0.5906900$$
$$0.5868128 < \sin\left(\frac{\pi}{5}\right) < 0.5906900.$$

**21.** Let $f(x) = \sin x$, then $f'(x) = \cos x$ and $f''(x) = -\sin x$. The linearization at $x = \pi$ gives:

$$\sin(3.14) \approx \sin\pi + \cos\pi(3.14 - \pi) = \pi - 3.14 \approx 0.001592654.$$

Since $f''(x) < 0$ between 3.14 and $\pi$, the error $E$ in the above approximation is negative: $\sin(3.14) < 0.001592654$. For $3.14 \leq t \leq \pi$, we have

$$|f''(t)| = \sin t \leq \sin(3.14) < 0.001592654.$$

Thus the error satisfies

$$|E| \leq \frac{0.001592654}{2}(3.14 - \pi)^2 < 0.000000002.$$

Therefore $0.001592652 < \sin(3.14) < 0.001592654$.

**22.** Let $f(x) = \sin x$, then $f'(x) = \cos x$ and $f''(x) = -\sin x$. The linearization at $x = 30° = \pi/6$ gives

$$\begin{aligned}\sin(33°) &= \sin\left(\tfrac{\pi}{6} + \tfrac{\pi}{60}\right) \\ &\approx \sin\frac{\pi}{6} + \cos\frac{\pi}{6}\left(\frac{\pi}{60}\right) \\ &= \frac{1}{2} + \frac{\sqrt{3}}{2}\left(\frac{\pi}{60}\right) \approx 0.545345.\end{aligned}$$

Since $f''(x) < 0$ between $30°$ and $33°$, the error $E$ in the above approximation is negative: $\sin(33°) < 0.545345$. For $30° \le t \le 33°$, we have

$$|f''(t)| = \sin t \le \sin(33°) < 0.545345.$$

Thus the error satisfies

$$|E| \le \frac{0.545345}{2}\left(\frac{\pi}{60}\right)^2 < 0.000747.$$

Therefore

$$\begin{aligned}0.545345 - 0.000747 < \sin(33°) < 0.545345 \\ 0.544598 < \sin(33°) < 0.545345.\end{aligned}$$

**23.** From the solution to Exercise 15, the linearization to $f(x) = x^{1/2}$ at $x = 49$ has value at $x = 50$ given by

$$L(50) = f(49) + f'(49)(50 - 49) \approx 7.071429.$$

Also, $7.071064 \le \sqrt{50} \le 7.071429$, and, since $f''(x) = -1/(4(\sqrt{x})^3)$,

$$\frac{-1}{4(7)^3} \le f''(x) \le \frac{-1}{4(\sqrt{50})^3} \le \frac{-1}{4(7.071429)^3}$$

for $49 \le x \le 50$. Thus, on that interval, $M \le f''(x) \le N$, where $M = -0.000729$ and $N = -0.000707$. By Corollary C,

$$\begin{aligned}&L(50) + \frac{M}{2}(50 - 49)^2 \le f(50) \le L(50) + \frac{N}{2}(50 - 49)^2 \\ &7.071064 \le \sqrt{50} \le 7.071075.\end{aligned}$$

Using the midpoint of this interval as a new approximation for $\sqrt{50}$ ensures that the error is no greater than half the length of the interval:

$$\sqrt{50} \approx 7.071070, \quad |\text{error}| \le 0.000006.$$

**24.** From the solution to Exercise 16, the linearization to $f(x) = x^{1/2}$ at $x = 49$ has value at $x = 47$ given by

$$L(47) = f(49) + f'(49)(47 - 49) \approx 6.8571429.$$

Also, $6.8548 \le \sqrt{47} \le 6.8572$, and, since $f''(x) = -1/(4(\sqrt{x})^3)$,

$$\frac{-1}{4(6.8548)^3} \le \frac{-1}{4(\sqrt{47})^3} \le f''(x) \le \frac{-1}{4(7)^3}$$

for $47 \le x \le 49$. Thus, on that interval, $M \le f''(x) \le N$, where $M = -0.000776$ and $N = -0.000729$. By Corollary C,

$$\begin{aligned}&L(47) + \frac{M}{2}(47 - 49)^2 \le f(47) \le L(47) + \frac{N}{2}(47 - 49)^2 \\ &6.855591 \le \sqrt{47} \le 6.855685.\end{aligned}$$

Using the midpoint of this interval as a new approximation for $\sqrt{47}$ ensures that the error is no greater than half the length of the interval:

$$\sqrt{47} \approx 6.855638, \quad |\text{error}| \le 0.000047.$$

**25.** From the solution to Exercise 17, the linearization to $f(x) = x^{1/4}$ at $x = 81$ has value at $x = 85$ given by

$$L(85) = f(81) + f'(81)(85 - 81) \approx 3.037037.$$

Also, $3.036351 \le 85^{1/4} \le 3.037037$, and, since $f''(x) = -3/(16(x^{1/4})^7)$,

$$\frac{-3}{16(3)^7} \le f''(x) \le \frac{-3}{16(85^{1/4})^7} \le \frac{-3}{16(3.037037)^7}$$

for $81 \le x \le 85$. Thus, on that interval, $M \le f''(x) \le N$, where $M = -0.000086$ and $N = -0.000079$. By Corollary C,

$$\begin{aligned}&L(85) + \frac{M}{2}(85 - 81)^2 \le f(85) \le L(85) + \frac{N}{2}(85 - 81)^2 \\ &3.036351 \le 85^{1/4} \le 3.036405.\end{aligned}$$

Using the midpoint of this interval as a new approximation for $85^{1/4}$ ensures that the error is no greater than half the length of the interval:

$$85^{1/4} \approx 3.036378, \quad |\text{error}| \le 0.000028.$$

**26.** From the solution to Exercise 22, the linearization to $f(x) = \sin x$ at $x = 30° = \pi/6$ has value at $x = 33° = \pi/6 + \pi/60$ given by

$$L(33°) = f(\pi/6) + f'(\pi/6)(\pi/60) \approx 0.545345.$$

Also, $0.544597 \le \sin(33°) \le 5.545345$, and, since $f''(x) = -\sin x$,

$$-\sin(33°) \le f''(x) \le -\sin(30°)$$

for $30° \le x \le 33°$. Thus, on that interval, $M \le f''(x) \le N$, where $M = -0.545345$ and $N = -0.5$. By Corollary C,

$$L(33°) + \frac{M}{2}(\pi/60)^2 \le \sin(33°) \le L(33°) + \frac{N}{2}(\pi/60)^2$$
$$0.544597 \le \sin(33°) \le 0.544660.$$

Using the midpoint of this interval as a new approximation for $\sin(33°)$ ensures that the error is no greater than half the length of the interval:

$$\sin(33°) \approx 0.544629, \quad |\text{error}| \le 0.000031.$$

**27.** $f(2) = 4, \quad f'(2) = -1, \quad 0 \le f''(x) \le \dfrac{1}{x}$ if $x > 0$.
$f(3) \approx f(2) + f'(2)(3-2) = 4 - 1 = 3.$
$f''(x) \ge 0 \Rightarrow$ error $\ge 0 \Rightarrow f(3) \ge 3.$
$|f''(x)| \le \dfrac{1}{x} \le \dfrac{1}{2}$ if $2 \le x \le 3$, so $|\text{Error}| \le \dfrac{1}{4}(3-2)^2$.
Thus $3 \le f(3) \le 3\frac{1}{4}$

**28.** The linearization of $f(x)$ at $x = 2$ is

$$L(x) = f(2) + f'(2)(x-2) = 4 - (x-2).$$

Thus $L(3) = 3$. Also, since $1/(2x) \le f''(x) \le 1/x$ for $x > 0$, we have for $2 \le x \le 3$, $(1/6) \le f''(x) \le (1/2)$. Thus

$$3 + \frac{1}{2}\left(\frac{1}{6}\right)(3-2)^2 \le f(3) \le 3 + \frac{1}{2}\left(\frac{1}{2}\right)(3-2)^2.$$

The best approximation for $f(3)$ is the midpoint of this interval: $f(3) \approx 3\frac{1}{6}$.

**29.** The linearization of $g(x)$ at $x = 2$ is

$$L(x) = g(2) + g'(2)(x-2) = 1 + 2(x-2).$$

Thus $L(1.8) = 0.6$.
If $|g''(x)| \le 1 + (x-2)^2$ for $x > 0$, then $|g''(x)| < 1 + (-0.2)^2 = 1.04$ for $1.8 \le x \le 2$. Hence $g(1.8) \approx 0.6$ with $|\text{error}| < \dfrac{1}{2}(1.04)(1.8-2)^2 = 0.0208$.

**30.** If $f(\theta) = \sin\theta$, then $f'(\theta) = \cos\theta$ and $f''(\theta) = -\sin\theta$. Since $f(0) = 0$ and $f'(0) = 1$, the linearization of $f$ at $\theta = 0$ is $L(\theta) = 0 + 1(\theta - 0) = \theta$.
If $0 \le t \le \theta$, then $f''(t) \le 0$, so $0 \le \sin\theta \le \theta$.
If $0 \ge t \ge \theta$, then $f''(t) \ge 0$, so $0 \ge \sin\theta \ge \theta$.
In either case, $|\sin t| \le |\sin\theta| \le |\theta|$ if $t$ is between 0 and $\theta$. Thus the error $E(\theta)$ in the approximation $\sin\theta \approx \theta$ satisfies

$$|E(\theta) \le \frac{|\theta|}{2}|\theta|^2 = \frac{|\theta|^3}{2}.$$

If $|\theta| \le 17° = 17\pi/180$, then

$$\frac{|E(\theta)|}{|\theta|} \le \frac{1}{2}\left(\frac{17\pi}{180}\right)^2 \approx 0.044.$$

Thus the percentage error is less than 5%.

**31.** $V = \frac{4}{3}\pi r^3 \Rightarrow \Delta V \approx 4\pi r^2\,\Delta r$
If $r = 20.00$ and $\Delta r = 0.20$, then
$\Delta V \approx 4\pi(20.00)^2(0.20) \approx 1005.$
The volume has increased by about 1005 cm$^2$.

## Section 4.8 Taylor Polynomials (page 264)

**1.** If $f(x) = e^{-x}$, then $f^{(k)}(x) = (-1)^k e^{-x}$, so $f^{(k)}(0) = (-1)^k$. Thus

$$P_4(x) = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!}.$$

**2.** If $f(x) = \cos x$, then $f'(x) = -\sin x$, $f''(x) = -\cos x$, and $f'''(x) = \sin x$. In particular, $f(\pi/4) = f'''(\pi/4) = 1/\sqrt{2}$ and $f'(\pi/4) = f''(\pi/4) = -1/\sqrt{2}$. Thus

$$P_3(x) = \frac{1}{\sqrt{2}}\left[1 - \left(x - \frac{\pi}{4}\right) - \frac{1}{2}\left(x - \frac{\pi}{4}\right)^2 + \frac{1}{6}\left(x - \frac{\pi}{4}\right)^3\right].$$

**3.**
$$\begin{aligned} f(x) &= \ln x & f(2) &= \ln 2 \\ f'(x) &= \frac{1}{x} & f'(2) &= \frac{1}{2} \\ f''(x) &= \frac{-1}{x^2} & f''(2) &= \frac{-1}{4} \\ f'''(x) &= \frac{2}{x^3} & f'''(2) &= \frac{2}{8} \\ f^{(4)}(x) &= \frac{-6}{x^4} & f^{(4)}(2) &= \frac{-6}{16} \end{aligned}$$

Thus

$$P_4(x) = \ln 2 + \frac{1}{2}(x-2) - \frac{1}{8}(x-2)^2 + \frac{1}{24}(x-2)^3 - \frac{1}{64}(x-2)^4.$$

**4.**
$$\begin{aligned} f(x) &= \sec x & f(0) &= 1 \\ f'(x) &= \sec x \tan x & f'(0) &= 0 \\ f''(x) &= 2\sec^3 x - \sec x & f''(0) &= 1 \\ f'''(x) &= (6\sec^2 x - 1)\sec x \tan x & f'''(0) &= 0 \end{aligned}$$

Thus $P_3(x) = 1 + (x^2/2)$.

