

## CHAPTER 5. INTEGRATION

Section 5.1 Sums and Sigma Notation  
(page 278)

1.  $\sum_{i=1}^4 i^3 = 1^3 + 2^3 + 3^3 + 4^3$
2.  $\sum_{j=1}^{100} \frac{j}{j+1} = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots + \frac{100}{101}$
3.  $\sum_{i=1}^n 3^i = 3 + 3^2 + 3^3 + \cdots + 3^n$
4.  $\sum_{i=0}^{n-1} \frac{(-1)^i}{i+1} = 1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{(-1)^{n-1}}{n}$
5.  $\sum_{j=3}^n \frac{(-2)^j}{(j-2)^2} = -\frac{2^3}{1^2} + \frac{2^4}{2^2} - \frac{2^5}{3^2} + \cdots + \frac{(-1)^n 2^n}{(n-2)^2}$
6.  $\sum_{j=1}^n \frac{j^2}{n^3} = \frac{1}{n^3} + \frac{4}{n^3} + \frac{9}{n^3} + \cdots + \frac{n^2}{n^3}$
7.  $5 + 6 + 7 + 8 + 9 = \sum_{i=5}^9 i$
8.  $2 + 2 + 2 + \cdots + 2$  (200 terms) equals  $\sum_{i=1}^{200} 2$
9.  $2^2 - 3^2 + 4^2 - 5^2 + \cdots - 99^2 = \sum_{i=2}^{99} (-1)^i i^2$
10.  $1 + 2x + 3x^2 + 4x^3 + \cdots + 100x^{99} = \sum_{i=1}^{100} ix^{i-1}$
11.  $1 + x + x^2 + x^3 + \cdots + x^n = \sum_{i=0}^n x^i$
12.  $1 - x + x^2 - x^3 + \cdots + x^{2n} = \sum_{i=0}^{2n} (-1)^i x^i$
13.  $1 - \frac{1}{4} + \frac{1}{9} - \cdots + \frac{(-1)^{n-1}}{n^2} = \sum_{i=1}^n \frac{(-1)^{i-1}}{i^2}$
14.  $\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \cdots + \frac{n}{2^n} = \sum_{i=1}^n \frac{i}{2^i}$
15.  $\sum_{j=0}^{99} \sin j = \sum_{i=1}^{100} \sin(i-1)$
16.  $\sum_{k=-5}^m \frac{1}{k^2+1} = \sum_{i=1}^{m+6} \frac{1}{((i-6)^2+1)}$
17.  $\sum_{i=1}^n (i^2+2i) = \frac{n(n+1)(2n+1)}{6} + 2\frac{n(n+1)}{2} = \frac{n(n+1)(2n+7)}{6}$
18.  $\sum_{j=1}^{1,000} (2j+3) = \frac{2(1,000)(1,001)}{2} + 3,000 = 1,004,000$
19.  $\sum_{k=1}^n (\pi^k - 3) = \frac{\pi(\pi^n - 1)}{\pi - 1} - 3n$
20.  $\sum_{i=1}^n (2^i - i^2) = 2^{n+1} - 2 - \frac{1}{6}n(n+1)(2n+1)$
21.  $\sum_{m=1}^n \ln m = \ln 1 + \ln 2 + \cdots + \ln n = \ln(n!)$
22.  $\sum_{i=0}^n e^{i/n} = \frac{e^{(n+1)/n} - 1}{e^{1/n} - 1}$
23.  $2 + 2 + \cdots + 2$  (200 terms) equals 400
24.  $1 + x + x^2 + \cdots + x^n = \begin{cases} \frac{1-x^{n+1}}{1-x} & \text{if } x \neq 1 \\ n+1 & \text{if } x = 1 \end{cases}$
25.  $1 - x + x^2 - x^3 + \cdots + x^{2n} = \begin{cases} \frac{1+x^{2n+1}}{1+x} & \text{if } x \neq -1 \\ 2n+1 & \text{if } x = -1 \end{cases}$
26. Let  $f(x) = 1 + x + x^2 + \cdots + x^{100} = \frac{x^{101} - 1}{x - 1}$  if  $x \neq 1$ .  
Then  
$$f'(x) = 1 + 2x + 3x^2 + \cdots + 100x^{99}$$
$$= \frac{d}{dx} \frac{x^{101} - 1}{x - 1} = \frac{100x^{101} - 101x^{100} + 1}{(x-1)^2}.$$
27.  $2^2 - 3^2 + 4^2 - 5^2 + \cdots + 98^2 - 99^2$ 
$$= \sum_{k=1}^{49} [(2k)^2 - (2k+1)^2] = \sum_{k=1}^{49} [4k^2 - 4k^2 - 4k - 1]$$
$$= -\sum_{k=1}^{49} [4k+1] = -4\frac{49 \times 50}{2} - 49 = -4,949$$
28. Let  $s = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \cdots + \frac{n}{2^n}$ . Then  
$$\frac{s}{2} = \frac{1}{4} + \frac{2}{8} + \frac{3}{16} + \cdots + \frac{n}{2^{n+1}}.$$

Subtracting these two sums, we get

$$\begin{aligned} \frac{s}{2} &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} - \frac{n}{2^{n+1}} \\ &= \frac{1}{2} \frac{1 - (1/2^n)}{1 - (1/2)} - \frac{n}{2^{n+1}} \\ &= 1 - \frac{n+2}{2^{n+1}}. \end{aligned}$$

Thus  $s = 2 + (n+2)/2^n$ .

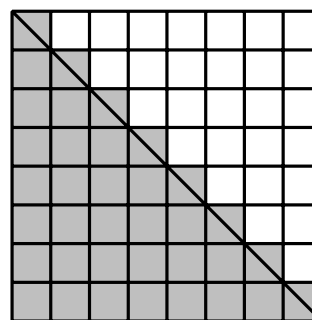


Fig. 5.1.34

$$\begin{aligned} 29. \quad \sum_{i=m}^n (f(i+1) - f(i)) &= \sum_{i=m}^n f(i+1) - \sum_{i=m}^n f(i) \\ &= \sum_{j=m+1}^{n+1} f(j) - \sum_{i=m}^n f(i) \\ &= f(n+1) - f(m), \end{aligned}$$

because each sum has only one term that is not cancelled by a term in the other sum. It is called “telescoping” because the sum “folds up” to a sum involving only part of the first and last terms.

$$30. \quad \sum_{n=1}^{10} (n^4 - (n-1)^4) = 10^4 - 0^4 = 10,000$$

$$31. \quad \sum_{j=1}^m (2^j - 2^{j-1}) = 2^m - 2^0 = 2^m - 1$$

$$32. \quad \sum_{i=m}^{2m} \left( \frac{1}{i} - \frac{1}{i+1} \right) = \frac{1}{m} - \frac{1}{2m+1} = \frac{m+1}{m(2m+1)}$$

$$33. \quad \sum_{j=1}^m \frac{1}{j(j+1)} = \sum_{j=1}^m \left( \frac{1}{j} - \frac{1}{j+1} \right) = 1 - \frac{1}{n+1} = \frac{n}{n+1}$$

34. The number of small shaded squares is  $1 + 2 + \cdots + n$ . Since each has area 1, the total area shaded is  $\sum_{i=1}^n i$ . But this area consists of a large right-angled triangle of area  $n^2/2$  (below the diagonal), and  $n$  small triangles (above the diagonal) each of area  $1/2$ . Equating these areas, we get

$$\sum_{i=1}^n i = \frac{n^2}{2} + n \frac{1}{2} = \frac{n(n+1)}{2}.$$

35. To show that

$$\sum_{i=1}^n i = \frac{n(n+1)}{2},$$

we write  $n$  copies of the identity

$$(k+1)^2 - k^2 = 2k + 1,$$

one for each  $k$  from 1 to  $n$ :

$$2^2 - 1^2 = 2(1) + 1$$

$$3^2 - 2^2 = 2(2) + 1$$

$$4^2 - 3^2 = 2(3) + 1$$

$\vdots$

$$(n+1)^2 - n^2 = 2(n) + 1.$$

Adding the left and right sides of these formulas we get

$$(n+1)^2 - 1^2 = 2 \sum_{i=1}^n i + n.$$

Hence,  $\sum_{i=1}^n i = \frac{1}{2}(n^2 + 2n + 1 - 1 - n) = \frac{n(n+1)}{2}$ .

36. The formula  $\sum_{i=1}^n i = n(n+1)/2$  holds for  $n = 1$ , since it says  $1 = 1$  in this case. Now assume that it holds for  $n =$  some number  $k \geq 1$ ; that is,  $\sum_{i=1}^k i = k(k+1)/2$ . Then for  $n = k + 1$ , we have

$$\sum_{i=1}^{k+1} i = \sum_{i=1}^k i + (k+1) = \frac{k(k+1)}{2} + (k+1) = \frac{(k+1)(k+2)}{2}.$$

Thus the formula also holds for  $n = k + 1$ . By induction, it holds for all positive integers  $n$ .

37. The formula  $\sum_{i=1}^n i^2 = n(n+1)(2n+1)/6$  holds for  $n = 1$ , since it says  $1 = 1$  in this case. Now assume that it holds for  $n = \text{some number } k \geq 1$ ; that is,  $\sum_{i=1}^k i^2 = k(k+1)(2k+1)/6$ . Then for  $n = k + 1$ , we have

$$\begin{aligned} \sum_{i=1}^{k+1} i^2 &= \sum_{i=1}^k i^2 + (k+1)^2 \\ &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\ &= \frac{k+1}{6} [2k^2 + k + 6k + 6] \\ &= \frac{k+1}{6} (k+2)(2k+3) \\ &= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}. \end{aligned}$$

Thus the formula also holds for  $n = k + 1$ . By induction, it holds for all positive integers  $n$ .

38. The formula  $\sum_{i=1}^n r^{i-1} = (r^n - 1)/(r - 1)$  (for  $r \neq 1$ ) holds for  $n = 1$ , since it says  $1 = 1$  in this case. Now assume that it holds for  $n = \text{some number } k \geq 1$ ; that is,  $\sum_{i=1}^k r^{i-1} = (r^k - 1)/(r - 1)$ . Then for  $n = k + 1$ , we have

$$\sum_{i=1}^{k+1} r^{i-1} = \sum_{i=1}^k r^{i-1} + r^k = \frac{r^k - 1}{r - 1} + r^k = \frac{r^{k+1} - 1}{r - 1}.$$

Thus the formula also holds for  $n = k + 1$ . By induction, it holds for all positive integers  $n$ .

39.

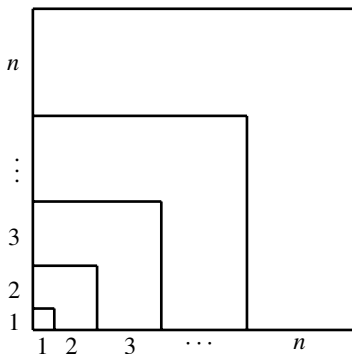


Fig. 5.1.39

The L-shaped region with short side  $i$  is a square of side  $i(i+1)/2$  with a square of side  $(i-1)i/2$  cut out. Since

$$\begin{aligned} \left(\frac{i(i+1)}{2}\right)^2 - \left(\frac{(i-1)i}{2}\right)^2 &= \frac{i^4 + 2i^3 + i^2 - (i^4 - 2i^3 + i^2)}{4} = i^3, \end{aligned}$$

that L-shaped region has area  $i^3$ . The sum of the areas of the  $n$  L-shaped regions is the area of the large square of side  $n(n+1)/2$ , so

$$\sum_{i=1}^n i^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

40. To show that

$$\sum_{j=1}^n j^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4},$$

we write  $n$  copies of the identity

$$(k+1)^4 - k^4 = 4k^3 + 6k^2 + 4k + 1,$$

one for each  $k$  from 1 to  $n$ :

$$\begin{aligned} 2^4 - 1^4 &= 4(1)^3 + 6(1)^2 + 4(1) + 1 \\ 3^4 - 2^4 &= 4(2)^3 + 6(2)^2 + 4(2) + 1 \\ 4^4 - 3^4 &= 4(3)^3 + 6(3)^2 + 4(3) + 1 \\ &\vdots \\ (n+1)^4 - n^4 &= 4(n)^3 + 6(n)^2 + 4(n) + 1. \end{aligned}$$

Adding the left and right sides of these formulas we get

$$\begin{aligned} (n+1)^4 - 1^4 &= 4 \sum_{j=1}^n j^3 + 6 \sum_{j=1}^n j^2 + 4 \sum_{j=1}^n j + n \\ &= 4 \sum_{j=1}^n j^3 + \frac{6n(n+1)(2n+1)}{6} + \frac{4n(n+1)}{2} + n. \end{aligned}$$

Hence,

$$\begin{aligned} 4 \sum_{j=1}^n j^3 &= (n+1)^4 - 1 - n(n+1)(2n+1) - 2n(n+1) - n \\ &= n^2(n+1)^2 \end{aligned}$$

$$\text{so } \sum_{j=1}^n j^3 = \frac{n^2(n+1)^2}{4}.$$

41. The formula  $\sum_{i=1}^n i^3 = n^2(n+1)^2/4$  holds for  $n = 1$ , since it says  $1 = 1$  in this case. Now assume that it holds for  $n = \text{some number } k \geq 1$ ; that is,  $\sum_{i=1}^k i^3 = k^2(k+1)^2/4$ . Then for  $n = k + 1$ , we have

$$\begin{aligned} \sum_{i=1}^{k+1} i^3 &= \sum_{i=1}^k i^3 + (k+1)^3 \\ &= \frac{k^2(k+1)^2}{4} + (k+1)^3 = \frac{(k+1)^2}{4} [k^2 + 4(k+1)] \\ &= \frac{(k+1)^2}{4} (k+2)^2. \end{aligned}$$

Thus the formula also holds for  $n = k + 1$ . By induction, it holds for all positive integers  $n$ .

42. To find  $\sum_{j=1}^n j^4 = 1^4 + 2^4 + 3^4 + \dots + n^4$ , we write  $n$  copies of the identity

$$(k + 1)^5 - k^5 = 5k^4 + 10k^3 + 10k^2 + 5k + 1,$$

one for each  $k$  from 1 to  $n$ :

$$\begin{aligned} 2^5 - 1^5 &= 5(1)^4 + 10(1)^3 + 10(1)^2 + 5(1) + 1 \\ 3^5 - 2^5 &= 5(2)^4 + 10(2)^3 + 10(2)^2 + 5(2) + 1 \\ 4^5 - 3^5 &= 5(3)^4 + 10(3)^3 + 10(3)^2 + 5(3) + 1 \\ &\vdots \\ (n + 1)^5 - n^5 &= 5(n)^4 + 10(n)^3 + 10(n)^2 + 5(n) + 1. \end{aligned}$$

Adding the left and right sides of these formulas we get

$$(n + 1)^5 - 1^5 = 5 \sum_{j=1}^n j^4 + 10 \sum_{j=1}^n j^3 + 10 \sum_{j=1}^n j^2 + 5 \sum_{j=1}^n j + n.$$

Substituting the known formulas for all the sums except  $\sum_{j=1}^n j^4$ , and solving for this quantity, gives

$$\sum_{j=1}^n j^4 = \frac{n(n + 1)(2n + 1)(3n^2 + 3n - 1)}{30}.$$

Of course we got Maple to do the donkey work!

43. 
$$\begin{aligned} \sum_{i=1}^n i^5 &= \frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{12}n^4 - \frac{1}{12}n^2 \\ \sum_{i=1}^n i^6 &= \frac{1}{7}n^7 + \frac{1}{2}n^6 + \frac{1}{2}n^5 - \frac{1}{6}n^3 + \frac{1}{42}n \\ \sum_{i=1}^n i^7 &= \frac{1}{8}n^8 + \frac{1}{2}n^7 + \frac{7}{12}n^6 - \frac{7}{24}n^4 + \frac{1}{12}n^2 \\ \sum_{i=1}^n i^8 &= \frac{1}{9}n^9 + \frac{1}{2}n^8 + \dots \end{aligned}$$

We would guess (correctly) that

$$\sum_{i=1}^n i^{10} = \frac{1}{11}n^{11} + \frac{1}{2}n^{10} + \dots$$

**Section 5.2 Areas as Limits of Sums (page 284)**

1. The area is the limit of the sum of the areas of the rectangles shown in the figure. It is

$$\begin{aligned} A &= \lim_{n \rightarrow \infty} \frac{1}{n} \left[ \frac{3}{n} + \frac{3 \times 2}{n} + \frac{3 \times 3}{n} + \dots + \frac{3n}{n} \right] \\ &= \lim_{n \rightarrow \infty} \frac{3}{n^2} (1 + 2 + 3 + \dots + n) \\ &= \lim_{n \rightarrow \infty} \frac{3}{n^2} \cdot \frac{n(n + 1)}{2} = \frac{3}{2} \text{sq. units.} \end{aligned}$$

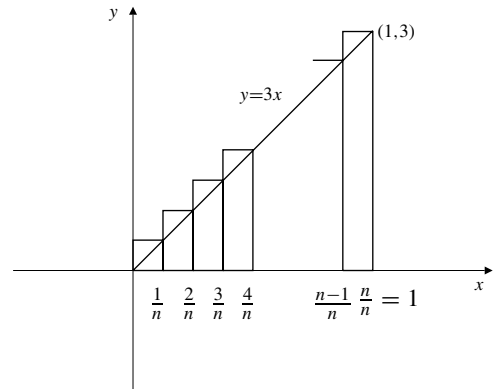


Fig. 5.2.1

2. This is similar to #1; the rectangles now have width  $3/n$  and the  $i$ th has height  $2(3i/n)+1$ , the value of  $2x+1$  at  $x = 3i/n$ . The area is

$$\begin{aligned} A &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{3}{n} \left( 2 \frac{3i}{n} + 1 \right) \\ &= \lim_{n \rightarrow \infty} \frac{18}{n^2} \sum_{i=1}^n i + \frac{3}{n} \\ &= \lim_{n \rightarrow \infty} \frac{18}{n^2} \frac{n(n + 1)}{2} + 3 = 9 + 3 = 12 \text{sq. units.} \end{aligned}$$

3. This is similar to #1; the rectangles have width  $(3-1)/n = 2/n$  and the  $i$ th has height the value of  $2x-1$  at  $x = 1 + (2i/n)$ . The area is

$$\begin{aligned} A &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{2}{n} \left( 2 + 2 \frac{2i}{n} - 1 \right) \\ &= \lim_{n \rightarrow \infty} \frac{8}{n^2} \sum_{i=1}^n i + \frac{2}{n} \\ &= \lim_{n \rightarrow \infty} \frac{8}{n^2} \frac{n(n + 1)}{2} + 2 = 4 + 2 = 6 \text{sq. units.} \end{aligned}$$

4. This is similar to #1; the rectangles have width  $(2 - (-1))/n = 3/n$  and the  $i$ th has height the value of  $3x + 4$  at  $x = -1 + (3i/n)$ . The area is

$$\begin{aligned} A &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{3}{n} \left( -3 + 3\frac{3i}{n} + 4 \right) \\ &= \lim_{n \rightarrow \infty} \frac{27}{n^2} \sum_{i=1}^n i + \frac{3}{n} \\ &= \lim_{n \rightarrow \infty} \frac{27}{n^2} \frac{n(n+1)}{2} + 3 = \frac{27}{2} + 3 = \frac{33}{2} \text{ sq. units.} \end{aligned}$$

5. The area is the limit of the sum of the areas of the rectangles shown in the figure. It is

$$\begin{aligned} A &= \lim_{n \rightarrow \infty} \frac{2}{n} \left[ \left(1 + \frac{2}{n}\right)^2 + \left(1 + \frac{4}{n}\right)^2 + \cdots + \left(1 + \frac{2n}{n}\right)^2 \right] \\ &= \lim_{n \rightarrow \infty} \frac{2}{n} \left[ 1 + \frac{4}{n} + \frac{4}{n^2} + 1 + \frac{8}{n} + \frac{16}{n^2} \right. \\ &\quad \left. + \cdots + 1 + \frac{4n}{n} + \frac{4n^2}{n^2} \right] \\ &= \lim_{n \rightarrow \infty} \left( 2 + \frac{8}{n^2} \cdot \frac{n(n+1)}{2} + \frac{8}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} \right) \\ &= 2 + 4 + \frac{8}{3} = \frac{26}{3} \text{ sq. units.} \end{aligned}$$

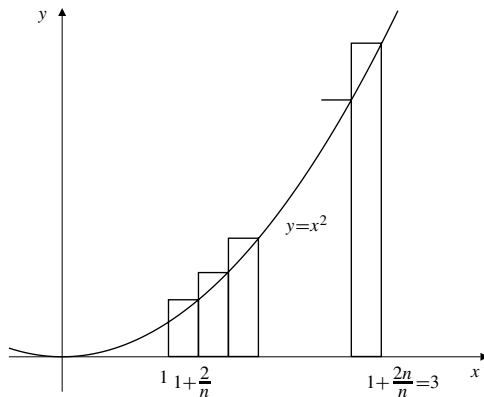


Fig. 5.2.5

6. Divide  $[0, a]$  into  $n$  equal subintervals of length  $\Delta x = \frac{a}{n}$  by points  $x_i = \frac{ia}{n}$ ,  $(0 \leq i \leq n)$ . Then

$$\begin{aligned} S_n &= \sum_{i=1}^n \left( \frac{a}{n} \right) \left[ \left( \frac{ia}{n} \right)^2 + 1 \right] \\ &= \left( \frac{a}{n} \right)^3 \sum_{i=1}^n i^2 + \frac{a}{n} \sum_{i=1}^n (1) \\ &\quad \text{(Use Theorem 1(a) and 1(c).)} \end{aligned}$$

$$\begin{aligned} &= \left( \frac{a}{n} \right)^3 \frac{n(n+1)(2n+1)}{6} + \frac{a}{n}(n) \\ &= \frac{a^3}{6} \frac{(n+1)(2n+1)}{n^2} + a. \end{aligned}$$

$$\text{Area} = \lim_{n \rightarrow \infty} S_n = \frac{a^3}{3} + a \text{ sq. units.}$$

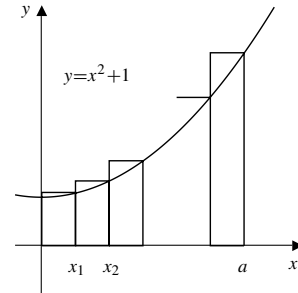


Fig. 5.2.6

7. The required area is (see the figure)

$$\begin{aligned} A &= \lim_{n \rightarrow \infty} \frac{3}{n} \left[ \left( -1 + \frac{3}{n} \right)^2 + 2 \left( -1 + \frac{3}{n} \right) + 3 \right. \\ &\quad \left. + \left( -1 + \frac{6}{n} \right)^2 + 2 \left( -1 + \frac{6}{n} \right) + 3 \right. \\ &\quad \left. + \cdots + \left( -1 + \frac{3n}{n} \right)^2 + 2 \left( -1 + \frac{3n}{n} \right) + 3 \right] \\ &= \lim_{n \rightarrow \infty} \frac{3}{n} \left[ \left( 1 - \frac{6}{n} + \frac{3^2}{n^2} - 2 + \frac{6}{n} + 3 \right) \right. \\ &\quad \left. + \left( 1 - \frac{12}{n} + \frac{6^2}{n^2} - 2 + \frac{12}{n} + 3 \right) \right. \\ &\quad \left. + \cdots + \left( 1 - \frac{6n}{n} + \frac{9n^2}{n^2} - 2 + \frac{6n}{n} + 3 \right) \right] \\ &= \lim_{n \rightarrow \infty} \left( 6 + \frac{27}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} \right) \\ &= 6 + 9 = 15 \text{ sq. units.} \end{aligned}$$

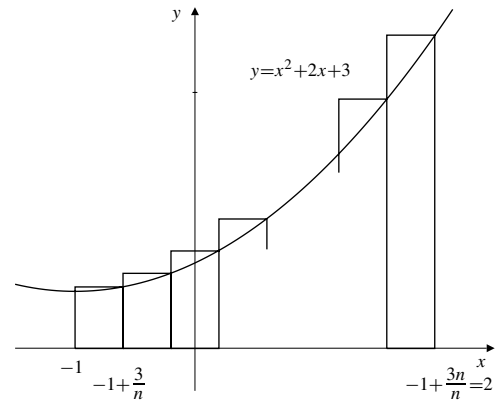


Fig. 5.2.7

8.

