CHAPTER 6. TECHNIQUES OF INTEGRATION

Section 6.1 Integration by Parts (page 321)

1. $\int x \cos x \, dx$

$$\begin{aligned} U &= x & dV &= \cos x \, dx \\ dU &= dx & V &= \sin x \end{aligned}$$

$$= x \sin x - \int \sin x \, dx$$
$$= x \sin x + \cos x + C.$$

2. $\int (x+3)e^{2x} \, dx$

$$\begin{aligned} U &= x+3 & dV &= e^{2x} \, dx \\ dU &= dx & V &= \tfrac{1}{2}e^{2x} \end{aligned}$$

$$= \frac{1}{2}(x+3)e^{2x} - \frac{1}{2}\int e^{2x} \, dx$$
$$= \frac{1}{2}(x+3)e^{2x} - \frac{1}{4}e^{2x} + C.$$

3. $\int x^2 \cos \pi x \, dx$

$$\begin{aligned} U &= x^2 & dV &= \cos \pi x \, dx \\ dU &= 2x \, dx & V &= \frac{\sin \pi x}{\pi} \end{aligned}$$

$$= \frac{x^2 \sin \pi x}{\pi} - \frac{2}{\pi}\int x \sin \pi x \, dx$$

$$\begin{aligned} U &= x & dV &= \sin \pi x \, dx \\ dU &= dx & V &= -\frac{\cos \pi x}{\pi} \end{aligned}$$

$$= \frac{x^2 \sin \pi x}{\pi} - \frac{2}{\pi}\left(-\frac{x \cos \pi x}{\pi} + \frac{1}{\pi}\int \cos \pi x \, dx\right)$$
$$= \frac{1}{\pi}x^2 \sin \pi x + \frac{2}{\pi^2}x \cos \pi x - \frac{2}{\pi^3}\sin \pi x + C.$$

4. $\int (x^2 - 2x)e^{kx} \, dx$

$$\begin{aligned} U &= x^2 - 2x & dV &= e^{kx} \\ dU &= (2x-2) \, dx & V &= \frac{1}{k}e^{kx} \end{aligned}$$

$$= \frac{1}{k}(x^2-2x)e^{kx} - \frac{1}{k}\int (2x-2)e^{kx} \, dx$$

$$\begin{aligned} U &= x-1 & dV &= e^{kx} \, dx \\ dU &= dx & V &= \frac{1}{k}e^{kx} \end{aligned}$$

$$= \frac{1}{k}(x^2-2x)e^{kx} - \frac{2}{k}\left[\frac{1}{k}(x-1)e^{kx} - \frac{1}{k}\int e^{kx} \, dx\right]$$
$$= \frac{1}{k}(x^2-2x)e^{kx} - \frac{2}{k^2}(x-1)e^{kx} + \frac{2}{k^3}e^{kx} + C.$$

5. $\int x^3 \ln x \, dx$

$$\begin{aligned} U &= \ln x & dV &= x^3 \, dx \\ dU &= \frac{dx}{x} & V &= \frac{x^4}{4} \end{aligned}$$

$$= \frac{1}{4}x^4 \ln x - \frac{1}{4}\int x^3 \, dx$$
$$= \frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + C.$$

6. $\int x(\ln x)^3 \, dx = I_3$ where

$$I_n = \int x(\ln x)^n \, dx$$

$$\begin{aligned} U &= (\ln x)^n & dV &= x \, dx \\ dU &= \frac{n}{x}(\ln x)^{n-1} \, dx & V &= \frac{1}{2}x^2 \end{aligned}$$

$$= \frac{1}{2}x^2(\ln x)^n - \frac{n}{2}\int x(\ln x)^{n-1} \, dx$$
$$= \frac{1}{2}x^2(\ln x)^n - \frac{n}{2}I_{n-1}$$

$$I_3 = \frac{1}{2}x^2(\ln x)^3 - \frac{3}{2}I_2$$
$$= \frac{1}{2}x^2(\ln x)^3 - \frac{3}{2}\left[\frac{1}{2}x^2(\ln x)^2 - \frac{2}{2}I_1\right]$$
$$= \frac{1}{2}x^2(\ln x)^3 - \frac{3}{4}x^2(\ln x)^2 + \frac{3}{2}\left[\frac{1}{2}x^2(\ln x) - \frac{1}{2}I_0\right]$$
$$= \frac{1}{2}x^2(\ln x)^3 - \frac{3}{4}x^2(\ln x)^2 + \frac{3}{4}x^2(\ln x) - \frac{3}{4}\int x \, dx$$
$$= \frac{x^2}{2}\left[(\ln x)^3 - \frac{3}{2}(\ln x)^2 + \frac{3}{2}(\ln x) - \frac{3}{4}\right] + C.$$

7. $\int \tan^{-1} x \, dx$

$$\begin{aligned} U &= \tan^{-1} x & dV &= dx \\ dU &= \frac{dx}{1+x^2} & V &= x \end{aligned}$$

$$= x \tan^{-1} x - \int \frac{x \, dx}{1+x^2}$$
$$= x \tan^{-1} x - \frac{1}{2}\ln(1+x^2) + C.$$

8. $\int x^2 \tan^{-1} x \, dx$

$$\begin{aligned} U &= \tan^{-1} x & dV &= x^2 \, dx \\ dU &= \frac{dx}{1+x^2} & V &= \frac{x^3}{3} \end{aligned}$$

$$= \frac{x^3}{3}\tan^{-1} x - \frac{1}{3}\int \frac{x^3}{1+x^2} \, dx$$
$$= \frac{x^3}{3}\tan^{-1} x - \frac{1}{3}\int \left(x - \frac{x}{1+x^2}\right) dx$$
$$= \frac{x^3}{3}\tan^{-1} x - \frac{x^2}{6} + \frac{1}{6}\ln(1+x^2) + C.$$

9. $\int x \sin^{-1} x\, dx$

$$U = \sin^{-1} x \qquad dV = x\,dx$$
$$dU = \frac{dx}{\sqrt{1-x^2}} \qquad V = \frac{x^2}{2}$$
$$= \frac{1}{2}x^2 \sin^{-1} x - \frac{1}{2}\int \frac{x^2\,dx}{\sqrt{1-x^2}} \quad \text{Let } x = \sin\theta,\ dx = \cos\theta\,d\theta$$
$$= \frac{1}{2}x^2 \sin^{-1} x - \frac{1}{2}\int \sin^2\theta\,d\theta$$
$$= \frac{1}{2}x^2 \sin^{-1} x - \frac{1}{4}(\theta - \sin\theta\cos\theta) + C$$
$$= \left(\frac{1}{2}x^2 - \frac{1}{4}\right)\sin^{-1} x + \frac{1}{4}x\sqrt{1-x^2} + C.$$

10. $\int x^5 e^{-x^2}\,dx = I_2$ where

$$I_n = \int x^{(2n+1)} e^{-x^2}\,dx$$

$$U = x^{2n} \qquad dV = xe^{-x^2}\,dx$$
$$dU = 2nx^{(2n-1)}\,dx \qquad V = -\tfrac{1}{2}e^{-x^2}$$
$$= -\frac{1}{2}x^{2n}e^{-x^2} + n\int x^{(2n-1)}e^{-x^2}\,dx$$
$$= -\frac{1}{2}x^{2n}e^{-x^2} + nI_{n-1}$$
$$I_2 = -\frac{1}{2}x^4 e^{-x^2} + 2\left[-\frac{1}{2}x^2e^{-x^2} + \int xe^{-x^2}\,dx\right]$$
$$= -\frac{1}{2}e^{-x^2}(x^4 + 2x^2 + 2) + C.$$

11. $I_n = \int_0^{\pi/4} \sec^n x\,dx$

$$U = \sec^{n-2} x \qquad dV = \sec^2 x\,dx$$
$$dU = (n-2)\sec^{n-2} x \tan x\,dx \qquad V = \tan x$$
$$= \tan x \sec^{n-2} x\Big|_0^{\pi/4} - (n-2)\int_0^{\pi/4} \sec^{n-2} x \tan^2 x\,dx$$
$$= (\sqrt{2})^{n-2} - (n-2)(I_n - I_{n-2}).$$
$$(n-1)I_n = (\sqrt{2})^{n-2} + (n-2)I_{n-2}.$$

Therefore

$$I_n = \frac{(\sqrt{2})^{n-2}}{n-1} + \frac{n-2}{n-1}I_{n-2}, \qquad (n \ge 2).$$

For $n = 5$ we have

$$\int_0^{\pi/4} \sec^5 x\,dx = I_5 = \frac{2\sqrt{2}}{4} + \frac{3}{4}I_3$$
$$= \frac{\sqrt{2}}{2} + \frac{3}{4}\left(\frac{\sqrt{2}}{2} + \frac{1}{2}I_1\right)$$
$$= \frac{7\sqrt{2}}{8} + \frac{3}{8}\ln|\sec x + \tan x|\Big|_0^{\pi/4}$$
$$= \frac{7\sqrt{2}}{8} + \frac{3}{8}\ln(1+\sqrt{2}).$$

12. $I = \int \tan^2 x \sec x\,dx$

$$U = \tan x \qquad dV = \sec x \tan x\,dx$$
$$dU = \sec^2 x\,dx \qquad V = \sec x$$
$$= \sec x \tan x - \int \sec^3 x\,dx$$
$$= \sec x \tan x - \int (1 + \tan^2 x)\sec x\,dx$$
$$= \sec x \tan x - \ln|\sec x + \tan x| - I$$

Thus, $I = \frac{1}{2}\sec x \tan x - \frac{1}{2}\ln|\sec x + \tan x| + C.$

13. $I = \int e^{2x}\sin 3x\,dx$

$$U = e^{2x} \qquad dV = \sin 3x\,dx$$
$$dU = 2e^{2x}\,dx \qquad V = -\tfrac{1}{3}\cos 3x$$
$$= -\frac{1}{3}e^{2x}\cos 3x + \frac{2}{3}\int e^{2x}\cos 3x\,dx$$

$$U = e^{2x} \qquad dV = \cos 3x\,dx$$
$$dU = 2e^{2x}\,dx \qquad V = \tfrac{1}{3}\sin 3x$$
$$= -\frac{1}{3}e^{2x}\cos 3x + \frac{2}{3}\left(\frac{1}{3}e^{2x}\sin 3x - \frac{2}{3}I\right)$$
$$\frac{13}{9}I = -\frac{1}{3}e^{2x}\cos 3x + \frac{2}{9}e^{2x}\sin 3x + C_1$$
$$I = \frac{1}{13}e^{2x}(2\sin 3x - 3\cos 3x) + C.$$

14. $I = \int xe^{\sqrt{x}}\,dx$ Let $x = w^2$, $dx = 2w\,dw$

$$= 2\int w^3 e^w\,dw = 2I_3 \text{ where}$$
$$I_n = \int w^n e^w\,dw$$
$$U = w^n \qquad dV = e^w\,dw$$
$$dU = nw^{n-1}\,dw \qquad V = e^w$$
$$= w^n e^w - nI_{n-1}.$$
$$I = 2I_3 = 2w^3e^w - 6[w^2e^w - 2(we^w - I_0)]$$
$$= e^{\sqrt{x}}(2x\sqrt{x} - 6x + 12\sqrt{x} - 12) + C.$$

15. $\displaystyle\int_{1/2}^{1} \frac{\sin^{-1} x}{x^2}\,dx$

$$U = \sin^{-1} x \qquad dV = \frac{dx}{x^2}$$
$$dU = \frac{dx}{\sqrt{1-x^2}} \qquad V = -\frac{1}{x}$$
$$= -\frac{1}{x}\sin^{-1} x\Big|_{1/2}^{1} + \int_{1/2}^{1} \frac{dx}{x\sqrt{1-x^2}} \quad \text{Let } x = \sin\theta, \ dx = \cos\theta\,d\theta$$
$$= -\frac{\pi}{2} + \frac{\pi}{3} + \int_{\pi/6}^{\pi/2} \csc\theta\,d\theta$$
$$= -\frac{\pi}{6} - \ln|\csc\theta + \cot\theta|\Big|_{\pi/6}^{\pi/2}$$
$$= -\frac{\pi}{6} - \ln 1 + \ln(2+\sqrt{3}) = \ln(2+\sqrt{3}) - \frac{\pi}{6}.$$

16. $\displaystyle\int_0^1 \sqrt{x}\sin(\pi\sqrt{x})\,dx$ Let $x = w^2$, $dx = 2w\,dw$

$$= 2\int_0^1 w^2 \sin(\pi w)\,dw$$
$$U = w^2 \qquad dV = \sin(\pi w)\,dw$$
$$dU = 2w\,dw \qquad V = -\frac{\cos(\pi w)}{\pi}$$
$$= -\frac{2}{\pi}w^2\cos(\pi w)\Big|_0^1 + \frac{4}{\pi}\int_0^1 w\cos(\pi w)\,dw$$
$$U = w \qquad dV = \cos(\pi w)\,dw$$
$$dU = dw \qquad V = \frac{\sin(\pi w)}{\pi}$$
$$= \frac{2}{\pi} + \frac{4}{\pi}\left[\frac{w}{\pi}\sin(\pi w)\right]\Big|_0^1 - \frac{4}{\pi^2}\int_0^1 \sin(\pi w)\,dw$$
$$= \frac{2}{\pi} + \frac{4}{\pi^3}\cos(\pi w)\Big|_0^1 = \frac{2}{\pi} + \frac{4}{\pi^3}(-2) = \frac{2}{\pi} - \frac{8}{\pi^3}.$$

17. $\displaystyle\int x\sec^2 x\,dx$

$$U = x \qquad dV = \sec^2 x\,dx$$
$$dU = dx \qquad V = \tan x$$
$$= x\tan x - \int \tan x\,dx$$
$$= x\tan x - \ln|\sec x| + C.$$

18. $\displaystyle\int x\sin^2 x\,dx = \frac{1}{2}\int (x - x\cos 2x)\,dx$

$$= \frac{x^2}{4} - \frac{1}{2}\int x\cos 2x\,dx$$
$$U = x \qquad dV = \cos 2x\,dx$$
$$dU = dx \qquad V = \tfrac{1}{2}\sin 2x$$
$$= \frac{x^2}{4} - \frac{1}{2}\left[\frac{1}{2}x\sin 2x - \frac{1}{2}\int \sin 2x\,dx\right]$$
$$= \frac{x^2}{4} - \frac{x}{4}\sin 2x - \frac{1}{8}\cos 2x + C.$$

19. $\displaystyle I = \int \cos(\ln x)\,dx$

$$U = \cos(\ln x) \qquad dV = dx$$
$$dU = -\frac{\sin(\ln x)}{x}\,dx \qquad V = x$$
$$= x\cos(\ln x) + \int \sin(\ln x)\,dx$$
$$U = \sin(\ln x) \qquad dV = dx$$
$$dU = \frac{\cos(\ln x)}{x}\,dx \qquad V = x$$
$$= x\cos(\ln x) + x\sin(\ln x) - I$$
$$I = \frac{1}{2}\Big(x\cos(\ln x) + x\sin(\ln x)\Big) + C.$$

20. $\displaystyle I = \int_1^e \sin(\ln x)\,dx$

$$U = \sin(\ln x) \qquad dV = dx$$
$$dU = \frac{\cos(\ln x)}{x}\,dx \qquad V = x$$
$$= x\sin(\ln x)\Big|_1^e - \int_1^e \cos(\ln x)\,dx$$
$$U = \cos(\ln x) \qquad dV = dx$$
$$dU = -\frac{\sin(\ln x)}{x}\,dx \qquad V = x$$
$$= e\sin(1) - \left[x\cos(\ln x)\Big|_1^e + I\right]$$

Thus, $I = \dfrac{1}{2}[e\sin(1) - e\cos(1) + 1]$.