**5.**
$$\begin{aligned} f(x) &= x^{1/2} & f(4) &= 2 \\ f'(x) &= \frac{1}{2}x^{-1/2} & f'(4) &= \frac{1}{4} \\ f''(x) &= \frac{-1}{4}x^{-3/2} & f''(4) &= \frac{-1}{32} \\ f'''(x) &= \frac{3}{8}x^{-5/2} & f'''(4) &= \frac{3}{256} \end{aligned}$$

Thus

$$P_3(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3.$$

**6.**
$$\begin{aligned} f(x) &= (1-x)^{-1} & f(0) &= 1 \\ f'(x) &= (1-x)^{-2} & f'(0) &= 1 \\ f''(x) &= 2(1-x)^{-3} & f''(0) &= 2 \\ f'''(x) &= 3!(1-x)^{-4} & f'''(0) &= 3! \\ &\vdots & &\vdots \\ f^{(n)}(x) &= n!(1-x)^{-(n+1)} & f^{(n)}(0) &= n! \end{aligned}$$

Thus

$$P_n(x) = 1 + x + x^2 + x^3 + \cdots + x^n.$$

**7.**
$$\begin{aligned} f(x) &= \frac{1}{2+x} & f(1) &= \frac{1}{3} \\ f'(x) &= \frac{-1}{(2+x)^2} & f'(1) &= \frac{-1}{9} \\ f''(x) &= \frac{2!}{(2+x)^3} & f''(1) &= \frac{2!}{27} \\ f'''(x) &= \frac{-3!}{(2+x)^4} & f'''(1) &= \frac{-3!}{3^4} \\ &\vdots & &\vdots \\ f^{(n)}(x) &= \frac{(-1)^n n!}{(2+x)^{n+1}} & f^{(n)}(1) &= \frac{(-1)^n n!}{3^{n+1}} \end{aligned}$$

Thus

$$P_n(x) = \frac{1}{3} - \frac{1}{9}(x-1) + \frac{1}{27}(x-1)^2 - \cdots + \frac{(-1)^n}{3^{n+1}}(x-1)^n.$$

**8.**
$$\begin{aligned} f(x) &= \sin(2x) & f(\pi/2) &= 0 \\ f'(x) &= 2\cos(2x) & f'(\pi/2) &= -2 \\ f''(x) &= -2^2\sin(2x) & f''(\pi/2) &= 0 \\ f'''(x) &= -2^3\cos(2x) & f'''(\pi/2) &= 2^3 \\ f^{(4)}(x) &= 2^4\sin(2x) = 2^4 f(x) & f^{(4)}(\pi/2) &= 0 \\ f^{(5)}(x) &= 2^4 f'(x) & f^{(5)}(\pi/2) &= -2^5 \\ &\vdots & &\vdots \end{aligned}$$

Evidently $f^{(2n)}(\pi/2) = 0$ and $f^{(2n-1)}(\pi/2) = (-1)^n 2^{2n-1}$. Thus

$$P_{2n-1}(x) = -2\left(x - \frac{\pi}{2}\right) + \frac{2^3}{3!}\left(x - \frac{\pi}{2}\right)^3 - \frac{2^5}{5!}\left(x - \frac{\pi}{2}\right)^5 + \cdots + (-1)^n \frac{2^{2\cdots}}{(2n\cdots}$$

**9.** $f(x) = x^{1/3}$, $f'(x) = \dfrac{1}{3}x^{-2/3}$,

$f''(x) = -\dfrac{2}{9}x^{-5/3}$, $f'''(x) = \dfrac{10}{27}x^{-8/3}$.

$$\begin{aligned} a = 8: \quad f(x) &\approx f(8) + f'(8)(x-8) + \frac{f''(8)}{2}(x-8)^2 \\ &= 2 + \frac{1}{12}(x-8) - \frac{1}{9\times 32}(x-8)^2 \end{aligned}$$

$9^{1/2} \approx 2 + \dfrac{1}{12} - \dfrac{1}{288} \approx 2.07986$

Error $= \dfrac{f'''(c)}{3!}(9-8)^3 = \dfrac{10}{27\times 6}\dfrac{1}{X^{8/3}}$ for some $c$ in $[8, 9]$.

For $8 \le c \le 9$ we have $c^{8/3} \ge 8^{8/3} = 2^8 = 256$ so

$$0 < \text{Error} \le \frac{5}{81 \times 256} < 0.000241.$$

Thus $2.07986 < 9^{1/3} < 2.08010$.

**10.** Since $f(x) = \sqrt{x}$, then $f'(x) = \frac{1}{2}x^{-1/2}$, $f''(x) = -\frac{1}{4}x^{-3/2}$ and $f'''(x) = \frac{3}{8}x^{-5/2}$. Hence,

$$\begin{aligned} \sqrt{61} &\approx f(64) + f'(64)(61-64) + \frac{1}{2}f''(64)(61-64)^2 \\ &= 8 + \frac{1}{16}(-3) - \frac{1}{2}\left(\frac{1}{2048}\right)(-3)^2 \approx 7.8103027. \end{aligned}$$

The error is $R_2 = R_2(f; 64, 61) = \dfrac{f'''(c)}{3!}(61-64)^3$ for some $c$ between 61 and 64. Clearly $R_2 < 0$. If $t \ge 49$, and in particular $61 \le t \le 64$, then

$$|f'''(t)| \le \tfrac{3}{8}(49)^{-5/2} = 0.0000223 = K.$$

Hence,

$$|R_2| \le \frac{K}{3!}|61 - 64|^3 = 0.0001004.$$

Since $R_2 < 0$, therefore,

$$\begin{aligned} 7.8103027 - 0.0001004 < \sqrt{61} &< 7.8103027 \\ 7.8102023 < \sqrt{61} &< 7.8103027. \end{aligned}$$

**11.** $f(x) = \dfrac{1}{x}$, $f'(x) = -\dfrac{1}{x^2}$,

$f''(x) = \dfrac{2}{x^3}$, $f'''(x) = \dfrac{-6}{x^4}$.

$a = 1: f(x) \approx 1 - (x-1) + \dfrac{2}{2}(x-1)^2$

$\dfrac{1}{1.02} \approx 1 - (0.02) + (0.02)^2 = 0.9804.$

Error $= \dfrac{f'''(c)}{3!}(0.02)^3 = -\dfrac{1}{X^4}(0.02)^3$ where $1 \le c \le 1.02$.

Therefore, $-(0.02)^3 \le \dfrac{1}{1.02} - 0.9804 < 0$,

i.e., $0.980392 \le \dfrac{1}{1.02} < 0.980400$.

**12.** Since $f(x) = \tan^{-1} x$, then

$$f'(x) = \frac{1}{1+x^2}, \quad f''(x) = \frac{-2x}{(1+x^2)^2}, \quad f'''(x) = \frac{-2+6x^2}{(1+x^2)^3}.$$

Hence,

$$\begin{aligned}\tan^{-1}(0.97) &\approx f(1) + f'(1)(0.97-1) + \tfrac{1}{2}f''(1)(0.97-1)^2 \\ &= \frac{\pi}{4} + \frac{1}{2}(-0.03) + \left(-\frac{1}{4}\right)(-0.03)^2 \\ &= 0.7701731.\end{aligned}$$

The error is $R_2 = \dfrac{f'''(c)}{3!}(-0.03)^3$ for some $c$ between 0.97 and 1. Note that $R_2 < 0$. If $0.97 \le t \le 1$, then

$$|f'''(t)| \le f'''(1) = \frac{-2+6}{(1.97)^3} < 0.5232 = K.$$

Hence,

$$|R_2| \le \frac{K}{3!}|0.97-1|^3 < 0.0000024.$$

Since $R_2 < 0$,

$$\begin{aligned}0.7701731 - 0.0000024 < \tan^{-1}(0.97) &< 0.7701731 \\ 0.7701707 < \tan^{-1}(0.97) &< 0.7701731.\end{aligned}$$

**13.** $f(x) = e^x, \quad f^{(k)}(x) = e^x$ for $k = 1, 2, 3 \ldots$

$a = 0: \qquad f(x) \approx 1 + x + \dfrac{x^2}{2}$

$e^{-0.5} \approx 1 - 0.5 + \dfrac{(0.5)^2}{2} = 0.625$

Error $= \dfrac{f'''(c)}{6}(0.5)^3 = \dfrac{e^c}{6}(-0.05)^3$ for some $c$ between $-0.5$ and 0. Thus

$$|\text{Error}| < \frac{(0.5)^3}{6} < 0.020834,$$

and $-0.020833 < e^{-0.5} - 0.625 < 0$, or $0.604 < e^{-0.5} < 0.625$.

**14.** Since $f(x) = \sin x$, then $f'(x) = \cos x$, $f''(x) = -\sin x$ and $f'''(x) = -\cos x$. Hence,

$$\begin{aligned}\sin(47^\circ) &= f\left(\frac{\pi}{4} + \frac{\pi}{90}\right) \\ &\approx f\left(\frac{\pi}{4}\right) + f'\left(\frac{\pi}{4}\right)\left(\frac{\pi}{90}\right) + \frac{1}{2}f''\left(\frac{\pi}{4}\right)\left(\frac{\pi}{90}\right)^2 \\ &= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\left(\frac{\pi}{90}\right) - \frac{1}{2\sqrt{2}}\left(\frac{\pi}{90}\right)^2 \\ &\approx 0.7313587.\end{aligned}$$

The error is $R_2 = \dfrac{f'''(c)}{3!}\left(\dfrac{\pi}{90}\right)^3$ for some $c$ between 45° and 47°. Observe that $R_2 < 0$. If $45^\circ \le t \le 47^\circ$, then

$$|f'''(t)| \le |-\cos 45^\circ| = \frac{1}{\sqrt{2}} = K.$$

Hence,

$$|R_2| \le \frac{K}{3!}\left(\frac{\pi}{90}\right)^3 < 0.0000051.$$

Since $R_2 < 0$, therefore

$$\begin{aligned}0.7313587 - 0.0000051 < \sin(47^\circ) &< 0.7313587 \\ 0.7313536 < \sin(47^\circ) &< 0.7313587.\end{aligned}$$

**15.**

$$\begin{aligned}f(x) &= \sin x \\ f'(x) &= \cos x \\ f''(x) &= -\sin x \\ f'''(x) &= -\cos x \\ f^{(4)}(x) &= \sin x\end{aligned}$$

$a = 0$; $n = 7$:

$$\begin{aligned}\sin x &= 0 + x - 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!} - 0 - \frac{x^7}{7!} + R_7, \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + R_7(x)\end{aligned}$$

where $R_7(x) = \dfrac{\sin c}{8!}x^8$ for some $c$ between 0 and $x$.