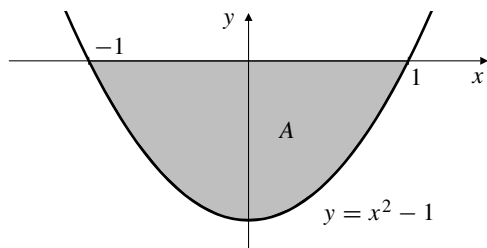


Fig. 5.2.8

The region in question lies between  $x = -1$  and  $x = 1$  and is symmetric about the  $y$ -axis. We can therefore double the area between  $x = 0$  and  $x = 1$ . If we divide this interval into  $n$  equal subintervals of width  $1/n$  and use the distance  $0 - (x^2 - 1) = 1 - x^2$  between  $y = 0$  and  $y = x^2 - 1$  for the heights of rectangles, we find that the required area is

$$\begin{aligned} A &= 2 \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{n} \left( 1 - \frac{i^2}{n^2} \right) \\ &= 2 \lim_{n \rightarrow \infty} \sum_{i=1}^n \left( \frac{1}{n} - \frac{i^2}{n^3} \right) \\ &= 2 \lim_{n \rightarrow \infty} \left( \frac{n}{n} - \frac{n(n+1)(2n+1)}{6n^3} \right) = 2 - \frac{4}{6} = \frac{4}{3} \text{ sq. units.} \end{aligned}$$

9.

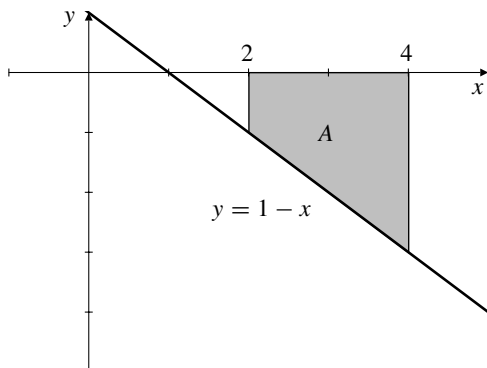


Fig. 5.2.9

The height of the region at position  $x$  is  $0 - (1 - x) = x - 1$ . The “base” is an interval of length 2, so we approximate using  $n$  rectangles of width  $2/n$ . The shaded area is

$$\begin{aligned} A &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{2}{n} \left( 2 + \frac{2i}{n} - 1 \right) \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left( \frac{2}{n} + \frac{4i}{n^2} \right) \\ &= \lim_{n \rightarrow \infty} \left( \frac{2n}{n} + 4 \frac{n(n+1)}{2n^2} \right) = 2 + 2 = 4 \text{ sq. units.} \end{aligned}$$

10.

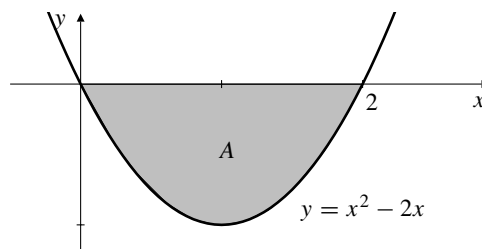


Fig. 5.2.10

The height of the region at position  $x$  is  $0 - (x^2 - 2x) = 2x - x^2$ . The “base” is an interval of length 2, so we approximate using  $n$  rectangles of width  $2/n$ . The shaded area is

$$\begin{aligned} A &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{2}{n} \left( 2 \frac{2i}{n} - \frac{4i^2}{n^2} \right) \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left( \frac{8i}{n^2} - \frac{8i^2}{n^3} \right) \\ &= \lim_{n \rightarrow \infty} \left( \frac{8}{n^2} \frac{n(n+1)}{2} - \frac{8}{n^3} \frac{n(n+1)(2n+1)}{6} \right) \\ &= 4 - \frac{8}{3} = \frac{4}{3} \text{ sq. units.} \end{aligned}$$

11.

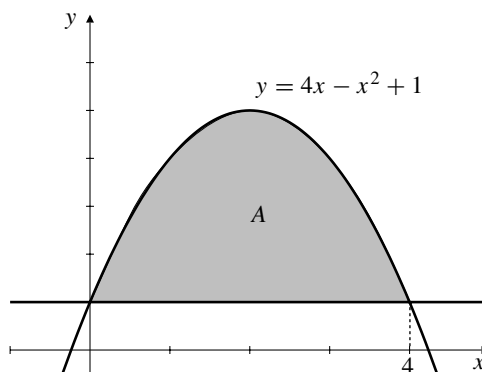


Fig. 5.2.11

The height of the region at position  $x$  is  $4x - x^2 + 1 - 1 = 4x - x^2$ . The “base” is an interval of length 4, so we approximate using  $n$  rectangles of width  $4/n$ . The shaded area is

$$\begin{aligned} A &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{4}{n} \left( 4 \frac{4i}{n} - \frac{16i^2}{n^2} \right) \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left( \frac{64i}{n^2} - \frac{64i^2}{n^3} \right) \\ &= \lim_{n \rightarrow \infty} \left( \frac{64}{n^2} \frac{n(n+1)}{2} - \frac{64}{n^3} \frac{n(n+1)(2n+1)}{6} \right) \\ &= 32 - \frac{64}{3} = \frac{32}{3} \text{ sq. units.} \end{aligned}$$

12. Divide  $[0, b]$  into  $n$  equal subintervals of length  $\Delta x = \frac{b}{n}$  by points  $x_i = \frac{ib}{n}$ ,  $(0 \leq i \leq n)$ . Then

$$\begin{aligned} S_n &= \sum_{i=1}^n \frac{b}{n} \left( e^{(ib/n)} \right) = \frac{b}{n} \sum_{i=1}^n \left( e^{(b/n)} \right)^i \\ &= \frac{b}{n} e^{(b/n)} \sum_{i=1}^n \left( e^{(b/n)} \right)^{i-1} \quad (\text{Use Thm. 6.1.2(d).}) \\ &= \frac{b}{n} e^{(b/n)} \frac{e^{(b/n)n} - 1}{e^{(b/n)} - 1} \\ &= \frac{b}{n} e^{(b/n)} \frac{e^b - 1}{e^{(b/n)} - 1}. \end{aligned}$$

$$\text{Let } r = \frac{b}{n}.$$

$$\begin{aligned} \text{Area} &= \lim_{n \rightarrow \infty} S_n = (e^b - 1) \lim_{r \rightarrow 0^+} e^r \lim_{r \rightarrow 0^+} \frac{r}{e^r - 1} \quad \left[ \frac{0}{0} \right] \\ &= (e^b - 1)(1) \lim_{r \rightarrow 0^+} \frac{1}{e^r} = e^b - 1 \text{ sq. units.} \end{aligned}$$

13. The required area is

$$\begin{aligned} A &= \lim_{n \rightarrow \infty} \frac{2}{n} \left[ 2^{-1+(2/n)} + 2^{-1+(4/n)} + \dots + 2^{-1+(2n/n)} \right] \\ &= \lim_{n \rightarrow \infty} \frac{2^{2/n}}{n} \left[ 1 + (2^{2/n}) + (2^{2/n})^2 + \dots + (2^{2/n})^{n-1} \right] \\ &= \lim_{n \rightarrow \infty} \frac{2^{2/n}}{n} \cdot \frac{(2^{2/n})^n - 1}{2^{2/n} - 1} \\ &= \lim_{n \rightarrow \infty} 2^{2/n} \times 3 \times \frac{1}{n(2^{2/n} - 1)} \\ &= 3 \lim_{n \rightarrow \infty} \frac{1}{n(2^{2/n} - 1)}. \end{aligned}$$

Now we can use l'Hôpital's rule to evaluate

$$\begin{aligned} \lim_{n \rightarrow \infty} n(2^{2/n} - 1) &= \lim_{n \rightarrow \infty} \frac{2^{2/n} - 1}{\frac{1}{n}} \quad \left[ \frac{0}{0} \right] \\ &= \lim_{n \rightarrow \infty} \frac{2^{2/n} \ln 2 \left( \frac{-2}{n^2} \right)}{\frac{-1}{n^2}} \\ &= \lim_{n \rightarrow \infty} 2^{(2/n)+1} \ln 2 = 2 \ln 2. \end{aligned}$$

Thus the area is  $\frac{3}{2 \ln 2}$  square units.

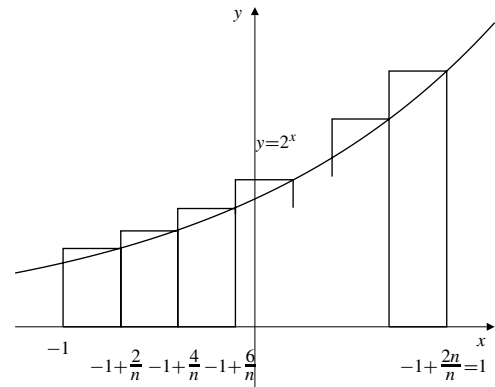


Fig. 5.2.13

14. Area =  $\lim_{n \rightarrow \infty} \frac{b}{n} \left[ \left( \frac{b}{n} \right)^3 + \left( \frac{2b}{n} \right)^3 + \dots + \left( \frac{nb}{n} \right)^3 \right]$

$$\begin{aligned} &= \lim_{n \rightarrow \infty} \frac{b^4}{n^4} (1^3 + 2^3 + 3^3 + \dots + n^3) \\ &= \lim_{n \rightarrow \infty} \frac{b^4}{n^4} \cdot \frac{n^2(n+1)^2}{4} = \frac{b^4}{4} \text{ sq. units.} \end{aligned}$$

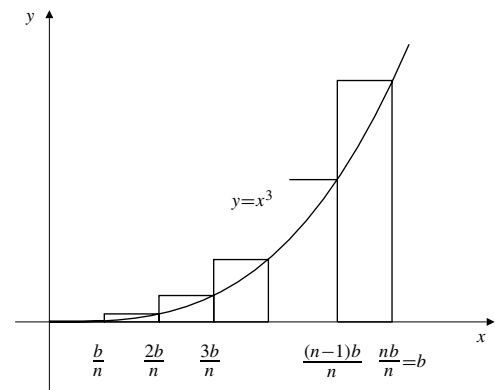


Fig. 5.2.14

15. Let  $t = \left( \frac{b}{a} \right)^{1/n}$  and let

$$x_0 = a, \quad x_1 = at, \quad x_2 = at^2, \quad \dots, \quad x_n = at^n = b.$$

The  $i$ th subinterval  $[x_{i-1}, x_i]$  has length  $\Delta x_i = at^{i-1}(t-1)$ . Since  $f(x_{i-1}) = \frac{1}{at^{i-1}}$ , we form the sum

$$S_n = \sum_{i=1}^n at^{i-1}(t-1) \left( \frac{1}{at^{i-1}} \right) = n(t-1) = n \left[ \left( \frac{b}{a} \right)^{1/n} - 1 \right].$$

Let  $r = \frac{1}{n}$  and  $c = \frac{b}{a}$ . The area under the curve is

$$A = \lim_{n \rightarrow \infty} S_n = \lim_{r \rightarrow 0^+} \frac{c^r - 1}{r} \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] = \lim_{r \rightarrow 0^+} \frac{c^r \ln c}{1} = \ln c = \ln \left( \frac{b}{a} \right) \text{ square units.}$$

This is not surprising because it follows from the *definition* of  $\ln$ .

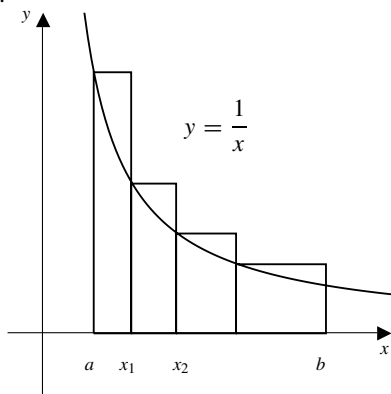


Fig. 5.2.15

16.

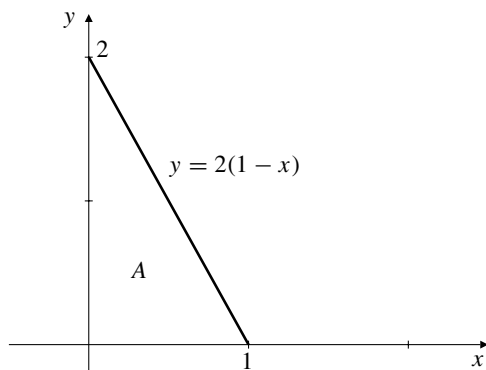


Fig. 5.2.16

$s_n = \sum_{i=1}^n \frac{2}{n} \left( 1 - \frac{i}{n} \right)$  represents a sum of areas of  $n$  rectangles each of width  $1/n$  and having heights equal to the height to the graph  $y = 2(1-x)$  at the points  $x = i/n$ . Thus  $\lim_{n \rightarrow \infty} S_n$  is the area  $A$  of the triangle in the figure above, and therefore has the value 1.

17.

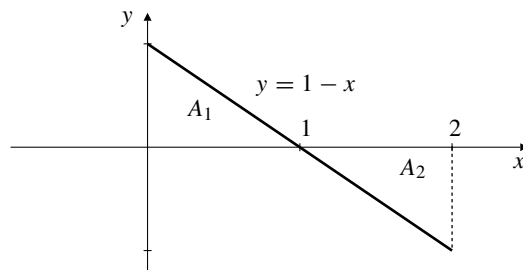


Fig. 5.2.17

$s_n = \sum_{i=1}^n \frac{2}{n} \left( 1 - \frac{2i}{n} \right)$  represents a sum of areas of  $n$  rectangles each of width  $2/n$  and having heights equal to the height to the graph  $y = 1-x$  at the points  $x = 2i/n$ . Half of these rectangles have negative height, and  $\lim_{n \rightarrow \infty} S_n$  is the difference  $A_1 - A_2$  of the areas of the two triangles in the figure above. It has the value 0 since the two triangles have the same area.

18.

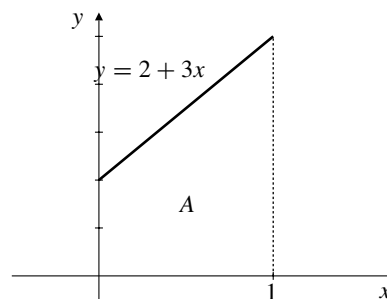


Fig. 5.2.18

$s_n = \sum_{i=1}^n \frac{2n+3i}{n^2} = \sum_{i=1}^n \frac{1}{n} \left( 2 + \frac{3i}{n} \right)$  represents a sum of areas of  $n$  rectangles each of width  $1/n$  and having heights equal to the height to the graph  $y = 2+3x$  at the points  $x = i/n$ . Thus  $\lim_{n \rightarrow \infty} S_n$  is the area of the trapezoid in the figure above, and has the value  $1(2+5)/2 = 7/2$ .

19. 
$$S_n = \sum_{j=1}^n \frac{1}{n} \sqrt{1 - \left( \frac{j}{n} \right)^2}$$

= sum of areas of rectangles in the figure.

Thus the limit of  $S_n$  is the area of a quarter circle of unit radius:

$$\lim_{n \rightarrow \infty} S_n = \frac{\pi}{4}.$$



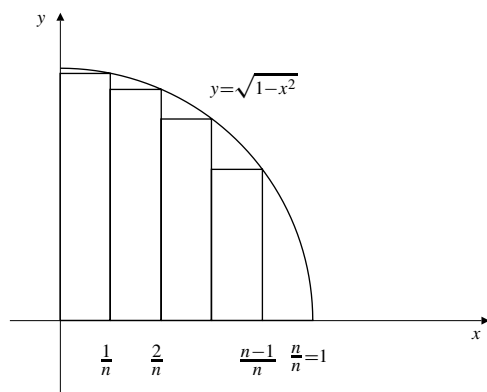


Fig. 5.2.19

### Section 5.3 The Definite Integral (page 290)

1.  $f(x) = x$  on  $[0, 2]$ ,  $n = 8$ .

$$P_8 = \left\{ 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}, \frac{3}{2}, \frac{7}{4}, 2 \right\}$$

$$L(f, P_8) = \frac{2-0}{8} \left[ 0 + \frac{1}{4} + \frac{1}{2} + \frac{3}{4} + 1 + \frac{5}{4} + \frac{3}{2} + \frac{7}{4} \right] = \frac{7}{4}$$

$$U(f, P_8) = \frac{2-0}{8} \left[ \frac{1}{4} + \frac{1}{2} + \frac{3}{4} + 1 + \frac{5}{4} + \frac{3}{2} + \frac{7}{4} + 2 \right] = \frac{9}{4}$$

2.  $f(x) = x^2$  on  $[0, 4]$ ,  $n = 4$ .

$$L(f, P_4) = \left( \frac{4-0}{4} \right) [0 + (1)^2 + (2)^2 + (3)^2] = 14.$$

$$U(f, P_4) = \left( \frac{4-0}{4} \right) [(1)^2 + (2)^2 + (3)^2 + (4)^2] = 30.$$

3.  $f(x) = e^x$  on  $[-2, 2]$ ,  $n = 4$ .

$$L(f, P_4) = 1(e^{-2} + e^{-1} + e^0 + e^1) = \frac{e^4 - 1}{e^2(e-1)} \approx 4.22$$

$$U(f, P_4) = 1(e^{-1} + e^0 + e^1 + e^2) = \frac{e^4 - 1}{e(e-1)} \approx 11.48.$$

4.  $f(x) = \ln x$  on  $[1, 2]$ ,  $n = 5$ .

$$L(f, P_5) = \left( \frac{2-1}{5} \right) \left[ \ln 1 + \ln \frac{6}{5} + \ln \frac{7}{5} + \ln \frac{8}{5} + \ln \frac{9}{5} \right] \approx 0.3153168.$$

$$U(f, P_5) = \left( \frac{2-1}{5} \right) \left[ \ln \frac{6}{5} + \ln \frac{7}{5} + \ln \frac{8}{5} + \ln \frac{9}{5} + \ln 2 \right] \approx 0.4539462.$$

5.  $f(x) = \sin x$  on  $[0, \pi]$ ,  $n = 6$ .

$$P_6 = \left\{ 0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{5\pi}{6}, \pi \right\}$$

$$L(f, P_6) = \frac{\pi}{6} \left[ 0 + \frac{1}{2} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + \frac{1}{2} + 0 \right] = \frac{\pi}{6} (1 + \sqrt{3}) \approx 1.43,$$

$$U(f, P_6) = \frac{\pi}{6} \left[ \frac{1}{2} + \frac{\sqrt{3}}{2} + 1 + 1 + \frac{\sqrt{3}}{2} + \frac{1}{2} \right] = \frac{\pi}{6} (3 + \sqrt{3}) \approx 2.48.$$

6.  $f(x) = \cos x$  on  $[0, 2\pi]$ ,  $n = 4$ .

$$L(f, P_4) = \left( \frac{2\pi}{4} \right) \left[ \cos \frac{\pi}{2} + \cos \pi + \cos \pi + \cos \frac{3\pi}{2} \right] = -\pi.$$

$$U(f, P_4) = \left( \frac{2\pi}{4} \right) \left[ \cos 0 + \cos \frac{\pi}{2} + \cos \frac{3\pi}{2} + \cos 2\pi \right] = \pi.$$

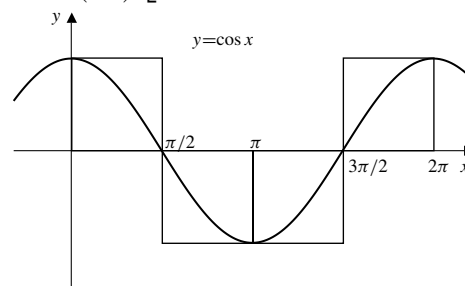


Fig. 5.3.6

7.  $f(x) = x$  on  $[0, 1]$ .  $P_n = \left\{ 0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, \frac{n}{n} \right\}$ . We have

$$L(f, P_n) = \frac{1}{n} \left( 0 + \frac{1}{n} + \frac{2}{n} + \dots + \frac{n-1}{n} \right) = \frac{1}{n^2} \cdot \frac{(n-1)n}{2} = \frac{n-1}{2n},$$

$$U(f, P_n) = \frac{1}{n} \left( \frac{1}{n} + \frac{2}{n} + \frac{3}{n} + \dots + \frac{n}{n} \right) = \frac{1}{n^2} \cdot \frac{n(n+1)n}{2} = \frac{n+1}{2n}.$$

Thus  $\lim_{n \rightarrow \infty} L(f, P_n) = \lim_{n \rightarrow \infty} U(f, P_n) = 1/2$ .