21. $\displaystyle\int \frac{\ln(\ln x)}{x}\,dx$ Let $u = \ln x$, $du = \dfrac{dx}{x}$

$$= \int \ln u\,du$$
$$U = \ln u \qquad dV = du$$
$$dU = \frac{du}{u} \qquad V = u$$
$$= u\ln u - \int du = u\ln u - u + C$$
$$= (\ln x)(\ln(\ln x)) - \ln x + C.$$

22. $\displaystyle\int_0^4 \sqrt{x}e^{\sqrt{x}}\,dx$ Let $x = w^2$, $dx = 2w\,dw$

$$= 2\int_0^2 w^2 e^w\,dw = 2I_2$$

See solution #16 for the formula $I_n = \int w^n e^w\,dw = w^n e^w - nI_{n-1}$.

$$= 2\Big(w^2e^w\Big|_0^2 - 2I_1\Big) = 8e^2 - 4\Big(we^w\Big|_0^2 - I_0\Big)$$
$$= 8e^2 - 8e^2 + 4\int_0^2 e^w\,dw = 4(e^2 - 1).$$

23. $\int \cos^{-1} x\, dx$

$$U = \cos^{-1} x \qquad dV = dx$$
$$dU = -\frac{dx}{\sqrt{1-x^2}} \qquad V = x$$
$$= x\cos^{-1} x + \int \frac{x\,dx}{\sqrt{1-x^2}}$$
$$= x\cos^{-1} x - \sqrt{1-x^2} + C.$$

24. $\int x \sec^{-1} x\, dx$

$$U = \sec^{-1} x \qquad dV = x\,dx$$
$$dU = \frac{dx}{|x|\sqrt{x^2-1}} \qquad V = \frac{1}{2}x^2$$
$$= \frac{1}{2}x^2 \sec^{-1} x - \frac{1}{2}\int \frac{|x|}{\sqrt{x^2-1}}\,dx$$
$$= \frac{1}{2}x^2 \sec^{-1} x - \frac{1}{2}\operatorname{sgn}(x)\sqrt{x^2-1} + C.$$

25. $\int_1^2 \sec^{-1} x\, dx$

$$= \int_1^2 \cos^{-1}\frac{1}{x}$$
$$U = \cos^{-1}\frac{1}{x} \qquad dV = dx$$
$$dU = -\frac{1}{\sqrt{1-\frac{1}{x^2}}}\left(-\frac{1}{x^2}\right)dx \qquad V = x$$
$$= x\cos^{-1}\frac{1}{x}\Big|_1^2 - \int_1^2 \frac{dx}{\sqrt{x^2-1}} \qquad \text{Let } x = \sec\theta,\ dx = \sec\theta\tan\theta\,d\theta$$
$$= \frac{2\pi}{3} - 0 - \int_0^{\pi/3} \sec\theta\,d\theta$$
$$= \frac{2\pi}{3} - \ln|\sec\theta + \tan\theta|\Big|_0^{\pi/3}$$
$$= \frac{2\pi}{3} - \ln(2+\sqrt{3}).$$

26. $\int (\sin^{-1} x)^2\, dx \qquad \text{Let } x = \sin\theta,\ dx = \cos\theta\,d\theta$

$$= \int \theta^2 \cos\theta\,d\theta$$
$$U = \theta^2 \qquad dV = \cos\theta\,d\theta$$
$$dU = 2\theta\,d\theta \qquad V = \sin\theta$$
$$= \theta^2\sin\theta - 2\int \theta\sin\theta\,d\theta$$
$$U = \theta \qquad dV = \sin\theta\,d\theta$$
$$dU = d\theta \qquad V = -\cos\theta$$
$$= \theta^2\sin\theta - 2(-\theta\cos\theta + \int \cos\theta\,d\theta)$$
$$= \theta^2\sin\theta + 2\theta\cos\theta - 2\sin\theta + C$$
$$= x(\sin^{-1} x)^2 + 2\sqrt{1-x^2}(\sin^{-1} x) - 2x + C.$$

27. $\int x(\tan^{-1} x)^2\, dx$

$$U = (\tan^{-1} x)^2 \qquad dV = x\,dx$$
$$dU = \frac{2\tan^{-1} x\,dx}{1+x^2} \qquad V = \frac{x^2}{2}$$
$$= \frac{x^2}{2}(\tan^{-1} x)^2 - \int \frac{x^2\tan^{-1} x}{1+x^2}\,dx \qquad \text{Let } u = \tan^{-1} x,\ du = \frac{dx}{1+x^2}$$
$$= \frac{x^2}{2}(\tan^{-1} x)^2 - \int u\tan^2 u\,du$$
$$= \frac{x^2}{2}(\tan^{-1} x)^2 + \int (u - u\sec^2 u)\,du$$
$$= \frac{x^2}{2}(\tan^{-1} x)^2 + \frac{u^2}{2} - \int u\sec^2 u\,du$$
$$U = u \qquad dV = \sec^2 u\,du$$
$$dU = du \qquad V = \tan u$$
$$= \frac{1}{2}(x^2+1)(\tan^{-1} x)^2 - u\tan u + \int \tan u\,du$$
$$= \frac{1}{2}(x^2+1)(\tan^{-1} x)^2 - x\tan^{-1} x + \ln|\sec u| + C$$
$$= \frac{1}{2}(x^2+1)(\tan^{-1} x)^2 - x\tan^{-1} x + \frac{1}{2}\ln(1+x^2) + C$$

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28. By the procedure used in Example 4 of Section 7.1,

$$\int e^x\cos x\,dx = \tfrac{1}{2}e^x(\sin x + \cos x) + C;$$
$$\int e^x\sin x\,dx = \tfrac{1}{2}e^x(\sin x - \cos x) + C.$$

Now

$$\int xe^x\cos x\,dx$$
$$U = x \qquad dV = e^x\cos x\,dx$$
$$dU = dx \qquad V = \tfrac{1}{2}e^x(\sin x + \cos x)$$
$$= \tfrac{1}{2}xe^x(\sin + \cos x) - \tfrac{1}{2}\int e^x(\sin x + \cos x)\,dx$$
$$= \tfrac{1}{2}xe^x(\sin + \cos x)$$
$$\quad - \tfrac{1}{4}e^x(\sin x - \cos x + \sin x + \cos x) + C$$
$$= \tfrac{1}{2}xe^x(\sin x + \cos x) - \tfrac{1}{2}e^x\sin x + C.$$

29. Area $= A = \int_0^\pi e^{-x}\sin x\,dx$

$$U = e^{-x} \qquad dV = \sin x\,dx$$
$$dU = -e^{-x}\,dx \qquad V = -\cos x$$
$$= -e^{-x}\cos x\Big|_0^\pi - \int_0^\pi e^{-x}\cos x\,dx$$
$$U = e^{-x} \qquad dV = \cos x\,dx$$
$$dU = -e^{-x}\,dx \qquad V = \sin x$$
$$= e^{-\pi} + 1 - \left(e^{-x}\sin x\Big|_0^\pi + A\right)$$

Thus Area $= A = \dfrac{1+e^{-\pi}}{2}$ units^2

30. The tangent line to $y = \ln x$ at $x = 1$ is $y = x - 1$, Hence,

$$\text{Shaded area } = \frac{1}{2}(1)(1) + (1)(e-2) - \int_1^e \ln x\,dx$$
$$= e - \frac{3}{2} - (x\ln x - x)\Big|_1^e$$
$$= e - \frac{3}{2} - e + e + 0 - 1 = e - \frac{5}{2} \text{ sq. units.}$$

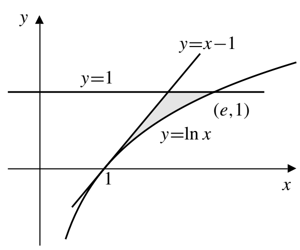

Fig. 6.1.30

31. $I_n = \int (\ln x)^n\,dx$

$$U = (\ln x)^n \qquad dV = dx$$
$$dU = n(\ln x)^{n-1}\frac{dx}{x} \qquad V = x$$
$$I_n = x(\ln x)^n - nI_{n-1}.$$
$$I_4 = x(\ln x)^4 - 4I_3$$
$$= x(\ln x)^4 - 4\Big(x(\ln x)^3 - 3I_2\Big)$$
$$= x(\ln x)^4 - 4x(\ln x)^3 + 12\Big(x(\ln x)^2 - 2I_1\Big)$$
$$= x(\ln x)^4 - 4x(\ln x)^3 + 12x(\ln x)^2 - 24\Big(x\ln x - x\Big) + C$$
$$= x\Big((\ln x)^4 - 4(\ln x)^3 + 12(\ln x)^2 - 24\ln x + 24\Big) + C.$$

32. $I_n = \int_0^{\pi/2} x^n \sin x\,dx$

$$U = x^n \qquad dV = \sin x\,dx$$
$$dU = nx^{n-1}\,dx \qquad V = -\cos x$$
$$= -x^n\cos x\Big|_0^{\pi/2} + n\int_0^{\pi/2} x^{n-1}\cos x\,dx$$
$$U = x^{n-1} \qquad dV = \cos x\,dx$$
$$dU = (n-1)x^{n-2}\,dx \qquad V = \sin x$$
$$= n\left[x^{n-1}\sin x\Big|_0^{\pi/2} - (n-1)\int_0^{\pi/2} x^{n-2}\sin x\,dx\right]$$
$$= n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)I_{n-2}, \qquad (n \ge 2).$$
$$I_0 = \int_0^{\pi/2} \sin x\,dx = -\cos x\Big|_0^{\pi/2} = 1.$$
$$I_6 = 6\left(\frac{\pi}{2}\right)^5 - 6(5)\left\{4\left(\frac{\pi}{2}\right)^3 - 4(3)\left[2\left(\frac{\pi}{2}\right) - 2(1)I_0\right]\right\}$$
$$= \frac{3}{16}\pi^5 - 15\pi^3 + 360\pi - 720.$$

33. $I_n = \int \sin^n x\,dx \quad (n \ge 2)$

$$U = \sin^{n-1} x \qquad dV = \sin x\,dx$$
$$dU = (n-1)\sin^{n-2} x\cos x\,dx \qquad V = -\cos x$$
$$= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\cos^2 x\,dx$$
$$= -\sin^{n-1} x\cos x + (n-1)(I_{n-2} - I_n)$$
$$nI_n = -\sin^{n-1} x\cos x + (n-1)I_{n-2}$$
$$I_n = -\frac{1}{n}\sin^{n-1} x\cos x + \frac{n-1}{n}I_{n-2}.$$

Note: $I_0 = x + C$, $I_1 = -\cos x + C$. Hence

$$I_6 = -\frac{1}{6}\sin^5 x\cos x + \frac{5}{6}I_4$$
$$= -\frac{1}{6}\sin^5 x\cos x + \frac{5}{6}\left(-\frac{1}{4}\sin^3 x\cos x + \frac{3}{4}I_2\right)$$
$$= -\frac{1}{6}\sin^5 x\cos x - \frac{5}{24}\sin^3 x\cos x + \frac{5}{8}\left(-\frac{1}{2}\sin x\cos x + \frac{1}{2}I_0\right)$$
$$= -\frac{1}{6}\sin^5 x\cos x - \frac{5}{24}\sin^3 x\cos x - \frac{5}{16}\sin x\cos x + \frac{5}{16}x + C$$
$$= \frac{5x}{16} - \cos x\left(\frac{\sin^5 x}{6} + \frac{5\sin^3 x}{24} + \frac{5\sin x}{16}\right) + C.$$

$$\begin{aligned}
I_7 &= -\frac{1}{7}\sin^6 x\cos x + \frac{6}{7}I_5 \\
&= -\frac{1}{7}\sin^6 x\cos x + \frac{6}{7}\left(-\frac{1}{5}\sin^4 x\cos x + \frac{4}{5}I_3\right) \\
&= -\frac{1}{7}\sin^6 x\cos x - \frac{6}{35}\sin^4 x\cos x \\
&\qquad + \frac{24}{35}\left(-\frac{1}{3}\sin^2 x\cos x + \frac{2}{3}I_1\right) \\
&= -\frac{1}{7}\sin^6 x\cos x - \frac{6}{35}\sin^4 x\cos x - \frac{8}{35}\sin^2 x\cos x \\
&\qquad - \frac{16}{35}\cos x + C \\
&= -\cos x\left(\frac{\sin^6 x}{7} + \frac{6\sin^4 x}{35} + \frac{8\sin^2 x}{35} + \frac{16}{35}\right) + C.
\end{aligned}$$

34. We have

$$\begin{aligned}
I_n &= \int \sec^n x\,dx \qquad (n \ge 3) \\
&\quad U = \sec^{n-2} x \qquad\qquad dV = \sec^2 x\,dx \\
&\quad dU = (n-2)\sec^{n-2} x\tan x\,dx \qquad V = \tan x \\
&= \sec^{n-2} x\,\tan x - (n-2)\int \sec^{n-2} x\,\tan^2 x\,dx \\
&= \sec^{n-2} x\,\tan x - (n-2)\int \sec^{n-2} x(\sec^2 x - 1)\,dx \\
&= \sec^{n-2} x\,\tan x - (n-2)I_n + (n-2)I_{n-2} + C \\
I_n &= \frac{1}{n-1}(\sec^{n-2} x\,\tan x) + \frac{n-2}{n-1}I_{n-2} + C. \\
I_1 &= \int \sec x\,dx = \ln|\sec x + \tan x| + C; \\
I_2 &= \int \sec^2 x\,dx = \tan x + C. \\
I_6 &= \frac{1}{5}(\sec^4 x\,\tan x) + \frac{4}{5}\left(\frac{1}{3}\sec^2 x\,\tan x + \frac{2}{3}I_2\right) + C \\
&= \frac{1}{5}\sec^4 x\,\tan x + \frac{4}{15}\sec^2 x\,\tan x + \frac{8}{15}\tan x + C. \\
I_7 &= \frac{1}{6}(\sec^5 x\,\tan x) + \frac{5}{6}\left[\frac{1}{4}\sec^3 x\,\tan x + \right. \\
&\qquad \left.\frac{3}{4}\left(\frac{1}{2}\sec x\,\tan x + \frac{1}{2}I_1\right)\right] + C \\
&= \frac{1}{6}\sec^5 x\,\tan x + \frac{5}{24}\sec^3 x\,\tan x + \frac{15}{48}\sec x\,\tan x + \\
&\qquad \frac{15}{48}\ln|\sec x + \tan x| + C.
\end{aligned}$$

35.

$$\begin{aligned}
I_n &= \int \frac{dx}{(x^2+a^2)^n} = \frac{1}{a^2}\int \frac{x^2+a^2-x^2}{(x^2+a^2)^n}\,dx \\
&= \frac{1}{a^2}\int \frac{dx}{(x^2+a^2)^{n-1}} - \frac{1}{a^2}\int \frac{x^2}{(x^2+a^2)^n}\,dx \\
&\quad U = x \qquad dV = \frac{x\,dx}{(x^2+a^2)^n} \\
&\quad dU = dx \qquad V = \frac{-1}{2(n-1)(x^2+a^2)^{n-1}} \\
&= \frac{1}{a^2}I_{n-1} - \frac{1}{a^2}\left(\frac{-x}{2(n-1)(x^2+a^2)^{n-1}}\right. \\
&\qquad \left. + \frac{1}{2(n-1)}\int \frac{dx}{(x^2+a^2)^{n-1}}.\right) \\
I_n &= \frac{x}{2(n-1)a^2(x^2+a^2)^{n-1}} + \frac{2n-3}{2(n-1)a^2}I_{n-1}.
\end{aligned}$$

Now $I_1 = \dfrac{1}{a}\tan^{-1}\dfrac{x}{a}$, so

$$\begin{aligned}
I_3 &= \frac{x}{4a^2(x^2+a^2)^2} + \frac{3}{4a^2}I_2 \\
&= \frac{x}{4a^2(x^2+a^2)^2} + \frac{3}{4a^2}\left(\frac{x}{2a^2(x^2+a^2)} + \frac{1}{2a^2}I_1\right) \\
&= \frac{x}{4a^2(x^2+a^2)^2} + \frac{3x}{8a^4(x^2+a^2)} + \frac{3}{8a^5}\tan^{-1}\frac{x}{a} + C.
\end{aligned}$$

36. Given that $f(a) = f(b) = 0$.

$$\begin{aligned}
&\int_a^b (x-a)(b-x)f''(x)\,dx \\
&\quad U = (x-a)(b-x) \qquad dV = f''(x)\,dx \\
&\quad dU = (b+a-2x)\,dx \qquad V = f'(x) \\
&= (x-a)(b-x)f'(x)\bigg|_a^b - \int_a^b (b+a-2x)f'(x)\,dx \\
&\quad U = b+a-2x \quad dV = f'(x)\,dx \\
&\quad dU = -2dx \qquad V = f(x) \\
&= 0 - \left[(b+a-2x)f(x)\bigg|_a^b + 2\int_a^b f(x)\,dx\right] \\
&= -2\int_a^b f(x)\,dx.
\end{aligned}$$

37. Given: f'' and g'' are continuous on $[a,b]$, and $f(a) = g(a) = f(b) = g(b) = 0$. We have

$$\begin{aligned}
&\int_a^b f(x)g''(x)\,dx \\
&\quad U = f(x) \qquad dV = g''(x)\,dx \\
&\quad dU = f'(x)\,dx \qquad V = g'(x) \\
&= f(x)g'(x)\bigg|_a^b - \int_a^b f'(x)g'(x)\,dx.
\end{aligned}$$

Similarly,

$$\int_a^b f''(x)g(x)\,dx = f'(x)g(x)\bigg|_a^b - \int_a^b f'(x)g'(x)\,dx.$$

Thus we have

$$\int_a^b f(x)g''(x)\,dx - \int_a^b f''(x)g(x)\,dx$$
$$= \Big(f(x)g'(x) - f'(x)g(x)\Big)\Big|_a^b = 0$$

by the assumptions on f and g. Thus

$$\int_a^b f(x)g''(x)\,dx = \int_a^b f''(x)g(x)\,dx.$$

This equation is also valid for any (sufficiently smooth) functions f and g for which

$$f(b)g'(b) - f'(b)g(b) = f(a)g'(a) - f'(a)g(a).$$

Examples are functions which are periodic with period $b-a$, or if $f(a) = f(b) = f'(a) = f'(b) = 0$, or if instead g satisfies such conditions. Other combinations of conditions on f and g will also do.