**16.** For $f(x) = \cos x$ we have

$$\begin{aligned}&f'(x) = -\sin x \qquad f''(x) = -\cos x \qquad f'''(x) = \sin x \\ &f^{(4)}(x) = \cos x \qquad f^{(5)}(x) = -\sin x \qquad f^{(6)}(x) = -\cos x.\end{aligned}$$

The Taylor's Formula for $f$ with $a = 0$ and $n = 6$ is

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + R_6(f; 0, x)$$

where the Lagrange remainder $R_6$ is given by

$$R_6 = R_6(f; 0, x) = \frac{f^{(7)}(c)}{7!}x^7 = \frac{\sin c}{7!}x^7,$$

for some $c$ between 0 and $x$.

**17.** $f(x) = \sin x \qquad a = \dfrac{\pi}{4}, \qquad n = 4$

$$\sin x = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\left(x - \frac{\pi}{4}\right) - \frac{1}{\sqrt{2}}\frac{1}{2!}\left(x - \frac{\pi}{4}\right)^2$$
$$-\frac{1}{\sqrt{2}}\frac{1}{3!}\left(x - \frac{\pi}{4}\right)^3 + \frac{1}{\sqrt{2}}\frac{1}{4!}\left(x - \frac{\pi}{4}\right)^4 + R_4(x)$$

where $R_4(x) = \dfrac{1}{5!}(\cos c)\left(x - \dfrac{\pi}{4}\right)^5$

for some $c$ between $\dfrac{\pi}{4}$ and $x$.

**18.** Given that $f(x) = \dfrac{1}{1-x}$, then

$$f'(x) = \frac{1}{(1-x)^2}, \quad f''(x) = \frac{2}{(1-x)^3}.$$

In general,

$$f^{(n)}(x) = \frac{n!}{(1-x)^{(n+1)}}.$$

Since $a = 0$, $f^{(n)}(0) = n!$. Hence, for $n = 6$, the Taylor's Formula is

$$\begin{aligned}\frac{1}{1-x} &= f(0) + \sum_{n=1}^{6} \frac{f^{(n)}(0)}{n!}x^n + R_6(f;0,x)\\ &= 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + R_6(f;0,x).\end{aligned}$$

The Langrange remainder is

$$R_6(f;0,x) = \frac{f^{(7)}(c)}{7!}x^7 = \frac{x^7}{(1-c)^8}$$

for some $c$ between 0 and $x$.

**19.**
$$\begin{aligned}f(x) &= \ln x\\ f'(x) &= \frac{1}{x}\\ f''(x) &= -\frac{-1}{x^2}\\ f'''(x) &= \frac{2!}{x^3}\\ f^{(4)}(x) &= \frac{-3!}{x^4}\\ f^{(5)}(x) &= \frac{4!}{x^5}\\ f^{(6)}(x) &= \frac{-5!}{x^6}\\ f^{(7)} &= \frac{6!}{x^7}\end{aligned}$$
$a = 1, \quad n = 6$
$$\begin{aligned}\ln x &= 0 + 1(x-1) - \frac{1}{2!}(x-1)^2 + \frac{2!}{3!}(x-1)^3\\ &\quad - \frac{3!}{4!}(x-1)^4 + \frac{4!}{5!}(x-1)^5 - \frac{5!}{6!}(x-1)^6 + R_6(x)\\ &= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4}\\ &\quad + \frac{(x-1)^5}{5} - \frac{(x-1)^6}{6} + R_6(x)\end{aligned}$$
where $R_6(x) = \dfrac{1}{7c^7}(x-1)^7$ for some $c$ between 1 and $x$.

**20.** Given that $f(x) = \tan x$, then

$$\begin{aligned}f'(x) &= \sec^2 x\\ f''(x) &= 2\sec^2 x \tan x\\ f^{(3)}(x) &= 6\sec^4 x - 4\sec^2 x\\ f^{(4)}(x) &= 8\tan x(3\sec^4 x - \sec^2 x).\end{aligned}$$

Given that $a = 0$ and $n = 3$, the Taylor's Formula is

$$\begin{aligned}\tan x &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + R_3(f;0,x)\\ &= x + \frac{2}{3!}x^3 + R_3(f;0,x)\\ &= x + \frac{1}{3}x^3 + \frac{2}{15}x^5.\end{aligned}$$

The Lagrange remainder is

$$R_3(f;0,x) = \frac{f^{(4)}(c)}{4!}x^4 = \frac{\tan c(3\sec^4 X - \sec^2 C)}{3}x^4$$

for some $c$ between 0 and $x$.

**21.** $e^{3x} = e^{3(x+1)}\, e^{-3}$
$$P_3(x) = e^{-3}\left[1 + 3(x+1) + \frac{9}{2}(x+1)^2 + \frac{9}{2}(x+1)^3\right].$$

**22.** For $e^u$, $P_4(u) = 1 + u + \dfrac{u^2}{2!} + \dfrac{u^3}{3!} + \dfrac{u^4}{4!}$. Let $u = -x^2$. Then for $e^{-x^2}$:

$$P_8(x) = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!}.$$

**23.** For $\sin^2 x = \dfrac{1}{2}\Big(1 - \cos(2x)\Big)$ at $x = 0$, we have

$$P_4(x) = \frac{1}{2}\left[1 - \left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!}\right)\right] = x^2 - \frac{x^4}{3}.$$

**24.** $\sin x = \sin\Big(\pi + (x-\pi)\Big) = -\sin(x-\pi)$
$$P_5(x) = -(x-\pi) + \frac{(x-\pi)^3}{3!} - \frac{(x-\pi)^5}{5!}$$

**25.** For $\dfrac{1}{1-u}$ at $u = 0$, $P_3(u) = 1 + u + u^2 + u^3$. Let $u = -2x^2$. Then for $\dfrac{1}{1+2x^2}$ at $x = 0$,

$$P_6(x) = 1 - 2x^2 + 4x^4 - 8x^6.$$

**26.** $\cos(3x - \pi) = -\cos(3x)$
$$P_8(x) = -1 + \frac{3^2x^2}{2!} - \frac{3^4x^4}{4!} + \frac{3^6x^6}{6!} - \frac{3^8x^8}{8!}.$$

**27.** Since $x^3 = 0 + 0x + 0x^2 + x^3 + 0x^4 + \cdots$ we have $P_n(x) = 0$ if $0 \le n \le 2$; $P_n(x) = x^3$ if $n \ge 3$

**28.** Let $t = x - 1$ so that

$$\begin{aligned}x^3 &= (1+t)^3 = 1 + 3t + 3t^2 + t^3\\ &= 1_3(x-1) + 3(x-1)^2 + (x-1)^3.\end{aligned}$$

Thus the Taylor polynomials for $x^3$ at $x = 1$ are

$$P_0(x) = 1$$
$$P_1(x) = 1 + 3(x-1)$$
$$P_1(x) = 1 + 3(x-1) + 3(x-1)^2$$
$$P_n(x) = 1 + 3(x-1) + 3(x-1)^2 + (x-1)^3 \quad \text{if } n \ge 3.$$

**29.**
$$\sinh x = \frac{1}{2}(e^x - e^{-x})$$
$$\begin{aligned} P_{2n+1}(x) =& \frac{1}{2}\left(1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{2n+1}}{(2n+1)!}\right) \\ & - \frac{1}{2}\left(1 - x + \frac{x^2}{2!} + \cdots - \frac{x^{2n+1}}{(2n+1)!}\right) \\ =& x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{x^{2n+1}}{(2n+1)!}. \end{aligned}$$

**30.** For $\ln(1+x)$ at $x = 0$ we have

$$P_{2n+1}(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{x^{2n+1}}{2n+1}.$$

For $\ln(1-x)$ at $x = 0$ we have

$$P_{2n+1}(x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots - \frac{x^{2n+1}}{2n+1}.$$

For $\tanh^{-1} x = \frac{1}{2}\ln(1+x) - \frac{1}{2}\ln(1-x)$,

$$P_{2n+1}(x) = x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots + \frac{x^{2n+1}}{2n+1}.$$

**31.** $f(x) = e^{-x}$
$$f^{(n)}(x) = \begin{cases} e^{-x} & \text{if } n \text{ is even} \\ -e^{-x} & \text{if } n \text{ is odd} \end{cases}$$
$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots + (-1)^n \frac{x^5}{n!} + R_n(x)$$
where $R_n(x) = (-1)^{n+1}\frac{X^{n+1}}{(n+1)!}$ for some $X$ between 0 and $x$.
For $x = 1$, we have
$$\frac{1}{e} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!} + R_n(1)$$
where $R_n(1) = (-1)^{n+1}\frac{e^{-X}x^{n+1}}{(n+1)!}$ for some $X$ between $-1$ and 0.
Therefore, $|R_n(1)| < \frac{1}{(n+1)!}$. We want $|R_n(1)| < 0.000005$ for 5 decimal places.
Choose $n$ so that $\frac{1}{(n+1)!} < 0.000005$. $n = 8$ will do since $1/9! \approx 0.0000027$.
Thus $$\frac{1}{e} \approx \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \frac{1}{8!}$$
$$\approx 0.36788 \qquad \text{(to 5 decimal places)}.$$

**32.** In Taylor's Formulas for $f(x) = \sin x$ with $a = 0$, only odd powers of $x$ have nonzero coefficients. Accordingly we can take terms up to order $x^{2n+1}$ but use the remainder after the next term $0x^{2n+2}$. The formula is

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + R_{2n+2},$$

where

$$R_{2n+2}(f; 0, x) = (-1)^{n+1}\frac{\cos c}{(2n+3)!}x^{2n+3}$$

for some $c$ between 0 and $x$.
In order to use the formula to approximate $\sin(1)$ correctly to 5 decimal places, we need $|R_{2n+2}(f; 0, 1)| < 0.000005$. Since $|\cos c| \le 1$, it is sufficient to have $1/(2n+3)! < 0.000005$. $n = 3$ will do since $1/9! \approx 0.000003$. Thus

$$\sin(1) \approx 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} \approx 0.84147$$

correct to five decimal places.

**33.** $f(x) = (x-1)^2$, $\quad f'(x) = 2(x-1)$, $\quad f''(x) = 2$.
$f(x) \approx 1 - 2x + \frac{2}{2}x^2 = 1 - 2x + x^2$
Error = 0
$g(x) = x^3 + 2x^2 + 3x + 4$
Quadratic approx.: $\quad g(x) \approx 4 + 3x + 2x^2$
Error $= x^3$
Since $g'''(c) = 6 = 3!$, error $= \frac{g'''(c)}{3!}x^3$
so that constant $\frac{1}{3!}$ in the error formula for the quadratic approximation cannot be improved.

**34.** $1 - x^{n+1} = (1-x)(1 + x + x^2 + x^3 + \cdots + x^n)$. Thus

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \frac{x^{n+1}}{1-x}.$$

If $|x| \le K < 1$, then $|1 - x| \ge 1 - K > 0$, so

$$\left|\frac{x^{n+1}}{1-x}\right| \le \frac{1}{1-K}|x^{n+1}| = O(x^{n+1})$$

as $x \to 0$. By Theorem 11, the $n$th-order Maclaurin polynomial for $1/(1-x)$ must be $P_n(x) = 1 + x + x^2 + x^3 + \cdots + x^n$.