If  $P$  is any partition of  $[0, 1]$ , then

$$L(f, P) \leq U(f, P_n) = \frac{n+1}{2n}$$

for every  $n$ , so  $L(f, P) \leq \lim_{n \rightarrow \infty} U(f, P_n) = 1/2$ .

Similarly,  $U(f, P) \geq 1/2$ . If there exists any number  $I$  such that  $L(f, P) \leq I \leq U(f, P)$  for all  $P$ , then  $I$  cannot be less than  $1/2$  (or there would exist a  $P_n$  such that  $L(f, P_n) > I$ ), and, similarly,  $I$  cannot be greater than  $1/2$  (or there would exist a  $P_n$  such that  $U(f, P_n) < I$ ).

Thus  $I = 1/2$  and  $\int_0^1 x \, dx = 1/2$ .

8.  $f(x) = 1 - x$  on  $[0, 2]$ .  $P_n = \{0, \frac{2}{n}, \frac{4}{n}, \dots, \frac{2n-2}{n}, \frac{2n}{n}\}$ . We have

$$\begin{aligned} L(f, P_n) &= \frac{2}{n} \left( \left(1 - \frac{2}{n}\right) + \left(1 - \frac{4}{n}\right) + \dots + \left(1 - \frac{2n}{n}\right) \right) \\ &= \frac{2}{n} - \frac{4}{n^2} \sum_{i=1}^n i \\ &= 2 - \frac{4}{n^2} \frac{n(n+1)}{2} = -\frac{2}{n} \rightarrow 0 \text{ as } n \rightarrow \infty, \\ U(f, P_n) &= \frac{2}{n} \left( \left(1 - \frac{0}{n}\right) + \left(1 - \frac{2}{n}\right) + \dots + \left(1 - \frac{2n-2}{n}\right) \right) \\ &= \frac{2}{n} - \frac{4}{n^2} \sum_{i=0}^{n-1} i \\ &= 2 - \frac{4}{n^2} \frac{(n-1)n}{2} = \frac{2}{n} \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

Thus  $\int_0^2 (1-x) dx = 0$ .

9.  $f(x) = x^3$  on  $[0, 1]$ .  $P_n = \{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, \frac{n}{n}\}$ . We have (using the result of Exercise 51 (or 52) of Section 6.1)

$$\begin{aligned} L(f, P_n) &= \frac{1}{n} \left( \left(\frac{0}{n}\right)^3 + \left(\frac{1}{n}\right)^3 + \dots + \left(\frac{n-1}{n}\right)^3 \right) \\ &= \frac{1}{n^4} \sum_{i=0}^{n-1} i^3 = \frac{1}{n^4} \frac{(n-1)^2 n^2}{4} \\ &= \frac{1}{4} \left(\frac{n-1}{n}\right)^2 \rightarrow \frac{1}{4} \text{ as } n \rightarrow \infty, \\ U(f, P_n) &= \frac{1}{n} \left( \left(\frac{1}{n}\right)^3 + \left(\frac{2}{n}\right)^3 + \dots + \left(\frac{n}{n}\right)^3 \right) \\ &= \frac{1}{n^4} \sum_{i=1}^n i^3 = \frac{1}{n^4} \frac{n^2(n+1)^2}{4} \\ &= \frac{1}{4} \left(\frac{n+1}{n}\right)^2 \rightarrow \frac{1}{4} \text{ as } n \rightarrow \infty. \end{aligned}$$

Thus  $\int_0^1 x^3 dx = \frac{1}{4}$ .

10.  $f(x) = e^x$  on  $[0, 3]$ .  $P_n = \{0, \frac{3}{n}, \frac{6}{n}, \dots, \frac{3n-3}{n}, \frac{3n}{n}\}$ . We have (using the result of Exercise 51 (or 52) of Section 6.1)

$$\begin{aligned} L(f, P_n) &= \frac{3}{n} \left( e^{0/n} + e^{3/n} + e^{6/n} + \dots + e^{3(n-1)/n} \right) \\ &= \frac{3}{n} \frac{e^{3n/n} - 1}{e^{3/n} - 1} = \frac{3(e^3 - 1)}{n(e^{3/n} - 1)}, \\ U(f, P_n) &= \frac{3}{n} \left( e^{3/n} + e^{6/n} + e^{9/n} + \dots + e^{3n/n} \right) = e^{3/n} L(f, P_n). \end{aligned}$$

By l'Hôpital's Rule,

$$\begin{aligned} \lim_{n \rightarrow \infty} n(e^{3/n} - 1) &= \lim_{n \rightarrow \infty} \frac{e^{3/n} - 1}{1/n} \\ &= \lim_{n \rightarrow \infty} \frac{e^{3/n}(-3/n^2)}{-1/n^2} = \lim_{n \rightarrow \infty} \frac{3e^{3/n}}{1} = 3. \end{aligned}$$

Thus

$$\lim_{n \rightarrow \infty} L(f, P_n) = \lim_{n \rightarrow \infty} U(f, P_n) = e^3 - 1 = \int_0^3 e^x dx.$$

11.  $\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{n} \sqrt{\frac{i}{n}} = \int_0^1 \sqrt{x} dx$

12.  $\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{n} \sqrt{\frac{i-1}{n}} = \int_0^1 \sqrt{x} dx$

13.  $\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{\pi}{n} \sin \frac{\pi i}{n} = \int_0^\pi \sin x dx$

14.  $\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{2}{n} \ln \left(1 + \frac{2i}{n}\right) = \int_0^2 \ln(1+x) dx$

15.  $\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{n} \tan^{-1} \left(\frac{2i-1}{2n}\right) = \int_0^1 \tan^{-1} x dx$

Note that  $\frac{2i-1}{2n}$  is the midpoint of  $\left[\frac{i-1}{n}, \frac{i}{n}\right]$ .

16.  $\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{n}{n^2 + i^2} = \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{n} \frac{1}{1 + (i/n)^2} = \int_0^1 \frac{dx}{1+x^2}$

17. Let  $\Delta x = \frac{b-a}{n}$  and  $x_i = a + i\Delta x$  where  $1 \leq i \leq n-1$ . Since  $f$  is continuous and nondecreasing,

$$\begin{aligned} L(f, P_n) &= f(a)\Delta x + f(x_1)\Delta x + \dots + f(x_{n-1})\Delta x \\ &= \frac{b-a}{n} \left[ f(a) + \sum_{i=1}^{n-1} f(x_i) \right], \\ U(f, P_n) &= f(x_1)\Delta x + f(x_2)\Delta x + \dots + f(x_{n-1})\Delta x + f(b)\Delta x \\ &= \frac{b-a}{n} \left[ \sum_{i=1}^{n-1} f(x_i) + f(b) \right]. \end{aligned}$$

Thus,

$$\begin{aligned} U(f, P_n) - L(f, P_n) &= \frac{b-a}{n} \left[ \sum_{i=1}^{n-1} f(x_i) + f(b) - f(a) - \sum_{i=1}^{n-1} f(x_i) \right] \\ &= \frac{(b-a)(f(b) - f(a))}{n}. \end{aligned}$$

Since

$$\lim_{n \rightarrow \infty} [U(f, P_n) - L(f, P - n)] = 0,$$

therefore  $f$  must be integrable on  $[a, b]$ .

18.  $P = \{x_0 < x_1 < \dots < x_n\}$ ,  
 $P' = \{x_0 < x_1 < \dots < x_{j-1} < x' < x_j < \dots < x_n\}$ .  
 Let  $m_i$  and  $M_i$  be, respectively, the minimum and maximum values of  $f(x)$  on the interval  $[x_{i-1}, x_i]$ , for  $1 \leq i \leq n$ . Then

$$L(f, P) = \sum_{i=1}^n m_i(x_i - x_{i-1}),$$

$$U(f, P) = \sum_{i=1}^n M_i(x_i - x_{i-1}).$$

If  $m'_j$  and  $M'_j$  are the minimum and maximum values of  $f(x)$  on  $[x_{j-1}, x']$ , and if  $m''_j$  and  $M''_j$  are the corresponding values for  $[x', x_j]$ , then

$$m'_j \geq m_j, \quad m''_j \geq m_j, \quad M'_j \leq M_j, \quad M''_j \leq M_j.$$

Therefore we have

$$m_j(x_j - x_{j-1}) \leq m'_j(x' - x_{j-1}) + m''_j(x_j - x'),$$

$$M_j(x_j - x_{j-1}) \geq M'_j(x' - x_{j-1}) + M''_j(x_j - x').$$

Hence  $L(f, P) \leq L(f, P')$  and  $U(f, P) \geq U(f, P')$ .

If  $P''$  is any refinement of  $P$  we can add the new points in  $P''$  to those in  $P$  one at a time, and thus obtain

$$L(f, P) \leq L(f, P''), \quad U(f, P'') \leq U(f, P).$$

### Section 5.4 Properties of the Definite Integral (page 296)

1. 
$$\int_a^b f(x) dx + \int_b^c f(x) dx + \int_c^a f(x) dx$$
  

$$= \int_a^c f(x) dx - \int_a^c f(x) dx = 0$$

2. 
$$\int_0^2 3f(x) dx + \int_1^3 3f(x) dx - \int_0^3 2f(x) dx$$
  

$$- \int_1^2 3f(x) dx$$
  

$$= \int_0^1 (3-2)f(x) dx + \int_1^2 (3+3-2-3)f(x) dx$$
  

$$+ \int_2^3 (3-2)f(x) dx$$
  

$$= \int_0^3 f(x) dx$$

3. 
$$\int_{-2}^2 (x+2) dx = \frac{1}{2}(4)(4) = 8$$

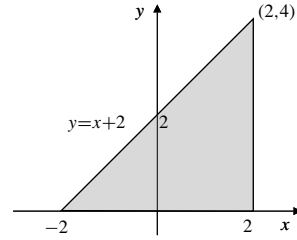


Fig. 5.4.3

4. 
$$\int_0^2 (3x+1) dx = \text{shaded area} = \frac{1}{2}(1+7)(2) = 8$$

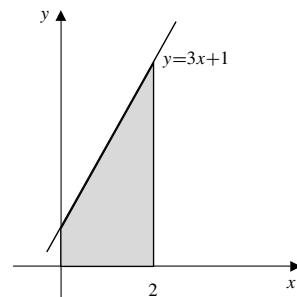


Fig. 5.4.4

5. 
$$\int_a^b x dx = \frac{b^2}{2} - \frac{a^2}{2}$$

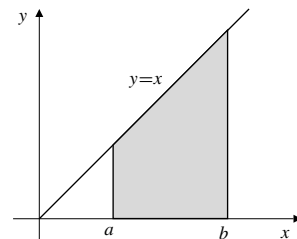


Fig. 5.4.5

6. 
$$\int_{-1}^2 (1-2x) dx = A_1 - A_2 = 0$$

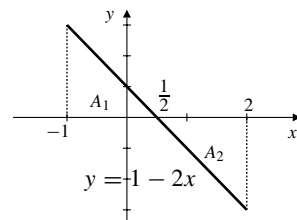


Fig. 5.4.6

7. 
$$\int_{-\sqrt{2}}^{\sqrt{2}} \sqrt{2-t^2} dt = \frac{1}{2}\pi(\sqrt{2})^2 = \pi$$

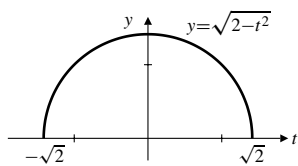


Fig. 5.4.7

8.  $\int_{-\sqrt{2}}^0 \sqrt{2-x^2} dx = \text{quarter disk} = \frac{1}{4}\pi(\sqrt{2})^2 = \frac{\pi}{2}$
9.  $\int_{-\pi}^{\pi} \sin(x^3) dx = 0$ . (The integrand is an odd function and the interval of integration is symmetric about  $x = 0$ .)
10.  $\int_{-a}^a (a - |s|) ds = \text{shaded area} = 2(\frac{1}{2}a^2) = a^2$

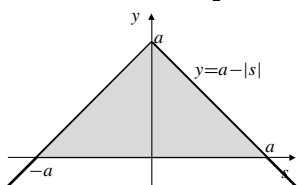


Fig. 5.4.10

11.  $\int_{-1}^1 (u^5 - 3u^3 + \pi) du = \pi \int_{-1}^1 du = 2\pi$

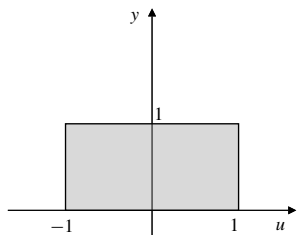


Fig. 5.4.11

12. Let  $y = \sqrt{2x - x^2} \Rightarrow y^2 + (x - 1)^2 = 1$ .  
 $\int_0^2 \sqrt{2x - x^2} dx = \text{shaded area} = \frac{1}{2}\pi(1)^2 = \frac{\pi}{2}$ .

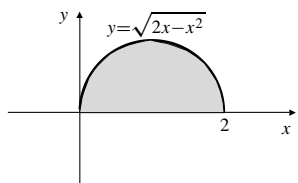


Fig. 5.4.12

13.  $\int_{-4}^4 (e^x - e^{-x}) dx = 0$  (odd function, symmetric interval)
14.  $\int_{-3}^3 (2+t)\sqrt{9-t^2} dt = 2 \int_{-3}^3 \sqrt{9-t^2} dt + \int_{-3}^3 t\sqrt{9-t^2} dt$   
 $= 2\left(\frac{1}{2}\pi 3^2\right) + 0 = 9\pi$

15.  $\int_0^1 \sqrt{4-x^2} dx = \text{area } A_1 \text{ in figure below}$   
 $= \frac{1}{4} \text{area of circle} - \text{area } A_2$   
 (see #14 below)  
 $= \frac{1}{4}(\pi 2^2) - \left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)$   
 $= \frac{\pi}{3} + \frac{\sqrt{3}}{2}$

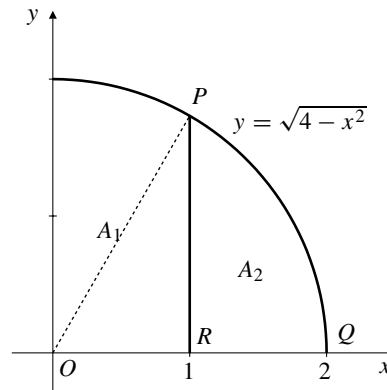


Fig. 5.4.15

16.  $\int_1^2 \sqrt{4-x^2} dx = \text{area } A_2 \text{ in figure above}$   
 $= \text{area sector } POQ - \text{area triangle } POR$   
 $= \frac{1}{6}(\pi 2^2) - \frac{1}{2}(1)\sqrt{3}$   
 $= \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$

17.  $\int_0^2 6x^2 dx = 6 \int_0^2 x^2 dx = 6 \frac{3^3}{3} = 16$
18.  $\int_2^3 (x^2 - 4) dx = \int_0^3 x^2 dx - \int_0^2 x^2 dx - 4(3 - 2)$   
 $= \frac{3^3}{3} - \frac{2^3}{3} - 4 = \frac{7}{3}$
19.  $\int_{-2}^2 (4 - t^2) dt = 2 \int_0^2 (4 - t^2) dt$   
 $= 2\left(2(4) - \frac{2^3}{3}\right) = \frac{32}{3}$

20.  $\int_0^2 (v^2 - v) dv = \frac{2^3}{3} - \frac{2^2}{2} = \frac{2}{3}$
21.  $\int_0^1 (x^2 + \sqrt{1-x^2}) dx = \frac{1^3}{3} + \frac{1}{4}(\pi 1^2)$   
 $= \frac{1}{3} + \frac{\pi}{4}$

$$\begin{aligned}
 22. \quad \int_{-6}^6 x^2(2 + \sin x) dx &= \int_{-6}^6 2x^2 dx + \int_{-6}^6 x^2 \sin x dx \\
 &= 4 \int_0^6 x^2 dx + 0 = \frac{4}{3}(6^3) = 288
 \end{aligned}$$

$$23. \quad \int_1^2 \frac{1}{x} dx = \ln 2$$

$$\begin{aligned}
 24. \quad \int_2^4 \frac{1}{t} dt &= \int_1^4 \frac{1}{t} dt - \int_1^2 \frac{1}{t} dt \\
 &= \ln 4 - \ln 2 = \ln(4/2) = \ln 2
 \end{aligned}$$

$$25. \quad \int_{1/3}^1 \frac{1}{t} dt = - \int_1^{1/3} \frac{1}{t} dt = -\ln \frac{1}{3} = \ln 3$$

$$\begin{aligned}
 26. \quad \int_{1/4}^3 \frac{1}{s} ds &= \int_1^3 \frac{1}{s} ds - \int_1^{1/4} \frac{1}{s} ds \\
 &= \ln 3 - \ln \frac{1}{4} = \ln 3 + \ln 4 = \ln 12
 \end{aligned}$$

$$\begin{aligned}
 27. \quad \text{Average} &= \frac{1}{4-0} \int_0^4 (x+2) dx \\
 &= \frac{1}{4} \left[ \frac{1}{2}(4^2) + 2(4) \right] = 4
 \end{aligned}$$

$$\begin{aligned}
 28. \quad \text{Average} &= \frac{1}{b-a} \int_a^b (x+2) dx \\
 &= \frac{1}{b-a} \left[ \frac{1}{2}(b^2 - a^2) + 2(b-a) \right] \\
 &= \frac{1}{2}(b+a) + 2 = \frac{4+a+b}{2}
 \end{aligned}$$

$$\begin{aligned}
 29. \quad \text{Average} &= \frac{1}{\pi - (-\pi)} \int_{-\pi}^{\pi} (1 + \sin t) dt \\
 &= \frac{1}{2\pi} \left[ \int_{-\pi}^{\pi} 1 dt + \int_{-\pi}^{\pi} \sin t dt \right] \\
 &= \frac{1}{2\pi} [2\pi + 0] = 1
 \end{aligned}$$

$$30. \quad \text{Average} = \frac{1}{3-0} \int_0^3 x^2 dx = \frac{1}{3} \frac{3^3}{3} = 3$$

$$\begin{aligned}
 31. \quad \text{Average value} &= \frac{1}{2-0} \int_0^2 (4-x^2)^{1/2} dx \\
 &= \frac{1}{2} (\text{shaded area}) \\
 &= \frac{1}{2} \left( \frac{1}{4} \pi (2)^2 \right) = \frac{\pi}{2}
 \end{aligned}$$

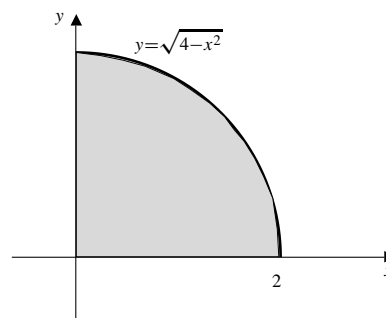


Fig. 5.4.31

$$\begin{aligned}
 32. \quad \text{Average value} &= \frac{1}{2 - (1/2)} \int_{1/2}^2 \frac{1}{s} ds \\
 &= \frac{2}{3} \left( \ln 2 - \ln \frac{1}{2} \right) = \frac{4}{3} \ln 2
 \end{aligned}$$

$$33. \quad \int_{-1}^2 \operatorname{sgn} x dx = 2 - 1 = 1$$

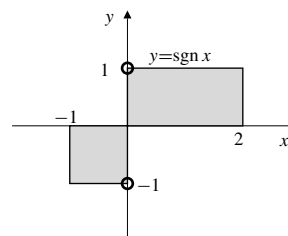


Fig. 5.4.33

34. Let

$$f(x) = \begin{cases} 1+x & \text{if } x < 0 \\ 2 & \text{if } x \geq 0. \end{cases}$$

Then

$$\begin{aligned}
 \int_{-3}^2 f(x) dx &= \text{area}(1) + \text{area}(2) - \text{area}(3) \\
 &= (2 \times 2) + \frac{1}{2}(1)(1) - \frac{1}{2}(2)(2) = 2\frac{1}{2}.
 \end{aligned}$$

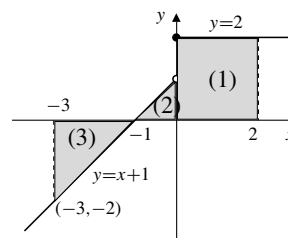


Fig. 5.4.34

$$\begin{aligned}
 35. \quad \int_0^2 g(x) dx &= \int_0^1 x^2 dx + \int_1^2 x dx \\
 &= \frac{1^3}{3} + \frac{2^2 - 1^2}{2} = \frac{11}{6}
 \end{aligned}$$

$$\begin{aligned}
 36. \int_0^3 |2-x| dx &= \int_0^2 (2-x) dx + \int_2^3 (x-2) dx \\
 &= \left(2x - \frac{x^2}{2}\right) \Big|_0^2 + \left(\frac{x^2}{2} - 2x\right) \Big|_2^3 \\
 &= 4 - 2 - 0 + \frac{9}{2} - 6 - 2 + 4 = \frac{5}{2}
 \end{aligned}$$

$$\begin{aligned}
 37. I &= \int_0^2 \sqrt{4-x^2} \operatorname{sgn}(x-1) dx \\
 &= \text{area } A_1 - \text{area } A_2. \\
 \text{Area } A_1 &= \frac{1}{6}\pi 2^2 - \frac{1}{2}(1)(\sqrt{3}) = \frac{2}{3}\pi - \frac{1}{2}\sqrt{3}. \\
 \text{Area } A_2 &= \frac{1}{4}\pi 2^2 - \text{area } A_1 = \frac{1}{3}\pi + \frac{1}{2}\sqrt{3}. \\
 \text{Therefore } I &= (\pi/3) - \sqrt{3}.
 \end{aligned}$$

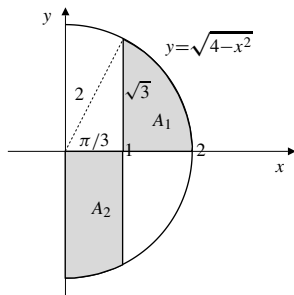


Fig. 5.4.37

$$38. \int_0^{3.5} [x] dx = \text{shaded area} = 1 + 2 + 1.5 = 4.5.$$

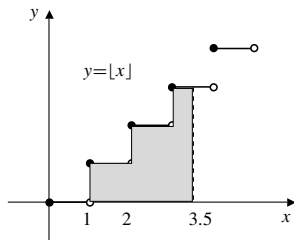


Fig. 5.4.38

39.