38. $I_n = \displaystyle\int_0^{\pi/2} \cos^n x\,dx.$

a) For $0 \le x \le \pi/2$ we have $0 \le \cos x \le 1$, and so $0 \le \cos^{2n+2} x \le \cos^{2n+1} x \le \cos^{2n} x$. Therefore $0 \le I_{2n+2} \le I_{2n+1} \le I_{2n}$.

b) Since $I_n = \dfrac{n-1}{n} I_{n-2}$, we have $I_{2n+2} = \dfrac{2n+1}{2n+2} I_{2n}$. Combining this with part (a), we get

$$\frac{2n+1}{2n+2} = \frac{I_{2n+2}}{I_{2n}} \le \frac{I_{2n+1}}{I_{2n}} \le 1.$$

The left side approaches 1 as $n \to \infty$, so, by the Squeeze Theorem,

$$\lim_{n\to\infty} \frac{I_{2n+1}}{I_{2n}} = 1.$$

c) By Example 6 we have, since $2n+1$ is odd and $2n$ is even,

$$I_{2n+1} = \frac{2n}{2n+1}\cdot\frac{2n-2}{2n-1}\cdots\frac{4}{5}\cdot\frac{2}{3}$$
$$I_{2n} = \frac{2n-1}{2n}\cdot\frac{2n-3}{2n-2}\cdots\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}.$$

Multiplying the expression for I_{2n+1} by $\pi/2$ and dividing by the expression for I_{2n}, we obtain, by part (b),

$$\lim_{n\to\infty} \frac{\dfrac{2n}{2n+1}\cdot\dfrac{2n-2}{2n-1}\cdots\dfrac{4}{5}\cdot\dfrac{2}{3}\cdot\dfrac{\pi}{2}}{\dfrac{2n-1}{2n}\cdot\dfrac{2n-3}{2n-2}\cdots\dfrac{3}{4}\cdot\dfrac{1}{2}\cdot\dfrac{\pi}{2}} = \frac{\pi}{2}\times 1 = \frac{\pi}{2},$$

or, rearranging the factors on the left,

$$\lim_{n\to\infty} \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdots\frac{2n}{2n-1}\cdot\frac{2n}{2n+1} = \frac{\pi}{2}.$$

Section 6.2 Inverse Substitutions (page 328)

1. $\displaystyle\int \frac{dx}{\sqrt{1-4x^2}}$ Let $u = 2x$, $du = 2\,dx$

$$= \frac{1}{2}\int \frac{du}{\sqrt{1-u^2}} = \frac{1}{2}\sin^{-1} u + C = \frac{1}{2}\sin^{-1}(2x) + C.$$

2. $\displaystyle\int \frac{x^2\,dx}{\sqrt{1-4x^2}}$ Let $2x = \sin u$, $2\,dx = \cos u\,du$

$$= \frac{1}{8}\int \frac{\sin^2 u \cos u\,du}{\cos u}$$
$$= \frac{1}{16}\int (1-\cos 2u)\,du = \frac{u}{16} - \frac{\sin 2u}{32} + C$$
$$= \frac{1}{16}\sin^{-1} 2x - \frac{1}{16}\sin u\cos u + C$$
$$= \frac{1}{16}\sin^{-1} 2x - \frac{1}{8}x\sqrt{1-4x^2} + C.$$

3. $\displaystyle\int \frac{x^2\,dx}{\sqrt{9-x^2}}$ Let $x = 3\sin\theta$, $dx = 3\cos\theta\,d\theta$

$$= \int \frac{9\sin^2\theta\, 3\cos\theta\,d\theta}{3\cos\theta}$$
$$= \frac{9}{2}(\theta - \sin\theta\cos\theta) + C$$
$$= \frac{9}{2}\sin^{-1}\frac{x}{3} - \frac{1}{2}x\sqrt{9-x^2} + C.$$

4. $\displaystyle\int \frac{dx}{x\sqrt{1-4x^2}}$ Let $x = \frac{1}{2}\sin\theta$, $dx = \frac{1}{2}\cos\theta\,d\theta$

$$= \int \frac{\cos\theta\,d\theta}{\sin\theta\sqrt{1-\sin^2\theta}} = \int \csc\theta\,d\theta$$
$$= \ln|\csc\theta - \cot\theta| + C = \ln\left|\frac{1}{2x} - \frac{\sqrt{1-4x^2}}{2x}\right| + C$$
$$= \ln\left|\frac{1-\sqrt{1-4x^2}}{x}\right| + C_1.$$

5. $\displaystyle\int \frac{dx}{x^2\sqrt{9-x^2}}$ Let $x = 3\sin\theta$, $dx = 3\cos\theta\,d\theta$

$$= \int \frac{3\cos\theta\,d\theta}{9\sin^2\theta\, 3\cos\theta}$$
$$= \frac{1}{9}\int \csc^2\theta\,d\theta$$
$$= -\frac{1}{9}\cot\theta + C = -\frac{1}{9}\frac{\sqrt{9-x^2}}{x} + C.$$

6. $\displaystyle\int \frac{dx}{x\sqrt{9-x^2}}$ Let $x = 3\sin\theta$, $dx = 3\cos\theta\,d\theta$

$$= \int \frac{3\cos\theta\,d\theta}{3\sin\theta\,3\cos\theta} = \frac{1}{3}\int \csc\theta\,d\theta$$
$$= \frac{1}{3}\ln|\csc\theta - \cot\theta| + C = \frac{1}{3}\ln\left|\frac{3}{x} - \frac{\sqrt{9-x^2}}{x}\right| + C$$
$$= \frac{1}{3}\ln\left|\frac{3-\sqrt{9-x^2}}{x}\right| + C.$$

7. $$\int \frac{x+1}{\sqrt{9-x^2}}\,dx = \int \frac{x\,dx}{\sqrt{9-x^2}} + \int \frac{dx}{\sqrt{9-x^2}}$$
$$= -\sqrt{9-x^2} + \sin^{-1}\frac{x}{3} + C.$$

8. $\displaystyle\int \frac{dx}{\sqrt{9+x^2}}$ Let $x = 3\tan\theta$, $dx = 3\sec^2\theta\,d\theta$

$$= \int \frac{3\sec^2\theta\,d\theta}{3\sec\theta} = \int \sec\theta\,d\theta$$
$$= \ln|\sec\theta + \tan\theta| + C = \ln(x + \sqrt{9+x^2}) + C_1.$$

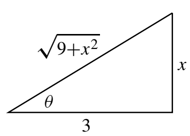

Fig. 6.2.8

9. $\displaystyle\int \frac{x^3\,dx}{\sqrt{9+x^2}}$ Let $u = 9 + x^2$, $du = 2x\,dx$

$$= \frac{1}{2}\int \frac{(u-9)\,du}{\sqrt{u}} = \frac{1}{2}\int (u^{1/2} - 9u^{-1/2})\,du$$
$$= \frac{1}{3}u^{3/2} - 9u^{1/2} + C$$
$$= \frac{1}{3}(9+x^2)^{3/2} - 9\sqrt{9+x^2} + C.$$

10. $\displaystyle\int \frac{\sqrt{9+x^2}}{x^4}\,dx$ Let $x = 3\tan\theta$, $dx = 3\sec^2\theta\,d\theta$

$$= \int \frac{(3\sec\theta)(3\sec^2\theta)\,d\theta}{81\tan^4\theta}$$
$$= \frac{1}{9}\int \frac{\sec^3\theta}{\tan^4\theta}\,d\theta = \frac{1}{9}\int \frac{\cos\theta}{\sin^4\theta}\,d\theta \quad \text{Let } u = \sin\theta,\ du = \cos\theta\,d\theta$$
$$= \frac{1}{9}\int \frac{du}{u^4} = -\frac{1}{27u^3} + C = -\frac{1}{27\sin^3\theta} + C$$
$$= -\frac{(9+x^2)^{3/2}}{27x^3} + C.$$

11. $\displaystyle\int \frac{dx}{(a^2-x^2)^{3/2}}$ Let $x = a\sin\theta$, $dx = a\cos\theta\,d\theta$

$$= \int \frac{a\cos\theta\,d\theta}{a^3\cos^3\theta} = \frac{1}{a^2}\int \sec^2\theta\,d\theta$$
$$= \frac{1}{a^2}\tan\theta + C = \frac{1}{a^2}\frac{x}{\sqrt{a^2-x^2}} + C.$$

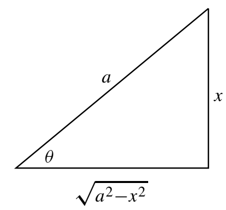

Fig. 6.2.11

12. $\displaystyle\int \frac{dx}{(a^2+x^2)^{3/2}}$ Let $x = a\tan\theta$, $dx = a\sec^2\theta\,d\theta$

$$= \int \frac{a\sec^2\theta\,d\theta}{(a^2+a^2\tan^2\theta)^{3/2}} = \int \frac{a\sec^2\theta\,d\theta}{a^3\sec^3\theta}$$
$$= \frac{1}{a^2}\int \cos\theta\,d\theta = \frac{1}{a^2}\sin\theta + C = \frac{x}{a^2\sqrt{a^2+x^2}} + C.$$

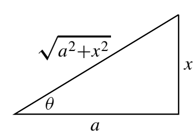

Fig. 6.2.12

13. $\displaystyle\int \frac{x^2\,dx}{(a^2-x^2)^{3/2}}$ Let $x = a\sin\theta$, $dx = a\cos\theta\,d\theta$

$$= \int \frac{a^2\sin^2\theta\,a\cos\theta\,d\theta}{a^3\cos^3\theta}$$
$$= \int \tan^2\theta\,d\theta = \int (\sec^2\theta - 1)\,d\theta$$
$$= \tan\theta - \theta + C \qquad \text{(see Fig. s6-5-17)}$$
$$= \frac{x}{\sqrt{a^2-x^2}} - \sin^{-1}\frac{x}{a} + C.$$

14. $\displaystyle\int \frac{dx}{(1+2x^2)^{5/2}}$ Let $x = \frac{1}{\sqrt{2}}\tan\theta$, $dx = \frac{1}{\sqrt{2}}\sec^2\theta\,d\theta$

$$= \frac{1}{\sqrt{2}}\int \frac{\sec^2\theta\,d\theta}{(1+\tan^2\theta)^{5/2}} = \frac{1}{\sqrt{2}}\int \cos^3\theta\,d\theta$$

$$= \frac{1}{\sqrt{2}}\int (1-\sin^2\theta)\cos\theta\,d\theta \quad \text{Let } u = \sin\theta,\ du = \cos\theta\,d\theta$$

$$= \frac{1}{\sqrt{2}}\int (1-u^2)\,du = \frac{1}{\sqrt{2}}\left(u - \frac{1}{3}u^3\right) + C$$

$$= \frac{1}{\sqrt{2}}\sin\theta - \frac{1}{3\sqrt{2}}\sin^3\theta + C$$

$$= \frac{\sqrt{2}x}{\sqrt{2}\sqrt{1+2x^2}} - \frac{1}{3\sqrt{2}}\left(\frac{\sqrt{2}x}{\sqrt{1+2x^2}}\right)^3 + C$$

$$= \frac{4x^3+3x}{3(1+2x^2)^{3/2}} + C.$$

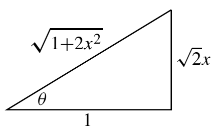

Fig. 6.2.14

15. $\displaystyle\int \frac{dx}{x\sqrt{x^2-4}}$ Let $x = 2\sec\theta$ $(x > 2)$, $dx = 2\sec\theta\tan\theta\,d\theta$

$$= \int \frac{2\sec\theta\tan\theta\,d\theta}{2\sec\theta\,2\tan\theta}$$

$$= \frac{1}{2}\int d\theta = \frac{\theta}{2} + C = \frac{1}{2}\sec^{-1}\frac{x}{2} + C.$$

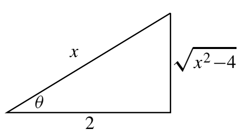

Fig. 6.2.15

16. $\displaystyle\int \frac{dx}{x^2\sqrt{x^2-a^2}}$ Let $x = a\sec\theta$ $(a > 0)$, $dx = a\sec\theta\tan\theta\,d\theta$

$$= \int \frac{a\sec\theta\ \tan\theta\,d\theta}{a^2\sec^2\theta\,a\tan\theta}$$

$$= \frac{1}{a^2}\int \cos\theta\,d\theta = \frac{1}{a^2}\sin\theta + C$$

$$= \frac{1}{a^2}\frac{\sqrt{x^2-a^2}}{x} + C.$$

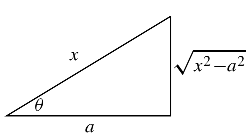

Fig. 6.2.16

17. $\displaystyle\int \frac{dx}{x^2+2x+10} = \int \frac{dx}{(x+1)^2+9} = \frac{1}{3}\tan^{-1}\frac{x+1}{3} + C.$

18. $\displaystyle\int \frac{dx}{x^2+x+1} = \int \frac{dx}{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$ Let $u = x + \frac{1}{2}$, $du = dx$

$$= \int \frac{du}{u^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2}{\sqrt{3}}u\right) + C$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C.$$

19. $\displaystyle\int \frac{dx}{(4x^2+4x+5)^2}$

$$= \int \frac{dx}{\left((2x+1)^2+4\right)^2} \quad \text{Let } 2x+1 = 2\tan\theta,\ 2\,dx = 2\sec^2\theta\,d\theta$$

$$= \int \frac{\sec^2\theta\,d\theta}{16\sec^4\theta} = \frac{1}{16}\int \cos^2\theta\,d\theta$$

$$= \frac{1}{32}\left(\theta + \sin\theta\cos\theta\right)$$

$$= \frac{1}{32}\tan^{-1}\frac{2x+1}{2} + \frac{1}{16}\frac{2x+1}{4x^2+4x+5} + C.$$

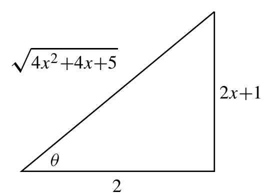

Fig. 6.2.19

20. $\displaystyle\int \frac{x\,dx}{x^2-2x+3} = \int \frac{(x-1)+1}{(x-1)^2+2}\,dx$ Let $u = x-1$, $du = dx$

$$= \int \frac{u\,du}{u^2+2} + \int \frac{du}{u^2+2}$$

$$= \frac{1}{2}\ln(u^2+2) + \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right) + C$$

$$= \frac{1}{2}\ln(x^2-2x+3) + \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x-1}{\sqrt{2}}\right) + C.$$

21. $\displaystyle\int \frac{x\,dx}{\sqrt{2ax-x^2}}$

$$= \int \frac{x\,dx}{\sqrt{a^2-(x-a)^2}} \quad \text{Let } x-a = a\sin\theta,\ dx = a\cos\theta\,d\theta$$

$$= \int \frac{(a+a\sin\theta)a\cos\theta\,d\theta}{a\cos\theta}$$

$$= a(\theta - \cos\theta) + C$$

$$= a\sin^{-1}\frac{x-a}{a} - \sqrt{2ax-x^2} + C.$$

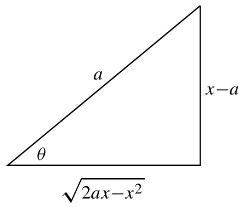

Fig. 6.2.21

22.
$$\int \frac{dx}{(4x-x^2)^{3/2}}$$
$$= \int \frac{dx}{[4-(2-x)^2]^{3/2}} \qquad \text{Let } 2-x = 2\sin u,\ -dx = 2\cos u\, du$$
$$= -\int \frac{2\cos u\, du}{8\cos^3 u} = -\frac{1}{4}\int \sec^2 u\, du$$
$$= -\frac{1}{4}\tan u + C = \frac{1}{4}\frac{x-2}{\sqrt{4x-x^2}} + C.$$

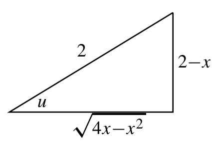

Fig. 6.2.22

23.
$$\int \frac{x\,dx}{(3-2x-x^2)^{3/2}}$$
$$= \int \frac{x\,dx}{\left(4-(x+1)^2\right)^{3/2}} \qquad \text{Let } x+1 = 2\sin\theta,\ dx = 2\cos\theta\, d\theta$$
$$= \int \frac{(2\sin\theta - 1)2\cos\theta\, d\theta}{8\cos^3\theta}$$
$$= \frac{1}{2}\int \sec\theta\tan\theta\, d\theta - \frac{1}{4}\int \sec^2\theta\, d\theta$$
$$= \frac{1}{2}\sec\theta - \frac{1}{4}\tan\theta + C$$
$$= \frac{1}{\sqrt{3-2x-x^2}} - \frac{1}{4}\frac{x+1}{\sqrt{3-2x-x^2}} + C$$
$$= \frac{1}{4}\cdot\frac{3-x}{\sqrt{3-2x-x^2}} + C.$$

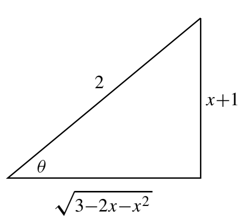

Fig. 6.2.23

24.
$$\int \frac{dx}{(x^2+2x+2)^2} = \int \frac{dx}{[(x+1)^2+1]^2} \qquad \text{Let } x+1 = \tan u,\ dx = \sec^2 u\, du$$
$$= \int \frac{\sec^2 u\, du}{\sec^4 u} = \int \cos^2 u\, du$$
$$= \frac{1}{2}\int (1+\cos 2u)\, du = \frac{u}{2} + \frac{\sin 2u}{4} + C$$
$$= \frac{1}{2}\tan^{-1}(x+1) + \frac{1}{2}\sin u\ \cos u + C$$
$$= \frac{1}{2}\tan^{-1}(x+1) + \frac{1}{2}\frac{x+1}{x^2+2x+2} + C.$$

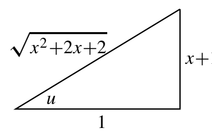

Fig. 6.2.24

25.
$$\int \frac{dx}{(1+x^2)^3} \qquad \text{Let } x = \tan\theta,\ dx = \sec^2\, d\theta$$
$$= \int \frac{\sec^2\theta}{\sec^6\theta}\, d\theta = \int \cos^4\theta\, d\theta$$
$$= \int \left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta$$
$$= \frac{1}{4}\int \left(1 + 2\cos 2\theta + \frac{1+\cos 4\theta}{2}\right) d\theta$$
$$= \frac{3\theta}{8} + \frac{\sin 2\theta}{4} + \frac{\sin 4\theta}{32} + C$$
$$= \frac{3\theta}{8} + \frac{\sin\theta\cos\theta}{2} + \frac{\sin 2\theta\cos 2\theta}{16} + C$$
$$= \frac{3\theta}{8} + \frac{\sin\theta\cos\theta}{2} + \frac{1}{8}\sin\theta\cos\theta(2\cos^2\theta - 1) + C$$
$$= \frac{3}{8}\tan^{-1}x + \frac{1}{2}\cdot\frac{x}{1+x^2} + \frac{1}{8}\cdot\frac{x}{1+x^2}\left(\frac{2}{1+x^2} - 1\right) + C$$
$$= \frac{3}{8}\tan^{-1}x + \frac{3}{8}\cdot\frac{x}{1+x^2} + \frac{1}{4}\cdot\frac{x}{(1+x^2)^2} + C$$
$$= \frac{3}{8}\tan^{-1}x + \frac{3x^3+5x}{8(1+x^2)^2} + C.$$