**35.** Differentiating

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \frac{x^{n+1}}{1-x}$$

with respect to $x$ gives

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots + nx^{n-1} + \frac{n+1-nx}{(1-x)^2}\,x^n.$$

Then replacing $n$ with $n+1$ gives

$$\frac{1}{(1-x)^2} = 1{+}2x{+}3x^2{+}\cdots{+}(n{+}1)x^n{+}\frac{n+2-(n+1)x}{(1-x)^2}\,x^{n+1}.$$

If $|x| \le K < 1$, then $|1-x| \ge 1-K > 0$, and so

$$\left|\frac{n+2-(n+1)x}{(1-x)^2}\,x^{n+1}\right| \le \frac{n+2}{(1-K)^2}|x^{n+1}| = O(x^{n+1})$$

as $x \to 0$. By Theorem 11, the $n$th-order Maclaurin polynomial for $1/(1-x)^2$ must be $P_n(x) == 1 + 2x + 3x^2 + \cdots + (n+1)x^n$.

## Section 4.9 Indeterminate Forms (page 269)

**1.** $\displaystyle \lim_{x\to 0} \frac{3x}{\tan 4x} \qquad \left[\frac{0}{0}\right]$
$\displaystyle = \lim_{x\to 0} \frac{3}{4\sec^2 4x} = \frac{3}{4}$

**2.** $\displaystyle \lim_{x\to 2} \frac{\ln(2x-3)}{x^2-4} \qquad \left[\frac{0}{0}\right]$
$\displaystyle = \frac{\left(\frac{2}{2x-3}\right)}{2x} = \frac{1}{2}.$

**3.** $\displaystyle \lim_{x\to 0} \frac{\sin ax}{\sin bx} \qquad \left[\frac{0}{0}\right]$
$\displaystyle = \lim_{x\to 0} \frac{a\cos ax}{b\cos bx} = \frac{a}{b}$

**4.** $\displaystyle \lim_{x\to 0} \frac{1-\cos ax}{1-\cos bx} \qquad \left[\frac{0}{0}\right]$
$\displaystyle = \lim_{x\to 0} \frac{a\sin ax}{b\sin bx} \qquad \left[\frac{0}{0}\right]$
$\displaystyle = \lim_{x\to 0} \frac{a^2\cos ax}{b^2\cos bx} = \frac{a^2}{b^2}.$

**5.** $\displaystyle \lim_{x\to 0} \frac{\sin^{-1} x}{\tan^{-1} x} \qquad \left[\frac{0}{0}\right]$
$\displaystyle = \lim_{x\to 0} \frac{1+x^2}{\sqrt{1-x^2}} = 1$

**6.** $\displaystyle \lim_{x\to 1} \frac{x^{1/3}-1}{x^{2/3}-1} \qquad \left[\frac{0}{0}\right]$
$\displaystyle = \lim_{x\to 1} \frac{(\frac{1}{3})x^{-2/3}}{(\frac{2}{3})x^{-1/3}} = \frac{1}{2}.$

**7.** $\displaystyle \lim_{x\to 0} x\cot x \qquad [0\times\infty]$
$\displaystyle = \lim_{x\to 0}\left(\frac{x}{\sin x}\right)\cos x$
$\displaystyle = 1 \times \lim_{x\to 0} \frac{x}{\sin x} \qquad \left[\frac{0}{0}\right]$
$\displaystyle = \lim_{x\to 0} \frac{1}{\cos x} = 1$

**8.** $\displaystyle \lim_{x\to 0} \frac{1-\cos x}{\ln(1+x^2)} \qquad \left[\frac{0}{0}\right]$
$\displaystyle = \lim_{x\to 0} \frac{\sin x}{\left(\frac{2x}{1+x^2}\right)}$
$\displaystyle = \lim_{x\to 0}(1+x^2)\lim_{x\to 0}\frac{\sin x}{2x}$
$\displaystyle = \lim_{x\to 0}\frac{\cos x}{2} = \frac{1}{2}.$

**9.** $\displaystyle \lim_{t\to\pi} \frac{\sin^2 t}{t-\pi} \qquad \left[\frac{0}{0}\right]$
$\displaystyle = \lim_{t\to\pi} \frac{2\sin t\cos t}{1} = 0$

**10.** $\displaystyle \lim_{x\to 0} \frac{10^x - e^x}{x} \qquad \left[\frac{0}{0}\right]$
$\displaystyle = \lim_{x\to 0} \frac{10^x \ln 10 - e^x}{1} = \ln 10 - 1.$

**11.** $\displaystyle \lim_{x\to\pi/2} \frac{\cos 3x}{\pi - 2x} \qquad \left[\frac{0}{0}\right]$
$\displaystyle = \lim_{x\to\pi/2} \frac{-3\sin 3x}{-2} = \frac{3}{2}(-1) = -\frac{3}{2}$

**12.** $\displaystyle \lim_{x\to 1} \frac{\ln(ex)-1}{\sin\pi x} \qquad \left[\frac{0}{0}\right]$
$\displaystyle = \lim_{x\to 1} \frac{\frac{1}{x}}{\pi\cos(\pi x)} = -\frac{1}{\pi}.$

**13.** $\displaystyle \lim_{x\to\infty} x\sin\frac{1}{x} \qquad [\infty\times 0]$
$\displaystyle = \lim_{x\to\infty} \frac{\sin\frac{1}{x}}{\frac{1}{x}} \qquad \left[\frac{0}{0}\right]$
$\displaystyle = \lim_{x\to\infty} \frac{-\frac{1}{x^2}\cos\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to\infty}\cos\frac{1}{x} = 1.$

**14.** $$\lim_{x\to 0}\frac{x-\sin x}{x^3}\quad\left[\frac{0}{0}\right]$$
$$=\lim_{x\to 0}\frac{1-\cos x}{3x^2}\quad\left[\frac{0}{0}\right]$$
$$=\lim_{x\to 0}\frac{\sin x}{6x}\quad\left[\frac{0}{0}\right]$$
$$=\lim_{x\to 0}\frac{\cos x}{6}=\frac{1}{6}.$$

**15.** $$\lim_{x\to 0}\frac{x-\sin x}{x-\tan x}\quad\left[\frac{0}{0}\right]$$
$$=\lim_{x\to 0}\frac{1-\cos x}{1-\sec^2 x}\quad\left[\frac{0}{0}\right]$$
$$=\lim_{x\to 0}(\cos^2 x)\frac{1-\cos x}{\cos^2 x-1}$$
$$=-1\times\lim_{x\to 0}\frac{\cos x-1}{(\cos x-1)(\cos x+1)}$$
$$=-\frac{1}{2}$$

**16.** $$\lim_{x\to 0}\frac{2-x^2-2\cos x}{x^4}\quad\left[\frac{0}{0}\right]$$
$$=\lim_{x\to 0}\frac{-2x+2\sin x}{4x^3}\quad\left[\frac{0}{0}\right]$$
$$=-\frac{1}{2}\lim_{x\to 0}\frac{x-\sin x}{x^3}$$
$$=-\frac{1}{2}\left(\frac{1}{6}\right)=-\frac{1}{12}\quad\text{(by Exercise 14).}$$

**17.** $$\lim_{x\to 0+}\frac{\sin^2 x}{\tan x-x}\quad\left[\frac{0}{0}\right]$$
$$=\lim_{x\to 0+}\frac{2\sin x\cos x}{\sec^2 x-1}\quad\left[\frac{0}{0}\right]$$
$$=2\times 1\times\lim_{x\to 0+}\frac{\cos x}{2\sec^2 x\tan x}=\infty$$

**18.** $$\lim_{r\to\pi/2}\frac{\ln\sin r}{\cos r}\quad\left[\frac{0}{0}\right]$$
$$=\lim_{r\to\pi/2}\frac{\left(\dfrac{\cos r}{\sin r}\right)}{-\sin r}=0.$$

**19.** $$\lim_{t\to\pi/2}\frac{\sin t}{t}=\frac{2}{\pi}$$

**20.** $$\lim_{x\to 1-}\frac{\cos^{-1} x}{x-1}\quad\left[\frac{0}{0}\right]$$
$$=\lim_{x\to 1-}\frac{-\left(\dfrac{1}{\sqrt{1-x^2}}\right)}{1}=-\infty.$$

**21.** $$\lim_{x\to\infty}x(2\tan^{-1}x-\pi)\qquad[0\times\infty]$$
$$=\lim_{x\to\infty}\frac{2\tan^{-1}x-\pi}{\dfrac{1}{x}}\quad\left[\frac{0}{0}\right]$$
$$=\lim_{x\to\infty}\frac{2}{1+x^2}\Big/-\frac{1}{x^2}$$
$$=\lim_{x\to\infty}-\frac{2x^2}{1+x^2}=-2$$

**22.** $$\lim_{t\to(\pi/2)-}(\sec t-\tan t)\quad[\infty-\infty]$$
$$=\lim_{t\to(\pi/2)-}\frac{1-\sin t}{\cos t}\quad\left[\frac{0}{0}\right]$$
$$=\lim_{t\to(\pi/2)-}\frac{-\cos t}{-\sin t}=0.$$

**23.** $$\lim_{t\to 0}\left(\frac{1}{t}-\frac{1}{te^{at}}\right)\qquad(\infty-\infty)$$
$$=\lim_{t\to 0}\frac{e^{at}-1}{te^{at}}\quad\left[\frac{0}{0}\right]$$
$$=\lim_{t\to 0}\frac{ae^{at}}{e^{at}+ate^{at}}=a$$

**24.** Since $$\lim_{x\to 0+}\sqrt{x}\ln x=\lim_{x\to 0+}\frac{\ln x}{x^{-1/2}}\quad\left[\frac{0}{0}\right]$$
$$=\lim_{x\to 0+}\frac{\left(\dfrac{1}{x}\right)}{\left(-\dfrac{1}{2}\right)x^{-3/2}}=0,$$
hence $\lim_{x\to 0+}x^{\sqrt{x}}$
$$=\lim_{x\to 0+}e^{\sqrt{x}\ln x}=e^0=1.$$

**25.** Let $y=(\csc x)^{\sin^2 x}$.
Then $\ln y=\sin^2 x\ln(\csc x)$
$$\lim_{x\to 0+}\ln y=\lim_{x\to 0+}\frac{\ln(\csc x)}{\csc^2 x}\quad\left[\frac{\infty}{\infty}\right]$$
$$=\lim_{x\to 0+}\frac{-\dfrac{\csc x\cot x}{\csc x}}{-2\csc^2 x\cot x}$$
$$=\lim_{x\to 0+}\frac{1}{2\csc^2 x}=0.$$
Thus $\lim_{x\to 0+}(\csc x)^{\sin^2 x}=e^0=1.$

**26.** $$\lim_{x\to 1+}\left(\frac{x}{x-1}-\frac{1}{\ln x}\right) \quad [\infty-\infty]$$
$$= \lim_{x\to 1+}\frac{x\ln x - x + 1}{(x-1)(\ln x)} \quad \left[\frac{0}{0}\right]$$
$$= \lim_{x\to 1+}\frac{\ln x}{\ln x + 1 - \frac{1}{x}} \quad \left[\frac{0}{0}\right]$$
$$= \lim_{x\to 1+}\frac{\frac{1}{x}}{\frac{1}{x}+\frac{1}{x^2}}$$
$$= \lim_{x\to 1+} = \frac{x}{x+1} = \frac{1}{2}.$$