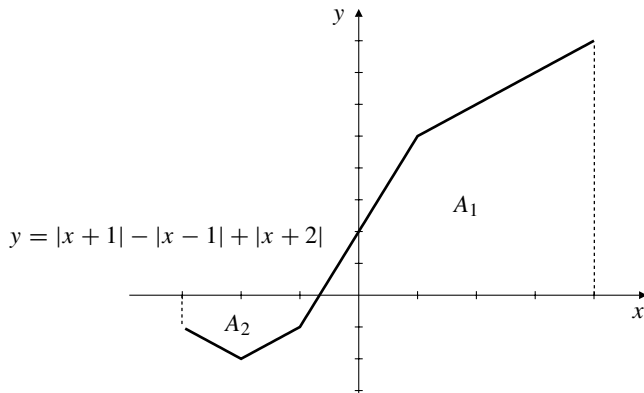


Fig. 5.4.39

$$\begin{aligned}
 &\int_{-3}^4 (|x+1| - |x-1| + |x+2|) dx \\
 &= \text{area } A_1 - \text{area } A_2 \\
 &= \frac{1}{2} \frac{5}{3} (5) + \frac{5+8}{2} (3) - \frac{1+2}{2} (1) - \frac{1+2}{2} (1) - \frac{1}{2} \frac{1}{3} (1) = \frac{41}{2}
 \end{aligned}$$

40.

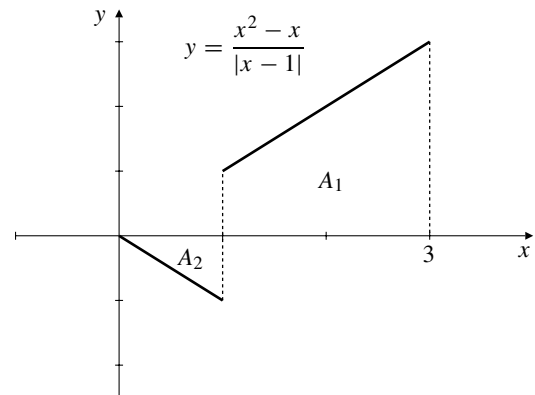


Fig. 5.4.40

$$\begin{aligned}
 &\int_0^3 \frac{x^2 - x}{|x-1|} dx \\
 &= \text{area } A_1 - \text{area } A_2 \\
 &= \frac{1+3}{2} (2) - \frac{1}{2} (1)(1) = \frac{7}{2}
 \end{aligned}$$

$$\begin{aligned}
 41. \text{Average} &= \frac{1}{4} \int_{-2}^2 |x+1| \operatorname{sgn} x dx \\
 &= \frac{1}{4} \left( \int_0^2 (x+1) dx - \int_{-2}^0 |x+1| dx \right) \\
 &= \frac{1}{4} \left( \frac{1+3}{2} \times 2 - 2 \times \frac{1}{2} \times 1 \times 1 \right) \\
 &= 1 - \frac{1}{4} = \frac{3}{4}.
 \end{aligned}$$

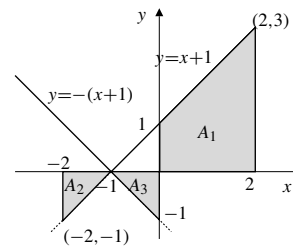


Fig. 5.4.41

$$\begin{aligned}
 42. \quad \int_a^b (f(x) - \bar{f}) dx &= \int_a^b f(x) dx - \int_a^b \bar{f} dx \\
 &= (b-a)\bar{f} - \bar{f} \int_a^b dx \\
 &= (b-a)\bar{f} - (b-a)\bar{f} = 0
 \end{aligned}$$

$$\begin{aligned}
 43. \quad \int_a^b (f(x) - k)^2 dx &= \int_a^b (f(x))^2 dx - 2k \int_a^b f(x) dx + k^2 \int_a^b dx \\
 &= \int_a^b (f(x))^2 dx - 2k(b-a)\bar{f} + k^2(b-a) \\
 &= (b-a)(k - \bar{f})^2 + \int_a^b (f(x))^2 dx - (b-a)\bar{f}^2
 \end{aligned}$$

This is minimum if  $k = \bar{f}$ .

### Section 5.5 The Fundamental Theorem of Calculus (page 301)

$$1. \quad \int_0^2 x^3 dx = \frac{x^4}{4} \Big|_0^2 = \frac{16-0}{4} = 4$$

$$2. \quad \int_0^4 \sqrt{x} dx = \frac{2}{3} x^{3/2} \Big|_0^4 = \frac{16}{3}$$

$$3. \quad \int_{1/2}^1 \frac{1}{x^2} dx = \frac{-1}{x} \Big|_{1/2}^1 = -1 - (-2) = 1$$

$$\begin{aligned}
 4. \quad \int_{-2}^{-1} \left( \frac{1}{x^2} - \frac{1}{x^3} \right) dx &= \left( -\frac{1}{x} + \frac{1}{2x^2} \right) \Big|_{-2}^{-1} \\
 &= 1 + \frac{1}{2} - \left( \frac{1}{2} + \frac{1}{8} \right) = \frac{7}{8}
 \end{aligned}$$

$$5. \quad \int_{-1}^2 (3x^2 - 4x + 2) dx = (x^3 - 2x^2 + 2x) \Big|_{-1}^2 = 9$$

$$6. \quad \int_1^2 \left( \frac{2}{x^3} - \frac{x^3}{2} \right) dx = \left( -\frac{1}{x^2} - \frac{x^4}{8} \right) \Big|_1^2 = -9/8$$

$$\begin{aligned}
 7. \quad \int_{-2}^2 (x^2 + 3)^2 dx &= 2 \int_0^2 (x^4 + 6x^2 + 9) dx \\
 &= 2 \left( \frac{x^5}{5} + 2x^3 + 9x \right) \Big|_0^2 \\
 &= 2 \left( \frac{32}{5} + 16 + 18 \right) = \frac{404}{5}
 \end{aligned}$$

$$\begin{aligned}
 8. \quad \int_4^9 \left( \sqrt{x} - \frac{1}{\sqrt{x}} \right) dx &= \frac{2}{3} x^{3/2} - 2\sqrt{x} \Big|_4^9 \\
 &= \left[ \frac{2}{3}(9)^{3/2} - 2\sqrt{9} \right] - \left[ \frac{2}{3}(4)^{3/2} - 2\sqrt{4} \right] = \frac{32}{3}
 \end{aligned}$$

$$\begin{aligned}
 9. \quad \int_{-\pi/4}^{-\pi/6} \cos x dx &= \sin x \Big|_{-\pi/4}^{-\pi/6} \\
 &= -\frac{1}{2} + \frac{1}{\sqrt{2}} = \frac{2 - \sqrt{2}}{2\sqrt{2}}
 \end{aligned}$$

$$10. \quad \int_0^{\pi/3} \sec^2 \theta d\theta = \tan \theta \Big|_0^{\pi/3} = \tan \frac{\pi}{3} = \sqrt{3}$$

$$11. \quad \int_{\pi/4}^{\pi/3} \sin \theta d\theta = -\cos \theta \Big|_{\pi/4}^{\pi/3} = \frac{\sqrt{2}-1}{2}$$

$$12. \quad \int_0^{2\pi} (1 + \sin u) du = (u - \cos u) \Big|_0^{2\pi} = 2\pi$$

$$13. \quad \int_{-\pi}^{\pi} e^x dx = e^x \Big|_{-\pi}^{\pi} = e^{\pi} - e^{-\pi}$$

$$14. \quad \int_{-2}^2 (e^x - e^{-x}) dx = 0 \text{ (odd function, symmetric interval)}$$

$$15. \quad \int_0^e a^x dx = \frac{a^x}{\ln a} \Big|_0^e = \frac{a^e - 1}{\ln a}$$

$$16. \quad \int_{-1}^1 2^x dx = \frac{2^x}{\ln 2} \Big|_{-1}^1 = \frac{2}{\ln 2} - \frac{1}{2\ln 2} = \frac{3}{2\ln 2}$$

$$17. \quad \int_{-1}^1 \frac{dx}{1+x^2} = \tan^{-1} x \Big|_{-1}^1 = \frac{\pi}{2}$$

$$18. \quad \int_0^{1/2} \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x \Big|_0^{1/2} = \frac{\pi}{6}$$

$$\begin{aligned}
 19. \quad \int_{-1}^1 \frac{dx}{\sqrt{4-x^2}} &= \sin^{-1} \frac{x}{2} \Big|_{-1}^1 \\
 &= \sin^{-1} \frac{1}{2} - \sin^{-1} \left( -\frac{1}{2} \right) \\
 &= \frac{\pi}{6} - \left( -\frac{\pi}{6} \right) = \frac{\pi}{3}
 \end{aligned}$$

$$20. \quad \int_{-2}^0 \frac{dx}{4+x^2} = \frac{1}{2} \tan^{-1} \frac{x}{2} \Big|_{-2}^0 = 0 - \frac{1}{2} \tan^{-1}(-1) = \frac{\pi}{8}$$

$$21. \quad \text{Area } R = \int_0^1 x^4 dx = \frac{x^5}{5} \Big|_0^1 = \frac{1}{5} \text{ sq. units.}$$

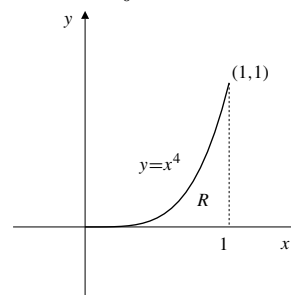


Fig. 5.5.21

22. 
$$\text{Area} = \int_e^{e^2} \frac{1}{x} dx = \ln x \Big|_e^{e^2}$$

$$= \ln e^2 - \ln e = 2 - 1 = 1 \text{ sq. units.}$$

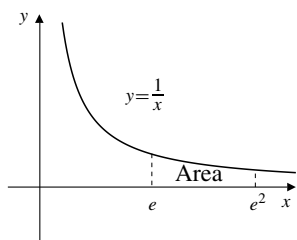


Fig. 5.5.22

23. 
$$\text{Area } R = - \int_0^4 (x^2 - 4x) dx$$

$$= - \left( \frac{x^3}{3} - 2x^2 \right) \Big|_0^4$$

$$= - \left( \frac{64}{3} - 32 \right) = \frac{32}{3} \text{ sq. units.}$$

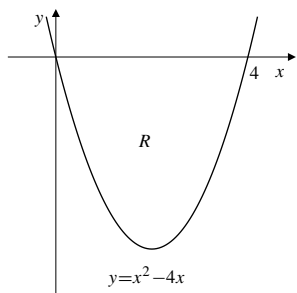


Fig. 5.5.23

24. Since  $y = 5 - 2x - 3x^2 = (5 + 3x)(1 - x)$ , therefore  $y = 0$  at  $x = -\frac{5}{3}$  and 1, and  $y > 0$  if  $-\frac{5}{3} < x < 1$ . Thus, the area is

$$\int_{-1}^1 (5 - 2x - 3x^2) dx = 2 \int_0^1 (5 - 3x^2) dx$$

$$= 2(5x - x^3) \Big|_0^1$$

$$= 2(5 - 1) = 8 \text{ sq. units.}$$

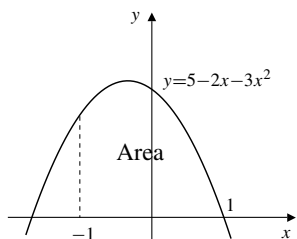


Fig. 5.5.24

25. For intersection of  $y = x^2 - 3x + 3$  and  $y = 1$ , we have

$$x^2 - 3x + 3 = 1$$

$$x^2 - 3x + 2 = 0$$

$$(x - 2)(x - 1) = 0.$$

Thus  $x = 1$  or  $x = 2$ . The indicated region has area

$$\text{Area } R = 1 - \int_1^2 (x^2 - 3x + 3) dx$$

$$= 1 - \left( \frac{x^3}{3} - \frac{3x^2}{2} + 3x \right) \Big|_1^2$$

$$= 1 - \left( \frac{8}{3} - 6 + 6 - \left[ \frac{1}{3} - \frac{3}{2} + 3 \right] \right) = \frac{1}{6} \text{ sq. units.}$$

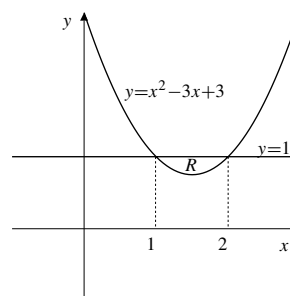


Fig. 5.5.25

26. Since  $y = \sqrt{x}$  and  $y = \frac{x}{2}$  intersect where  $\sqrt{x} = \frac{x}{2}$ , that is, at  $x = 0$  and  $x = 4$ , thus,

$$\text{Area} = \int_0^4 \sqrt{x} dx - \int_0^4 \frac{x}{2} dx$$

$$= \frac{2}{3} x^{3/2} \Big|_0^4 - \frac{x^2}{4} \Big|_0^4$$

$$= \frac{16}{3} - \frac{16}{4} = \frac{4}{3} \text{ sq. units.}$$

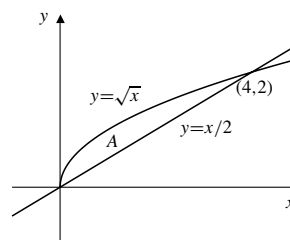


Fig. 5.5.26

27. Area  $R = 2 \times$  shaded area

$$= 2 \left( \frac{1}{2} - \int_0^1 x^2 dx \right)$$

$$= 2 \left( \frac{1}{2} - \frac{1}{3} \right) = \frac{1}{3} \text{ sq. units.}$$



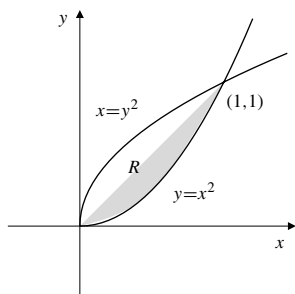


Fig. 5.5.27

28. The two graphs intersect at  $(\pm 3, 3)$ , thus

$$\begin{aligned} \text{Area} &= 2 \int_0^3 (12 - x^2) dx - 2 \int_0^3 x dx \\ &= 2 \left( 12x - \frac{1}{3}x^3 \right) \Big|_0^3 - 2 \left( \frac{1}{2}x^2 \right) \Big|_0^3 \\ &= 2(36 - 9) - 9 = 45 \text{ sq. units.} \end{aligned}$$

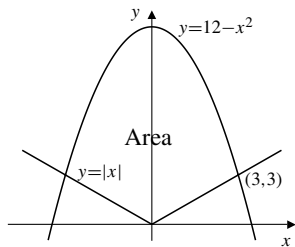


Fig. 5.5.28

29. 
$$\text{Area } R = \int_0^1 x^{1/3} dx - \int_0^1 x^{1/2} dx$$

$$= \frac{3}{4}x^{4/3} \Big|_0^1 - \frac{2}{3}x^{3/2} \Big|_0^1 = \frac{3}{4} - \frac{2}{3} = \frac{1}{12} \text{ sq. units.}$$

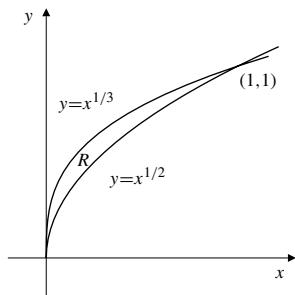


Fig. 5.5.29

30. 
$$\text{Area} = \int_{-a}^0 e^{-x} dx = -e^{-x} \Big|_{-a}^0 = e^a - 1 \text{ sq. units.}$$

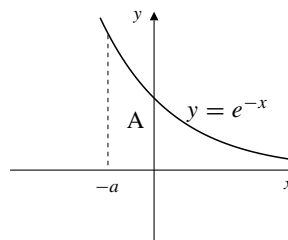


Fig. 5.5.30

31. 
$$\text{Area } R = \int_0^{2\pi} (1 - \cos x) dx$$

$$= (x - \sin x) \Big|_0^{2\pi} = 2\pi \text{ sq. units.}$$

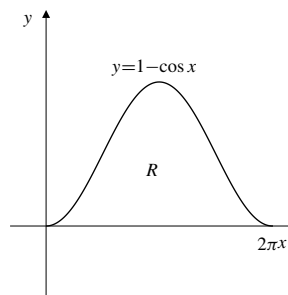


Fig. 5.5.31

32. 
$$\text{Area} = \int_1^{27} x^{-1/3} dx = \frac{3}{2}x^{2/3} \Big|_1^{27}$$

$$= \frac{3}{2}(27)^{2/3} - \frac{3}{2} = 12 \text{ sq. units.}$$

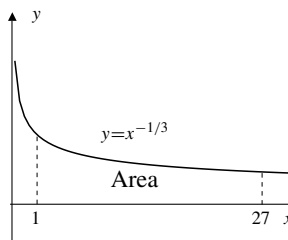


Fig. 5.5.32

33. 
$$\int_0^{3\pi/2} |\cos x| dx = \int_0^{\pi/2} \cos x dx - \int_{\pi/2}^{3\pi/2} \cos x dx$$

$$= \sin x \Big|_0^{\pi/2} - \sin x \Big|_{\pi/2}^{3\pi/2}$$

$$= 1 + 1 + 1 = 3$$

34. 
$$\int_1^3 \frac{\text{sgn}(x-2)}{x^2} dx = -\int_1^2 \frac{dx}{x^2} + \int_2^3 \frac{dx}{x^2}$$

$$= \frac{1}{x} \Big|_1^2 - \frac{1}{x} \Big|_2^3 = -\frac{1}{3}$$

35. Average value

$$\begin{aligned} &= \frac{1}{2} \int_0^2 (1 + x + x^2 + x^3) dx \\ &= \frac{1}{2} \left( x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} \right) \Big|_0^2 \\ &= \frac{1}{2} \left( 2 + 2 + \frac{8}{3} + 4 \right) = \frac{16}{3}. \end{aligned}$$

36. Average value =  $\frac{1}{2 - (-2)} \int_{-2}^2 e^{3x} dx$

$$\begin{aligned} &= \frac{1}{4} \left( \frac{1}{3} e^{3x} \right) \Big|_{-2}^2 \\ &= \frac{1}{12} (e^6 - e^{-6}). \end{aligned}$$

37. Avg. =  $\frac{1}{1/\ln 2} \int_0^{1/\ln 2} 2^x dx = (\ln 2) \frac{2^x}{\ln 2} \Big|_0^{1/\ln 2} = e - 1$

38. Since

$$g(t) = \begin{cases} 0, & \text{if } 0 \leq t \leq 1, \\ 1, & \text{if } 1 < t \leq 3, \end{cases}$$

the average value of  $g(t)$  over  $[0,3]$  is

$$\begin{aligned} \frac{1}{3} \left[ \int_0^1 (0) dt + \int_1^3 1 dt \right] &= \frac{1}{3} \left[ 0 + t \Big|_1^3 \right] \\ &= \frac{1}{3} (3 - 1) = \frac{2}{3}. \end{aligned}$$

39.  $\frac{d}{dx} \int_2^x \frac{\sin t}{t} dt = \frac{\sin x}{x}$

40.  $\frac{d}{dt} \int_t^3 \frac{\sin x}{x} dx = \frac{d}{dt} \left[ - \int_3^t \frac{\sin x}{x} dx \right] = -\frac{\sin t}{t}$

41.  $\frac{d}{dx} \int_{x^2}^0 \frac{\sin t}{t} dt = -\frac{d}{dx} \int_0^{x^2} \frac{\sin t}{t} dt$   
 $= -2x \frac{\sin x^2}{x^2} = -2 \frac{\sin x^2}{x}$

42.  $\frac{d}{dx} x^2 \int_0^{x^2} \frac{\sin u}{u} du$   
 $= 2x \int_0^{x^2} \frac{\sin u}{u} du + x^2 \frac{d}{dx} \int_0^{x^2} \frac{\sin u}{u} du$   
 $= 2x \int_0^{x^2} \frac{\sin u}{u} du + x^2 \left[ \frac{2x \sin x^2}{x^2} \right]$   
 $= 2x \int_0^{x^2} \frac{\sin u}{u} du + 2x \sin(x^2)$

43.  $\frac{d}{dt} \int_{-\pi}^t \frac{\cos y}{1 + y^2} dy = \frac{\cos t}{1 + t^2}$

44.  $\frac{d}{d\theta} \int_{\sin \theta}^{\cos \theta} \frac{1}{1 - x^2} dx$   
 $= \frac{d}{d\theta} \left[ \int_a^{\cos \theta} \frac{1}{1 - x^2} dx - \int_a^{\sin \theta} \frac{1}{1 - x^2} dx \right]$   
 $= \frac{-\sin \theta}{1 - \cos^2 \theta} - \frac{\cos \theta}{1 - \sin^2 \theta}$   
 $= \frac{-1}{\sin \theta} - \frac{1}{\cos \theta} = -\csc \theta - \sec \theta$

45.  $F(t) = \int_0^t \cos(x^2) dx$

$$\begin{aligned} F(\sqrt{x}) &= \int_0^{\sqrt{x}} \cos(u^2) du \\ \frac{d}{dx} F(\sqrt{x}) &= \cos x \frac{1}{2\sqrt{x}} = \frac{\cos x}{2\sqrt{x}} \end{aligned}$$

46.  $H(x) = 3x \int_4^{x^2} e^{-\sqrt{t}} dt$

$$H'(x) = 3 \int_4^{x^2} e^{-\sqrt{t}} dt + 3x(2x e^{-|x|})$$

$$\begin{aligned} H'(2) &= 3 \int_4^4 e^{-\sqrt{t}} dt + 3(2)(4e^{-2}) \\ &= 3(0) + 24e^{-2} = \frac{24}{e^2} \end{aligned}$$

47.  $f(x) = \pi + \pi \int_1^x f(t) dt$

$$\begin{aligned} f'(x) &= \pi f(x) \implies f(x) = C e^{\pi x} \\ \pi &= f(1) = C e^{\pi} \implies C = \pi e^{-\pi} \\ f(x) &= \pi e^{\pi(x-1)}. \end{aligned}$$

48.  $f(x) = 1 - \int_0^x f(t) dt$

$$\begin{aligned} f'(x) &= -f(x) \implies f(x) = C e^{-x} \\ 1 &= f(0) = C \\ f(x) &= e^{-x}. \end{aligned}$$

49. The function  $1/x^2$  is not *defined* (and therefore not continuous) at  $x = 0$ , so the Fundamental Theorem of Calculus cannot be applied to it on the interval  $[-1, 1]$ . Since  $1/x^2 > 0$  wherever it is defined, we would expect  $\int_{-1}^1 \frac{dx}{x^2}$  to be *positive* if it exists at all (which it doesn't).