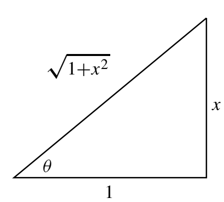

Fig. 6.2.25

26. $\displaystyle\int \frac{x^2\,dx}{(1+x^2)^2}$ Let $x = \tan u$, $dx = \sec^2 u\,du$

$$= \int \frac{\tan^2 u\,\sec^2 u\,du}{\sec^4 u} = \int \frac{\tan^2 u\,du}{\sec^2 u}$$
$$= \int \sin^2 u\,du = \frac{1}{2}\int (1-\cos 2u)\,du$$
$$= \frac{u}{2} - \frac{\sin u\,\cos u}{2} + C$$
$$= \frac{1}{2}\tan^{-1} x - \frac{1}{2}\frac{x}{1+x^2} + C.$$

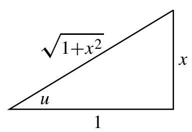

Fig. 6.2.26

27. $\displaystyle\int \frac{\sqrt{1-x^2}}{x^3}\,dx$ Let $x = \sin\theta$, $dx = \cos\theta\,d\theta$

$$= \int \frac{\cos^2\theta}{\sin^3\theta}\,d\theta = I, \text{ where}$$
$$I = \int \cot^2\theta\csc\theta\,d\theta$$

$U = \cot\theta$ $\quad dV = \cot\theta\csc\theta\,d\theta$
$dU = -\csc^2\theta\,d\theta$ $\quad V = -\csc\theta$

$$= -\csc\theta\cot\theta - \int \csc^3\theta\,d\theta$$
$$= -\csc\theta\cot\theta - \int \csc\theta\,d\theta - I.$$

Therefore

$$I = -\frac{1}{2}\csc\theta\cot\theta + \frac{1}{2}\ln|\csc\theta+\cot\theta| + C$$
$$= -\frac{1}{2}\frac{\sqrt{1-x^2}}{x^2} + \frac{1}{2}\ln\left|\frac{1}{x} + \frac{\sqrt{1-x^2}}{x}\right| + C$$
$$= \frac{1}{2}\ln(1+\sqrt{1-x^2}) - \frac{1}{2}\ln|x| - \frac{1}{2}\frac{\sqrt{1-x^2}}{x^2} + C.$$

28. $\displaystyle I = \int \sqrt{9+x^2}\,dx$ Let $x = 3\tan\theta$, $dx = 3\sec^2\theta\,d\theta$

$$= \int 3\sec\theta\,3\sec^2\theta\,d\theta$$
$$= 9\int \sec^3\theta\,d\theta$$

$U = \sec\theta$ $\quad dV = \sec^2\theta\,d\theta$
$dU = \sec\theta\,\tan\theta\,d\theta$ $\quad V = \tan\theta$

$$= 9\sec\theta\,\tan\theta - 9\int \sec\theta\,\tan^2\theta\,d\theta$$
$$= 9\sec\theta\,\tan\theta - 9\int \sec\theta(\sec^2\theta - 1)\,d\theta$$
$$= 9\sec\theta\,\tan\theta + 9\int \sec\theta\,d\theta - 9\int \sec^3\theta\,d\theta$$
$$= 9\sec\theta\,\tan\theta + 9\ln|\sec\theta + \tan\theta| - I$$
$$I = \frac{9}{2}\left[\left(\frac{\sqrt{9+x^2}}{3}\right)\left(\frac{x}{3}\right)\right] + \frac{9}{2}\ln\left|\frac{\sqrt{9+x^2}}{3} + \frac{x}{3}\right| + C$$
$$= \frac{1}{2}x\sqrt{9+x^2} + \frac{9}{2}\ln\left(\sqrt{9+x^2}+x\right) + C_1.$$

(where $C_1 = C - \frac{9}{2}\ln 3$)

29. $\displaystyle\int \frac{dx}{2+\sqrt{x}}$ Let $x = u^2$, $dx = 2u\,du$

$$= \int \frac{2u\,du}{2+u} = 2\int \left(1 - \frac{2}{u+2}\right)du$$
$$= 2u - 4\ln|u+2| + C = 2\sqrt{x} - 4\ln(2+\sqrt{x}) + C.$$

30. $\displaystyle\int \frac{dx}{1+x^{1/3}}$ Let $x = u^3$, $dx = 3u^2\,du$

$$= 3\int \frac{u^2\,du}{1+u} \quad \text{Let } v = 1+u,\ dv = du$$
$$= 3\int \frac{v^2-2v+1}{v}\,dv = 3\int \left(v - 2 + \frac{1}{v}\right)dv$$
$$= 3\left(\frac{v^2}{2} - 2v + \ln|v|\right) + C$$
$$= \frac{3}{2}(1+x^{1/3})^2 - 6(1+x^{1/3}) + 3\ln|1+x^{1/3}| + C.$$

31. $\displaystyle I = \int \frac{1+x^{1/2}}{1+x^{1/3}}\,dx$ Let $x = u^6$, $dx = 6u^5\,du$.

$$= \int \frac{1+u^3}{1+u^2}\,6u^5\,du = 6\int \frac{u^8+u^5}{1+u^2}\,du.$$

Division is required to render the last integrand as a polynomial with a remainder fraction of simpler form: observe that

$$u^8 = u^8 + u^6 - u^6 - u^4 + u^4 + u^2 - u^2 - 1 + 1$$
$$= (u^2+1)(u^6 - u^4 + u^2 - 1) + 1$$
$$u^5 = u^5 + u^3 - u^3 - u + u$$
$$= (u^2+1)(u^3 - u) + u.$$

Thus

$$\frac{u^8+u^5}{u^2+1} = u^6-u^4+u^3+u^2-u-1+\frac{u+1}{u^2+1}.$$

Therefore

$$\begin{aligned}
I &= 6\int\left(u^6-u^4+u^3+u^2-u-1+\frac{u+1}{u^2+1}\right)du\\
&= 6\left(\frac{u^7}{7}-\frac{u^5}{5}+\frac{u^4}{4}+\frac{u^3}{3}-\frac{u^2}{2}-u\right.\\
&\qquad\left.+\frac{1}{2}\ln(u^2+1)+\tan^{-1}u\right)+C\\
&= \frac{6}{7}x^{7/6}-\frac{6}{5}x^{5/6}+\frac{3}{2}x^{2/3}+2x^{1/2}-3x^{1/3}-6x^{1/6}\\
&\qquad+3\ln(1+x^{1/3})+6\tan^{-1}x^{1/6}+C.
\end{aligned}$$

32. $$\begin{aligned}
&\int\frac{x\sqrt{2-x^2}}{\sqrt{x^2+1}}\,dx \quad \text{Let } u^2=x^2+1,\ 2u\,du=2x\,dx\\
&= \int\frac{u\sqrt{3-u^2}\,du}{u}\\
&= \int\sqrt{3-u^2}\,du \quad \text{Let } u=\sqrt{3}\sin v,\ du=\sqrt{3}\cos v\,dv\\
&= \int(\sqrt{3}\cos v)\sqrt{3}\cos v\,dv = 3\int\cos^2 v\,dv\\
&= \frac{3}{2}(v+\sin v\,\cos v)+C\\
&= \frac{3}{2}\sin^{-1}\left(\frac{u}{\sqrt{3}}\right)+\frac{3}{2}\frac{u\sqrt{3-u^2}}{3}+C\\
&= \frac{3}{2}\sin^{-1}\left(\sqrt{\frac{x^2+1}{3}}\right)+\frac{1}{2}\sqrt{(x^2+1)(2-x^2)}+C.
\end{aligned}$$

33. $$\begin{aligned}
&\int_{-\ln 2}^{0} e^x\sqrt{1-e^{2x}}\,dx \quad \text{Let } e^x=\sin\theta,\ e^x\,dx=\cos\theta\,d\theta\\
&= \int_{\pi/6}^{\pi/2}\cos^2\theta\,d\theta = \frac{1}{2}(\theta+\sin\theta\cos\theta)\Big|_{\pi/6}^{\pi/2}\\
&= \frac{1}{2}\left(\frac{\pi}{3}-\frac{\sqrt{3}}{4}\right) = \frac{\pi}{6}-\frac{\sqrt{3}}{8}.
\end{aligned}$$

34. $$\begin{aligned}
&\int_0^{\pi/2}\frac{\cos x}{\sqrt{1+\sin^2 x}}\,dx \quad \text{Let } u=\sin x,\ du=\cos x\,dx\\
&= \int_0^1\frac{du}{\sqrt{1+u^2}} \quad \text{Let } u=\tan w,\ du=\sec^2 w\,dw\\
&= \int_0^{\pi/4}\frac{\sec^2 w\,dw}{\sec w} = \int_0^{\pi/4}\sec w\,dw\\
&= \ln|\sec w+\tan w|\Big|_0^{\pi/4}\\
&= \ln|\sqrt{2}+1|-\ln|1+0| = \ln(\sqrt{2}+1).
\end{aligned}$$

35. $$\begin{aligned}
&\int_{-1}^{\sqrt{3}-1}\frac{dx}{x^2+2x+2}\\
&= \int_{-1}^{\sqrt{3}-1}\frac{dx}{(x+1)^2+1} \quad \text{Let } u=x+1,\ du=dx\\
&= \int_0^{\sqrt{3}}\frac{du}{u^2+1} = \tan^{-1}u\Big|_0^{\sqrt{3}} = \frac{\pi}{3}.
\end{aligned}$$

36. $$\begin{aligned}
&\int_1^2\frac{dx}{x^2\sqrt{9-x^2}} \quad \text{Let } x=3\sin u,\ dx=3\cos u\,du\\
&= \int_{x=1}^{x=2}\frac{3\cos u\,du}{9\sin^2 u(3\cos u)} = \frac{1}{9}\int_{x=1}^{x=2}\csc^2 u\,du\\
&= \frac{1}{9}(-\cot u)\Big|_{x=1}^{x=2} = -\frac{1}{9}\left(\frac{\sqrt{9-x^2}}{x}\right)\Big|_{x=1}^{x=2}\\
&= -\frac{1}{9}\left(\frac{\sqrt{5}}{2}-\frac{\sqrt{8}}{1}\right) = \frac{2\sqrt{2}}{9}-\frac{\sqrt{5}}{18}.
\end{aligned}$$

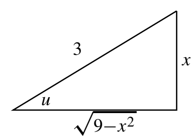

Fig. 6.2.36

37. $$\begin{aligned}
&\int\frac{d\theta}{2+\sin\theta} \quad \text{Let } x=\tan(\theta/2),\ \sin\theta=\frac{2x}{1+x^2},\quad d\theta=\frac{2\,dx}{1+x^2}\\
&= \int\frac{\frac{2\,dx}{1+x^2}}{2+\frac{2x}{1+x^2}} = \int\frac{dx}{1+x+x^2}\\
&= \int\frac{dx}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}} = \frac{2}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}+C\\
&= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2\tan(\theta/2)+1}{\sqrt{3}}\right)+C.
\end{aligned}$$

38. $$\begin{aligned}
&\int_0^{\pi/2}\frac{d\theta}{1+\cos\theta+\sin\theta} \quad \text{Let } x=\tan\frac{\theta}{2},\quad d\theta=\frac{2}{1+x^2}\,dx,\\
&\qquad \cos\theta=\frac{1-x^2}{1+x^2},\quad \sin\theta=\frac{2x}{1+x^2}.\\
&= \int_0^1\frac{\left(\frac{2}{1+x^2}\right)dx}{1+\left(\frac{1-x^2}{1+x^2}\right)+\left(\frac{2x}{1+x^2}\right)}\\
&= 2\int_0^1\frac{dx}{2+2x} = \int_0^1\frac{dx}{1+x}\\
&= \ln|1+x|\Big|_0^1 = \ln 2.
\end{aligned}$$

39. $\displaystyle\int \frac{d\theta}{3+2\cos\theta}$ Let $x=\tan(\theta/2)$, $\cos\theta = \dfrac{1-x^2}{1+x^2}$, $d\theta = \dfrac{2\,dx}{1+x^2}$

$$= \int \frac{\frac{2\,dx}{1+x^2}}{3+\frac{2-2x^2}{1+x^2}} = \int \frac{2\,dx}{5+x^2}$$
$$= \frac{2}{\sqrt{5}}\tan^{-1}\frac{x}{\sqrt{5}} + C = \frac{2}{\sqrt{5}}\tan^{-1}\left(\frac{\tan(\theta/2)}{\sqrt{5}}\right) + C.$$

40. $\displaystyle \text{Area} = \int_{1/2}^{1} \frac{dx}{\sqrt{2x-x^2}} = \int_{1/2}^{1}\frac{dx}{\sqrt{1-(x-1)^2}}$

Let $u = x-1$
$du = dx$

$$= \int_{-1/2}^{0}\frac{du}{\sqrt{1-u^2}} = \sin^{-1} u\Big|_{-1/2}^{0}$$
$$= 0 - \left(-\frac{\pi}{6}\right) = \frac{\pi}{6} \text{ sq. units.}$$

41. For intersection of $y = \dfrac{9}{x^4+4x^2+4}$ and $y=1$ we have

$$x^4+4x^2+4 = 9$$
$$x^4+4x^2-5=0$$
$$(x^2+5)(x^2-1)=0,$$

so the intersections are at $x=\pm 1$. The required area is

$$A = 2\int_0^1\left(\frac{9\,dx}{x^4+4x^2+4} - 1\right)dx$$
$$= 18\int_0^1\frac{dx}{(x^2+2)^2} - 2 \quad \text{Let } x=\sqrt{2}\tan\theta,\ dx = \sqrt{2}\sec^2\theta$$
$$= 18\int_{x=0}^{x=1}\frac{\sqrt{2}\sec^2\theta\,d\theta}{4\sec^4\theta} - 2$$
$$= \frac{9}{\sqrt{2}}\int_{x=0}^{x=1}\cos^2\theta\,d\theta \ -2$$
$$= \frac{9}{2\sqrt{2}}\Big(\theta+\sin\theta\cos\theta\Big)\Big|_{x=0}^{x=1} - 2$$
$$= \frac{9}{2\sqrt{2}}\left(\tan^{-1}\frac{x}{\sqrt{2}} + \frac{\sqrt{2}x}{x^2+2}\right)\Big|_0^1 - 2$$
$$= \frac{9}{2\sqrt{2}}\left(\tan^{-1}\frac{1}{\sqrt{2}}\right) - \frac{1}{2} \text{ units}^2$$

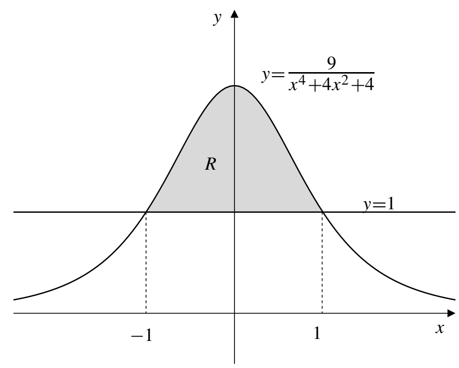

Fig. 6.2.41

42. $\displaystyle \text{Average value} = \frac{1}{4}\int_0^4\frac{dx}{(x^2-4x+8)^{3/2}}$

$$= \frac{1}{4}\int_0^4\frac{dx}{[(x-2)^2+4]^{3/2}}$$

Let $x-2 = 2\tan u$
$dx = 2\sec^2 u\,du$

$$= \frac{1}{4}\int_{-\pi/4}^{\pi/4}\frac{2\sec^2 u\,du}{8\sec^3 u}$$
$$= \frac{1}{16}\int_{-\pi/4}^{\pi/4}\cos u\,du = \frac{1}{16}\sin u\Big|_{-\pi/4}^{\pi/4}$$
$$= \frac{1}{16}\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right) = \frac{\sqrt{2}}{16}.$$

43. Area of R

$$= 2\int_0^{\sqrt{a^2-b^2}}\left(\sqrt{a^2-x^2}-b\right)dx \quad \text{Let } x = a\sin\theta,\ dx = a\cos\theta\,d\theta$$
$$= 2\int_{x=0}^{x=\sqrt{a^2-b^2}} a^2\cos^2\theta\,d\theta - 2b\sqrt{a^2-b^2}$$
$$= a^2(\theta+\sin\theta\cos\theta)\Big|_{x=0}^{x=\sqrt{a^2-b^2}} - 2b\sqrt{a^2-b^2}$$
$$= \left(a^2\sin^{-1}\frac{x}{a} + x\sqrt{a^2-x^2}\right)\Big|_0^{\sqrt{a^2-b^2}} - 2b\sqrt{a^2-b^2}$$
$$= a^2\sin^{-1}\sqrt{1-\frac{b^2}{a^2}} + b\sqrt{a^2-b^2} - 2b\sqrt{a^2-b^2}$$
$$= a^2\cos^{-1}\frac{b}{a} - b\sqrt{a^2-b^2} \text{ units}^2$$