**27.** $$\lim_{t\to 0}\frac{3\sin t - \sin 3t}{3\tan t - \tan 3t} \quad \left[\frac{0}{0}\right]$$
$$= \lim_{t\to 0}\frac{3(\cos t - \cos 3t)}{3(\sec^2 t - \sec^2 3t)} \quad \left[\frac{0}{0}\right]$$
$$= \lim_{t\to 0}\frac{\cos t - \cos 3t}{\frac{\cos^2 3t - \cos^2 t}{\cos^2 t\cos^2 3t}}$$
$$= -\lim_{t\to 0}\frac{\cos 3t - \cos t}{\cos^2 3t - \cos^2 t}$$
$$= -\lim_{t\to 0}\frac{1}{\cos 3t + \cos t} = -\frac{1}{2}$$

**28.** Let $y = \left(\frac{\sin x}{x}\right)^{1/x^2}$.
$$\lim_{x\to 0}\ln y = \lim_{x\to 0}\frac{\ln\left(\frac{\sin x}{x}\right)}{x^2} \quad \left[\frac{0}{0}\right]$$
$$= \lim_{x\to 0}\frac{\left(\frac{x}{\sin x}\right)\left(\frac{x\cos x - \sin x}{x^2}\right)}{2x}$$
$$= \lim_{x\to 0}\frac{x\cos x - \sin x}{2x^2\sin x} \quad \left[\frac{0}{0}\right]$$
$$= \lim_{x\to 0}\frac{-x\sin x}{4x\sin x + 2x^2\cos x}$$
$$= \lim_{x\to 0}\frac{-\sin x}{4\sin x + 2x\cos x} \quad \left[\frac{0}{0}\right]$$
$$= \lim_{x\to 0}\frac{-\cos x}{6\cos x - 2x\sin x} = -\frac{1}{6}.$$
Thus, $\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{1/x^2} = e^{-1/6}$.

**29.** Let $y = (\cos 2t)^{1/t^2}$.
Then $\ln y = \frac{\ln(\cos 2t)}{t^2}$. We have
$$\lim_{t\to 0}\ln y = \lim_{t\to 0}\frac{\ln(\cos 2t)}{t^2} \quad \left[\frac{0}{0}\right]$$
$$= \lim_{t\to 0}\frac{-2\tan 2t}{2t} \quad \left[\frac{0}{0}\right]$$
$$= -\lim_{t\to 0}\frac{2\sec^2 2t}{1} = -2.$$
Therefore $\lim_{t\to 0}(\cos 2t)^{1/t^2} = e^{-2}$.

**30.** $$\lim_{x\to 0+}\frac{\csc x}{\ln x} \quad \left[-\frac{\infty}{\infty}\right]$$
$$= \lim_{x\to 0+}\frac{-\csc x\cot x}{\frac{1}{x}} \quad \left[-\frac{\infty}{\infty}\right]$$
$$= \lim_{x\to 0+}\frac{-x\cos x}{\sin^2 x} \quad \left[\frac{0}{0}\right]$$
$$= -\left(\lim_{x\to 0+}\cos x\right)\lim_{x\to 0+}\frac{1}{2\sin x\cos x}$$
$$= -\infty.$$

**31.** $$\lim_{x\to 1-}\frac{\ln\sin\pi x}{\csc\pi x} \quad \left[\frac{\infty}{\infty}\right]$$
$$= \lim_{x\to 1-}\frac{\frac{\pi\cos\pi x}{\sin\pi x}}{-\pi\csc\pi x\cot\pi x}$$
$$= \frac{-\pi}{\pi}\lim_{x\to 1-}\tan\pi x = 0$$

**32.** Let $y = (1+\tan x)^{1/x}$.
$$\lim_{x\to 0}\ln y = \lim_{x\to 0}\frac{\ln(1+\tan x)}{x} \quad \left[\frac{0}{0}\right]$$
$$= \lim_{x\to 0}\frac{\sec^2 x}{1+\tan x} = 1.$$
Thus, $\lim_{x\to 0}(1+\tan x)^{1/x} = e$.

**33.** $$\lim_{h\to 0}\frac{f(x+h) - 2f(x) + f(x-h)}{h^2} \quad \left[\frac{0}{0}\right]$$
$$= \lim_{h\to 0}\frac{f'(x+h) - f'(x-h)}{2h} \quad \left[\frac{0}{0}\right]$$
$$= \lim_{h\to 0}\frac{f''(x+h) + f''(x-h)}{2}$$
$$= \frac{2f(x)}{2} = f''(x)$$

**34.** $$\lim_{h\to 0}\frac{f(x+3h) - 3f(x+h) + 3f(x-h) - f(x-3h)}{h^3}$$
$$= \lim_{h\to 0}\frac{3f'(x+3h) - 3f'(x+h) - 3f'(x-h) + 3f'(x-3h)}{3h^2}$$
$$= \lim_{h\to 0}\frac{3f''(x+3h) - f''(x+h) + f''(x-h) - 3f''(x-3h)}{2h}$$
$$= \lim_{h\to 0}\frac{9f'''(x+3h) - f'''(x+h) - f'''(x-h) + 9f'''(x-3h)}{2}$$
$$=8f'''(x).$$

**35.** Suppose that $f$ and $g$ are continuous on $[a,b]$ and differentiable on $(a,b)$ and $g(x) \neq 0$ there. Let $a < x < t < b$, and apply the Generalized Mean-Value Theorem; there exists $c$ in $(x,t)$ such that

$$\begin{aligned}
&\frac{f(x)-f(t)}{g(x)-g(t)} = \frac{f'(c)}{g'(c)} \\
\Rightarrow\quad &\left[\frac{f(x)-f(t)}{g(x)}\right]\left[\frac{g(x)}{g(x)-g(t)}\right] = \frac{f'(c)}{g'(c)} \\
\Rightarrow\quad &\frac{f(x)}{g(x)} - \frac{f(t)}{g(x)} = \frac{f'(c)}{g'(c)}\left[\frac{g(x)-g(t)}{g(x)}\right] \\
\Rightarrow\quad &\frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)} - \frac{g(t)}{g(x)}\frac{f'(c)}{g'(c)} + \frac{f(t)}{g(x)} \\
\Rightarrow\quad &\frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)} + \frac{1}{g(x)}\left[f(t) - g(t)\frac{f'(c)}{g'(c)}\right] \\
\Rightarrow\quad &\frac{f(x)}{g(x)} - L = \frac{f'(c)}{g'(c)} - L + \frac{1}{g(x)}\left[f(t) - g(t)\frac{f'(c)}{g'(c)}\right].
\end{aligned}$$

Since $|m+n| \le |m| + |n|$, therefore,

$$\left|\frac{f(x)}{g(x)} - L\right| \le \left|\frac{f'(c)}{g'(c)} - L\right| + \frac{1}{|g(x)|}\left[|f(t)| + |g(t)|\left|\frac{f'(c)}{g'(c)}\right|\right].$$

Now suppose that $\epsilon$ is an arbitrary small positive number. Since $\lim_{c\to a+} f'(c)/g'(c) = L$, and since $a < x < c < t$, we can choose $t$ sufficiently close to $a$ to ensure that

$$\left|\frac{f'(c)}{g'(c)} - L\right| < \frac{\epsilon}{2}.$$

In particular,

$$\left|\frac{f'(c)}{g'(c)}\right| < |L| + \frac{\epsilon}{2}.$$

Since $\lim_{x\to a+} |g(x)| = \infty$, we can choose $x$ between $a$ and $t$ sufficiently close to $a$ to ensure that

$$\frac{1}{|g(x)|}\left[|f(t)| + |g(t)|\left(|L| + \frac{\epsilon}{2}\right)\right] < \frac{\epsilon}{2}.$$

It follows that

$$\left|\frac{f(x)}{g(x)} - L\right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus $\lim_{x\to a+} \dfrac{f(x)}{g(x)} = L$.

## Review Exercises 4 (page 270)

**1.** Since $dr/dt = 2r/100$ and $V = (4/3)\pi r^3$, we have

$$\frac{dV}{dt} = \frac{4\pi}{3} 3r^2 \frac{dr}{dt} = 3V \frac{2}{100} = \frac{6V}{100}.$$

Hence The volume is increasing at 6%/min.

**2.** a) Since $F$ must be continuous at $r = R$, we have

$$\frac{mgR^2}{R^2} = mkR, \quad \text{or} \quad k = \frac{g}{R}.$$

b) The rate of change of $F$ as $r$ decreases from $R$ is

$$\left(-\frac{d}{dr}(mkr)\right)\bigg|_{r=R} = -mk = -\frac{mg}{R}.$$

The rate of change of $F$ as $r$ increases from $R$ is

$$\left(-\frac{d}{dr}\frac{mgR^2}{r^2}\right)\bigg|_{r=R} = -\frac{2mgR^2}{R^3} = -2\frac{mg}{R}.$$

Thus $F$ decreases as $r$ increases from $R$ at twice the rate at which it decreases as $r$ decreases from $R$.

**3.** $1/R = 1/R_1 + 1/R_2$. If $R_1 = 250$ ohms and $R_2 = 1,000$ ohms, then $1/R = (1/250) + (1/1,000) = 1/200$, so $R = 200$ ohms. If $dR_1/dt = 100$ ohms/min, then

$$\begin{aligned}
-\frac{1}{R^2}\frac{dR}{dt} &= -\frac{1}{R_1^2}\frac{dR_1}{dt} - \frac{1}{R_2^2}\frac{dR_2}{dt} \\
\frac{1}{200^2}\frac{dR}{dt} &= \frac{1}{250^2}(100) + \frac{1}{1,000^2}\frac{dR_2}{dt}.
\end{aligned}$$

a) If $R$ remains constant, then $dR/dt = 0$, so

$$\frac{dR_2}{dt} = -\frac{1,000^2 \times 100}{250^2} = -1,600.$$

$R_2$ is decreasing at 1,600 ohms/min.

b) If $R$ is increasing at 10 ohms/min, then then $dR/dt = 10$, and

$$\frac{dR_2}{dt} = 1,000^2\left(\frac{10}{200^2} - \frac{100}{250^2}\right) = -1,350.$$

$R_2$ is decreasing at 1,350 ohms/min.

**4.** If $pV = 5.0T$, then

$$\frac{dp}{dt}V + p\frac{dV}{dt} = 5.0\frac{dT}{dt}.$$

a) If $T = 400$ K, $dT/dt = 4$ K/min, and $V = 2.0$ m$^3$, then $dV/dt = 0$, so $dp/dt = 5.0(4)/2.0 = 10$. The pressure is increasing at 10 kPa/min.

b) If $T = 400$ K, $dT/dt = 0$, $V = 2$ m$^3$, and $dV/dt = 0.05$ m$^3$/min, then $p = 5.0(400)/2 = 1,000$ kPa, and $2\,dp/dt + 1,000(0.05) = 0$, so $dp/dt = -25$. The pressure is decreasing at 25 kPa/min.

**5.** If $x$ copies of the book are printed, the cost of printing each book is

$$C = \frac{10,000}{x} + 8 + 6.25 \times 10^{-7}x^2.$$

Since $C \to \infty$ as $x \to 0+$ or $x \to \infty$, $C$ will be minimum at a critical point. For CP:

$$0 = \frac{dC}{dx} = -\frac{10,000}{x^2} + 12.5 \times 10^{-7}x,$$

so $x^3 = 8 \times 10^9$ and $x = 2 \times 10^3$. 2,000 books should be printed.