50. If  $F(x) = \int_{17}^x \frac{\sin t}{1 + t^2} dt$ , then  $F'(x) = \frac{\sin x}{1 + x^2}$  and  $F(17) = 0$ .

51.  $F(x) = \int_0^{2x-x^2} \cos\left(\frac{1}{1+t^2}\right) dt$ .

Note that  $0 < \frac{1}{1+t^2} \leq 1$  for all  $t$ , and hence

$$0 < \cos(1) \leq \cos\left(\frac{1}{1+t^2}\right) \leq 1.$$

The integrand is continuous for all  $t$ , so  $F(x)$  is defined and differentiable for all  $x$ . Since

$$\lim_{x \rightarrow \pm\infty} (2x - x^2) = -\infty, \text{ therefore}$$

$$\lim_{x \rightarrow \pm\infty} F(x) = -\infty. \text{ Now}$$

$$F'(x) = (2 - 2x) \cos\left(\frac{1}{1 + (2x - x^2)^2}\right) = 0$$

only at  $x = 1$ . Therefore  $F$  must have a maximum value at  $x = 1$ , and no minimum value.

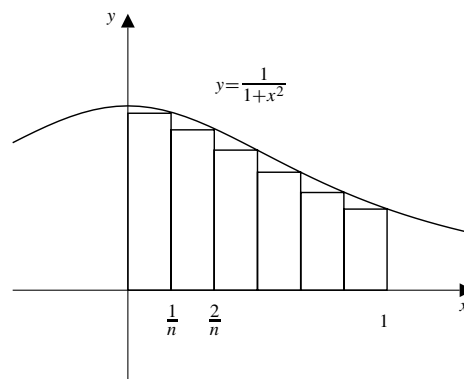


Fig. 5.5.54

$$\begin{aligned} 52. \quad & \lim_{n \rightarrow \infty} \frac{1}{n} \left[ \left(1 + \frac{1}{n}\right)^5 + \left(1 + \frac{2}{n}\right)^5 + \cdots + \left(1 + \frac{n}{n}\right)^5 \right] \\ &= \text{area below } y = x^5, \text{ above } y = 0, \\ & \quad \text{between } x = 1 \text{ and } x = 2 \\ &= \int_1^2 x^5 dx = \frac{1}{6} x^6 \Big|_1^2 = \frac{1}{6} (2^6 - 1) = \frac{21}{2} \end{aligned}$$

$$\begin{aligned} 53. \quad & \lim_{n \rightarrow \infty} \frac{\pi}{n} \left( \sin \frac{\pi}{n} + \sin \frac{2\pi}{n} + \cdots + \sin \frac{n\pi}{n} \right) \\ &= \lim_{n \rightarrow \infty} \text{sum of areas of rectangles shown in figure} \\ &= \int_0^\pi \sin x dx = -\cos x \Big|_0^\pi = 2 \end{aligned}$$

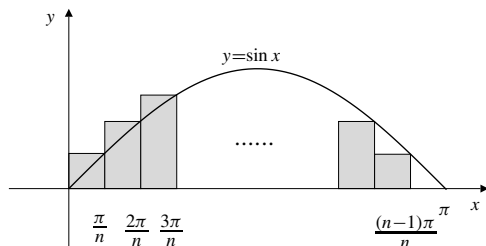


Fig. 5.5.53

$$\begin{aligned} 54. \quad & \lim_{n \rightarrow \infty} \left( \frac{n}{n^2+1} + \frac{n}{n^2+4} + \frac{n}{n^2+9} + \cdots + \frac{n}{2n^2} \right) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \left( \frac{n^2}{n^2+1} + \frac{n^2}{n^2+4} + \frac{n^2}{n^2+9} + \cdots + \frac{n^2}{2n^2} \right) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \left( \frac{1}{1 + \left(\frac{1}{n}\right)^2} + \frac{1}{1 + \left(\frac{2}{n}\right)^2} + \cdots + \frac{1}{1 + \left(\frac{n}{n}\right)^2} \right) \\ &= \text{area below } y = \frac{1}{1+x^2}, \text{ above } y = 0, \\ & \quad \text{between } x = 0 \text{ and } x = 1 \\ &= \int_0^1 \frac{1}{1+x^2} dx = \tan^{-1} x \Big|_0^1 = \frac{\pi}{4} \end{aligned}$$

### Section 5.6 The Method of Substitution (page 308)

- $$\int e^{5-2x} dx \quad \text{Let } u = 5 - 2x$$

$$du = -2 dx$$

$$= -\frac{1}{2} \int e^u du = -\frac{1}{2} e^u + C = -\frac{1}{2} e^{5-2x} + C.$$
- $$\int \cos(ax + b) dx \quad \text{Let } u = ax + b$$

$$du = a dx$$

$$= \frac{1}{a} \int \cos u du = \frac{1}{a} \sin u + C$$

$$= \frac{1}{a} \sin(ax + b) + C.$$
- $$\int \sqrt{3x+4} dx \quad \text{Let } u = 3x + 4$$

$$du = 3 dx$$

$$= \frac{1}{3} \int u^{1/2} du = \frac{2}{9} u^{3/2} + C = \frac{2}{9} (3x+4)^{3/2} + C.$$
- $$\int e^{2x} \sin(e^{2x}) dx \quad \text{Let } u = e^{2x}$$

$$du = 2e^{2x} dx$$

$$= \frac{1}{2} \int \sin u du = -\frac{1}{2} \cos u + C$$

$$= -\frac{1}{2} \cos(e^{2x}) + C.$$
- $$\int \frac{x dx}{(4x^2+1)^5} \quad \text{Let } u = 4x^2 + 1$$

$$du = 8x dx$$

$$= \frac{1}{8} \int u^{-5} du = -\frac{1}{32} u^{-4} + C = \frac{-1}{32(4x^2+1)^4} + C.$$
- $$\int \frac{\sin \sqrt{x}}{\sqrt{x}} dx \quad \text{Let } u = \sqrt{x}$$

$$du = \frac{dx}{2\sqrt{x}}$$

$$= 2 \int \sin u du = -2 \cos u + C$$

$$= -2 \cos \sqrt{x} + C.$$

7.  $\int x e^{x^2} dx$  Let  $u = x^2$   
 $du = 2x dx$   
 $= \frac{1}{2} \int e^u du = \frac{1}{2} e^u + C = \frac{1}{2} e^{x^2} + C.$
8.  $\int x^2 2^{x^3+1} dx$  Let  $u = x^3 + 1$   
 $du = 3x^2 dx$   
 $= \frac{1}{3} \int 2^u du = \frac{1}{3} \frac{2^u}{\ln 2} + C$   
 $= \frac{2^{x^3+1}}{3 \ln 2} + C.$
9.  $\int \frac{\cos x}{4 + \sin^2 x} dx$  Let  $u = \sin x$   
 $du = \cos x dx$   
 $= \int \frac{du}{4 + u^2}$   
 $= \frac{1}{2} \tan^{-1} \frac{u}{2} + C = \frac{1}{2} \tan^{-1} \left( \frac{1}{2} \sin x \right) + C.$
10.  $\int \frac{\sec^2 x}{\sqrt{1 - \tan^2 x}} dx$  Let  $u = \tan x$   
 $du = \sec^2 x dx$   
 $= \frac{du}{\sqrt{1 - u^2}}$   
 $= \sin^{-1} u + C$   
 $= \sin^{-1}(\tan x) + C.$
11.  $\int \frac{e^x + 1}{e^x - 1} dx$   
 $= \int \frac{e^{x/2} + e^{-x/2}}{e^{x/2} - e^{-x/2}} dx$  Let  $u = e^{x/2} - e^{-x/2}$   
 $du = \frac{1}{2} (e^{x/2} + e^{-x/2}) dx$   
 $= 2 \int \frac{du}{u} = 2 \ln |u| + C$   
 $= 2 \ln |e^{x/2} - e^{-x/2}| + C = \ln |e^x + e^{-x} - 2| + C.$
12.  $\int \frac{\ln t}{t} dt$  Let  $u = \ln t$   
 $du = \frac{dt}{t}$   
 $= \int u du = \frac{1}{2} u^2 + C = \frac{1}{2} (\ln t)^2 + C.$
13.  $\int \frac{ds}{\sqrt{4 - 5s}}$  Let  $u = 4 - 5s$   
 $du = -5 ds$   
 $= -\frac{1}{5} \int \frac{du}{\sqrt{u}}$   
 $= -\frac{2}{5} u^{1/2} + C = -\frac{2}{5} \sqrt{4 - 5s} + C.$
14.  $\int \frac{x+1}{\sqrt{x^2+2x+3}} dx$  Let  $u = x^2 + 2x + 3$   
 $du = 2(x+1) dx$   
 $= \frac{1}{2} \int \frac{1}{\sqrt{u}} du = \sqrt{u} + C = \sqrt{x^2 + 2x + 3} + C$
15.  $\int \frac{t}{\sqrt{4-t^4}} dt$  Let  $u = t^2$   
 $du = 2t dt$   
 $= \frac{1}{2} \int \frac{du}{\sqrt{4-u^2}}$   
 $= \frac{1}{2} \sin^{-1} \frac{u}{2} + C = \frac{1}{2} \sin^{-1} \left( \frac{t^2}{2} \right) + C.$
16.  $\int \frac{x^2}{2+x^6} dx$  Let  $u = x^3$   
 $du = 3x^2 dx$   
 $= \frac{1}{3} \int \frac{du}{2+u^2} = \frac{1}{3\sqrt{2}} \tan^{-1} \left( \frac{u}{\sqrt{2}} \right) + C$   
 $= \frac{1}{3\sqrt{2}} \tan^{-1} \left( \frac{x^3}{\sqrt{2}} \right) + C.$
17.  $\int \frac{dx}{e^x + 1} = \int \frac{e^{-x} dx}{1 + e^{-x}}$  Let  $u = 1 + e^{-x}$   
 $du = -e^{-x} dx$   
 $= - \int \frac{du}{u} = -\ln |u| + C = -\ln(1 + e^{-x}) + C.$
18.  $\int \frac{dx}{e^x + e^{-x}} = \int \frac{e^x dx}{e^{2x} + 1}$  Let  $u = e^x$   
 $du = e^x dx$   
 $= \int \frac{du}{u^2 + 1} = \tan^{-1} u + C$   
 $= \tan^{-1} e^x + C.$
19.  $\int \tan x \ln \cos x dx$  Let  $u = \ln \cos x$   
 $du = -\tan x dx$   
 $= - \int u du = -\frac{1}{2} u^2 + C = -\frac{1}{2} (\ln \cos x)^2 + C.$
20.  $\int \frac{x+1}{\sqrt{1-x^2}} dx$   
 $= \int \frac{x dx}{\sqrt{1-x^2}} + \int \frac{dx}{\sqrt{1-x^2}}$  Let  $u = 1 - x^2$   
 $du = -2x dx$   
in the first integral only  
 $= -\frac{1}{2} \int \frac{du}{\sqrt{u}} + \sin^{-1} x = -\sqrt{u} + \sin^{-1} x + C$   
 $= -\sqrt{1-x^2} + \sin^{-1} x + C.$
21.  $\int \frac{dx}{x^2 + 6x + 13} = \int \frac{dx}{(x+3)^2 + 4}$  Let  $u = x + 3$   
 $du = dx$   
 $= \int \frac{du}{u^2 + 4} = \frac{1}{2} \tan^{-1} \frac{u}{2} + C$   
 $= \frac{1}{2} \tan^{-1} \frac{x+3}{2} + C.$
22.  $\int \frac{dx}{\sqrt{4+2x-x^2}} = \int \frac{dx}{\sqrt{5-(1-x)^2}}$  Let  $u = 1 - x$   
 $du = -dx$   
 $= - \int \frac{du}{\sqrt{5-u^2}} = -\sin^{-1} \left( \frac{u}{\sqrt{5}} \right) + C$   
 $= -\sin^{-1} \left( \frac{1-x}{\sqrt{5}} \right) + C = \sin^{-1} \left( \frac{x-1}{\sqrt{5}} \right) + C.$

$$\begin{aligned}
 23. \quad & \int \sin^3 x \cos^5 x \, dx \\
 &= \int \sin x (\cos^5 x - \cos^7 x) \, dx \quad \text{Let } u = \cos x \\
 & \quad \quad \quad du = -\sin x \, dx \\
 &= \int (u^7 - u^5) \, du \\
 &= \frac{u^8}{8} - \frac{u^6}{6} + C = \frac{\cos^8 x}{8} - \frac{\cos^6 x}{6} + C.
 \end{aligned}$$

$$\begin{aligned}
 24. \quad & \int \sin^4 t \cos^5 t \, dt \\
 &= \int \sin^4 t (1 - \sin^2 t)^2 \cos t \, dt \quad \text{Let } u = \sin t \\
 & \quad \quad \quad du = \cos t \, dt \\
 &= \int (u^4 - 2u^6 + u^8) \, du = \frac{u^5}{5} - \frac{2u^7}{7} + \frac{u^9}{9} + C \\
 &= \frac{1}{5} \sin^5 t - \frac{2}{7} \sin^7 t + \frac{1}{9} \sin^9 t + C.
 \end{aligned}$$

$$\begin{aligned}
 25. \quad & \int \sin ax \cos^2 ax \, dx \quad \text{Let } u = \cos ax \\
 & \quad \quad \quad du = -a \sin ax \, dx \\
 &= -\frac{1}{a} \int u^2 \, du \\
 &= -\frac{u^3}{3a} + C = -\frac{1}{3a} \cos^3 ax + C.
 \end{aligned}$$

$$\begin{aligned}
 26. \quad & \int \sin^2 x \cos^2 x \, dx = \int \left( \frac{1 - \cos 2x}{2} \right)^2 dx \\
 &= \frac{1}{4} \int \frac{1 - \cos 4x}{2} \, dx = \frac{x}{8} - \frac{\sin 4x}{32} + C.
 \end{aligned}$$

$$\begin{aligned}
 27. \quad & \int \sin^6 x \, dx = \int \left( \frac{1 - \cos 2x}{2} \right)^3 dx \\
 &= \frac{1}{8} \int (1 - 3 \cos 2x + 3 \cos^2 2x - \cos^3 2x) \, dx \\
 &= \frac{x}{8} - \frac{3 \sin 2x}{16} + \frac{3}{16} \int (1 + \cos 4x) \, dx \\
 & \quad - \frac{1}{8} \int \cos 2x (1 - \sin^2 2x) \, dx \quad \text{Let } u = \sin 2x \\
 & \quad \quad \quad du = 2 \cos 2x \, dx \\
 &= \frac{5x}{16} - \frac{3 \sin 2x}{16} + \frac{3 \sin 4x}{64} - \frac{1}{16} \int (1 - u^2) \, du \\
 &= \frac{5x}{16} - \frac{3 \sin 2x}{16} + \frac{3 \sin 4x}{64} - \frac{\sin 2x}{16} + \frac{\sin^3 2x}{48} + C \\
 &= \frac{5x}{16} - \frac{\sin 2x}{4} + \frac{3 \sin 4x}{64} + \frac{\sin^3 2x}{48} + C.
 \end{aligned}$$

$$\begin{aligned}
 28. \quad & \int \cos^4 x \, dx = \int \frac{[1 + \cos(2x)]^2}{4} \, dx \\
 &= \frac{1}{4} \int [1 + 2 \cos(2x) + \cos^2(2x)] \, dx \\
 &= \frac{x}{4} + \frac{\sin(2x)}{4} + \frac{1}{8} \int 1 + \cos(4x) \, dx \\
 &= \frac{x}{4} + \frac{\sin(2x)}{4} + \frac{x}{8} + \frac{\sin(4x)}{32} + C \\
 &= \frac{3x}{8} + \frac{\sin(2x)}{4} + \frac{\sin(4x)}{32} + C.
 \end{aligned}$$

$$\begin{aligned}
 29. \quad & \int \sec^5 x \tan x \, dx \quad \text{Let } u = \sec x \\
 & \quad \quad \quad du = \sec x \tan x \, dx \\
 &= \int u^4 \, du = \frac{u^5}{5} + C = \frac{\sec^5 x}{5} + C.
 \end{aligned}$$

$$\begin{aligned}
 30. \quad & \int \sec^6 x \tan^2 x \, dx \\
 &= \int \sec^2 x \tan^2 x (1 + \tan^2 x)^2 \, dx \quad \text{Let } u = \tan x \\
 & \quad \quad \quad du = \sec^2 x \, dx \\
 &= \int (u^2 + 2u^4 + u^6) \, du = \frac{1}{3} u^3 + \frac{2}{5} u^5 + \frac{1}{7} u^7 + C \\
 &= \frac{1}{3} \tan^3 x + \frac{2}{5} \tan^5 x + \frac{1}{7} \tan^7 x + C.
 \end{aligned}$$

$$\begin{aligned}
 31. \quad & \int \sqrt{\tan x} \sec^4 x \, dx \\
 &= \int \sqrt{\tan x} (1 + \tan^2 x) \sec^2 x \, dx \quad \text{Let } u = \tan x \\
 & \quad \quad \quad du = \sec^2 x \, dx \\
 &= \int (u^{1/2} + u^{5/2}) \, du \\
 &= \frac{2u^{3/2}}{3} + \frac{2u^{7/2}}{7} + C \\
 &= \frac{2}{3} (\tan x)^{3/2} + \frac{2}{7} (\tan x)^{7/2} + C.
 \end{aligned}$$

$$\begin{aligned}
 32. \quad & \int \sin^{-2/3} x \cos^3 x \, dx \quad \text{Let } u = \sin x \\
 & \quad \quad \quad du = \cos x \, dx \\
 &= \int \frac{1 - u^2}{u^{2/3}} \, du = 3u^{1/3} - \frac{3}{7} u^{7/3} + C \\
 &= 3 \sin^{1/3} x - \frac{3}{7} \sin^{7/3} x + C.
 \end{aligned}$$

$$\begin{aligned}
 33. \quad & \int \cos x \sin^4(\sin x) \, dx \quad \text{Let } u = \sin x \\
 & \quad \quad \quad du = \cos x \, dx \\
 &= \int \sin^4 u \, du = \int \left( \frac{1 - \cos 2u}{2} \right)^2 du \\
 &= \frac{1}{4} \int \left( 1 - 2 \cos 2u + \frac{1 + \cos 4u}{2} \right) du \\
 &= \frac{3u}{8} - \frac{\sin 2u}{4} + \frac{\sin 4u}{32} + C \\
 &= \frac{3}{8} \sin x - \frac{1}{4} \sin(2 \sin x) + \frac{1}{32} \sin(4 \sin x) + C.
 \end{aligned}$$

$$\begin{aligned}
 34. \quad & \int \frac{\sin^3(\ln x) \cos^3(\ln x)}{x} dx \quad \text{Let } u = \sin(\ln x) \\
 & \qquad \qquad \qquad du = \frac{\cos(\ln x)}{x} dx \\
 & = \int u^3(1-u^2) du = \frac{1}{4}u^4 - \frac{1}{6}u^6 + C \\
 & = \frac{1}{4} \sin^4(\ln x) - \frac{1}{6} \sin^6(\ln x) + C.
 \end{aligned}$$

$$\begin{aligned}
 35. \quad & \int \frac{\sin^2 x}{\cos^4 x} dx \\
 & = \int \tan^2 x \sec^2 x dx \quad \text{Let } u = \tan x \\
 & \qquad \qquad \qquad du = \sec^2 x dx \\
 & = \int u^2 du = \frac{u^3}{3} + C = \frac{1}{3} \tan^3 x + C.
 \end{aligned}$$

$$\begin{aligned}
 36. \quad & \int \frac{\sin^3 x}{\cos^4 x} dx = \int \tan^3 x \sec x dx \\
 & = \int (\sec^2 x - 1) \sec x \tan x dx \quad \text{Let } u = \sec x \\
 & \qquad \qquad \qquad du = \sec x \tan x dx \\
 & = \int (u^2 - 1) du = \frac{1}{3}u^3 - u + C \\
 & = \frac{1}{3} \sec^3 x - \sec x + C.
 \end{aligned}$$

$$\begin{aligned}
 37. \quad & \int \csc^5 x \cot^5 x dx \\
 & = \int \csc x \cot x \csc^4 x (\csc^2 x - 1)^2 dx \\
 & \qquad \qquad \text{Let } u = \csc x \\
 & \qquad \qquad du = -\csc x \cot x dx \\
 & = -\int (u^8 - 2u^6 + u^4) du \\
 & = -\frac{u^9}{9} + \frac{2u^7}{7} - \frac{u^5}{5} + C \\
 & = -\frac{1}{9} \csc^9 x + \frac{2}{7} \csc^7 x - \frac{1}{5} \csc^5 x + C.
 \end{aligned}$$

$$\begin{aligned}
 38. \quad & \int \frac{\cos^4 x}{\sin^8 x} dx = \int \cot^4 x \csc^4 x dx \\
 & = \int \cot^4 x (1 + \cot^2 x) \csc^2 x dx \quad \text{Let } u = \cot x \\
 & \qquad \qquad \qquad du = -\csc^2 x dx \\
 & = -\int u^4(1+u^2) du = -\frac{u^5}{5} - \frac{u^7}{7} + C \\
 & = -\frac{1}{5} \cot^5 x - \frac{1}{7} \cot^7 x + C.
 \end{aligned}$$

$$\begin{aligned}
 39. \quad & \int_0^4 x^3(x^2+1)^{-1/2} dx \quad \text{Let } u = x^2+1, \quad x^2 = u-1 \\
 & \qquad \qquad \qquad du = 2x dx \\
 & = \frac{1}{2} \int_1^{17} (u-1)u^{-1/2} du \\
 & = \frac{1}{2} \left( \frac{2}{3}u^{3/2} - 2u^{1/2} \right) \Big|_1^{17} \\
 & = \frac{17\sqrt{17}-1}{3} - (\sqrt{17}-1) = \frac{14\sqrt{17}}{3} + \frac{2}{3}.
 \end{aligned}$$

$$\begin{aligned}
 40. \quad & \int_1^{\sqrt{e}} \frac{\sin(\pi \ln x)}{x} dx \quad \text{Let } u = \pi \ln x \\
 & \qquad \qquad \qquad du = \frac{\pi}{x} dx \\
 & = \frac{1}{\pi} \int_0^{\pi/2} \sin u du = -\frac{1}{\pi} \cos u \Big|_0^{\pi/2} \\
 & = -\frac{1}{\pi}(0-1) = \frac{1}{\pi}.
 \end{aligned}$$