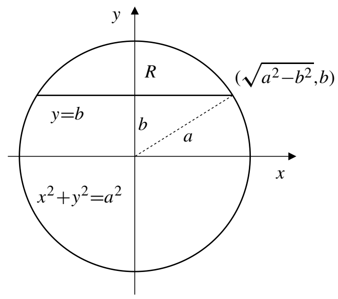

Fig. 6.2.43

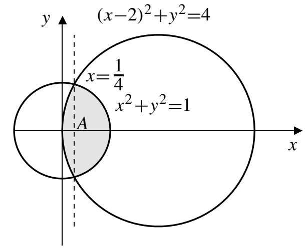

Fig. 6.2.44

44. The circles intersect at $x = \frac{1}{4}$, so the common area is $A_1 + A_2$ where

$$\begin{aligned}
A_1 &= 2\int_{1/4}^{1} \sqrt{1-x^2}\,dx \quad \text{Let } x = \sin u \\
&\qquad\qquad\qquad\qquad\quad dx = \cos u\,du \\
&= 2\int_{x=1/4}^{x=1} \cos^2 u\,du \\
&= (u + \sin u \cos u)\bigg|_{x=1/4}^{x=1} \\
&= (\sin^{-1} x + x\sqrt{1-x^2})\bigg|_{x=1/4}^{x=1} \\
&= \frac{\pi}{2} - \sin^{-1}\frac{1}{4} - \frac{\sqrt{15}}{16} \text{ sq. units.}
\end{aligned}$$

$$\begin{aligned}
A_2 &= 2\int_{0}^{1/4} \sqrt{4-(x-2)^2}\,dx \quad \text{Let } x-2 = 2\sin v \\
&\qquad\qquad\qquad\qquad\qquad\quad dx = 2\cos v\,dv \\
&= 8\int_{x=0}^{x=1/4} \cos^2 v\,dv \\
&= 4(v + \sin v \cos v)\bigg|_{x=0}^{x=1/4} \\
&= 4\left[\sin^{-1}\left(\frac{x-2}{2}\right) + \left(\frac{x-2}{2}\right)\frac{\sqrt{4x-x^2}}{2}\right]\Bigg|_{x=0}^{x=1/4} \\
&= 4\left[\sin^{-1}\left(-\frac{7}{8}\right) - \frac{7\sqrt{15}}{64} + \frac{\pi}{2}\right] \\
&= -4\sin^{-1}\left(\frac{7}{8}\right) - \frac{7\sqrt{15}}{16} + 2\pi \text{ sq. units.}
\end{aligned}$$

Hence, the common area is

$$\begin{aligned}
A_1 + A_2 &= \frac{5\pi}{2} - \frac{\sqrt{15}}{2} \\
&\qquad - \sin^{-1}\left(\frac{1}{4}\right) - 4\sin^{-1}\left(\frac{7}{8}\right) \text{ sq. units.}
\end{aligned}$$

45. Required area $= \displaystyle\int_3^4 \left(\sqrt{25-x^2} - \frac{12}{x}\right) dx$

$$\begin{aligned}
&= \int_3^4 \sqrt{25-x^2}\,dx - \int_3^4 \frac{12}{x}\,dx \\
&\qquad \text{Let } x = 5\sin u,\ dx = 5\cos u\,du \text{ in the first integral.} \\
&= \int_{x=3}^{x=4} 25\cos^2 u\,du - 12\ln x\bigg|_3^4 \\
&= \frac{25}{2}(u + \sin u \cos u)\bigg|_{x=3}^{x=4} - 12\ln\frac{4}{3} \\
&= \frac{25}{2}\left(\sin^{-1}\frac{x}{5}\right) + \frac{1}{2}x\sqrt{25-x^2}\bigg|_3^4 - 12\ln\frac{4}{3} \\
&= \frac{25}{2}\left(\sin^{-1}\frac{4}{5} - \sin^{-1}\frac{3}{5}\right) - 12\ln\frac{4}{3} \text{ sq. units.}
\end{aligned}$$

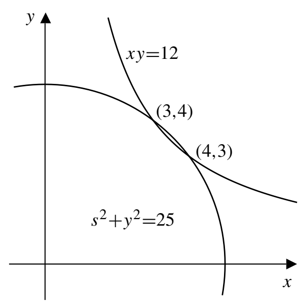

Fig. 6.2.45

46. Shaded area $= 2\displaystyle\int_c^a b\sqrt{1-\left(\frac{x}{a}\right)^2}\,dx$ Let $x = a\sin u$, $dx = a\cos u\,du$

$$\begin{aligned}
&= 2ab\int_{x=c}^{x=a} \cos^2 u\,du \\
&= ab(u + \sin u \cos u)\bigg|_{x=c}^{x-a} \\
&= \left(ab\sin^{-1}\frac{x}{a} + \frac{b}{a}x\sqrt{a^2-x^2}\right)\bigg|_c^a \\
&= ab\left(\frac{\pi}{2} - \sin^{-1}\frac{c}{a}\right) - \frac{cb}{a}\sqrt{a^2-c^2} \text{ sq. units.}
\end{aligned}$$

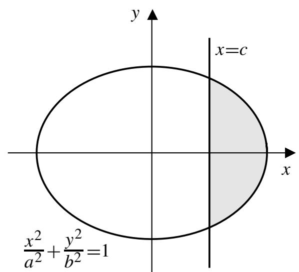

Fig. 6.2.46

47. Area of R

$$= \frac{Y}{2}\sqrt{1+Y^2} - \int_1^{\sqrt{1+Y^2}} \sqrt{x^2-1}\,dx \qquad \begin{aligned}&\text{Let } x = \sec\theta\\ &dx = \sec\theta\tan\theta\,d\theta\end{aligned}$$

$$= \frac{Y}{2}\sqrt{1+Y^2} - \int_0^{\tan^{-1}Y} \sec\theta\tan^2\theta\,d\theta$$

$$= \frac{Y}{2}\sqrt{1+Y^2} - \int_0^{\tan^{-1}Y} \sec^3\theta\,d\theta + \int_0^{\tan^{-1}Y} \sec\theta\,d\theta$$

$$= \frac{Y}{2}\sqrt{1+Y^2} + \left(-\frac{1}{2}\sec\theta\tan\theta - \frac{1}{2}\ln|\sec\theta+\tan\theta| + \ln|\sec\theta+\tan\theta|\right)\Bigg|_0^{\tan^{-1}Y}$$

$$= \frac{Y}{2}\sqrt{1+Y^2} - \frac{Y}{2}\sqrt{1+Y^2} + \frac{1}{2}\ln(Y+\sqrt{1+Y^2})$$

$$= \frac{1}{2}\ln(Y+\sqrt{1+Y^2}) \text{ units}^2$$

If $Y = \sinh t$, then we have

$$\text{Area } = \frac{1}{2}\ln(\sinh t + \cosh t) = \frac{1}{2}\ln e^t = \frac{t}{2} \text{ units}^2$$

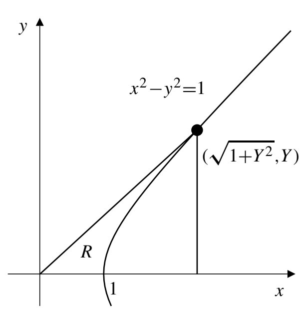

Fig. 6.2.47

48. $$\int \frac{dx}{\sqrt{x^2-a^2}} \qquad \begin{aligned}&\text{Let } x = a\cosh u\\ &dx = a\sinh u\,du\end{aligned}$$

$$= \int \frac{a\sinh u\,du}{a\sinh u} = u + C$$

$$= \cosh^{-1}\frac{x}{a} + C = \ln(x+\sqrt{x^2-a^2}) + C, \quad (x \ge a).$$

$$\int \frac{dx}{x^2\sqrt{x^2-a^2}} = \int \frac{a\sinh u\,du}{a^2\cosh^2 u\, a\sinh u}$$

$$= \frac{1}{a^2}\int \operatorname{sech}^2 u\,du = \frac{1}{a^2}\tanh u + C$$

$$= \frac{1}{a^2}\tanh\left(\cosh^{-1}\frac{x}{a}\right) + C$$

$$= \frac{1}{a^2}\cdot\frac{\dfrac{x}{a}+\sqrt{\dfrac{x^2}{a^2}-1} - \dfrac{1}{\dfrac{x}{a}-\sqrt{\dfrac{x^2}{a^2}-1}}}{\dfrac{x}{a}+\sqrt{\dfrac{x^2}{a^2}-1} + \dfrac{1}{\dfrac{x}{a}-\sqrt{\dfrac{x^2}{a^2}-1}}} + C$$

$$= \frac{\sqrt{x^2-a^2}}{a^2x} + C_1.$$

Section 6.3 Integrals of Rational Functions (page 336)

1. $$\int \frac{2\,dx}{2x-3} = \ln|2x-3| + C.$$

2. $$\int \frac{dx}{5-4x} = -\frac{1}{4}\ln|5-4x| + C.$$

3. $$\int \frac{x\,dx}{\pi x+2} = \frac{1}{\pi}\int \frac{\pi x+2-2}{\pi x+2}\,dx$$

$$= \frac{x}{\pi} - \frac{2}{\pi^2}\ln|\pi x+2| + C.$$

4. $$\int \frac{x^2}{x-4}\,dx = \int\left(x+4+\frac{16}{x-4}\right)dx$$

$$= \frac{x^2}{2} + 4x + 16\ln|x-4| + C.$$

5. $$\frac{1}{x^2-9} = \frac{A}{x-3} + \frac{B}{x+3}$$

$$= \frac{Ax+3A+Bx-3B}{x^2-9}$$

$$\Rightarrow \begin{cases} A+B & =0\\ 3(A-B) & =1\end{cases} \Rightarrow A = \frac{1}{6}, \quad B = -\frac{1}{6}.$$

$$\int \frac{dx}{x^2-9} = \frac{1}{6}\int\frac{dx}{x-3} - \frac{1}{6}\int\frac{dx}{x+3}$$

$$= \frac{1}{6}\Big(\ln|x-3| - \ln|x+3|\Big) + C$$

$$= \frac{1}{6}\ln\left|\frac{x-3}{x+3}\right| + C.$$

6.
$$\frac{1}{5-x^2} = \frac{A}{\sqrt{5}-x} + \frac{B}{\sqrt{5}+x} = \frac{(A+B)\sqrt{5}+(A-B)x}{5-x^2}$$
$$\Rightarrow \begin{cases} A+B = \frac{1}{\sqrt{5}} \\ A-B=0 \end{cases} \Rightarrow A = B = \frac{1}{2\sqrt{5}}.$$
$$\int \frac{1}{5-x^2}\,dx = \frac{1}{2\sqrt{5}}\int\left(\frac{1}{\sqrt{5}-x}+\frac{1}{\sqrt{5}+x}\right)dx = \frac{1}{2\sqrt{5}}\Big(-\ln|\sqrt{5}-x|+\ln|\sqrt{5}+x|\Big)+C = \frac{1}{2\sqrt{5}}\ln\left|\frac{\sqrt{5}+x}{\sqrt{5}-x}\right|+C.$$

7.
$$\frac{1}{a^2-x^2} = \frac{A}{a-x}+\frac{B}{a+x} = \frac{Aa+Ax+Ba-Bx}{a^2-x^2}$$
$$\Rightarrow \begin{cases} Aa+Ba=1 \\ A-B=0 \end{cases} \Rightarrow A=B=\frac{1}{2a}.$$
Thus
$$\int\frac{dx}{a^2-x^2} = \frac{1}{2a}\int\frac{dx}{a-x}+\frac{1}{2a}\int\frac{dx}{a+x} = \frac{1}{2a}\Big(-\ln|a-x|+\ln|a+x|\Big)+C = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right|+C.$$

8.
$$\frac{1}{b^2-a^2x^2} = \frac{A}{b-ax}+\frac{B}{b+ax} = \frac{(A+B)b+(A-B)ax}{b^2-a^2x^2}$$
$$\Rightarrow A=B=\frac{1}{2b}$$
$$\int\frac{dx}{b^2-a^2x^2} = \frac{1}{2b}\int\left(\frac{1}{b-ax}+\frac{1}{b+ax}\right)dx = \frac{1}{2b}\left(\frac{-\ln|b-ax|}{a}+\frac{\ln|b+ax|}{a}\right)+C = \frac{1}{2ab}\ln\left|\frac{b+ax}{b-ax}\right|+C.$$

9.
$$\int\frac{x^2\,dx}{x^2+x-2} = \int\left(1-\frac{x-2}{x^2+x-2}\right)dx = x-\int\frac{x-2}{x^2+x-2}\,dx.$$
If $\frac{x-2}{x^2+x-2} = \frac{A}{x+2}+\frac{B}{x-1} = \frac{Ax-A+Bx+2B}{x^2+x-2}$, then $A+B=1$ and $-A+2B=-2$, so that $A=4/3$ and $B=-1/3$. Thus
$$\int\frac{x^2\,dx}{x^2+x-2} = x-\frac{4}{3}\int\frac{dx}{x+2}+\frac{1}{3}\int\frac{dx}{x-1} = x-\frac{4}{3}\ln|x+2|+\frac{1}{3}\ln|x-1|+C.$$

10.
$$\frac{x}{3x^2+8x-3} = \frac{A}{3x-1}+\frac{B}{x+3} = \frac{(A+3B)x+(3A-B)}{3x^2+8x-3}$$
$$\Rightarrow \begin{cases} A+3B=1 \\ 3A-B=0 \end{cases} \Rightarrow A=\frac{1}{10},\ \ B=\frac{3}{10}.$$
$$\int\frac{x\,dx}{3x^2+8x-3} = \frac{1}{10}\int\left(\frac{1}{3x-1}+\frac{3}{x+3}\right)dx = \frac{1}{30}\ln|3x-1|+\frac{3}{10}\ln|x+3|+C.$$

11.
$$\frac{x-2}{x^2+x} = \frac{A}{x}+\frac{B}{x+1} = \frac{Ax+A+Bx}{x^2+x}$$
$$\Rightarrow \begin{cases} A+B=1 \\ A=-2 \end{cases} \Rightarrow A=-2,\quad B=3.$$
$$\int\frac{x-2}{x^2+x}\,dx = 3\int\frac{dx}{x+1}-2\int\frac{dx}{x} = 3\ln|x+1|-2\ln|x|+C.$$

12.
$$\frac{1}{x^3+9x} = \frac{A}{x}+\frac{Bx+C}{x^2+9} = \frac{Ax^2+9A+Bx^2+Cx}{x^3+9x}$$
$$\Rightarrow \begin{cases} A+B=0 \\ C=0 \\ 9A=1 \end{cases} \Rightarrow A=\frac{1}{9},\ \ B=-\frac{1}{9},\ \ C=0.$$
$$\int\frac{dx}{x^3+9x} = \frac{1}{9}\int\left(\frac{1}{x}-\frac{x}{x^2+9}\right)dx = \frac{1}{9}\ln|x|-\frac{1}{18}\ln(x^2+9)+K.$$

13.
$$\int\frac{dx}{1-6x+9x^2} = \int\frac{dx}{(1-3x)^2} = \frac{1}{3(1-3x)}+C.$$

14.
$$\int\frac{x}{2+6x+9x^2}\,dx = \int\frac{x}{(3x+1)^2+1}\,dx \quad \text{Let } u=3x+1,\ du=3\,dx$$
$$\frac{1}{9}\int\frac{u-1}{u^2+1}\,du = \frac{1}{9}\int\frac{u}{u^2+1}\,du-\frac{1}{9}\int\frac{1}{u^2+1}\,du$$
$$= \frac{1}{18}\ln(u^2+1)-\frac{1}{9}\tan^{-1}u+C$$
$$= \frac{1}{18}\ln(2+6x+9x^2)-\frac{1}{9}\tan^{-1}(3x+1)+C.$$

15.
$$\int\frac{x^2+1}{6x-9x^2}\,dx = \frac{1}{9}\int\frac{9x^2-6x+6x+9}{6x-9x^2}\,dx = -\frac{x}{9}+\frac{1}{9}\int\frac{2x+3}{x(2-3x)}\,dx.$$
Now
$$\frac{2x+3}{x(2-3x)} = \frac{A}{x}+\frac{B}{2-3x} = \frac{2A-3Ax+Bx}{x(2-3x)}$$
$$\Rightarrow 2A=3,\qquad -3A+B=2$$
$$\Rightarrow A=\frac{3}{2},\qquad B=\frac{13}{2}.$$

Therefore we have

$$\begin{aligned}&\int \frac{x^2+1}{6x-9x^2}\,dx\\&= -\frac{x}{9}+\frac{1}{6}\int\frac{dx}{x}+\frac{13}{18}\int\frac{dx}{2-3x}\\&= -\frac{x}{9}+\frac{1}{6}\ln|x|-\frac{13}{54}\ln|2-3x|+C.\end{aligned}$$

16. First divide to obtain

$$\begin{aligned}\frac{x^3+1}{x^2+7x+12} &= x-7+\frac{37x+85}{(x+4)(x+3)}\\ \frac{37x+85}{(x+4)(x+3)} &= \frac{A}{x+4}+\frac{B}{x+3}\\ &= \frac{(A+B)x+3A+4B}{x^2+7x+12}\\ &\Rightarrow \begin{cases}A+B=37\\3A+4B=85\end{cases}\Rightarrow A=63,\ \ B=-26.\end{aligned}$$

Now we have

$$\begin{aligned}&\int\frac{x^3+1}{12+7x+x^2}\,dx=\int\left(x-7+\frac{63}{x+4}-\frac{26}{x+3}\right)dx\\&=\frac{x^2}{2}-7x+63\ln|x+4|-26\ln|x+3|+C.\end{aligned}$$

17. $$\begin{aligned}\frac{1}{x(x^2-a^2)} &= \frac{A}{x}+\frac{B}{x-a}+\frac{C}{x+a}\\ &= \frac{Ax^2-Aa^2+Bx^2+Bax+Cx^2-Cax}{x(x^2-a^2)}\\ &\Rightarrow\begin{cases}A+B+C=0\\B-C=0\\-Aa^2=1\end{cases}\Rightarrow\begin{cases}A=-1/a^2\\B=C=1/(2a^2).\end{cases}\end{aligned}$$

Thus we have

$$\begin{aligned}&\int\frac{dx}{x(x^2-a^2)}\\&=\frac{1}{2a^2}\left(-2\int\frac{dx}{x}+\int\frac{dx}{x-a}+\int\frac{dx}{x+a}\right)\\&=\frac{1}{2a^2}\left(-2\ln|x|+\ln|x-a|+\ln|x+a|\right)+K\\&=\frac{1}{2a^2}\ln\frac{|x^2-a^2|}{x^2}+K.\end{aligned}$$