**6.** If she charges $\$x$ per bicycle, her total profit is $\$P$, where

$$P = (x - 75)N(x) = 4.5 \times 10^6 \frac{x - 75}{x^2}.$$

Evidently $P \le 0$ if $x \le 75$, and $P \to 0$ as $x \to \infty$. $P$ will therefore have a maximum value at a critical point in $(75, \infty)$. For CP:

$$0 = \frac{dP}{dx} = 4.5 \times 10^6 \frac{x^2 - (x - 75)2x}{x^4},$$

from which we obtain $x = 150$. She should charge \$150 per bicycle and order $N(150) = 200$ of them from the manufacturer.

**7.**

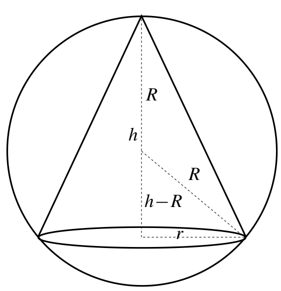



Fig. R-4.7

Let $r$, $h$ and $V$ denote the radius, height, and volume of the cone respectively. The volume of a cone is one-third the base area times the height, so

$$V = \frac{1}{3}\pi r^2 h.$$

From the small right-angled triangle in the figure,

$$(h - R)^2 + r^2 = R^2.$$

Thus $r^2 = R^2 - (h - R)^2$ and

$$V = V(h) = \frac{\pi}{3} h\Big(R^2 - (h - R)^2\Big) = \frac{\pi}{3}\Big(2Rh^2 - h^3\Big).$$

The height of any inscribed cone cannot exceed the diameter of the sphere, so $0 \le h \le 2R$. Being continuous, $V(h)$ must have a maximum value on this interval. Since $V = 0$ when $h = 0$ or $h = 2R$, and $V > 0$ if $0 < h < 2R$, the maximum value of $V$ must occur at a critical point. ($V$ has no singular points.) For a critical point,

$$0 = V'(h) = \frac{\pi}{3}(4Rh - 3h^2) = \frac{\pi}{3}h(4R - 3h),$$
$$h = 0 \quad \text{or} \quad h = \frac{4R}{3}.$$

$V'(h) > 0$ if $0 < h < 4R/3$ and $V'(h) < 0$ if $4R/3 < h < 2R$. Hence $h = 4R/3$ does indeed give the maximum value for $V$. The volume of the largest cone can be inscribed in a sphere of radius $R$ is

$$V\left(\frac{4R}{3}\right) = \frac{\pi}{3}\left(2R\left(\frac{4R}{3}\right)^2 - \left(\frac{4R}{3}\right)^3\right)$$
$$= \frac{32}{81}\pi R^3 \text{ cubic units.}$$

**8.**

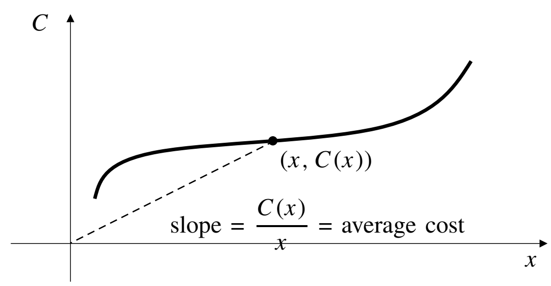



Fig. R-4.8

a) For minimum $C(x)/x$, we need

$$0 = \frac{d}{dx}\frac{C(x)}{x} = \frac{xC'(x) - C(x)}{x^2},$$

so $C'(x) = C(x)/x$; the marginal cost equals the average cost.

b) The line from $(0, 0)$ to $(x, C(x))$ has smallest slope at a value of $x$ which makes it tangent to the graph of $C(x)$. Thus $C'(x) = C(x)/x$, the slope of the line.

c) The line from $(0, 0)$ to $(x, C(x))$ can be tangent to the graph of $C(x)$ at more than one point. Not all such points will provide a minimum value for the average cost. (In the figure, one such line will make the average cost maximum.)

**9.**

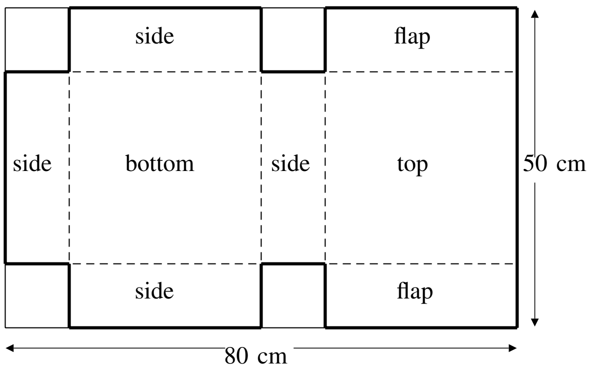



Fig. R-4.9

If the edge of the cutout squares is $x$ cm, then the volume of the folded box is

$$\begin{aligned} V(x) &= x(50-2x)(40-x) \\ &= 2x^3 - 130x^2 + 2,000x, \end{aligned}$$

and is valid for $0 \le x \le 25$. Since $V(0) = V(25) = 0$, and $V(x) > 0$ if $0 < x < 25$, the maximum will occur at a CP:

$$\begin{aligned} 0 = V'(x) &= 6x^2 - 260x + 2,000 \\ &= 2(3x^2 - 130x + 1,000) \\ &= 2(3x-100)(x-10). \end{aligned}$$

Thus $x = 10$ or $x = 100/3$. The latter CP is not in the interval $[0, 25]$, so the maximum occurs at $x = 10$. The maximum volume of the box is $V(10) = 9,000$ cm$^3$.

**10.** If $x$ more trees are planted, the yield of apples will be

$$\begin{aligned} Y &= (60+x)(800-10x) \\ &= 10(60+x)(80-x) \\ &= 10(4,800 + 20x - x^2). \end{aligned}$$

This is a quadratic expression with graph opening downward; its maximum occurs at a CP:

$$0 = \frac{dY}{dx} = 10(20-2x) = 20(10-x).$$

Thus 10 more trees should be planted to maximize the yield.

**11.**

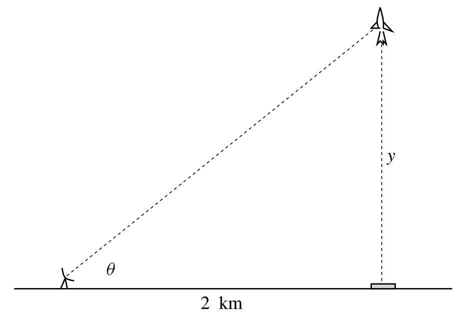



Fig. R-4.11

It was shown in the solution to Exercise 41 in Section 3.2 that at time $t$ s after launch, the tracking antenna rotates upward at rate

$$\frac{d\theta}{dt} = \frac{800t}{400^2 + t^4} = f(t), \text{ say.}$$

Observe that $f(0) = 0$ and $f(t) \to 0$ as $t \to \infty$. For critical points,

$$\begin{aligned} 0 = f'(t) &= 800\left[\frac{(400^2 + t^4) - 4t^4}{(400^2 + t^4)^2}\right] \\ &\Rightarrow \quad 3t^4 = 400^2, \qquad \text{or} \quad t \approx 15.197. \end{aligned}$$

The maximum rate at which the antenna must turn is $f(15.197) \approx 0.057$ rad/s.

**12.** The narrowest hallway in which the table can be turned horizontally through 180° has width equal to twice the greatest distance from the origin (the centre of the table) to the curve $x^2 + y^4 = 1/8$ (the edge of the table). We maximize the square of this distance, which we express as a function of $y$:

$$S(y) = x^2 + y^2 = y^2 + \frac{1}{8} - y^4, \quad (0 \le y \le (1/8)^{1/4}).$$

Note that $S(0) = 1/8$ and $S((1/8)^{1/4}) = 1/\sqrt{8} > S(0)$. For CP:

$$0 = \frac{dS}{dy} = 2y - 4y^3 = 2y(1-2y^2).$$

The CPs are given by $y = 0$ (already considered), and $y^2 = 1/2$, where $S(y) = 3/8$. Since $3/8 > 1/\sqrt{8}$, this is the maximum value of $S$. The hallway must therefore be at least $2\sqrt{3/8} \approx 1.225$ m wide.

**13.** Let the ball have radius $r$ cm. Its weight is proportional to the volume of metal it contains, so the condition of the problem states that

$$\frac{4\pi}{3}r^3 - \frac{4\pi}{3}(r-2)^3 = \frac{1}{2}\frac{4\pi}{3}r^3$$
$$r^3 - 12r^2 + 24r - 16 = 0.$$

Graphing the left side of this latter equation with a graphics calculator shows a root between 9 and 10. A "solve routine" or Newton's Method then refines an initial guess of, say, $r = 9.5$ to give $r = 9.69464420373$ cm for the radius of the ball.

**14.**

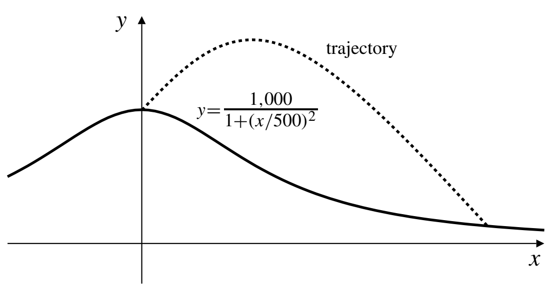



Fig. R-4.14

If the origin is at sea level under the launch point, and $x(t)$ and $y(t)$ are the horizontal and vertical coordinates of the cannon ball's position at time $t$ s after it is fired, then

$$\frac{d^2x}{dt^2} = 0, \qquad \frac{d^2y}{dt^2} = -32.$$

At $t = 0$, we have $dx/dt = dy/dt = 200/\sqrt{2}$, so

$$\frac{dx}{dt} = \frac{200}{\sqrt{2}}, \quad \frac{dy}{dt} = -32t + \frac{200}{\sqrt{2}}.$$

At $t = 0$, we have $x = 0$ and $y = 1,000$. Thus the position of the ball at time $t$ is given by

$$x = \frac{200t}{\sqrt{2}}, \quad y = -16t^2 + \frac{200t}{\sqrt{2}} + 1,000.$$

We can obtain the Cartesian equation for the path of the cannon ball by solving the first equation for $t$ and substituting into the second equation:

$$y = -16\,\frac{2x^2}{200^2} + x + 1,000.$$

The cannon ball strikes the ground when

$$-16\,\frac{2x^2}{200^2} + x + 1,000 = \frac{1,000}{1+(x/500)^2}.$$

Graphing both sides of this equation suggests a solution near $x = 1,900$. Newton's Method or a solve routine then gives $x \approx 1,873$. The horizontal range is about 1,873 ft.