$$\begin{aligned}
 41. \quad & \int_0^{\pi/2} \sin^4 x dx = \int_0^{\pi/2} \left( \frac{1-\cos 2x}{2} \right)^2 dx \\
 & = \frac{1}{4} \int_0^{\pi/2} \left( 1 - 2\cos 2x + \frac{1+\cos 4x}{2} \right) dx \\
 & = \frac{3x}{8} \Big|_0^{\pi/2} - \frac{\sin 2x}{4} \Big|_0^{\pi/2} + \frac{\sin 4x}{32} \Big|_0^{\pi/2} = \frac{3\pi}{16}.
 \end{aligned}$$

$$\begin{aligned}
 42. \quad & \int_{\pi/4}^{\pi} \sin^5 x dx \\
 & = \int_{\pi/4}^{\pi} (1-\cos^2 x)^2 \sin x dx \quad \text{Let } u = \cos x \\
 & \qquad \qquad \qquad du = -\sin x dx \\
 & = -\int_{1/\sqrt{2}}^{-1} (1-2u^2+u^4) du = u - \frac{2}{3}u^3 + \frac{1}{5}u^5 \Big|_{1/\sqrt{2}}^{-1} \\
 & = \frac{1}{\sqrt{2}} - \frac{1}{3\sqrt{2}} + \frac{1}{20\sqrt{2}} - \left( -1 + \frac{2}{3} - \frac{1}{5} \right) = \frac{43}{60\sqrt{2}} + \frac{8}{15}.
 \end{aligned}$$

$$\begin{aligned}
 43. \quad & \int_e^{e^2} \frac{dt}{t \ln t} \quad \text{Let } u = \ln t \\
 & \qquad \qquad \qquad du = \frac{dt}{t} \\
 & = \int_1^2 \frac{du}{u} = \ln u \Big|_1^2 = \ln 2 - \ln 1 = \ln 2.
 \end{aligned}$$

$$\begin{aligned}
 44. \quad & \int_{\pi^2/16}^{\pi^2/9} \frac{2^{\sin \sqrt{x}} \cos \sqrt{x}}{\sqrt{x}} dx \quad \text{Let } u = \sin \sqrt{x} \\
 & \qquad \qquad \qquad du = \frac{\cos \sqrt{x}}{2\sqrt{x}} dx \\
 & = 2 \int_{1/\sqrt{2}}^{\sqrt{3}/2} 2^u du = \frac{2(2^u)}{\ln 2} \Big|_{1/\sqrt{2}}^{\sqrt{3}/2} \\
 & = \frac{2}{\ln 2} (2^{\sqrt{3}/2} - 2^{1/\sqrt{2}}).
 \end{aligned}$$

$$\begin{aligned}
 45. \quad \int_0^{\pi/2} \sqrt{1 + \cos x} \, dx &= \int_0^{\pi/2} \sqrt{2 \cos^2 \frac{x}{2}} \, dx \\
 &= \sqrt{2} \int_0^{\pi/2} \cos \frac{x}{2} \, dx = 2\sqrt{2} \sin \frac{x}{2} \Big|_0^{\pi/2} = 2. \\
 \int_0^{\pi/2} \sqrt{1 - \sin x} \, dx \\
 &= \int_0^{\pi/2} \sqrt{1 - \cos(\frac{\pi}{2} - x)} \, dx \quad \text{Let } u = \frac{\pi}{2} - x \\
 &\qquad\qquad\qquad du = -dx \\
 &= - \int_{\pi/2}^0 \sqrt{1 - \cos u} \, du \\
 &= \int_0^{\pi/2} \sqrt{2 \sin^2 \frac{u}{2}} \, du = \sqrt{2} \left( -2 \cos \frac{u}{2} \right) \Big|_0^{\pi/2} \\
 &= -2 + 2\sqrt{2} = 2(\sqrt{2} - 1).
 \end{aligned}$$

$$\begin{aligned}
 46. \quad \text{Area} &= \int_0^2 \frac{x}{x^2 + 16} \, dx \quad \text{Let } u = x^2 + 16 \\
 &\qquad\qquad\qquad du = 2x \, dx \\
 &= \frac{1}{2} \int_{16}^{20} \frac{du}{u} = \frac{1}{2} \ln u \Big|_{16}^{20} \\
 &= \frac{1}{2} (\ln 20 - \ln 16) = \frac{1}{2} \ln \left( \frac{5}{4} \right) \text{ sq. units.}
 \end{aligned}$$

$$\begin{aligned}
 47. \quad \text{Area } R &= \int_0^2 \frac{x \, dx}{x^4 + 16} \quad \text{Let } u = x^2 \\
 &\qquad\qquad\qquad du = 2x \, dx \\
 &= \frac{1}{2} \int_0^4 \frac{du}{u^2 + 16} = \frac{1}{8} \tan^{-1} \frac{u}{4} \Big|_0^4 = \frac{\pi}{32} \text{ sq. units.}
 \end{aligned}$$

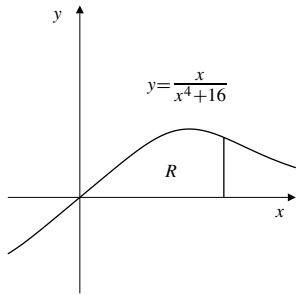


Fig. 5.6.47

48. The area bounded by the ellipse  $(x^2/a^2) + (y^2/b^2) = 1$  is

$$\begin{aligned}
 4 \int_0^a b \sqrt{1 - \frac{x^2}{a^2}} \, dx &\quad \text{Let } x = au \\
 &\qquad\qquad\qquad dx = a \, du \\
 &= 4ab \int_0^1 \sqrt{1 - u^2} \, du.
 \end{aligned}$$

The integral is the area of a quarter circle of radius 1.  
Hence

$$\text{Area} = 4ab \left( \frac{\pi(1)^2}{4} \right) = \pi ab \text{ sq. units.}$$

49. We start with the addition formulas

$$\begin{aligned}
 \cos(x + y) &= \cos x \cos y - \sin x \sin y \\
 \cos(x - y) &= \cos x \cos y + \sin x \sin y
 \end{aligned}$$

and take half their sum and half their difference to obtain

$$\begin{aligned}
 \cos x \cos y &= \frac{1}{2} (\cos(x + y) + \cos(x - y)) \\
 \sin x \sin y &= \frac{1}{2} (\cos(x - y) - \cos(x + y)).
 \end{aligned}$$

Similarly, taking half the sum of the formulas

$$\begin{aligned}
 \sin(x + y) &= \sin x \cos y + \cos x \sin y \\
 \sin(x - y) &= \sin x \cos y - \cos x \sin y,
 \end{aligned}$$

we obtain

$$\sin x \cos y = \frac{1}{2} (\sin(x + y) + \sin(x - y)).$$

50. We have

$$\begin{aligned}
 &\int \cos ax \cos bx \, dx \\
 &= \frac{1}{2} \int [\cos(ax - bx) + \cos(ax + bx)] \, dx \\
 &= \frac{1}{2} \int \cos[(a - b)x] \, dx + \frac{1}{2} \int \cos[(a + b)x] \, dx \\
 &\quad \text{Let } u = (a - b)x, \, du = (a - b) \, dx \text{ in the first integral;} \\
 &\quad \text{let } v = (a + b)x, \, dv = (a + b) \, dx \text{ in the second integral.} \\
 &= \frac{1}{2(a - b)} \int \cos u \, du + \frac{1}{2(a + b)} \int \cos v \, dv \\
 &= \frac{1}{2} \left[ \frac{\sin[(a - b)x]}{(a - b)} + \frac{\sin[(a + b)x]}{(a + b)} \right] + C. \\
 &\int \sin ax \sin bx \, dx \\
 &= \frac{1}{2} \int [\cos(ax - bx) - \cos(ax + bx)] \, dx \\
 &= \frac{1}{2} \left[ \frac{\sin[(a - b)x]}{(a - b)} - \frac{\sin[(a + b)x]}{(a + b)} \right] + C. \\
 &\int \sin ax \cos bx \, dx \\
 &= \frac{1}{2} \int [\sin(ax + bx) + \sin(ax - bx)] \, dx \\
 &= \frac{1}{2} \left[ \int \sin[(a + b)x] \, dx + \int \sin[(a - b)x] \, dx \right] \\
 &= -\frac{1}{2} \left[ \frac{\cos[(a + b)x]}{(a + b)} + \frac{\cos[(a - b)x]}{(a - b)} \right] + C.
 \end{aligned}$$

51. If  $m$  and  $n$  are integers, and  $m \neq n$ , then

$$\begin{aligned} & \int_{-\pi}^{\pi} \left\{ \begin{array}{l} \cos mx \cos nx \\ \sin mx \sin nx \end{array} \right\} dx \\ &= \frac{1}{2} \int_{-\pi}^{\pi} (\cos(m-n)x \pm \cos(m+n)x) dx \\ &= \frac{1}{2} \left( \frac{\sin(m-n)x}{m-n} \pm \frac{\sin(m+n)x}{m+n} \right) \Big|_{-\pi}^{\pi} \\ &= 0 \pm 0 = 0. \\ & \int_{-\pi}^{\pi} \sin mx \cos nx dx \\ &= \frac{1}{2} \int_{-\pi}^{\pi} (\sin(m+n)x + \sin(m-n)x) dx \\ &= -\frac{1}{2} \left( \frac{\cos(m+n)x}{m+n} + \frac{\cos(m-n)x}{m-n} \right) \Big|_{-\pi}^{\pi} \\ &= 0 \text{ (by periodicity).} \end{aligned}$$

If  $m = n \neq 0$  then

$$\begin{aligned} & \int_{-\pi}^{\pi} \sin mx \cos mx dx \\ &= \frac{1}{2} \int_{-\pi}^{\pi} \sin 2mx dx \\ &= -\frac{1}{4m} \cos 2mx \Big|_{-\pi}^{\pi} = 0 \text{ (by periodicity).} \end{aligned}$$

52. If  $1 \leq m \leq k$ , we have

$$\begin{aligned} \int_{-\pi}^{\pi} f(x) \cos mx dx &= \frac{a_0}{2} \int_{-\pi}^{\pi} \cos mx dx \\ &+ \sum_{n=1}^k a_n \int_{-\pi}^{\pi} \cos nx \cos mx dx \\ &+ \sum_{n=1}^k b_n \int_{-\pi}^{\pi} \sin nx \cos mx dx. \end{aligned}$$

By the previous exercise, all the integrals on the right side are zero except the one in the first sum having  $n = m$ . Thus the whole right side reduces to

$$\begin{aligned} a_m \int_{-\pi}^{\pi} \cos^2(mx) dx &= a_m \int_{-\pi}^{\pi} \frac{1 + \cos(2mx)}{2} dx \\ &= \frac{a_m}{2} (2\pi + 0) = \pi a_m. \end{aligned}$$

Thus

$$a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos mx dx.$$

A similar argument shows that

$$b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin mx dx.$$

For  $m = 0$  we have

$$\begin{aligned} \int_{-\pi}^{\pi} f(x) \cos mx dx &= \int_{-\pi}^{\pi} f(x) dx \\ &= \frac{a_0}{2} \int_{-\pi}^{\pi} dx \\ &+ \sum_{n=1}^k (a_n \cos(nx) + b_n \sin(nx)) dx \\ &= \frac{a_0}{2} (2\pi) + 0 + 0 = a_0\pi, \end{aligned}$$

so the formula for  $a_m$  holds for  $m = 0$  also.

### Section 5.7 Areas of Plane Regions (page 313)

1. Area of  $R = \int_0^1 (x - x^2) dx$   
 $= \left( \frac{x^2}{2} - \frac{x^3}{3} \right) \Big|_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$  sq. units.

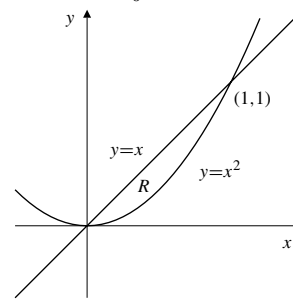


Fig. 5.7.1

2. Area of  $R = \int_0^1 (\sqrt{x} - x^2) dx$   
 $= \left( \frac{2}{3}x^{3/2} - \frac{1}{3}x^3 \right) \Big|_0^1 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$  sq. units.

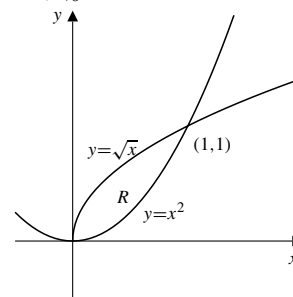


Fig. 5.7.2

3. Area of  $R = 2 \int_0^2 (8 - 2x^2) dx$   
 $= \left( 16x - \frac{4}{3}x^3 \right) \Big|_0^2 = \frac{64}{3}$  sq. units.



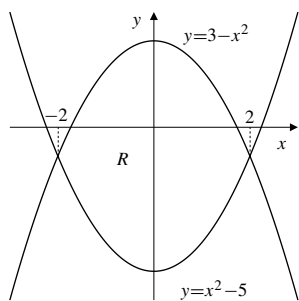


Fig. 5.7.3

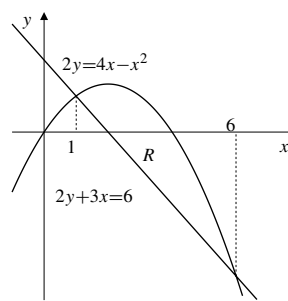


Fig. 5.7.5

4. For intersections:

$$x^2 - 2x = 6x - x^2 \Rightarrow 2x^2 - 8x = 0$$

i.e.,  $x = 0$  or  $4$ .

$$\text{Area of } R = \int_0^4 [6x - x^2 - (x^2 - 2x)] dx$$

$$= \int_0^4 (8x - 2x^2) dx$$

$$= \left(4x^2 - \frac{2}{3}x^3\right)\Big|_0^4 = \frac{64}{3} \text{ sq. units.}$$

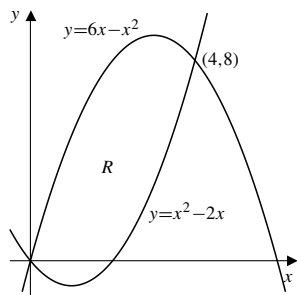


Fig. 5.7.4

6. For intersections:

$$7 + y = 2y^2 - y + 3 \Rightarrow 2y^2 - 2y - 4 = 0$$

$$2(y - 2)(y + 1) = 0 \Rightarrow \text{i.e., } y = -1 \text{ or } 2.$$

$$\text{Area of } R = \int_{-1}^2 [(7 + y) - (2y^2 - y + 3)] dy$$

$$= 2 \int_{-1}^2 (2 + y - y^2) dy$$

$$= 2 \left(2y + \frac{1}{2}y^2 - \frac{1}{3}y^3\right)\Big|_{-1}^2 = 9 \text{ sq. units.}$$

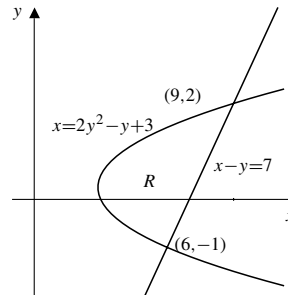


Fig. 5.7.6

5. For intersections:

$$\left. \begin{array}{l} 2y = 4x - x^2 \\ 2y + 3x = 6 \end{array} \right\} \Rightarrow \begin{array}{l} 4x - x^2 = 6 - 3x \\ x^2 - 7x + 6 = 0 \\ (x - 1)(x - 6) = 0 \end{array}$$

Thus intersections of the curves occur at  $x = 1$  and  $x = 6$ . We have

$$\begin{aligned} \text{Area of } R &= \int_1^6 \left(2x - \frac{x^2}{2} - 3 + \frac{3x}{2}\right) dx \\ &= \left(\frac{7x^2}{4} - \frac{x^3}{6} - 3x\right)\Big|_1^6 \\ &= \frac{245}{4} - 36 + \frac{1}{6} - 15 = \frac{125}{12} \text{ sq. units.} \end{aligned}$$

7. Area of
- $R = 2 \int_0^1 (x - x^3) dx$

$$= 2 \left(\frac{x^2}{2} - \frac{x^4}{4}\right)\Big|_0^1 = \frac{1}{2} \text{ sq. units.}$$

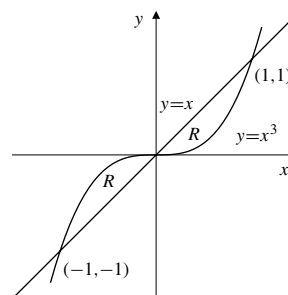


Fig. 5.7.7

8. 4 Shaded area  $= \int_0^1 (x^2 - x^3) dx$   
 $= \left( \frac{1}{3}x^3 - \frac{1}{4}x^4 \right) \Big|_0^1 = \frac{1}{12}$  sq. units.

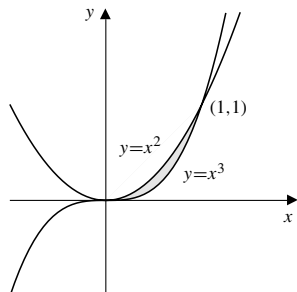


Fig. 5.7.8

9. Area of  $R = \int_0^1 (\sqrt{x} - x^3) dx$   
 $= \left( \frac{2}{3}x^{3/2} - \frac{x^4}{4} \right) \Big|_0^1 = \frac{5}{12}$  sq. units.

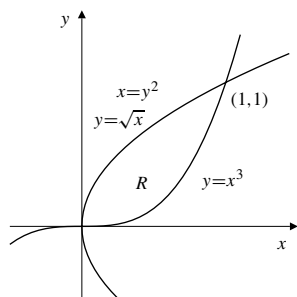


Fig. 5.7.9

10. For intersections:

$y^2 = 2y^2 - y - 2 \Rightarrow y^2 - y - 2 = 0$   
 $(y - 2)(y + 1) = 0 \Rightarrow$  i.e.,  $y = -1$  or  $2$ .

Area of  $R = \int_{-1}^2 [y^2 - (2y^2 - y - 2)] dy$   
 $= \int_{-1}^2 [2 + y - y^2] dy = \left( 2y + \frac{1}{2}y^2 - \frac{1}{3}y^3 \right) \Big|_{-1}^2$   
 $= \frac{9}{2}$  sq. units.

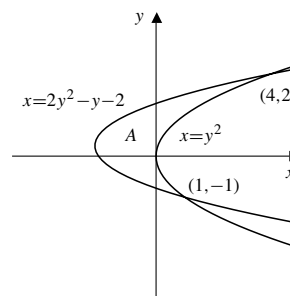


Fig. 5.7.10

11. For intersections:  $\frac{1}{x} = y = \frac{5-2x}{2}$ .  
 Thus  $2x^2 - 5x + 2 = 0$ , i.e.,  $(2x - 1)(x - 2) = 0$ . The graphs intersect at  $x = 1/2$  and  $x = 2$ . Thus

Area of  $R = \int_{1/2}^2 \left( \frac{5-2x}{2} - \frac{1}{x} \right) dx$   
 $= \left( \frac{5x}{2} - \frac{x^2}{2} - \ln x \right) \Big|_{1/2}^2$   
 $= \frac{15}{8} - 2 \ln 2$  sq. units.

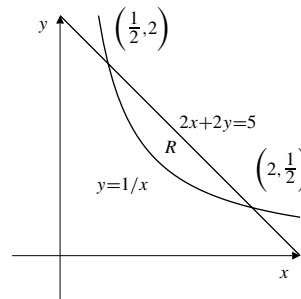


Fig. 5.7.11

12. Area of shaded region  $= 2 \int_0^1 [(1-x^2) - (x^2-1)^2] dx$   
 $= 2 \int_0^1 (x^2 - x^4) dx = 2 \left( \frac{1}{3}x^3 - \frac{1}{5}x^5 \right) \Big|_0^1 = \frac{4}{15}$  sq. units.

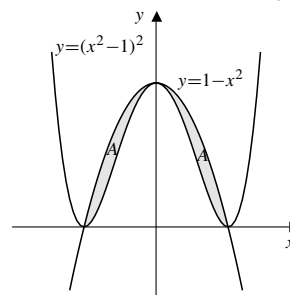


Fig. 5.7.12

13. The curves  $y = \frac{x^2}{2}$  and  $y = \frac{1}{1+x^2}$  intersect at  $x = \pm 1$ . Thus

$$\begin{aligned} \text{Area of } R &= 2 \int_0^1 \left( \frac{1}{1+x^2} - \frac{x^2}{2} \right) dx \\ &= 2 \left( \tan^{-1} x - \frac{x^3}{6} \right) \Big|_0^1 = \frac{\pi}{2} - \frac{1}{3} \text{ sq. units.} \end{aligned}$$

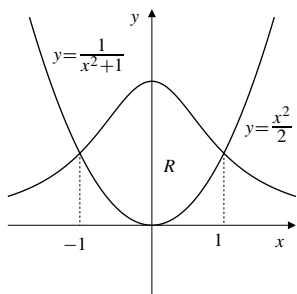


Fig. 5.7.13

14. For intersections:

$$\frac{4x}{3+x^2} = 1 \Rightarrow x^2 - 4x + 3 = 0$$

i.e.,  $x = 1$  or  $3$ .

$$\begin{aligned} \text{Shaded area} &= \int_1^3 \left[ \frac{4x}{3+x^2} - 1 \right] dx \\ &= [2 \ln(3+x^2) - x] \Big|_1^3 = 2 \ln 3 - 2 \text{ sq. units.} \end{aligned}$$

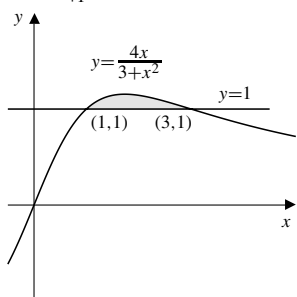


Fig. 5.7.14

15. The curves  $y = \frac{4}{x^2}$  and  $y = 5 - x^2$  intersect where  $x^4 - 5x^2 + 4 = 0$ , i.e., where  $(x^2 - 4)(x^2 - 1) = 0$ . Thus the intersections are at  $x = \pm 1$  and  $x = \pm 2$ . We have

$$\begin{aligned} \text{Area of } R &= 2 \int_1^2 \left( 5 - x^2 - \frac{4}{x^2} \right) dx \\ &= 2 \left( 5x - \frac{x^3}{3} + \frac{4}{x} \right) \Big|_1^2 = \frac{4}{3} \text{ sq. units.} \end{aligned}$$

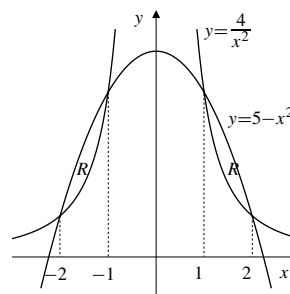


Fig. 5.7.15

16. Area  $A = \int_{-\pi}^{\pi} (\sin y - (y^2 - \pi^2)) dy$
- $$= \left( -\cos y + \pi^2 y - \frac{y^3}{3} \right) \Big|_{-\pi}^{\pi} = \frac{4\pi^3}{3} \text{ sq. units.}$$

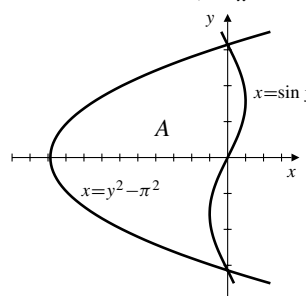


Fig. 5.7.16

17. Area of  $R = \int_{\pi/4}^{5\pi/4} (\sin x - \cos x) dx$
- $$\begin{aligned} &= -(\cos x + \sin x) \Big|_{\pi/4}^{5\pi/4} \\ &= \sqrt{2} + \sqrt{2} = 2\sqrt{2} \text{ sq. units.} \end{aligned}$$

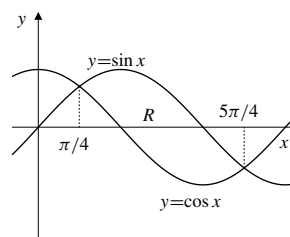


Fig. 5.7.17

18. Area  $= \int_{-\pi/2}^{\pi/2} (1 - \sin^2 x) dx$
- $$\begin{aligned} &= 2 \int_0^{\pi/2} \frac{1 + \cos(2x)}{2} dx \\ &= \left( x + \frac{\sin(2x)}{2} \right) \Big|_0^{\pi/2} = \frac{\pi}{2} \text{ sq. units.} \end{aligned}$$

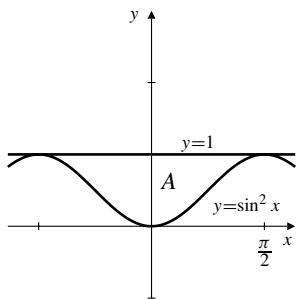


Fig. 5.7.18

19. Area  $A = \int_0^{\pi/2} (\sin x - \sin^2 x) dx$   
 $= \left( -\cos x + \frac{\sin x \cos x - x}{2} \right) \Big|_0^{\pi/2} = 1 - \frac{\pi}{4}$  sq. units.