18. The partial fraction decomposition is

$$\begin{aligned}\frac{1}{x^4-a^4} &= \frac{A}{x-a}+\frac{B}{x+a}+\frac{Cx+D}{x^2+a^2}\\ &= \frac{A(x^3+ax^2+a^2x+a^3)+B(x^3-ax^2+a^2x-a^3)}{x^4-a^4}\\ &\quad+\frac{C(x^3-a^2x)+D(x^2-a^2)}{x^4-a^4}\\ &\Rightarrow\begin{cases}A+B+C=0\\aA-aB+D=0\\a^2A+a^2B-a^2C=0\\a^3A-a^3B-a^2D=1\end{cases}\\ &\Rightarrow A=\frac{1}{4a^3},\ \ B=-\frac{1}{4a^3},\ \ C=0,\ \ D=-\frac{1}{2a^2}.\end{aligned}$$

$$\begin{aligned}\int\frac{dx}{x^4-a^4} &= \frac{1}{4a^3}\int\left(\frac{1}{x-a}-\frac{1}{x+a}-\frac{2a}{x^2+a^2}\right)dx\\ &= \frac{1}{4a^3}\ln\left|\frac{x-a}{x+a}\right|-\frac{1}{2a^3}\tan^{-1}\left(\frac{x}{a}\right)+K.\end{aligned}$$

19. $$\begin{aligned}\frac{a^3}{x^3-a^3} &= \frac{A}{x-a}+\frac{Bx+C}{x^2+ax+a^2}\\ &= \frac{Ax^2+Aax+Aa^2+Bx^2-Bax+Cx-Ca}{x^3-a^3}\\ &\Rightarrow\begin{cases}A+B=0\\Aa-Ba+C=0\\Aa^2-Ca=a^3\end{cases}\Rightarrow\begin{cases}A=a/3\\B=-a/3\\C=-2a^2/3.\end{cases}\end{aligned}$$

Therefore we have

$$\begin{aligned}&\int\frac{x^3}{x^3-a^3}=\int\left(1+\frac{a^3}{x^3-a^3}\right)dx\\&=x+\frac{a}{3}\int\frac{dx}{x-a}-\frac{a}{3}\int\frac{x+2a}{x^2+ax+a^2}\\&=x+\frac{a}{3}\ln|x-a|-\frac{a}{6}\int\frac{2x+a+3a}{x^2+ax+a^2}\\&=x+\frac{a}{3}\ln|x-a|-\frac{a}{6}\ln(x^2+ax+a^2)\\&\qquad-\frac{a^2}{2}\int\frac{dx}{\left(x+\frac{a}{2}\right)^2+\frac{3}{4}a^2}\\&=x+\frac{a}{3}\ln|x-a|-\frac{a}{6}\ln(x^2+ax+a^2)\\&\qquad-\frac{a^2}{2}\frac{2}{\sqrt{3}a}\tan^{-1}\frac{x+(a/2)}{(\sqrt{3}a)/2}+K\\&=x+\frac{a}{3}\ln|x-a|-\frac{a}{6}\ln(x^2+ax+a^2)\\&\qquad-\frac{a}{\sqrt{3}}\tan^{-1}\frac{2x+a}{\sqrt{3}a}+K.\end{aligned}$$

20. Here the expansion is

$$\begin{aligned}\frac{1}{x^3+2x^2+2x} &= \frac{A}{x} + \frac{Bx+C}{x^2+2x+2}\\ &= \frac{A(x^2+2x+2)+Bx^2+Cx}{x^3+2x^2+2}\\ &\Rightarrow \begin{cases} A+B=0 \\ 2A+C=0 \\ 2A=1 \end{cases} \Rightarrow A=-B=\frac{1}{2},\ C=-1,\end{aligned}$$

so we have

$$\begin{aligned}\int \frac{dx}{x^3+2x^2+2x} &= \frac{1}{2}\int \frac{dx}{x} - \frac{1}{2}\int \frac{x+2}{x^2+2x+2}\,dx\\ &\quad \text{Let } u=x+1\\ &\quad du=dx\\ &= \frac{1}{2}\ln|x| - \frac{1}{2}\int \frac{u+1}{u^2+1}\,du\\ &= \frac{1}{2}\ln|x| - \frac{1}{4}\ln(u^2+1) - \frac{1}{2}\tan^{-1}u + K\\ &= \frac{1}{2}\ln|x| - \frac{1}{4}\ln(x^2+2x+2) - \frac{1}{2}\tan^{-1}(x+1) + K.\end{aligned}$$

21. $$\begin{aligned}&\frac{1}{x^3-4x^2+3x} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x-3}\\ &= \frac{A(x^2-4x+3)+B(x^2-3x)+C(x^2-x)}{x^3-4x^2+3x}\\ &\Rightarrow \begin{cases} A+B+C=0 \\ -4A-3B-C=0 \\ 3A=1 \end{cases}\\ &\Rightarrow A=\frac{1}{3},\quad B=-\frac{1}{2},\quad C=\frac{1}{6}.\end{aligned}$$

Therefore we have

$$\begin{aligned}&\int \frac{dx}{x^3-4x^2+3x}\\ &= \frac{1}{3}\int \frac{dx}{x} - \frac{1}{2}\int \frac{dx}{x-1} + \frac{1}{6}\int \frac{dx}{x-3}\\ &= \frac{1}{3}\ln|x| - \frac{1}{2}\ln|x-1| + \frac{1}{6}\ln|x-3| + K.\end{aligned}$$

22. Here the expansion is

$$\begin{aligned}&\frac{x^2+1}{x^3+8} = \frac{A}{x+2} + \frac{Bx+C}{x^2-2x+4}\\ &= \frac{A(x^2-2x+4)+B(x^2+2x)+C(x+2)}{x^3+8}\\ &\Rightarrow \begin{cases} A+B=1 \\ -2A+2B+C=0 \\ 4A+2C=1 \end{cases} \Rightarrow A=\frac{5}{12}\ B=\frac{7}{12},\ C=-\frac{1}{3},\end{aligned}$$

so we have

$$\begin{aligned}\int \frac{x^2+1}{x^3+8}\,dx &= \frac{5}{12}\int \frac{dx}{x+2} + \frac{1}{12}\int \frac{7x-4}{(x-1)^2+3}\,dx\\ &\quad \text{Let } u=x-1\\ &\quad du=dx\\ &= \frac{5}{12}\ln|x+2| + \frac{1}{12}\int \frac{7u+3}{u^2+3}\,du\\ &= \frac{5}{12}\ln|x+2| + \frac{7}{24}\ln(x^2-2x+4)\\ &\quad + \frac{1}{4\sqrt{3}}\tan^{-1}\frac{x-1}{\sqrt{3}} + K.\end{aligned}$$

23. $$\begin{aligned}&\frac{1}{(x^2-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2}\\ &= \frac{1}{(x^2-1)^2}\Big(A(x-1)(x+1)^2 + B(x+1)^2\\ &\quad + C(x+1)(x-1)^2 + D(x-1)^2\Big)\\ &\Rightarrow \begin{cases} A+C=0 \\ A+B-C+D=0 \\ -A+2B-C-2D=0 \\ -A+B+C+D=1 \end{cases} \Rightarrow \begin{cases} A=-\frac{1}{4} \\ B=C=D=\frac{1}{4}. \end{cases}\end{aligned}$$

Thus

$$\begin{aligned}&\int \frac{dx}{(x^2-1)^2}\\ &= \frac{1}{4}\Big(-\int \frac{dx}{x-1} + \int \frac{dx}{(x-1)^2}\\ &\quad + \int \frac{dx}{x+1} + \int \frac{dx}{(x+1)^2}\Big)\\ &= \frac{1}{4}\left(\ln|x+1| - \ln|x-1| - \frac{1}{x-1} - \frac{1}{x+1}\right) + K\\ &= \frac{1}{4}\ln\left|\frac{x+1}{x-1}\right| - \frac{x}{2(x^2-1)} + K.\end{aligned}$$

24. The expansion is

$$\begin{aligned}&\frac{x^2}{(x^2-1)(x^2-4)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x-2} + \frac{D}{x+2}\\ &A = \lim_{x\to1}\frac{x^2}{(x+1)(x^2-4)} = \frac{1}{2(-3)} = -\frac{1}{6}\\ &B = \lim_{x\to-1}\frac{x^2}{(x-1)(x^2-4)} = \frac{1}{-2(-3)} = \frac{1}{6}\\ &C = \lim_{x\to2}\frac{x^2}{(x^2-1)(x+2)} = \frac{4}{3(4)} = \frac{1}{3}\\ &D = \lim_{x\to-2}\frac{x^2}{(x^2-1)(x-2)} = \frac{4}{3(-4)} = -\frac{1}{3}.\end{aligned}$$

Therefore

$$\begin{aligned}\int \frac{x^2}{(x^2-1)(x^2-4)}\,dx &= -\frac{1}{6}\ln|x-1| + \frac{1}{6}\ln|x+1| +\\ &\quad \frac{1}{3}\ln|x-2| - \frac{1}{3}\ln|x+2| + K.\end{aligned}$$

25.
$$\frac{1}{x^4-3x^3} = \frac{1}{x^3(x-3)}$$
$$= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x-3}$$
$$= \frac{A(x^3-3x^2)+B(x^2-3x)+C(x-3)+Dx^3}{x^3(x-3)}$$
$$\Rightarrow \begin{cases} A+D=0 \\ -3A+B=0 \\ -3B+C=0 \\ -3C=1 \end{cases}$$
$$\Rightarrow \begin{cases} A=-1/27 \\ B=-1/9 \\ C=-1/3 \\ D=1/27. \end{cases}$$

Therefore
$$\int \frac{dx}{x^4-3x^3}$$
$$= -\frac{1}{27}\int\frac{dx}{x} - \frac{1}{9}\int\frac{dx}{x^2} - \frac{1}{3}\int\frac{dx}{x^3} + \frac{1}{27}\int\frac{dx}{x-3}$$
$$= \frac{1}{27}\ln\left|\frac{x-3}{x}\right| + \frac{1}{9x} + \frac{1}{6x^2} + K.$$

26. We have
$$\int\frac{x\,dx}{(x^2-x+1)^2} = \int\frac{x\,dx}{\left[(x-\frac{1}{2})^2+\frac{3}{4}\right]^2} \quad \text{Let } u = x-\tfrac{1}{2},\ du = dx$$
$$= \int\frac{u\,du}{(u^2+\frac{3}{4})^2} + \frac{1}{2}\int\frac{du}{(u^2+\frac{3}{4})^2}$$

Let $u = \frac{\sqrt{3}}{2}\tan v$,

$du = \frac{\sqrt{3}}{2}\sec^2 v\,dv$ in the second integral.
$$= -\frac{1}{2}\left(\frac{1}{u^2+\frac{3}{4}}\right) + \frac{1}{2}\int\frac{\frac{\sqrt{3}}{2}\sec^2 v\,dv}{\frac{9}{16}\sec^4 v}$$
$$= \frac{-1}{2(x^2-x+1)} + \frac{4}{3\sqrt{3}}\int\cos^2 v\,dv$$
$$= \frac{-1}{2(x^2-x+1)} + \frac{2}{3\sqrt{3}}(v+\sin v\cos v)+C$$
$$= \frac{-1}{2(x^2-x+1)} + \frac{2}{3\sqrt{3}}\tan^{-1}\frac{2x-1}{\sqrt{3}} + \frac{2}{3\sqrt{3}}\frac{2(x-\frac{1}{2})\sqrt{3}}{(2\sqrt{x^2-x+1})^2}+C$$
$$= \frac{2}{3\sqrt{3}}\tan^{-1}\frac{2x-1}{\sqrt{3}} + \frac{x-2}{3(x^2-x+1)}+C.$$

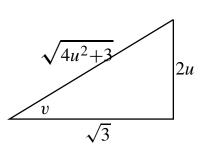

Fig. 6.3.26

27.
$$\frac{t}{(t+1)(t^2+1)^2} = \frac{A}{t+1} + \frac{Bt+C}{t^2+1} + \frac{Dt+E}{(t^2+1)^2}$$
$$= \frac{1}{(t+1)(t^2+1)^2}\Big(A(t^4+2t^2+1)+B(t^4+t^3+t^2+t)$$
$$+C(t^3+t^2+t+1)+D(t^2+t)+E(t+1)\Big)$$
$$\Rightarrow \begin{cases} A+B=0 \\ B+C=0 \\ 2A+B+C+D=0 \\ B+C+D+E=1 \\ A+C+E=0 \end{cases} \Rightarrow \begin{cases} B=-A \\ C=A \\ D=-2A \\ 2D=1 \\ E=-2A. \end{cases}$$

Thus $A=-1/4=C$, $B=1/4$, $D=E=1/2$. We have
$$\int\frac{t\,dt}{(t+1)(t^2+1)^2}$$
$$= -\frac{1}{4}\int\frac{dt}{t+1} + \frac{1}{4}\int\frac{(t-1)\,dt}{t^2+1} + \frac{1}{2}\int\frac{(t+1)\,dt}{(t^2+1)^2}$$
$$= -\frac{1}{4}\ln|t+1| + \frac{1}{8}\ln(t^2+1) - \frac{1}{4}\tan^{-1}t$$
$$-\frac{1}{4(t^2+1)} + \frac{1}{2}\int\frac{dt}{(t^2+1)^2} \quad \text{Let } t=\tan\theta,\ dt=\sec^2\theta\,d\theta$$
$$= -\frac{1}{4}\ln|t+1| + \frac{1}{8}\ln(t^2+1) - \frac{1}{4}\tan^{-1}t$$
$$-\frac{1}{4(t^2+1)} + \frac{1}{2}\int\cos^2\theta\,d\theta$$
$$= -\frac{1}{4}\ln|t+1| + \frac{1}{8}\ln(t^2+1) - \frac{1}{4}\tan^{-1}t$$
$$-\frac{1}{4(t^2+1)} + \frac{1}{4}(\theta+\sin\theta\cos\theta)+K$$
$$= -\frac{1}{4}\ln|t+1| + \frac{1}{8}\ln(t^2+1) - \frac{1}{4}\tan^{-1}t$$
$$-\frac{1}{4(t^2+1)} + \frac{1}{4}\tan^{-1}t + \frac{1}{4}\frac{t}{t^2+1}+K$$
$$= \frac{1}{4}\frac{t-1}{t^2+1} - \frac{1}{4}\ln|t+1| + \frac{1}{8}\ln(t^2+1)+K.$$

28. We have
$$\int\frac{dt}{(t-1)(t^2-1)^2}$$
$$= \int\frac{dt}{(t-1)^3(t+1)^2} \quad \text{Let } u=t-1,\ du=dt$$
$$= \int\frac{du}{u^3(u+2)^2}$$
$$\frac{1}{u^3(u+2)^2} = \frac{A}{u} + \frac{B}{u^2} + \frac{C}{u^3} + \frac{D}{u+2} + \frac{E}{(u+2)^2}$$
$$= \frac{A(u^4+4u^3+4u^2)+B(u^3+4u^2+4u)}{u^3(u+2)^2}$$
$$\frac{C(u^2+4u+4)+D(u^4+2u^3)+Eu^3}{u^3(u+2)^2}$$

$$\Rightarrow \begin{cases} A + D = 0 \\ 4A + B + 2D + E = 0 \\ 4A + 4B + C = 0 \\ 4B + 4C = 0 \\ 4C = 1 \end{cases}$$

$$\Rightarrow A = \frac{3}{16},\ B = -\frac{1}{4},\ C = \frac{1}{4},\ D = -\frac{3}{16},\ E = -\frac{1}{8}.$$

$$\int \frac{du}{u^3(u+2)^2}$$
$$= \frac{3}{16}\int \frac{du}{u} - \frac{1}{4}\int \frac{du}{u^2} + \frac{1}{4}\int \frac{du}{u^3}$$
$$\quad - \frac{3}{16}\int \frac{du}{u+2} - \frac{1}{8}\int \frac{du}{(u+2)^2}$$
$$= \frac{3}{16}\ln|t-1| + \frac{1}{4(t-1)} - \frac{1}{8(t-1)^2} -$$
$$\quad \frac{3}{16}\ln|t+1| + \frac{1}{8(t+1)} + K.$$

29. $I \int \dfrac{dx}{x(3+x^2)\sqrt{1-x^2}}$ Let $1 - x^2 = u^2$, $-2x\,dx = 2u\,du$

$$= -\int \frac{du}{(1-u^2)(4-u^2)}.$$

$$\frac{1}{(1-u^2)(4-u^2)} = \frac{A}{1-u} + \frac{B}{1+u} + \frac{C}{2-u} + \frac{D}{2+u}$$

$$A = \left.\frac{1}{(1+u)(4-u^2)}\right|_{u=1} = \frac{1}{6}$$

$$B = \left.\frac{1}{(1-u)(4-u^2)}\right|_{u=-1} = \frac{1}{6}$$

$$C = \left.\frac{1}{(1-u^2)(2+u)}\right|_{u=2} = -\frac{1}{12}$$

$$D = \left.\frac{1}{(1-u^2)(2-u)}\right|_{u=-2} = -\frac{1}{12}.$$

Thus

$$I = -\left(\frac{1}{6}\int \frac{du}{1-u} + \frac{1}{6}\int \frac{du}{1+u} - \frac{1}{12}\int \frac{du}{2-u} - \frac{1}{12}\int \frac{du}{2+u}\right)$$
$$= \frac{1}{6}\ln\left|\frac{1-u}{1+u}\right| + \frac{1}{12}\ln\left|\frac{2+u}{2-u}\right| + K$$
$$= \frac{1}{6}\ln\left|\frac{1-\sqrt{1-x^2}}{1+\sqrt{1-x^2}}\right| + \frac{1}{12}\ln\left|\frac{2+\sqrt{1-x^2}}{2-\sqrt{1-x^2}}\right| + K$$
$$= \frac{1}{6}\ln\frac{(1-\sqrt{1-x^2})^2}{x^2} + \frac{1}{12}\ln\frac{(2+\sqrt{1-x^2})^2}{3+x^2} + K.$$