**15.** The percentage error in the approximation $-(g/L)\sin\theta \approx -(g.L)\theta$ is

$$100\left|\frac{\sin\theta - \theta}{\sin\theta}\right| = 100\left(\frac{\theta}{\sin\theta} - 1\right).$$

Since $\lim_{\theta\to 0}\theta/(\sin\theta) = 1$, the percentage error $\to 0$ as $\theta \to 0$. Also, $\theta/\sin\theta$ grows steadily larger as $|\theta|$ increases from 0 towards $\pi/2$. Thus the maximum percentage error for $|\theta| \le 20° = \pi/9$ will occur at $\theta = \pi/9$. This maximum percentage error is

$$100\left(\frac{\pi/9}{\sin(\pi/9)} - 1\right) \approx 2.06\%.$$

**16.**
$$\begin{aligned}
\sin^2 x &= \frac{1}{2}\Big(1 - \cos(2x)\Big)\\
&= \frac{1}{2}\left[1 - \left(1 - \frac{2^2x^2}{2!} + \frac{2^4x^4}{4!} - \frac{2^6x^6}{6!} + O(x^8)\right)\right]\\
&= x^2 - \frac{x^4}{3} + \frac{2x^6}{45} + O(x^8)
\end{aligned}$$
$$\begin{aligned}
\lim_{x\to 0}&\frac{3\sin^2 x - 3x^2 + x^4}{x^6}\\
&= \lim_{x\to 0}\frac{3x^2 - x^4 + \frac{2}{15}x^6 + O(x^8) - 3x^2 - x^4}{x^6}\\
&= \lim_{x\to 0}\frac{2}{15} + O(x^2) = \frac{2}{15}.
\end{aligned}$$

**17.** $f(x) = \tan^{-1}x$, $f'(x) = \dfrac{1}{1+x^2}$, $f''(x) = \dfrac{-2x}{(1+x^2)^2}$, $f'''(x) = \dfrac{6x^2-2}{(1+x^2)^3}$.

About $x = 1$, $P_2(x) = \dfrac{\pi}{4} + \dfrac{x-1}{2} - \dfrac{(x-1)^2}{4}$.

Thus $\tan^{-1}(1.1) \approx \dfrac{\pi}{4} + \dfrac{1}{20} - \dfrac{1}{400} \approx 0.832898$. On [1, 1.1], we have

$$|f'''(x)| \le \frac{6(1.1)^2 - 2}{(1+1)^3} = 0.6575.$$

Thus the error does not exceed $\dfrac{0.6575}{3!}(1.1-1)^3 \approx .00011$ in absolute value.

**18.** The second approximation $x_1$ is the $x$-intercept of the tangent to $y = f(x)$ at $x = x_0 = 2$; it is the $x$-intercept of the line $2y = 10x - 19$. Thus $x_1 = 19/10 = 1.9$.

**19.**

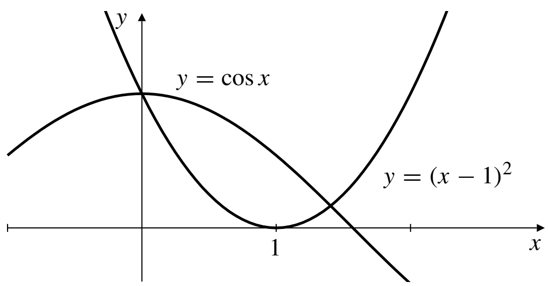



Fig. R-4.19

$y = \cos x$ and $y = (x-1)^2$ intersect at $x = 0$ and at a point $x$ between $x = 1$ and $x = \pi/2 \approx 1.57$. Starting with an initial guess $x_0 = 1.3$, and iterating the Newton's Method formula

$$x_{n+1} = x_n - \frac{(x_n - 1)^2 - \cos x_n}{2(x_n - 1) + \sin x_n},$$

we get $x_4 = x_5 = 1.40556363276$. To 10 decimal places the two roots of the equation are $x = 0$ (exact), and $x = 1.4055636328$.

**20.** The square of the distance from $(2, 0)$ to $(x, \ln x)$ is $S(x) = (x-2)^2 + (\ln x)^2$, for $x > 0$. Since $S(x) \to \infty$ as $x \to \infty$ or $x \to 0+$, the minimum value of $S(x)$ will occur at a critical point. For CP:

$$0 = S'(x) = 2\left(x - 2 + \frac{\ln x}{x}\right).$$

We solve this equation using a TI-85 solve routine; $x \approx 1.6895797$. The minimum distance from the origin to $y = e^x$ is $\sqrt{S(x)} \approx 0.6094586$.

**21.** If the car is at $(a, e^a)$, then its headlight beam lies along the tangent line to $y = e^x$ there, namely

$$y = e^a + e^a(x - a) = e^a(1 + x - a).$$

This line passes through $(1, 1)$ if $1 = e^a(2 - a)$. A solve routine gives $a \approx -1.1461932$. The corresponding value of $e^a$ is about 0.3178444. The car is at $(a, e^a)$.

## Challenging Problems 4 (page 272)

**1.** $\dfrac{dV}{dt} = kx^2(V_0 - V)$.

a) If $V = x^3$, then $3x^2\dfrac{dx}{dt} = \dfrac{dV}{dt} = kx^2(V_0 - x^3)$, so

$$\frac{dx}{dt} = \frac{k}{3}(V_0 - x^3).$$

b) The rate of growth of the edge is $(k/3)(V_0 - x^3)$, which is positive if $0 \le x < x_0 = V_0^{1/3}$. The time derivative of this rate is

$$-kx^2\frac{dx}{dt} = -\frac{k^2}{3}x^2(V_0 - x^3) < 0$$

for $0 < x < x_0$. Thus the edge length is increasing at a decreasing rate.

c) Initially, $x$ grows at rate $kV_0/3$. The rate of growth of $x$ will be half of this if

$$\frac{k}{3}(V_0 - x^3) = \frac{kV_0}{6},$$

that is, if $x = (V_0/2)^{1/3}$. Then $V = V_0/2$.

**2.** Let the speed of the tank be $v$ where $v = \dfrac{dy}{dt} = ky$. Thus, $y = Ce^{kt}$. Given that at $t = 0$, $y = 4$, then $4 = y(0) = C$. Also given that at $t = 10$, $y = 2$, thus,

$$2 = y(10) = 4e^{10k} \Rightarrow k = -\tfrac{1}{10}\ln 2.$$

Hence, $y = 4e^{(-\frac{1}{10}\ln 2)t}$ and $v = \dfrac{dy}{dt} = (-\dfrac{1}{10}\ln 2)y$. The slope of the curve $xy = 1$ is $m = \dfrac{dy}{dx} = -\dfrac{1}{x^2}$. Thus, the equation of the tangent line at the point $\left(\dfrac{1}{y_0}, y_0\right)$ is

$$y = y_0 - \frac{1}{\left(\frac{1}{y_0}\right)^2}\left(x - \frac{1}{y_0}\right), \quad \text{i.e.,} \quad y = 2y_0 - xy_0^2.$$

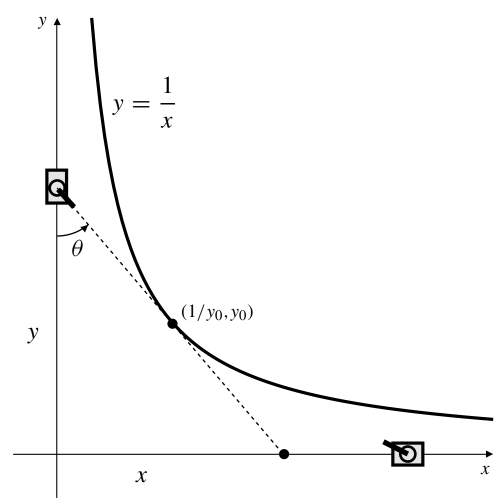



Fig. C-4.2

Hence, the $x$-intercept is $x = \dfrac{2}{y_0}$ and the $y$-intercept is $y = 2y_0$. Let $\theta$ be the angle between the gun and the $y$-axis. We have

$$\begin{aligned} \tan\theta &= \frac{x}{y} = \frac{\left(\frac{2}{y_0}\right)}{2y_0} = \frac{1}{y_0^2} = \frac{4}{y^2} \\ \Rightarrow \qquad \sec^2\theta \frac{d\theta}{dt} &= \frac{-8}{y^3}\frac{dy}{dt}. \end{aligned}$$

Now

$$\sec^2\theta = 1 + \tan^2\theta = 1 + \frac{16}{y^4} = \frac{y^4+16}{y^4},$$

so

$$\frac{d\theta}{dt} = -\frac{8y}{y^4+16}\frac{dy}{dt} = -\frac{8ky^2}{y^4+16}.$$

The maximum value of $\dfrac{y^2}{y^4+16}$ occurs at a critical point:

$$\begin{aligned} 0 &= \frac{(y^4+16)2y - y^2(4y^3)}{(y^4+16)^2} \\ \Leftrightarrow \qquad 2y^5 &= 32y, \end{aligned}$$

or $y = 2$. Therefore the maximum rate of rotation of the gun turret must be

$$-8k\frac{2^2}{2^4+16} = -k = \frac{1}{10}\ln 2 \approx 0.0693 \text{ rad/m},$$

and occurs when your tank is 2 km from the origin.

**3.** a) If $q = 0.99$, the number of tests required is $T = N\left((1/x) + 1 - 0.99^x\right)$. $T$ is a decreasing function for small values of $x$ because the term $1/x$ dominates. It is increasing for large $x$ because $-0.99^x$ dominates. Thus $T$ will have a minimum value at a critical point, provided $N$ is sufficiently large that the CP is in $(0, N)$. For CP:

$$\begin{aligned} 0 &= \frac{dT}{dx} = N\left(-\frac{1}{x^2} - 0.99^x\ln(0.99)\right) \\ x^2 &= \frac{(0.99)^{-x}}{-\ln(0.99)} \\ x &= \frac{(0.99)^{-x/2}}{\sqrt{-\ln(0.99)}} = f(x), \text{ say.} \end{aligned}$$

b) Starting with $x_0 = 20$, we iterate $x_{n+1} = f(x_n)$. The first three iterations give

$$x_1 \approx 11.03, \quad x_2 \approx 10.54, \quad x_3 \approx 10.51.$$

This suggests the CP is near 10.5. Since $x$ must be an integer, we test $x = 10$ and $x = 11$: $T(10) \approx 0.19562$ and $T(11) \approx 0.19557$. The minimum cost should arise by using groups of 11 individuals.

**4.** $P = 2\pi\sqrt{L/g} = 2\pi L^{1/2}g^{-1/2}$.

a) If $L$ remains constant, then

$$\begin{aligned} \Delta P &\approx \frac{dP}{dg}\Delta g = -\pi L^{1/2}g^{-3/2}\Delta g \\ \frac{\Delta P}{P} &\approx \frac{-\pi L^{1/2}g^{-3/2}}{2\pi L^{1/2}g^{-1/2}}\Delta g = -\frac{1}{2}\frac{\Delta g}{g}. \end{aligned}$$

If $g$ increases by 1%, then $\Delta g/g = 1/100$, and $\Delta P/P = -1/200$. Thus $P$ decreases by 0.5%.

b) If $g$ remains constant, then

$$\begin{aligned} \Delta P &\approx \frac{dP}{dL}\Delta L = \pi L^{-1/2}g^{-1/2}\Delta L \\ \frac{\Delta P}{P} &\approx \frac{\pi L^{-1/2}g^{-1/2}}{2\pi L^{1/2}g^{-1/2}}\Delta L = \frac{1}{2}\frac{\Delta L}{L}. \end{aligned}$$

If $L$ increases by 2%, then $\Delta L/L = 2/100$, and $\Delta P/P = 1/100$. Thus $P$ increases by 1%.