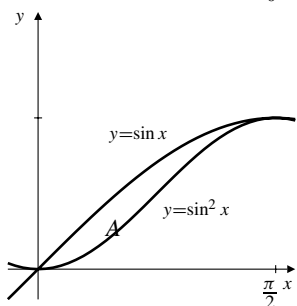


Fig. 5.7.19

20. Area  $A = 2 \int_0^{\pi/4} (\cos^2 x - \sin^2 x) dx$   
 $= 2 \int_0^{\pi/4} \cos(2x) dx = \sin(2x) \Big|_0^{\pi/4} = 1$  sq. units.

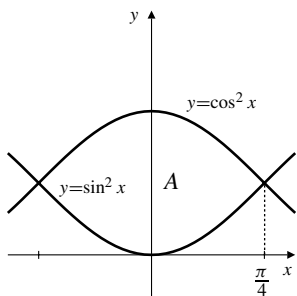


Fig. 5.7.20

21. For intersections:  $\frac{4x}{\pi} = \tan x \Rightarrow x = 0$  or  $\frac{\pi}{4}$ .  
 Area  $= \int_0^{\pi/4} \left( \frac{4x}{\pi} - \tan x \right) dx$   
 $= \left( \frac{2}{\pi} x^2 - \ln |\sec x| \right) \Big|_0^{\pi/4} = \frac{\pi}{8} - \frac{1}{2} \ln 2$  sq. units.

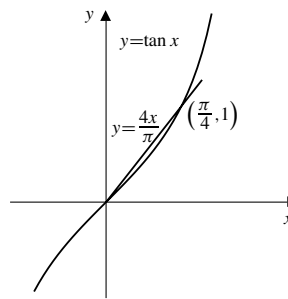


Fig. 5.7.21

22. For intersections:  $x^{1/3} = \tan(\pi x/4)$ . Thus  $x = \pm 1$ .

Area  $A = 2 \int_0^1 \left( x^{1/3} - \tan \frac{\pi x}{4} \right) dx$   
 $= 2 \left( \frac{3}{4} x^{4/3} - \frac{4}{\pi} \ln \left| \sec \frac{\pi x}{4} \right| \right) \Big|_0^1$   
 $= \frac{3}{2} - \frac{8}{\pi} \ln \sqrt{2} = \frac{3}{2} - \frac{4}{\pi} \ln 2$  sq. units.

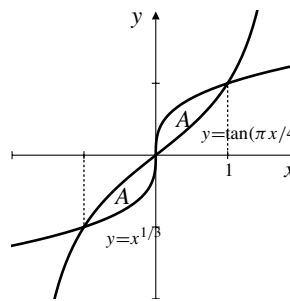


Fig. 5.7.22

23. For intersections:  $\sec x = 2$ . Thus  $x = \pm \pi/3$ .

Area  $A = 2 \int_0^{\pi/3} (2 - \sec x) dx$   
 $= (4x - 2 \ln |\sec x + \tan x|) \Big|_0^{\pi/3}$   
 $= \frac{4\pi}{3} - 2 \ln(2 + \sqrt{3})$  sq. units.

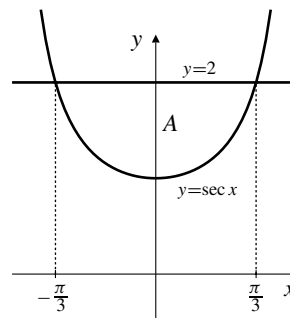


Fig. 5.7.23

24. For intersections:  $|x| = \sqrt{2} \cos(\pi x/4)$ . Thus  $x = \pm 1$ .

$$\begin{aligned} \text{Area } A &= 2 \int_0^1 \left( \sqrt{2} \cos \frac{\pi x}{4} - x \right) dx \\ &= \left( \frac{8\sqrt{2}}{\pi} \sin \frac{\pi x}{4} - x^2 \right) \Big|_0^1 \\ &= \frac{8}{\pi} - 1 \text{ sq. units.} \end{aligned}$$

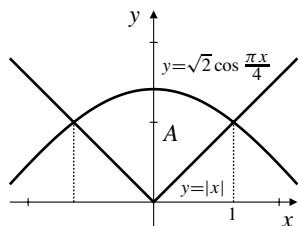


Fig. 5.7.24

25. For intersections:  $x = \sin(\pi x/2)$ . Thus  $x = \pm 1$ .

$$\begin{aligned} \text{Area } A &= 2 \int_0^1 \left( \sin \frac{\pi x}{2} - x \right) dx \\ &= \left( -\frac{4}{\pi} \cos \frac{\pi x}{2} - x^2 \right) \Big|_0^1 \\ &= \frac{4}{\pi} - 1 \text{ sq. units.} \end{aligned}$$

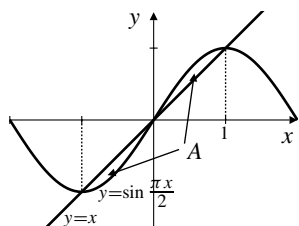


Fig. 5.7.25

26. For intersections:  $e^x = x + 2$ . There are two roots, both of which must be found numerically. We used a TI-85 solve routine to get  $x_1 \approx -1.841406$  and  $x_2 \approx 1.146193$ . Thus

$$\begin{aligned} \text{Area } A &= \int_{x_1}^{x_2} (x + 2 - e^x) dx \\ &= \left( \frac{x^2}{2} + 2x - e^x \right) \Big|_{x_1}^{x_2} \\ &\approx 1.949091 \text{ sq. units.} \end{aligned}$$

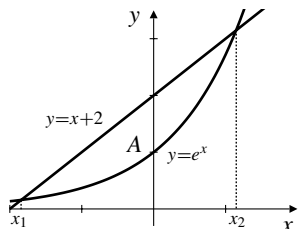


Fig. 5.7.26

27. Area of  $R = 4 \int_0^1 \sqrt{x^2 - x^4} dx$   
 $= 4 \int_0^1 x \sqrt{1 - x^2} dx$  Let  $u = 1 - x^2$   
 $du = -2x dx$   
 $= 2 \int_0^1 u^{1/2} du = \frac{4}{3} u^{3/2} \Big|_0^1 = \frac{4}{3}$  sq. units.

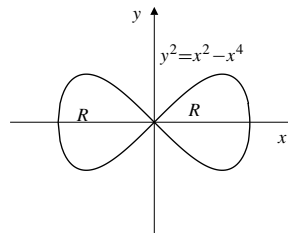


Fig. 5.7.27

28. Loop area  $= 2 \int_{-2}^0 x^2 \sqrt{2+x} dx$  Let  $u^2 = 2+x$   
 $2u du = dx$   
 $= 2 \int_0^{\sqrt{2}} (u^2 - 2)^2 u (2u) du = 4 \int_0^{\sqrt{2}} (u^6 - 4u^4 + 4u^2) du$   
 $= 4 \left( \frac{1}{7} u^7 - \frac{4}{5} u^5 + \frac{4}{3} u^3 \right) \Big|_0^{\sqrt{2}} = \frac{256\sqrt{2}}{105}$  sq. units.

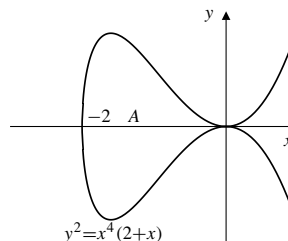


Fig. 5.7.28

29. The tangent line to  $y = e^x$  at  $x = 1$  is  $y - e = e(x - 1)$ , or  $y = ex$ . Thus

$$\begin{aligned} \text{Area of } R &= \int_0^1 (e^x - ex) dx \\ &= \left( e^x - \frac{ex^2}{2} \right) \Big|_0^1 = \frac{e}{2} - 1 \text{ sq. units.} \end{aligned}$$

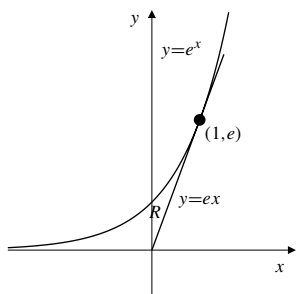


Fig. 5.7.29

30. The tangent line to  $y = x^3$  at  $(1, 1)$  is  $y - 1 = 3(x - 1)$ , or  $y = 3x - 2$ . The intersections of  $y = x^3$  and this tangent line occur where  $x^3 - 3x + 2 = 0$ . Of course  $x = 1$  is a (double) root of this cubic equation, which therefore factors to  $(x - 1)^2(x + 2) = 0$ . The other intersection is at  $x = -2$ . Thus

$$\begin{aligned} \text{Area of } R &= \int_{-2}^1 (x^3 - 3x + 2) dx \\ &= \left( \frac{x^4}{4} - \frac{3x^2}{2} + 2x \right) \Big|_{-2}^1 \\ &= -\frac{15}{4} - \frac{3}{2} + 6 + 2 + 4 = \frac{27}{4} \text{ sq. units.} \end{aligned}$$

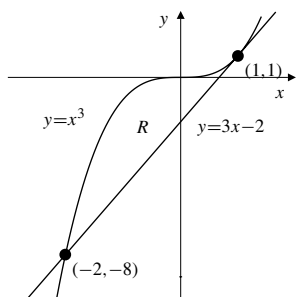


Fig. 5.7.30

Review Exercises 5 (page 314)

1. 
$$\frac{1}{j^2} - \frac{1}{(j+1)^2} = \frac{j^2 + 2j + 1 - j^2}{j^2(j+1)^2} = \frac{2j+1}{j^2(j+1)^2}$$

$$\sum_{j=1}^n \frac{2j+1}{j^2(j+1)^2} = \sum_{j=1}^n \left( \frac{1}{j^2} - \frac{1}{(j+1)^2} \right)$$

$$= \frac{1}{1^2} - \frac{1}{(n+1)^2} = \frac{n^2 + 2n}{(n+1)^2}$$

2. The number of balls is

$$\begin{aligned} &40 \times 30 + 39 \times 29 + \dots + 12 \times 2 + 11 \times 1 \\ &= \sum_{i=1}^{30} i(i+10) = \frac{(30)(31)(61)}{6} + 10 \frac{(30)(31)}{2} = 14,105. \end{aligned}$$

3.  $x_i = 1 + (2i/n)$ ,  $(i = 0, 1, 2, \dots, n)$ ,  $\Delta x_i = 2/n$ .

$$\begin{aligned} \int_1^3 f(x) dx &= \lim_{n \rightarrow \infty} \sum_{i=1}^n (x_i^2 - 2x_i + 3) \frac{2}{n} \\ &= \lim_{n \rightarrow \infty} \frac{2}{n} \sum_{i=1}^n \left[ \left( 1 + \frac{4i}{n} + \frac{4i^2}{n^2} \right) - \left( 2 + \frac{4i}{n} \right) + 3 \right] \\ &= \lim_{n \rightarrow \infty} \frac{2}{n} \sum_{i=1}^n \left[ 2 + \frac{4}{n^2} i^2 \right] \\ &= \lim_{n \rightarrow \infty} \left( \frac{4}{n} + \frac{8}{n^3} \frac{n(n+1)(2n+1)}{6} \right) \\ &= 4 + \frac{8}{3} = \frac{20}{3} \end{aligned}$$

4.  $R_n = \sum_{i=1}^n (1/n) \sqrt{1 + (i/n)}$  is a Riemann sum for  $f(x) = \sqrt{1+x}$  on the interval  $[0, 1]$ . Thus

$$\begin{aligned} \lim_{n \rightarrow \infty} R_n &= \int_0^1 \sqrt{1+x} dx \\ &= \frac{2}{3} (1+x)^{3/2} \Big|_0^1 = \frac{4\sqrt{2}-2}{3}. \end{aligned}$$

5.  $\int_{-\pi}^{\pi} (2 - \sin x) dx = 2(2\pi) - \int_{-\pi}^{\pi} \sin x dx = 4\pi - 0 = 4\pi$

6.  $\int_0^{\sqrt{5}} \sqrt{5-x^2} dx = 1/4$  of the area of a circle of radius  $\sqrt{5}$
- $$= \frac{1}{4} \pi (\sqrt{5})^2 = \frac{5\pi}{4}$$

7.  $\int_1^3 \left( 1 - \frac{x}{2} \right) dx = \text{area } A_1 - \text{area } A_2 = 0$

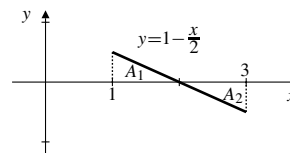


Fig. R-5.7

8.  $\int_0^{\pi} \cos x dx = \text{area } A_1 - \text{area } A_2 = 0$

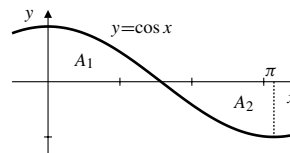


Fig. R-5.8

9.  $\bar{f} = \frac{1}{2\pi} \int_{-\pi}^{\pi} (2 - \sin(x^3)) dx = \frac{1}{2\pi} [2(2\pi) - 0] = 2$

10.  $\bar{h} = \frac{1}{3} \int_0^3 |x-2| dx = \frac{1}{3} \frac{5}{2} = \frac{5}{6}$  (via #9)

11.  $f(t) = \int_{13}^t \sin(x^2) dx, \quad f'(t) = \sin(t^2)$

12.  $f(x) = \int_{-13}^{\sin x} \sqrt{1+t^2} dt, \quad f'(x) = \sqrt{1+\sin^2 x}(\cos x)$

13.  $g(s) = \int_{4s}^1 e^{\sin u} du, \quad g'(s) = -4e^{\sin(4s)}$

14.  $g(\theta) = \int_{e^{\sin \theta}}^{e^{\cos \theta}} \ln x dx$   
 $g'(\theta) = (\ln(e^{\cos \theta}))e^{\cos \theta}(-\sin \theta) - (\ln(e^{\sin \theta}))e^{\sin \theta} \cos \theta$   
 $= -\sin \theta \cos \theta (e^{\cos \theta} + e^{\sin \theta})$

15.  $2f(x) + 1 = 3 \int_x^1 f(t) dt$   
 $2f'(x) = -3f(x) \implies f(x) = Ce^{-3x/2}$   
 $2f(1) + 1 = 0$   
 $-\frac{1}{2} = f(1) = Ce^{-3/2} \implies C = -\frac{1}{2}e^{3/2}$   
 $f(x) = -\frac{1}{2}e^{(3/2)(1-x)}$

16.  $I = \int_0^\pi xf(\sin x) dx$  Let  $x = \pi - u$   
 $dx = -du$   
 $= -\int_\pi^0 (\pi - u)f(\sin(\pi - u)) du$  (but  $\sin(\pi - u) = \sin u$ )  
 $= \pi \int_0^\pi f(\sin u) du - \int_0^\pi uf(\sin u) du$   
 $= \pi \int_0^\pi f(\sin x) dx - I$

Now, solving for  $I$ , we get

$$\int_0^\pi xf(\sin x) dx = I = \frac{\pi}{2} \int_0^\pi f(\sin x) dx.$$

17.  $y = 2 + x - x^2$  and  $y = 0$  intersect where  $2 + x - x^2 = 0$ , that is, where  $(2 - x)(1 + x) = 0$ , namely at  $x = -1$  and  $x = 2$ . Since  $2 + x - x^2 \geq 0$  on  $[-1, 2]$ , the required area is

$$\int_{-1}^2 (2 + x - x^2) dx = \left(2x + \frac{x^2}{2} - \frac{x^3}{3}\right) \Big|_{-1}^2 = \frac{9}{2} \text{ sq. units..}$$

18. The area bounded by  $y = (x - 1)^2$ ,  $y = 0$ , and  $x = 0$  is

$$\int_0^1 (x - 1)^2 dx = \frac{(x - 1)^3}{3} \Big|_0^1 = \frac{1}{3} \text{ sq. units..}$$

19.  $x = y - y^4$  and  $x = 0$  intersect where  $y - y^4 = 0$ , that is, at  $y = 0$  and  $y = 1$ . Since  $y - y^4 \geq 0$  on  $[0, 1]$ , the required area is

$$\int_0^1 (y - y^4 - 0) dy = \left(\frac{y^2}{2} - \frac{y^5}{5}\right) \Big|_0^1 = \frac{3}{10} \text{ sq. units.}$$

20.  $y = 4x - x^2$  and  $y = 3$  meet where  $x^2 - 4x + 3 = 0$ , that is, at  $x = 1$  and  $x = 3$ . Since  $4x - x^2 \geq 3$  on  $[1, 3]$ , the required area is

$$\int_1^3 (4x - x^2 - 3) dx = \left(2x^2 - \frac{x^3}{3} - 3x\right) \Big|_1^3 = \frac{4}{3} \text{ sq. units.}$$

21.  $y = \sin x$  and  $y = \cos(2x)$  intersect at  $x = \pi/6$ , but nowhere else in the interval  $[0, \pi/6]$ . The area between the curves in that interval is

$$\int_0^{\pi/6} (\cos(2x) - \sin x) dx = \left(\frac{1}{2} \sin(2x) + \cos x\right) \Big|_0^{\pi/6}$$

$$= \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{2} - 1 = \frac{3\sqrt{3}}{4} - 1 \text{ sq. units..}$$

22.  $y = 5 - x^2$  and  $y = 4/x^2$  meet where  $5 - x^2 = 4/x^2$ , that is, where

$$x^4 - 5x^2 + 4 = 0$$

$$(x^2 - 1)(x^2 - 4) = 0.$$

There are four intersections:  $x = \pm 1$  and  $x = \pm 2$ . By symmetry (see the figure) the total area bounded by the curves is

$$2 \int_1^2 \left(5 - x^2 - \frac{4}{x^2}\right) dx = 2 \left(5x - \frac{x^3}{3} + \frac{4}{x}\right) \Big|_1^2 = \frac{4}{3} \text{ sq. units.}$$

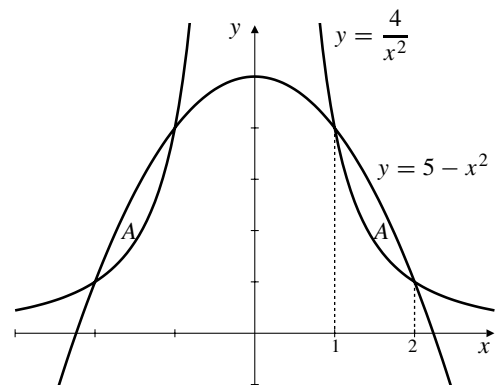


Fig. R-5.22

23.  $\int x^2 \cos(2x^3 + 1) dx$  Let  $u = 2x^3 + 1$   
 $du = 6x^2 dx$   
 $= \frac{1}{6} \int \cos u du = \frac{\sin u}{6} + C = \frac{\sin(2x^3 + 1)}{6} + C$

24.  $\int_1^e \frac{\ln x}{x} dx$  Let  $u = \ln x$   
 $du = dx/x$   
 $= \int_0^1 u du = \frac{u^2}{2} \Big|_0^1 = \frac{1}{2}$

25.  $\int_0^4 \sqrt{9t^2 + t^4} dt$   
 $= \int_0^4 t\sqrt{9 + t^2} dt$  Let  $u = 9 + t^2$   
 $du = 2t dt$   
 $= \frac{1}{2} \int_9^{25} \sqrt{u} du = \frac{1}{3} u^{3/2} \Big|_9^{25} = \frac{98}{3}$

26.  $\int \sin^3(\pi x) dx$   
 $= \int \sin(\pi x)(1 - \cos^2(\pi x)) dx$  Let  $u = \cos(\pi x)$   
 $du = -\pi \sin(\pi x) dx$   
 $= -\frac{1}{\pi} \int (1 - u^2) du$   
 $= \frac{1}{\pi} \left( \frac{u^3}{3} - u \right) + C = \frac{1}{3\pi} \cos^3(\pi x) - \frac{1}{\pi} \cos(\pi x) + C$

27.  $\int_0^{\ln 2} \frac{e^u}{4 + e^{2u}} du$  Let  $v = e^u$   
 $dv = e^u du$   
 $= \int_1^2 \frac{dv}{4 + v^2}$   
 $= \frac{1}{2} \tan^{-1} \frac{v}{2} \Big|_1^2 = \frac{\pi}{8} - \frac{1}{2} \tan^{-1} \frac{1}{2}$

28.  $\int_1^{\sqrt[4]{e}} \frac{\tan^2(\pi \ln x)}{x} dx$  Let  $u = \pi \ln x$   
 $du = (\pi/x) dx$   
 $= \frac{1}{\pi} \int_0^{\pi/4} \tan^2 u du = \frac{1}{\pi} \int_0^{\pi/4} (\sec^2 u - 1) du$   
 $= \frac{1}{\pi} (\tan u - u) \Big|_0^{\pi/4} = \frac{1}{\pi} - \frac{1}{4}$

29.  $\int \frac{\sin \sqrt{2s+1}}{\sqrt{2s+1}} ds$  Let  $u = \sqrt{2s+1}$   
 $du = ds/\sqrt{2s+1}$   
 $= \int \sin u du = -\cos u + C = -\cos \sqrt{2s+1} + C$

30.  $\int \cos^2 \frac{t}{5} \sin^2 \frac{t}{5} dt = \frac{1}{4} \int \sin^2 \frac{2t}{5} dt$   
 $= \frac{1}{8} \int \left( 1 - \cos \frac{4t}{5} \right) dt$   
 $= \frac{1}{8} \left( t - \frac{5}{4} \sin \frac{4t}{5} \right) + C$

31.  $F(x) = \int_0^{x^2-2x} \frac{1}{1+t^2} dt.$

Since  $1/(1+t^2) > 0$  for all  $t$ ,  $F(x)$  will be minimum when

$$x^2 - 2x = (x-1)^2 - 1$$

is minimum, that is, when  $x = 1$ . The minimum value is

$$F(1) = \int_0^{-1} \frac{dt}{1+t^2} = \tan^{-1} t \Big|_0^{-1} = -\frac{\pi}{4}.$$

$F$  has no maximum value;  $F(x) < \pi/2$  for all  $x$ , but  $F(x) \rightarrow \pi/2$  if  $x^2 - 2x \rightarrow \infty$ , which happens as  $x \rightarrow \pm\infty$ .