30. $\displaystyle\int \frac{dx}{e^{2x} - 4e^x + 4} = \int \frac{dx}{(e^x-2)^2}$ Let $u = e^x$, $du = e^x\,dx$

$$= \int \frac{du}{u(u-2)^2}$$

$$\frac{1}{u(u-2)^2} = \frac{A}{u} + \frac{B}{u-2} + \frac{C}{(u-2)^2}$$
$$= \frac{A(u^2-4u+4) + B(u^2-2u) + +Cu}{u(u-2)^2}$$

$$\Rightarrow \begin{cases} A + B = 0 \\ -4A - 2B + C = 0 \\ 4A = 1 \end{cases} \Rightarrow A = \frac{1}{4},\ B = -\frac{1}{4},\ C = \frac{1}{2}.$$

$$\int \frac{du}{u(u-2)^2} = \frac{1}{4}\int \frac{du}{u} - \frac{1}{4}\int \frac{du}{u-2} + \frac{1}{2}\int \frac{du}{(u-2)^2}$$
$$= \frac{1}{4}\ln|u| - \frac{1}{4}\ln|u-2| - \frac{1}{2}\frac{1}{(u-2)} + K$$
$$= \frac{x}{4} - \frac{1}{4}\ln|e^x - 2| - \frac{1}{2(e^x-2)} + K.$$

31. $I = \displaystyle\int \frac{dx}{x(1+x^2)^{3/2}}$ Let $x = \tan\theta$, $dx = \sec^2\theta\,d\theta$

$$= \int \frac{\sec^2\theta\,d\theta}{\tan\theta\sec^3\theta} = \int \frac{\cos^2\theta\,d\theta}{\sin\theta}$$

$$= \int \frac{\cos^2\theta\sin\theta\,d\theta}{\sin^2\theta} \quad \text{Let } u = \cos\theta,\ du = -\sin\theta\,d\theta$$

$$= -\int \frac{u^2\,du}{1-u^2} = u + \int \frac{du}{u^2-1}.$$

We have

$$\frac{1}{u^2-1} = \frac{1}{2}\left(\frac{1}{u-1} - \frac{1}{u+1}\right).$$

Thus

$$I = u + \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right| + C$$
$$= \cos\theta + \frac{1}{2}\ln\left|\frac{\cos\theta - 1}{\cos\theta + 1}\right| + C$$
$$= \frac{1}{\sqrt{1+x^2}} + \frac{1}{2}\ln\left|\frac{\frac{1}{\sqrt{1+x^2}} - 1}{\frac{1}{\sqrt{1+x^2}} + 1}\right| + C$$
$$= \frac{1}{\sqrt{1+x^2}} + \frac{1}{2}\ln\left(\frac{\sqrt{1+x^2} - 1}{\sqrt{1+x^2} + 1}\right) + C.$$

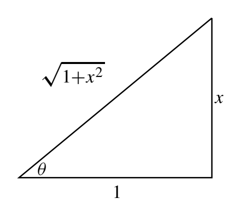

Fig. 6.3.31

32. We have

$$\begin{aligned}
I &= \int \frac{dx}{x(1-x^2)^{3/2}} \quad \begin{array}{l}\text{Let } u^2 = 1-x^2\\ 2u\,du = -2x\,dx\end{array}\\
&= -\int \frac{u\,du}{(1-u^2)u^3} = -\int \frac{du}{(1-u^2)u^2}
\end{aligned}$$

$$\begin{aligned}
\frac{1}{u^2(1-u^2)} &= \frac{A}{u} + \frac{B}{u^2} + \frac{C}{1-u} + \frac{D}{1+u}\\
&= \frac{A(u-u^3) + B(1-u^2) + C(u^2+u^3) + D(u^2-u^3)}{u^2(1-u^2)}
\end{aligned}$$

$$\Rightarrow \begin{cases} -A+C-D=0\\ -B+C+D=0\\ A=0\\ B=1 \end{cases}$$

$$\Rightarrow A=0,\ \ B=1,\ \ C=\frac{1}{2},\ \ D=\frac{1}{2}.$$

$$\begin{aligned}
I &= -\int \frac{du}{(1-u^2)u^2} = -\int \frac{du}{u^2} - \frac{1}{2}\int \frac{du}{1-u} - \frac{1}{2}\int \frac{du}{1+u}\\
&= \frac{1}{u} + \frac{1}{2}\ln|1-u| - \frac{1}{2}\ln|1+u| + K\\
&= \frac{1}{\sqrt{1-x^2}} + \frac{1}{2}\ln\left|\frac{1-\sqrt{1-x^2}}{1+\sqrt{1-x^2}}\right| + K\\
&= \frac{1}{\sqrt{1-x^2}} + \ln\Big(1-\sqrt{1-x^2}\Big) - \ln|x| + K.
\end{aligned}$$

33.

$$\begin{aligned}
&\int \frac{dx}{x^2(x^2-1)^{3/2}} \quad \begin{array}{l}\text{Let } x = \sec\theta\\ dx = \sec\theta\tan\theta\,d\theta\end{array}\\
&= \int \frac{\sec\theta\tan\theta\,d\theta}{\sec^2\theta\tan^3\theta} = \int \frac{\cos^3\theta\,d\theta}{\sin^2\theta}\\
&= \int \frac{1-\sin^2\theta}{\sin^2\theta}\cos\theta\,d\theta \quad \begin{array}{l}\text{Let } u = \sin\theta\\ du = \cos\theta\,d\theta\end{array}\\
&= \int \frac{1-u^2}{u^2}\,du = -\frac{1}{u} - u + C\\
&= -\left(\frac{1}{\sin\theta} + \sin\theta\right) + C\\
&= -\left(\frac{x}{\sqrt{x^2-1}} + \frac{\sqrt{x^2-1}}{x}\right) + C.
\end{aligned}$$

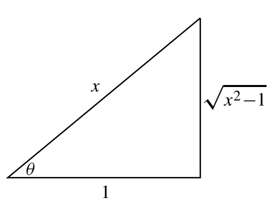

Fig. 6.3.33

34.

$$\begin{aligned}
&\int \frac{d\theta}{\cos\theta(1+\sin\theta)} \quad \begin{array}{l}\text{Let } u = \sin\theta\\ du = \cos\theta\,d\theta\end{array}\\
&= \int \frac{du}{(1-u^2)(1+u)} = \int \frac{du}{(1-u)(1+u)^2}
\end{aligned}$$

$$\begin{aligned}
\frac{1}{(1-u)(1+u)^2} &= \frac{A}{1-u} + \frac{B}{1+u} + \frac{C}{(1+u)^2}\\
&= \frac{A(1+2u+u^2) + B(1-u^2) + C(1-u)}{(1-u)(1+u)^2}
\end{aligned}$$

$$\Rightarrow \begin{cases} A-B=0\\ 2A-C=0\\ A+B+C=1 \end{cases} \Rightarrow A=\frac{1}{4},\ \ B=\frac{1}{4},\ \ C=\frac{1}{2}.$$

$$\begin{aligned}
&\int \frac{du}{(1-u)(1+u)^2}\\
&= \frac{1}{4}\int \frac{du}{1-u} + \frac{1}{4}\int \frac{du}{1+u} + \frac{1}{2}\int \frac{du}{(1+u)^2}\\
&= \frac{1}{4}\ln\left|\frac{1+\sin\theta}{1-\sin\theta}\right| - \frac{1}{2(1+\sin\theta)} + C.
\end{aligned}$$

35. Since $Q(x) = (x-a_1)(x-a_2)\cdots(x-a_n)$, we have

$$\ln Q(x) = \ln(x-a_1) + \ln(x-a_2) + \cdots + \ln(x-a_n),$$

and, differentiating both sides,

$$\begin{aligned}
\frac{Q'(x)}{Q(x)} &= \frac{d}{dx}[\ln Q(x)] = \frac{1}{x-a_1} + \frac{1}{x-a_2} + \cdots + \frac{1}{x-a_n}\\
\frac{1}{Q(x)} &= \frac{1}{Q'(x)}\left[\frac{1}{x-a_1} + \frac{1}{x-a_2} + \cdots + \frac{1}{x-a_n}\right].
\end{aligned}$$

Since

$$\frac{P(x)}{Q(x)} = \frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + \cdots + \frac{A_n}{x-a_n},$$

we have

$$\begin{aligned}
&\frac{P(x)}{Q'(x)}\left[\frac{1}{x-a_1} + \frac{1}{x-a_2} + \cdots + \frac{1}{x-a_n}\right]\\
&= \frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + \cdots + \frac{A_n}{x-a_n}.
\end{aligned}$$

Multiply both sides by $x-a_1$ and get

$$\begin{aligned}
&\frac{P(x)}{Q'(x)}\left[1 + \frac{x-a_1}{x-a_2} + \cdots + \frac{x-a_1}{x-a_n}\right]\\
&= A_1 + \frac{A_2(x-a_1)}{x-a_2} + \cdots + \frac{A_n(x-a_1)}{x-a_n}.
\end{aligned}$$

Now let $x = a_1$ and obtain $\dfrac{P(a_1)}{Q'(a_1)} = A_1$.

Similarly, $A_j = \dfrac{P(a_j)}{Q'(a_j)}$ for $1 \le j \le n$.

Section 6.4 Integration Using Computer Algebra or Tables (page 340)

2. According to Maple

$$\begin{aligned}&\int \frac{1+x+x^2}{(x^4-1)(x^4-16)^2}\,dx\\ &= \frac{\ln(x-1)}{300} - \frac{\ln(x+1)}{900} - \frac{7}{15{,}360(x-2)}\\ &\quad - \frac{613}{460{,}800}\ln(x-2) - \frac{1}{5{,}120(x+2)} + \frac{79}{153{,}600}\ln(x+2)\\ &\quad - \frac{\ln(x^2+1)}{900} + \frac{47}{115{,}200}\ln(x^2+4)\\ &\quad - \frac{23}{25{,}600}\tan^{-1}(x/2) - \frac{6x+8}{15{,}360(x^2+4)}\end{aligned}$$

One suspects it has forgotten to use absolute values in some of the logarithms.

3. Neither the author's version of Maple nor his version of Mathematics would do

$$I = \int \frac{t^5}{\sqrt{3-2t^4}}\,dt$$

as presented. Both did an integration by parts and left an unevaluated integral. Both managed to evaluate the integral after the substitution $u = t^2$ was made. (See Exercise 4.) However, Derive had no trouble doing the integral in its original form, giving as the answer

$$\frac{3\sqrt{2}}{16}\sin^{-1}\frac{\sqrt{6}t^2}{3} - \frac{t^2\sqrt{3-2t^4}}{8}.$$

4. Maple, Mathematica, and Derive readily gave

$$\int_0^1 \frac{1}{(x^2+1)^3}\,dx = \frac{3\pi}{32} + \frac{1}{4}.$$

5. Use the 6th integral in the list involving $\sqrt{x^2 \pm a^2}$.

$$\int \frac{x^2\,dx}{\sqrt{x^2-2}} = \frac{x}{2}\sqrt{x^2-2} + \ln|x+\sqrt{x^2-2}| + C$$

6. Use the last integral in the list involving $\sqrt{x^2 \pm a^2}$.

$$\int \sqrt{(x^2+4)^3}\,dx = \frac{x}{4}(x^2+10)\sqrt{x^2+4} + 6\ln|x+\sqrt{x^2+4}| + C$$

7. Use the 8th integral in the list involving $\sqrt{x^2 \pm a^2}$ after making the change of variable $x = \sqrt{3}t$.

$$\begin{aligned}&\int \frac{dt}{t^2\sqrt{3t^2+5}} \quad \begin{array}{l}\text{Let } x = \sqrt{3}t\\ dx = \sqrt{3}\,dt\end{array}\\ &= \frac{3}{\sqrt{3}}\int \frac{dx}{x^2\sqrt{x^2+5}}\\ &= -\sqrt{3}\,\frac{\sqrt{x^2+5}}{5x} + C = -\frac{\sqrt{3t^2+5}}{5t} + C\end{aligned}$$

8. Use the 8th integral in the miscellaneous algebraic set.

$$\int \frac{dt}{t\sqrt{3t-5}} = \frac{2}{\sqrt{5}}\tan^{-1}\sqrt{\frac{3t-5}{5}} + C$$

9. The 5th and 4th integrals in the exponential/logarithmic set give

$$\begin{aligned}&\int x^4(\ln x)^4\,dx = \frac{x^5(\ln x)^4}{5} - \frac{4}{5}\int x^4(\ln x)^3\,dx\\ &= \frac{x^5(\ln x)^4}{5} - \frac{4}{5}\left(\frac{x^5(\ln x)^3}{5} - \frac{3}{2}\int x^4(\ln x)^2\,dx\right)\\ &= x^5\left(\frac{(\ln x)^4}{5} - \frac{4(\ln x)^3}{25}\right)\\ &\qquad + \frac{12}{25}\left(\frac{x^5(\ln x)^2}{5} - \frac{2}{5}\int x^5\ln x\,dx\right)\\ &= x^5\left(\frac{(\ln x)^4}{5} - \frac{4(\ln x)^3}{25} + \frac{12(\ln x)^2}{125} - \frac{24\ln x}{625} + \frac{24}{3{,}125}\right) + C.\end{aligned}$$

10. We make a change of variable and then use the first two integrals in the exponential/logarithmic set.

$$\begin{aligned}&\int x^7 e^{x^2}\,dx \quad \begin{array}{l}\text{Let } u = x^2\\ du = 2x\,dx\end{array}\\ &= \frac{1}{2}\int u^3 e^u\,du\\ &= \frac{1}{2}\left(u^3e^u - 3\int u^2e^u\,du\right)\\ &= \frac{u^3e^u}{2} - \frac{3}{2}\left(u^2e^u - 2\int ue^u\,du\right)\\ &= \left(\frac{u^3}{2} - \frac{3u^2}{2} + 3(u-1)\right)e^u + C\\ &= \left(\frac{x^6}{2} - \frac{3x^4}{2} + 3x^2 - 3\right)e^{x^2} + C\end{aligned}$$

11. Use integrals 14 and 12 in the miscellaneous algebraic set.

$$\int x\sqrt{2x-x^2}\,dx$$
$$= -\frac{(2x-x^2)^{3/2}}{3} + \frac{3}{3}\int \sqrt{2x-x^2}\,dx$$
$$= -\frac{(2x-x^2)^{3/2}}{3} + \frac{x-1}{2}\sqrt{2x-x^2} + \frac{1}{2}\sin^{-1}(x-1) + C$$

12. Use integrals 17 and 16 in the miscellaneous algebraic set.

$$\int \frac{\sqrt{2x-x^2}}{x^2}\,dx$$
$$= -\frac{(2x-x^2)^{3/2}}{x^2} - \frac{1}{1}\int \frac{\sqrt{2x-x^2}}{x}\,dx$$
$$= -\frac{(2x-x^2)^{3/2}}{x^2} - \sqrt{2x-x^2} - \sin^{-1}(x-1) + C$$

13. Use the last integral in the miscellaneous algebraic set.

$$\int \frac{dx}{(\sqrt{4x-x^2})^3} = \frac{x-2}{4}\frac{1}{\sqrt{4x-x^2}} + C$$

14. Use the last integral in the miscellaneous algebraic set. Then complete the square, change variables, and use the second last integral in the elementary list.

$$\int \frac{dx}{(\sqrt{4x-x^2})^4}$$
$$= \frac{x-2}{8}(\sqrt{4x-x^2})^{-2} + \frac{1}{8}\int \frac{dx}{4x-x^2}$$
$$= \frac{x-2}{8(4x-x^2)} + \frac{1}{8}\int \frac{dx}{4-(x-2)^2} \quad \text{Let } u = x-2, \ du = dx$$
$$= \frac{x-2}{8(4x-x^2)} + \frac{1}{8}\int \frac{du}{4-u^2}$$
$$= \frac{x-2}{8(4x-x^2)} + \frac{1}{32}\ln\left|\frac{u+2}{u-2}\right| + C$$
$$= \frac{x-2}{8(4x-x^2)} + \frac{1}{32}\ln\left|\frac{x}{x-4}\right| + C$$

Section 6.5 Improper Integrals (page 347)

1. $$\int_2^\infty \frac{1}{(x-1)^3}\,dx \quad \text{Let } u = x-1, \ du = dx$$
$$= \int_1^\infty \frac{du}{u^3} = \lim_{R\to\infty}\int_1^R \frac{du}{u^3}$$
$$= \lim_{R\to\infty} \frac{-1}{2u^2}\bigg|_1^R = \lim_{R\to\infty}\left(\frac{1}{2} - \frac{1}{2R^2}\right) = \frac{1}{2}$$

2. $$\int_3^\infty \frac{1}{(2x-1)^{2/3}}\,dx \quad \text{Let } u = 2x-1, \ du = 2\,dx$$
$$= \frac{1}{2}\int_5^\infty \frac{du}{u^{2/3}} = \frac{1}{2}\lim_{R\to\infty}\int_5^R u^{-2/3}\,du$$
$$= \frac{1}{2}\lim_{R\to\infty} 3u^{1/3}\bigg|_5^R = \infty \quad \text{(diverges)}$$

3. $$\int_0^\infty e^{-2x}\,dx = \lim_{R\to\infty}\frac{e^{-2x}}{-2}\bigg|_0^R$$
$$= \lim_{R\to\infty}\left(\frac{1}{2} - \frac{1}{2e^R}\right)$$
$$= \frac{1}{2}. \quad \text{This integral converges.}$$

4. $$\int_{-\infty}^{-1}\frac{dx}{x^2+1} = \lim_{R\to-\infty}\int_R^{-1}\frac{dx}{x^2+1}$$
$$= \lim_{R\to-\infty}\left[\tan^{-1}(-1) - \tan^{-1}(R)\right]$$
$$= -\frac{\pi}{4} - \left(-\frac{\pi}{2}\right) = \frac{\pi}{4}.$$
This integral converges.