**5.** $\dfrac{dV}{dt} = -k\sqrt{y}, \quad V = Ay.$

a) $A\dfrac{dy}{dt} = \dfrac{dV}{dt} = -k\sqrt{y}$, so $\dfrac{dy}{dt} = -\dfrac{k}{A}\sqrt{y}$.

b) If $y(t) = \left(\sqrt{y_0} - \dfrac{kt}{2A}\right)^2$, then $y(0) = y_0$, and

$$\begin{aligned} \frac{dy}{dt} &= 2\left(\sqrt{y_0} - \frac{kt}{2A}\right)\left(-\frac{k}{2A}\right) \\ &= -\frac{k}{A}\sqrt{y(t)}. \end{aligned}$$

Thus the given expression does solve the initial-value problem for $y$.

c) If $y(T) = 0$, then $\dfrac{kT}{2A} = \sqrt{y_0}$, so $k = 2A\sqrt{y_0}/T$. Thus

$$y(t) = \left(\sqrt{y_0} - \frac{2A\sqrt{y_0}t}{2AT}\right)^2 = y_0\left(1 - \frac{t}{T}\right)^2.$$

d) Half the liquid drains out in time $t_1$, where

$$y_0\left(1 - \frac{t_1}{T}\right)^2 = \frac{y_0}{2}.$$

Thus $t_1 = T(1 - (1/\sqrt{2}))$.

**6.** If the depth of liquid in the tank at time $t$ is $y(t)$, then the surface of the liquid has radius $r(t) = Ry(t)/H$, and the volume of liquid in the tank at that time is

$$V(t) = \frac{\pi}{3}\left(\frac{Ry(t)}{H}\right)^2 y(t) = \frac{\pi R^2}{3H^2}\Big(y(t)\Big)^3.$$

By Torricelli's law, $dV/dt = -k\sqrt{y}$. Thus

$$\frac{\pi R^2}{3H^2}3y^2\frac{dy}{dt} = \frac{dV}{dt} = -k\sqrt{y},$$

or, $dy/dt = -k_1 y^{-3/2}$, where $k_1 = kH^2/(\pi R^2)$.
If $y(t) = y_0\left(1 - \frac{t}{T}\right)^{2/5}$, then $y(0) = y_0$, $y(T) = 0$, and

$$\frac{dy}{dt} = \frac{2}{5}y_0\left(1 - \frac{t}{T}\right)^{-3/5}\left(-\frac{1}{T}\right) = -k_1 y^{-3/2},$$

where $k_1 = 2y_0/(5T)$. Thus this function $y(t)$ satisfies the conditions of the problem.

**7.** If the triangle has legs $x$ and $y$ and hypotenuse $\sqrt{x^2+y^2}$, then

$$\begin{aligned}
&P = x + y + \sqrt{x^2+y^2}\\
&(P-x-y)^2 = x^2+y^2\\
&P^2 + x^2 + y^2 + 2xy - 2Px - 2Py = x^2 + y^2\\
&y(2P-2x) = P^2 - 2Px\\
&y = \frac{P(P-2x)}{2(P-x)}.
\end{aligned}$$

The area of the triangle is

$$A = \frac{xy}{2} = \frac{P}{4}\,\frac{Px - 2x^2}{P-x}.$$

$A = 0$ if $x = 0$ or $x = P/2$ and $A > 0$ between these values of $x$. The maximum area will therefore occur at a critical point.

$$\begin{aligned}
&0 = \frac{dA}{dx} = \frac{P}{4}\,\frac{(P-x)(P-4x) - x(P-2x)(-1)}{(P-x)^2}\\
&0 = P^2 - 5Px + 4x^2 + Px - 2x^2\\
&2x^2 - 4Px + P^2 = 0.
\end{aligned}$$

This quadratic has two roots, but the only one in $[0, P/2]$ is

$$x = \frac{4P - \sqrt{16P^2 - 8P^2}}{4} = P\left(1 - \frac{1}{\sqrt{2}}\right).$$

This value of $x$ gives $A(x) = \frac{1}{2}P^2\left(1 - \frac{1}{\sqrt{2}}\right)^2$ un$^2$ for the maximum area of the triangle. (Note that the maximal triangle is isosceles, as we might have guessed.)

**8.** The slope of $y = x^3 + ax^2 + bx + c$ is

$$y' = 3x^2 + 2ax + b,$$

which $\to\ \infty$ as $x\ \to\ \pm\infty$. The quadratic expression $y'$ takes each of its values at two different points except its minimum value, which is achieved only at one point given by $y'' = 6x + 2a = 0$. Thus the tangent to the cubic at $x = -a/3$ is not parallel to any other tangent. This tangent has equation

$$\begin{aligned}
y &= -\frac{a^3}{27} + \frac{a^3}{9} - \frac{ab}{3} + c\\
&\quad + \left(\frac{a^2}{3} - \frac{2a^2}{3} + b\right)\left(x + \frac{a}{3}\right)\\
&= -\frac{a^3}{27} + c + \left(b - \frac{a^2}{3}\right)x.
\end{aligned}$$

**9.**

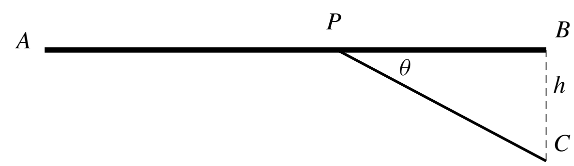



Fig. C-4.9

a) The total resistance of path $APC$ is

$$\begin{aligned}
R &= \frac{k|AP|}{r_1^2} + \frac{k|PC|}{r_2^2}\\
&= k\left(\frac{L - h\cot\theta}{r_1^2} + \frac{h\csc\theta}{r_2^2}\right).
\end{aligned}$$

We have

$$\frac{dR}{d\theta} = kh\left(\frac{\csc^2\theta}{r_1^2} - \frac{\csc\theta\cot\theta}{r_2^2}\right),$$

so the CP of $R$ is given by $\dfrac{\csc\theta}{\cot\theta} = \dfrac{r_1^2}{r_2^2}$, that is, $\cos\theta = (r_2/r_1)^2$ or $\theta = \cos^{-1}((r_2/r_1)^2)$. This CP will give the minimum resistance if it is in the interval of possible values of $\theta$, namely $[\tan^{-1}(h/L), \pi/2]$; otherwise the minimum will occur for $P = A$. Thus, for large $L$, $P$ should be chosen to make $\cos\theta = (r_2/r_1)^2$.

b) This is the same problem as that in (a) except that $r_1$ and $r_2$ are replaced with $r_1^2$ and $r_2^2$, respectively. Thus the minimum resistance corresponds to choosing $P$ so that $\cos\theta = (r_2/r_1)^4$. This puts $P$ closer to $B$ than it was in part (a), which is reasonable since the resistance ratio between the thin and thick pipes is greater than for the wires in part (a).

**10.**

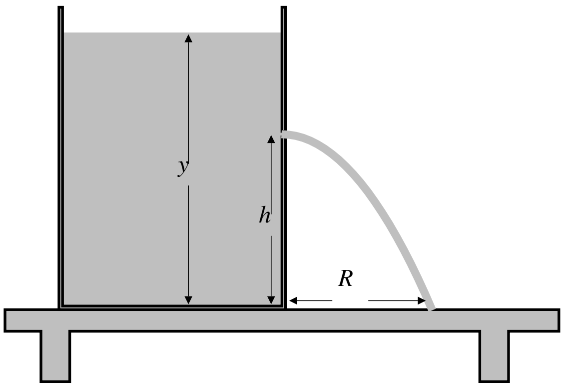



Fig. C-4.10

a) Let the origin be at the point on the table directly under the hole. If a water particle leaves the tank with horizontal velocity $v$, then its position $(X(t), Y(t))$, $t$ seconds later, is given by

$$\begin{array}{ll} \dfrac{d^2X}{dt^2} = 0 & \dfrac{d^2Y}{dt^2} = -g \\ \dfrac{dX}{dt} = v & \dfrac{dY}{dt} = -gt \\ X = vt & Y = -\dfrac{1}{2}gt^2 + h. \end{array}$$

The range $R$ of the particle (i.e., of the spurt) is the value of $X$ when $Y = 0$, that is, at time $t = \sqrt{2h/g}$. Thus $R = v\sqrt{2h/g}$.

b) Since $v = k\sqrt{y-h}$, the range $R$ is a function of $y$, the depth of water in the tank.

$$R = k\sqrt{\frac{2}{g}}\sqrt{h(y-h)}.$$

For a given depth $y$, $R$ will be maximum if $h(y-h)$ is maximum. This occurs at the critical point $h = y/2$ of the quadratic $Q(h) = h(y-h)$.

c) By the result of part (c) of Problem 3 (with $y$ replaced by $y - h$, the height of the surface of the water above the drain in the current problem), we have

$$y(t) - h = (y_0 - h)\left(1 - \frac{t}{T}\right)^2, \quad \text{for } 0 \le t \le T.$$

As shown above, the range of the spurt at time $t$ is

$$R(t) = k\sqrt{\frac{2}{g}}\sqrt{h\Big(y(t) - h\Big)}.$$

Since $R = R_0$ when $y = y_0$, we have

$$k = \frac{R_0}{\sqrt{\dfrac{2}{g}}\sqrt{h(y_0 - h)}}.$$

Therefore $R(t) = R_0 \dfrac{\sqrt{h\Big(y(t) - h\Big)}}{\sqrt{h(y_0 - h)}} = R_0\left(1 - \dfrac{t}{T}\right)$.

**11.**

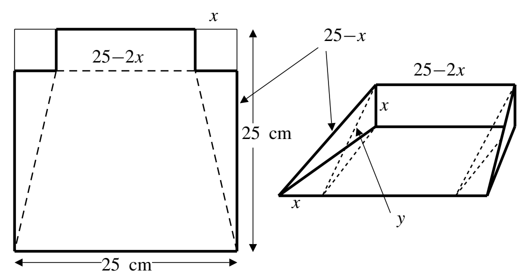



Fig. C-4.11

Note that the vertical back wall of the dustpan is perpendicular to the plane of the *top* of the pan, not the bottom. The volume of the pan is made up of three parts:

> a triangular prism (the centre part) having height $x$, width $25 - 2x$, and depth $y$ (all distances in cm), where $y^2 + x^2 = (25 - x)^2$, and so $y = \sqrt{625 - 50x} = 5\sqrt{25 - 2x}$, and
>
> two triangular pyramids (one on each side) each having height $x$ and a right-triangular top with dimensions $x$ and $y$.

The volume of the pan is, therefore,

$$\begin{aligned} V &= \frac{1}{2}xy(25 - 2x) + 2\left(\frac{1}{3}\right)\left(\frac{1}{2}xy\right)x \\ &= \frac{1}{2}xy\left(25 - 2x + \frac{2}{3}x\right) \\ &= \frac{5}{6}x\sqrt{25 - 2x}(75 - 4x) = V(x). \end{aligned}$$

The appropriate values for $x$ are $0 \le x \le 25/2$. Note that $V(0) = V(25/2) = 0$ and $V(x) > 0$ in $(0, 25/2)$. The maximum volume will therefore occur at a critical point:

$$0 = \frac{dV}{dx} = -\frac{25}{6}\frac{4x^2 - 85x + 375}{\sqrt{25 - 2x}}$$

(after simplification). The quadratic in the numerator factors to $(x - 15)(4x - 25)$, so the CPs are $x = 15$ and $x = 25/4$. Only $x = 25/4$ is in the required interval. The maximum volume of the dustpan is $V(25/4) \approx 921$ cm$^3$.