32.  $f(x) = 4x - x^2 \geq 0$  if  $0 \leq x \leq 4$ , and  $f(x) < 0$  otherwise. If  $a < b$ , then  $\int_a^b f(x) dx$  will be maximum if  $[a, b] = [0, 4]$ ; extending the interval to the left of 0 or to the right of 4 will introduce negative contributions to the integral. The maximum value is

$$\int_0^4 (4x - x^2) dx = \left( 2x^2 - \frac{x^3}{3} \right) \Big|_0^4 = \frac{32}{3}.$$

33. The average value of  $v(t) = dx/dt$  over  $[t_0, t_1]$  is

$$\frac{1}{t_1 - t_0} \int_{t_0}^{t_1} \frac{dx}{dt} dt = \frac{1}{t_1 - t_0} x(t) \Big|_{t_0}^{t_1} = \frac{x(t_1) - x(t_0)}{t_1 - t_0} = v_{av}.$$

34. If  $y(t)$  is the distance the object falls in  $t$  seconds from its release time, then

$$y''(t) = g, \quad y(0) = 0, \quad \text{and } y'(0) = 0.$$

Antidifferentiating twice and using the initial conditions leads to

$$y(t) = \frac{1}{2}gt^2.$$

The average height during the time interval  $[0, T]$  is

$$\frac{1}{T} \int_0^T \frac{1}{2}gt^2 dt = \frac{g}{2T} \frac{T^3}{3} = \frac{gT^2}{6} = y\left(\frac{T}{\sqrt{3}}\right).$$

35. Let  $f(x) = ax^3 + bx^2 + cx + d$  so that

$$\int_0^1 f(x) dx = \frac{a}{4} + \frac{b}{3} + \frac{c}{2} + d.$$

We want this integral to be  $(f(x_1) + f(x_2))/2$  for all choices of  $a, b, c$ , and  $d$ . Thus we require that

$$\begin{aligned} a(x_1^3 + x_2^3) + b(x_1^2 + x_2^2) + c(x_1 + x_2) + 2d \\ = 2 \int_0^1 f(x) dx = \frac{a}{2} + \frac{2b}{3} + c + 2d. \end{aligned}$$



It follows that  $x_1$  and  $x_2$  must satisfy

$$x_1^3 + x_2^3 = \frac{1}{2} \quad (1)$$

$$x_1^2 + x_2^2 = \frac{2}{3} \quad (2)$$

$$x_2 + x_1 = 1. \quad (3)$$

At first glance this system may seem overdetermined; there are three equations in only two unknowns. However, they do admit a solution as we now show. Squaring equation (3) and subtracting equation (2) we get  $2x_1x_2 = 1/3$ . Subtracting this latter equation from equation (2) then gives  $(x_2 - x_1)^2 = 1/3$ , so that  $x_2 - x_1 = 1/\sqrt{3}$  (the positive square root since we want  $x_1 < x_2$ ). Adding and subtracting this equation and equation (3) then produces the values  $x_2 = (\sqrt{3} + 1)/(2\sqrt{3})$  and  $x_1 = (\sqrt{3} - 1)/(2\sqrt{3})$ . These values also satisfy equation (1) since

$$x_1^3 + x_2^3 = (x_2 + x_1)(x_1^2 - x_1x_2 + x_2^2) = 1 \times \left(\frac{2}{3} - \frac{1}{6}\right) = \frac{1}{2}.$$

### Challenging Problems 5 (page 315)

1.  $x_i = 2^{i/n}$ ,  $0 \leq i \leq n$ ,  $f(x) = 1/x$  on  $[1, 2]$ . Since  $f$  is decreasing,  $f$  is largest at the left endpoint and smallest at the right endpoint of any interval  $[2^{(i-1)/n}, 2^{i/n}]$  of the partition. Thus

$$\begin{aligned} U(f, P_n) &= \sum_{i=1}^n \frac{1}{2^{(i-1)/n}} (2^{i/n} - 2^{(i-1)/n}) \\ &= \sum_{i=1}^n (2^{1/n} - 1) = n(2^{1/n} - 1) \\ L(f, P_n) &= \sum_{i=1}^n \frac{1}{2^{i/n}} (2^{i/n} - 2^{(i-1)/n}) \\ &= \sum_{i=1}^n (1 - 2^{-1/n}) = n(1 - 2^{-1/n}) = \frac{U(f, P_n)}{2^{1/n}}. \end{aligned}$$

Now, by l'Hôpital's rule,

$$\begin{aligned} \lim_{n \rightarrow \infty} n(2^{1/n} - 1) &= \lim_{x \rightarrow \infty} \frac{2^{1/x} - 1}{1/x} \quad \left[ \frac{0}{0} \right] \\ &= \lim_{x \rightarrow \infty} \frac{2^{1/x} \ln 2 (-1/x^2)}{-1/x^2} = \ln 2. \end{aligned}$$

Thus  $\lim_{n \rightarrow \infty} U(f, P_n) = \lim_{n \rightarrow \infty} L(f, P_n) = \ln 2$ .

2. a)  $\cos\left((j + \frac{1}{2})t\right) - \cos\left((j - \frac{1}{2})t\right)$   
 $= \cos(jt) \cos(\frac{1}{2}t) - \sin(jt) \sin(\frac{1}{2}t)$   
 $\quad - \cos(jt) \cos(\frac{1}{2}t) - \sin(jt) \sin(\frac{1}{2}t)$   
 $= -2 \sin(jt) \sin(\frac{1}{2}t).$

Therefore, we obtain a telescoping sum:

$$\begin{aligned} &\sum_{j=1}^n \sin(jt) \\ &= -\frac{1}{2 \sin(\frac{1}{2}t)} \sum_{j=1}^n \left[ \cos\left((j + \frac{1}{2})t\right) - \cos\left((j - \frac{1}{2})t\right) \right] \\ &= -\frac{1}{2 \sin(\frac{1}{2}t)} \left[ \cos\left((n + \frac{1}{2})t\right) - \cos(\frac{1}{2}t) \right] \\ &= \frac{1}{2 \sin(\frac{1}{2}t)} \left[ \cos(\frac{1}{2}t) - \cos\left((n + \frac{1}{2})t\right) \right]. \end{aligned}$$

- b) Let  $P_n = \{0, \frac{\pi}{2n}, \frac{2\pi}{2n}, \frac{3\pi}{2n}, \dots, \frac{n\pi}{2n}\}$  be the partition of  $[0, \pi/2]$  into  $n$  subintervals of equal length  $\Delta x = \pi/2n$ . Using  $t = \pi/2n$  in the formula obtained in part (a), we get

$$\begin{aligned} &\int_0^{\pi/2} \sin x \, dx \\ &= \lim_{n \rightarrow \infty} \sum_{j=1}^n \sin\left(\frac{j\pi}{2n}\right) \frac{\pi}{2n} \\ &= \lim_{n \rightarrow \infty} \frac{\pi}{2n} \frac{1}{2 \sin(\pi/(4n))} \left( \cos \frac{\pi}{4n} - \cos \frac{(2n+1)\pi}{4n} \right) \\ &= \lim_{n \rightarrow \infty} \frac{\pi/(4n)}{\sin(\pi/(4n))} \lim_{n \rightarrow \infty} \left( \cos \frac{\pi}{4n} - \cos \frac{(2n+1)\pi}{4n} \right) \\ &= 1 \times (\cos 0 - \cos \frac{\pi}{2}) = 1. \end{aligned}$$

3. a)  $\sin\left((j + \frac{1}{2})t\right) - \sin\left((j - \frac{1}{2})t\right)$   
 $= \sin(jt) \cos(\frac{1}{2}t) + \cos(jt) \sin(\frac{1}{2}t)$   
 $\quad - \sin(jt) \cos(\frac{1}{2}t) + \cos(jt) \sin(\frac{1}{2}t)$   
 $= 2 \cos(jt) \sin(\frac{1}{2}t).$

Therefore, we obtain a telescoping sum:

$$\begin{aligned} &\sum_{j=1}^n \cos(jt) \\ &= \frac{1}{2 \sin(\frac{1}{2}t)} \sum_{j=1}^n \left[ \sin\left((j + \frac{1}{2})t\right) - \sin\left((j - \frac{1}{2})t\right) \right] \\ &= \frac{1}{2 \sin(\frac{1}{2}t)} \left[ \sin\left((n + \frac{1}{2})t\right) - \sin(\frac{1}{2}t) \right]. \end{aligned}$$

- b) Let  $P_n = \{0, \frac{\pi}{3n}, \frac{2\pi}{3n}, \frac{3\pi}{3n}, \dots, \frac{n\pi}{3n}\}$  be the partition of  $[0, \pi/3]$  into  $n$  subintervals of equal length  $\Delta x = \pi/3n$ . Using  $t = \pi/3n$  in the formula obtained in part (a), we get

$$\begin{aligned} & \int_0^{\pi/2} \cos x \, dx \\ &= \lim_{n \rightarrow \infty} \sum_{j=1}^n \cos\left(\frac{j\pi}{3n}\right) \frac{\pi}{3n} \\ &= \lim_{n \rightarrow \infty} \frac{\pi}{3n} \frac{1}{2 \sin(\pi/(6n))} \left( \sin \frac{(2n+1)\pi}{6n} - \sin \frac{\pi}{6n} \right) \\ &= \lim_{n \rightarrow \infty} \frac{\pi/(6n)}{\sin(\pi/(6n))} \lim_{n \rightarrow \infty} \left( \sin \frac{(2n+1)\pi}{6n} - \sin \frac{\pi}{6n} \right) \\ &= 1 \times \left( \sin \frac{\pi}{3} - \sin 0 \right) = \frac{\sqrt{3}}{2}. \end{aligned}$$

4.  $f(x) = 1/x^2$ ,  $1 = x_0 < x_1 < x_2 < \dots < x_n = 2$ . If  $c_i = \sqrt{x_{i-1}x_i}$ , then

$$x_{i-1}^2 < x_{i-1}x_i = c_i^2 < x_i^2,$$

so  $x_{i-1} < c_i < x_i$ . We have

$$\begin{aligned} \sum_{i=1}^n f(c_i) \Delta x_i &= \sum_{i=1}^n \frac{1}{x_{i-1}x_i} (x_i - x_{i-1}) \\ &= \sum_{i=1}^n \left( \frac{1}{x_{i-1}} - \frac{1}{x_i} \right) \quad (\text{telescoping}) \\ &= \frac{1}{x_0} - \frac{1}{x_n} = 1 - \frac{1}{2} = \frac{1}{2}. \end{aligned}$$

Thus  $\int_1^2 \frac{dx}{x^2} = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(c_i) \Delta x_i = \frac{1}{2}$ .

5. We want to prove that for each positive integer  $k$ ,

$$\sum_{j=1}^n j^k = \frac{n^{k+1}}{k+1} + \frac{n^k}{2} + P_{k-1}(n),$$

where  $P_{k-1}$  is a polynomial of degree at most  $k-1$ . First check the case  $k=1$ :

$$\sum_{j=1}^n j = \frac{n(n+1)}{2} = \frac{n^{1+1}}{1+1} + \frac{n}{2} + P_0(n),$$

where  $P_0(n) = 0$  certainly has degree  $\leq 0$ . Now assume that the formula above holds for  $k=1, 2, 3, \dots, m$ . We will show that it also holds for  $k=m+1$ . To this end, sum the the formula

$$(j+1)^{m+2} - j^{m+2} = (m+2)j^{m+1} + \frac{(m+2)(m+1)}{2}j^m + \dots + 1$$

(obtained by the Binomial Theorem) for  $j=1, 2, \dots, n$ . The left side telescopes, and we get

$$\begin{aligned} (n+1)^{m+2} - 1^{m+2} &= (m+2) \sum_{j=1}^n j^{m+1} \\ &+ \frac{(m+2)(m+1)}{2} \sum_{j=1}^n j^m + \dots + \sum_{j=1}^n 1. \end{aligned}$$

Expanding the binomial power on the left and using the induction hypothesis on the other terms we get

$$\begin{aligned} n^{m+2} + (m+2)n^{m+1} + \dots &= (m+2) \sum_{j=1}^n j^{m+1} \\ &+ \frac{(m+2)(m+1)}{2} \frac{n^{m+1}}{m+1} + \dots, \end{aligned}$$

where the  $\dots$  represent terms of degree  $m$  or lower in the variable  $n$ . Solving for the remaining sum, we get

$$\begin{aligned} & \sum_{j=1}^n j^{m+1} \\ &= \frac{1}{m+2} \left( n^{m+2} + (m+2)n^{m+1} + \dots - \frac{m+2}{2}n^{m+1} - \dots \right) \\ &= \frac{n^{m+2}}{m+2} + \frac{n^{m+1}}{2} + \dots \end{aligned}$$

so that the formula is also correct for  $k=m+1$ . Hence it is true for all positive integers  $k$  by induction.

- b) Using the technique of Example 2 in Section 6.2 and the result above,

$$\begin{aligned} \int_0^a x^k \, dx &= \lim_{n \rightarrow \infty} \frac{a}{n} \sum_{j=1}^n \left( \frac{a}{n} \right)^j \\ &= a^{k+1} \lim_{n \rightarrow \infty} \frac{1}{n^{k+1}} \sum_{j=1}^n j^k \\ &= a^{k+1} \lim_{n \rightarrow \infty} \left( \frac{1}{k+1} + \frac{1}{2n} + \frac{P_{k-1}(n)}{n^{k+1}} \right) \\ &= \frac{a^{k+1}}{k+1}. \end{aligned}$$

6. Let  $f(x) = ax^3 + bx^2 + cx + d$ . We used Maple to calculate the following:

The tangent to  $y = f(x)$  at  $P = (p, f(p))$  has equation

$$y = g(x) = ap^3 + bp^2 + cp + d + (3ap^2 + 2bp + c)(x - p).$$

This line intersects  $y = f(x)$  at  $x = p$  (double root) and at  $x = q$ , where

$$q = -\frac{2ap + b}{a}.$$

Similarly, the tangent to  $y = f(x)$  at  $x = q$  has equation

$$y = h(x) = aq^3 + bq^2 + cq + d + (3aq^2 + 2bq + c)(x - q),$$

and intersects  $y = f(x)$  at  $x = q$  (double root) and  $x = r$ , where

$$r = -\frac{2aq + b}{a} = \frac{4ap + b}{a}.$$

The area between  $y = f(x)$  and the tangent line at  $P$  is the absolute value of

$$\begin{aligned} & \int_p^q (f(x) - g(x)) dx \\ &= -\frac{1}{12} \left( \frac{81a^4 p^4 + 108a^3 b p^3 + 54a^2 b^2 p^2 + 12ab^3 p + b^4}{a^3} \right). \end{aligned}$$

The area between  $y = f(x)$  and the tangent line at  $Q = (q, f(q))$  is the absolute value of

$$\begin{aligned} & \int_q^r (f(x) - h(x)) dx \\ &= -\frac{4}{3} \left( \frac{81a^4 p^4 + 108a^3 b p^3 + 54a^2 b^2 p^2 + 12ab^3 p + b^4}{a^3} \right), \end{aligned}$$

which is 16 times the area between  $y = f(x)$  and the tangent at  $P$ .

7. We continue with the calculations begun in the previous problem.  $P$  and  $Q$  are as they were in that problem, but  $R = (r, f(r))$  is now the inflection point of  $y = f(x)$ , given by  $f''(r) = 0$ . Maple gives

$$r = -\frac{b}{3a}.$$

Since

$$p - r = \frac{b + 3ap}{a} \text{ and } r - q = \frac{2(b + 3ap)}{a}$$

have the same sign,  $R$  must lie between  $Q$  and  $P$  on the curve  $y = f(x)$ . The line  $QR$  has a rather complicated equation  $y = k(x)$ , which we won't reproduce here, but the area between this line and the curve  $y = f(x)$  is the absolute value of  $\int_r^q (f(x) - k(x)) dx$ , which Maple evaluates to be

$$-\frac{4}{81} \left( \frac{81a^4 p^4 + 108a^3 b p^3 + 54a^2 b^2 p^2 + 12ab^3 p + b^4}{a^3} \right),$$

which is 16/27 of the area between the curve and its tangent at  $P$ . This leaves 11/27 of that area to lie between the curve,  $QR$ , and the tangent, so  $QR$  divides the area between  $y = f(x)$  and its tangent at  $P$  in the ratio 16/11.

8. Let  $f(x) = ax^4 + bx^3 + cx^2 + dx + e$ . The tangent to  $y = f(x)$  at  $P = (p, f(p))$  has equation

$$y = g(x) = ap^4 + bp^3 + cp^2 + dp + e + (4ap^3 + 3bp^2 + 2cp + d)(x - p),$$

and intersects  $y = f(x)$  at  $x = p$  (double root) and at the two points

$$x = \frac{-2ap - b \pm \sqrt{b^2 - 4ac - 4abp - 8a^2 p^2}}{2a}.$$

If these latter two points coincide, then the tangent is a "double tangent." This happens if

$$8a^2 p^2 + 4abp + 4ac - b^2 = 0,$$

which has two solutions, which we take to be  $p$  and  $q$ :

$$\begin{aligned} p &= \frac{-b + \sqrt{3b^2 - 8ac}}{4a} \\ q &= \frac{-b - \sqrt{3b^2 - 8ac}}{4a} = -p - \frac{b}{2a}. \end{aligned}$$

(Both roots exist and are distinct provided  $3b^2 > 8ac$ .) The point  $T$  corresponds to  $x = t = (p + q)/2 = -b/4a$ . The tangent to  $y = f(x)$  at  $x = t$  has equation

$$y = h(x) = -\frac{3b^4}{256a^3} + \frac{b^2 c}{16a^2} - \frac{bd}{4a} + e + \left( \frac{b^3}{8a^2} - \frac{bc}{2a} + d \right) \left( x + \frac{b}{4a} \right)$$

and it intersects  $y = f(x)$  at the points  $U$  and  $V$  with  $x$ -coordinates

$$\begin{aligned} u &= \frac{-b - \sqrt{2}\sqrt{3b^2 - 8ac}}{4a}, \\ v &= \frac{-b + \sqrt{2}\sqrt{3b^2 - 8ac}}{4a}. \end{aligned}$$

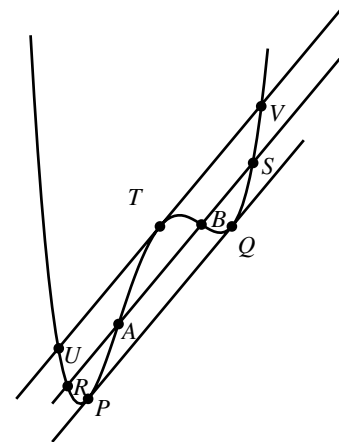


Fig. C-5.8

- a) The areas between the curve  $y = f(x)$  and the lines  $PQ$  and  $UV$  are, respectively, the absolute values of

$$A_1 = \int_p^q (f(x) - g(x)) dx \text{ and } A_2 = \int_u^v (h(x) - f(x)) dx.$$

Maple calculates these two integrals and simplifies the ratio  $A_1/A_2$  to be  $1/\sqrt{2}$ .

- b) The two inflection points  $A$  and  $B$  of  $f$  have  $x$ -coordinates shown by Maple to be

$$\alpha = \frac{-3b - \sqrt{3(3b^2 - 8ac)}}{12a} \text{ and}$$

$$\beta = \frac{-3b + \sqrt{3(3b^2 - 8ac)}}{12a}.$$

It then determines the four points of intersection of the line  $y = k(x)$  through these inflection points and the curve. The other two points have  $x$ -coordinates

$$r = \frac{-3b - \sqrt{15(3b^2 - 8ac)}}{12a} \text{ and}$$

$$s = \frac{-3b + \sqrt{15(3b^2 - 8ac)}}{12a}.$$

The region bounded by  $RS$  and the curve  $y = f(x)$  is divided into three parts by  $A$  and  $B$ . The areas of these three regions are the absolute values of

$$A_1 = \int_r^\alpha (k(x) - f(x)) dx$$

$$A_2 = \int_\alpha^\beta (f(x) - k(x)) dx$$

$$A_3 = \int_\beta^s (k(x) - f(x)) dx.$$

The expressions calculated by Maple for  $k(x)$  and for these three areas are very complicated, but Maple simplifies the ratios  $A_3/A_1$  and  $A_2/A_1$  to 1 and 2 respectively, as was to be shown.