5. $$\int_{-1}^1 \frac{dx}{(x+1)^{2/3}} = \lim_{c\to-1+} 3(x+1)^{1/3}\bigg|_c^1$$
$$= \lim_{c\to-1+} 3\left(2^{1/3} - (1+c)^{1/3}\right)$$
$$= 3\sqrt[3]{2}. \quad \text{This integral converges.}$$

6. $$\int_0^a \frac{dx}{a^2-x^2} = \lim_{C\to a-}\int_0^C \frac{dx}{a^2-x^2}$$
$$= \lim_{C\to a-}\frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right|\bigg|_0^C$$
$$= \lim_{C\to a-}\frac{1}{2a}\ln\frac{a+C}{a-C} = \infty.$$
The integral diverges to infinity.

7. $$\int_0^1 \frac{dx}{(1-x)^{1/3}} \quad \text{Let } u = 1-x, \ du = -dx$$
$$= \int_0^1 \frac{du}{u^{1/3}} = \lim_{c\to0+}\int_c^1 \frac{du}{u^{1/3}}$$
$$= \lim_{c\to0+}\frac{3}{2}u^{2/3}\bigg|_c^1 = \frac{3}{2}$$

8. $$\int_0^1 \frac{dx}{x\sqrt{1-x}} \quad \text{Let } u^2 = 1-x, \ 2u\,du = -dx$$
$$= \int_0^1 \frac{2u\,du}{(1-u^2)u} = 2\lim_{c\to1-}\int_0^c \frac{du}{1-u^2}$$
$$= 2\lim_{c\to1-}\frac{1}{2}\ln\left|\frac{u+1}{u-1}\right|\bigg|_0^c = \infty \quad \text{(diverges)}$$

9. $\displaystyle\int_0^{\pi/2} \frac{\cos x\,dx}{(1-\sin x)^{2/3}}$ Let $u = 1-\sin x$, $du = -\cos x\,dx$

$\displaystyle = \int_0^1 u^{-2/3}\,du = \lim_{c\to 0+} 3u^{1/3}\Big|_c^1 = 3.$

The integral converges.

10. $\displaystyle\int_0^\infty xe^{-x}\,dx$

$\displaystyle = \lim_{R\to\infty}\int_0^R xe^{-x}\,dx$

$U = x \quad dV = e^{-x}\,dx$

$dU = dx \quad V = -e^{-x}$

$\displaystyle = \lim_{R\to\infty}\left(-xe^{-x}\Big|_0^R + \int_0^R e^{-x}\,dx\right)$

$\displaystyle = \lim_{R\to\infty}\left(-\frac{R}{e^R} - \frac{1}{e^R} + 1\right) = 1.$

The integral converges.

11. $\displaystyle\int_0^1 \frac{dx}{\sqrt{x(1-x)}} = 2\int_0^{1/2} \frac{dx}{\sqrt{\frac14 - \left(x-\frac12\right)^2}}$

$\displaystyle = 2\lim_{c\to 0+}\int_c^{1/2} \frac{dx}{\sqrt{\frac14 - \left(x-\frac12\right)^2}}$

$\displaystyle = 2\lim_{c\to 0+}\sin^{-1}(2x-1)\Big|_c^{1/2} = \pi.$

The integral converges.

12. $\displaystyle\int_0^\infty \frac{x}{1+2x^2}\,dx = \lim_{R\to\infty}\int_0^R \frac{x}{1+2x^2}\,dx$

$\displaystyle = \lim_{R\to\infty}\frac14\ln(1+2x^2)\Big|_0^R$

$\displaystyle = \lim_{R\to\infty}\left[\frac14\ln(1+2R^2) - \frac14\ln 1\right] = \infty.$

This integral diverges to infinity.

13. $\displaystyle\int_0^\infty \frac{x\,dx}{(1+2x^2)^{3/2}}$ Let $u = 1+2x^2$, $du = 4x\,dx$

$\displaystyle = \frac14\int_1^\infty \frac{du}{u^{3/2}} = \frac14\lim_{R\to\infty}\left(-\frac{2}{\sqrt{u}}\right)\Bigg|_1^R$

$\displaystyle = \frac12.$ The integral converges.

14. $\displaystyle\int_0^{\pi/2}\sec x\,dx = \lim_{C\to(\pi/2)-}\ln|\sec x + \tan x|\Big|_0^C$

$\displaystyle = \lim_{C\to(\pi/2)-}\ln|\sec C + \tan C| = \infty.$

This integral diverges to infinity.

15. $\displaystyle\int_0^{\pi/2}\tan x\,dx = \lim_{c\to(\pi/2)-}\ln|\sec x|\Big|_0^c$

$\displaystyle = \lim_{c\to(\pi/2)-}\ln\sec c = \infty.$

This integral diverges to infinity.

16. $\displaystyle\int_e^\infty \frac{dx}{x(\ln x)}$ Let $u = \ln x$, $\displaystyle du = \frac{dx}{x}$

$\displaystyle = \lim_{R\to\infty}\int_1^{\ln R}\frac{du}{u} = \lim_{R\to\infty}\ln|u|\Big|_1^{\ln R}$

$\displaystyle = \lim_{R\to\infty}\ln(\ln R) - \ln 1 = \infty.$

This integral diverges to infinity.

17. $\displaystyle\int_1^e \frac{dx}{x\sqrt{\ln x}}$ Let $u = \ln x$, $du = dx/x$

$\displaystyle = \int_0^1 \frac{du}{\sqrt{u}} = \lim_{c\to 0+} 2\sqrt{u}\Big|_c^1 = 2.$

This integral converges.

18. $\displaystyle\int_e^\infty \frac{dx}{x(\ln x)^2}$ Let $u = \ln x$, $\displaystyle du = \frac{dx}{x}$

$\displaystyle = \lim_{R\to\infty}\int_1^{\ln R}\frac{du}{u^2} = \lim_{R\to\infty}\left(-\frac{1}{\ln R} + 1\right) = 1.$

The integral converges.

19. $\displaystyle I = \int_{-\infty}^\infty \frac{x\,dx}{1+x^2} = \int_{-\infty}^0 + \int_0^\infty = I_1 + I_2$

$\displaystyle I_2 = \int_0^\infty \frac{x\,dx}{1+x^2}$ Let $u = 1+x^2$, $du = 2x\,dx$

$\displaystyle = \lim_{R\to\infty}\frac12\int_1^R \frac{du}{u} = \infty$ (diverges)

20. $\displaystyle I = \int_{-\infty}^\infty \frac{x\,dx}{1+x^4} = \int_{-\infty}^0 + \int_0^\infty = I_1 + I_2$

$\displaystyle I_2 = \int_0^\infty \frac{x\,dx}{1+x^4}$ Let $u = x^2$, $du = 2x\,dx$

$\displaystyle = \frac12\int_0^\infty \frac{du}{1+u^2} = \frac12\lim_{R\to\infty}\tan^{-1}u\Big|_0^R = \frac{\pi}{4}$

Similarly, $I_1 = -\dfrac{\pi}{4}$. Therefore, $I = 0$.

21. $\displaystyle I = \int_{-\infty}^\infty xe^{-x^2}\,dx = \int_{-\infty}^0 + \int_0^\infty = I_1 + I_2$

$\displaystyle I_2 = \int_0^\infty xe^{-x^2}\,dx$ Let $u = x^2$, $du = 2x\,dx$

$\displaystyle = \frac12\int_0^\infty e^{-u}\,du = \frac12\lim_{R\to\infty} -e^{-u}\Big|_0^R = \frac12$

Similarly, $I_1 = -\frac{1}{2}$. Therefore, $I = 0$.

22. $I = \int_{-\infty}^{\infty} e^{-|x|}\,dx = \int_{-\infty}^{0} e^x\,dx + \int_0^{\infty} e^{-x}\,dx = I_1 + I_2$

$I_2 = \int_0^{\infty} e^{-x}\,dx = 1$

Similarly, $I_1 = 1$. Therefore, $I = 2$.

23. Area of $R = -\int_0^1 \ln x\,dx = -\lim_{c\to 0+}(x\ln x - x)\Big|_0^1$

$$= -(0-1) + \lim_{c\to 0+}(c\ln c - c)$$
$$= 1 - 0 = 1 \text{ units}^2$$

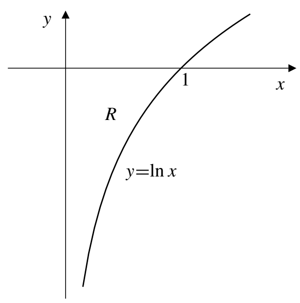

Fig. 6.5.23

24. Area of shaded region $= \int_0^{\infty}(e^{-x} - e^{-2x})\,dx$

$$= \lim_{R\to\infty}\left(-e^{-x} + \frac{1}{2}e^{-2x}\right)\Big|_0^R$$
$$= \lim_{R\to\infty}\left(-e^{-R} + \frac{1}{2}e^{-2R} + 1 - \frac{1}{2}\right) = \frac{1}{2} \text{ sq. units.}$$

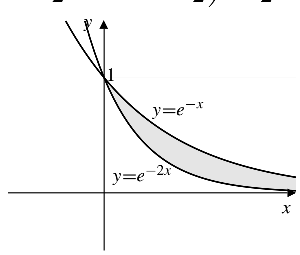

Fig. 6.5.24

25. Area $= \int_1^{\infty}\left(\frac{4}{2x+1} - \frac{2}{x+2}\right)dx$

$$= \lim_{R\to\infty} 2\Big(\ln(2x+1) - \ln(x+2)\Big)\Big|_1^R$$
$$= \lim_{R\to\infty} 2\ln\left(\frac{2R+1}{R+2}\right) - 0 = 2\ln 2 \text{ sq. units.}$$

26. The required area is

$$\text{Area} = \int_0^{\infty} x^{-2}e^{-1/x}\,dx$$
$$= \int_0^1 x^{-2}e^{-1/x}\,dx + \int_1^{\infty} x^{-2}e^{-1/x}\,dx$$
$$= I_1 + I_2.$$

Then let $u = -\frac{1}{x}$ and $du = x^{-2}\,dx$ in both I_1 and I_2:

$$I_1 = \lim_{C\to 0+}\int_C^1 x^{-2}e^{-1/x}\,dx = \lim_{C\to 0+}\int_{-1/C}^{-1} e^u\,du$$
$$= \lim_{C\to 0+}(e^{-1} - e^{-1/C}) = \frac{1}{e}.$$
$$I_2 = \lim_{R\to\infty}\int_1^R x^{-2}e^{-1/x}\,dx = \lim_{R\to\infty}\int_{-1}^{-1/R} e^u\,du$$
$$= \lim_{R\to\infty}(e^{-1/R} - e^{-1}) = 1 - \frac{1}{e}.$$

Hence, the total area is $I_1 + I_2 = 1$ square unit.

27. First assume that $p \neq 1$. Since $a > 0$ we have

$$\int_a^{\infty} x^{-p}\,dx = \lim_{R\to\infty}\frac{x^{-p+1}}{-p+1}\Big|_a^R$$
$$= \frac{-a^{-p+1}}{1-p} + \lim_{R\to\infty}\frac{1}{(1-p)R^{p-1}}$$
$$= \begin{cases} \dfrac{1}{(p-1)a^{p-1}} & \text{if } p > 1 \\ \infty & \text{if } p < 1 \end{cases}$$
$$\int_0^a x^{-p}\,dx = \lim_{c\to 0+}\frac{x^{-p+1}}{-p+1}\Big|_c^a$$
$$= \frac{a^{-p+1}}{1-p} + \lim_{c\to 0+}\frac{c^{1-p}}{p-1}$$
$$= \begin{cases} \dfrac{a^{1-p}}{1-p} & \text{if } p < 1 \\ \infty & \text{if } p > 1. \end{cases}$$

If $p = 1$ both integrals diverge as shown in Examples 2 and 6(a).

28. $\int_{-1}^1 \frac{x\,\mathrm{sgn}\,x}{x+2}\,dx = \int_{-1}^0 \frac{-x}{x+2}\,dx + \int_0^1 \frac{x}{x+2}\,dx$

$$= \int_{-1}^0\left(-1 + \frac{2}{x+2}\right)dx + \int_0^1\left(1 - \frac{2}{x+2}\right)dx$$
$$= (-x + 2\ln|x+2|)\Big|_{-1}^0 + (x - 2\ln|x+2|)\Big|_0^1 = \ln\frac{16}{9}.$$

29. $\int_0^2 x^2\,\mathrm{sgn}\,(x-1)\,dx$

$$= \int_0^1 -x^2\,dx + \int_1^2 x^2\,dx$$
$$= -\frac{x^3}{3}\Big|_0^1 + \frac{x^3}{3}\Big|_1^2 = -\frac{1}{3} + \frac{8}{3} - \frac{1}{3} = 2.$$

30. Since $\dfrac{x^2}{x^5+1} \le \dfrac{1}{x^3}$ for all $x \ge 0$, therefore

$$\begin{aligned} I &= \int_0^\infty \frac{x^2}{x^5+1}\,dx \\ &= \int_0^1 \frac{x^2}{x^5+1}\,dx + \int_1^\infty \frac{x^2}{x^5+1}\,dx \\ &\le \int_0^1 \frac{x^2}{x^5+1}\,dx + \int_1^\infty \frac{dx}{x^3} \\ &= I_1 + I_2. \end{aligned}$$

Since I_1 is a proper integral (finite) and I_2 is a convergent improper integral, (see Theorem 2), therefore I converges.

31. $\dfrac{1}{1+\sqrt{x}} \ge \dfrac{1}{2\sqrt{x}}$ on $[1,\infty)$.
Since $\displaystyle\int_1^\infty \frac{dx}{\sqrt{x}}$ diverges to infinity, so must $\displaystyle\int_1^\infty \frac{dx}{1+\sqrt{x}}$.
Therefore $\displaystyle\int_0^\infty \frac{dx}{1+\sqrt{x}}$ also diverges to infinity.

32. Since $\dfrac{x\sqrt{x}}{x^2-1} \ge \dfrac{1}{\sqrt{x}}$ for all $x > 1$, therefore

$$I = \int_2^\infty \frac{x\sqrt{x}}{x^2-1}\,dx \ge \int_2^\infty \frac{dx}{\sqrt{x}} = I_1 = \infty.$$

Since I_1 is a divergent improper integral, I diverges.

33. $\displaystyle\int_0^\infty e^{-x^3}\,dx = \left(\int_0^1 + \int_1^\infty\right) e^{-x^3}\,dx.$
Now $\displaystyle\int_0^1 e^{-x^3}\,dx$ is a proper integral, and is therefore finite. Since $x^3 \ge x$ on $[1,\infty)$, we have

$$\int_1^\infty e^{-x^3}\,dx \le \int_1^\infty e^{-x}\,dx = \frac{1}{e}.$$

Thus $\displaystyle\int_0^\infty e^{-x^3}\,dx$ converges.

34. On [0,1], $\dfrac{1}{\sqrt{x}+x^2} \le \dfrac{1}{\sqrt{x}}$. On $[1,\infty)$, $\dfrac{1}{\sqrt{x}+x^2} \le \dfrac{1}{x^2}$.
Thus,

$$\begin{aligned} \int_0^1 \frac{dx}{\sqrt{x}+x^2} &\le \int_0^1 \frac{dx}{\sqrt{x}} \\ \int_1^\infty \frac{dx}{\sqrt{x}+x^2} &\le \int_1^\infty \frac{dx}{x^2}. \end{aligned}$$

Since both of these integrals are convergent, therefore so is their sum $\displaystyle\int_0^\infty \frac{dx}{\sqrt{x}+x^2}$.

35. $\dfrac{e^x}{x+1} \ge \dfrac{e^{-1}}{x+1}$ on $[-1,1]$. Thus

$$\int_{-1}^1 \frac{e^x}{x+1}\,dx \ge \frac{1}{e}\int_{-1}^1 \frac{dx}{x+1} = \infty.$$

The given integral diverges to infinity.

36. Since $\sin x \le x$ for all $x \ge 0$, thus $\dfrac{\sin x}{x} \le 1$. Then

$$I = \int_0^\pi \frac{\sin x}{x}\,dx = \lim_{\epsilon\to 0+}\int_\epsilon^\pi \frac{\sin x}{x}\,dx \le \int_0^\pi (1)\,dx = \pi.$$

Hence, I converges.

37. Since $\sin x \ge \dfrac{2x}{\pi}$ on $[0,\pi/2]$, we have

$$\begin{aligned} \int_0^\infty \frac{|\sin x|}{x^2}\,dx &\ge \int_0^{\pi/2} \frac{\sin x}{x^2}\,dx \\ &\ge \frac{2}{\pi}\int_0^{\pi/2} \frac{dx}{x} = \infty. \end{aligned}$$

The given integral diverges to infinity.

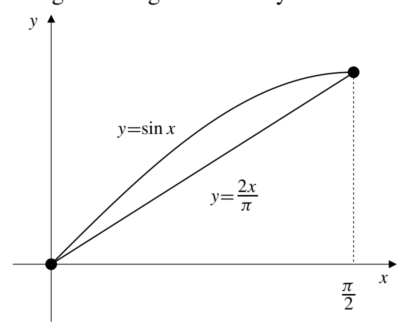

Fig. 6.5.37

38. Since $0 \le 1-\cos\sqrt{x} = 2\sin^2\left(\dfrac{\sqrt{x}}{2}\right) \le 2\left(\dfrac{\sqrt{x}}{2}\right)^2 = \dfrac{x}{2}$,
for $x \ge 0$, therefore $\displaystyle\int_0^{\pi^2} \frac{dx}{1-\cos\sqrt{x}} \ge 2\int_0^{\pi^2} \frac{dx}{x}$, which diverges to infinity.

39. On $(0,\pi/2]$, $\sin x < x$, and so $\csc x \ge 1/x$. Thus

$$\int_0^{\pi/2} \csc x\,dx > \int_0^{\pi/2} \frac{dx}{x} = \infty.$$

Therefore $\displaystyle\int_{-\pi/2}^{\pi/2} \csc x\,dx$ must diverge. (It is of the form $\infty - \infty$.)

40. Since $\ln x$ grows more slowly than any positive power of x, therefore we have $\ln x \le kx^{1/4}$ for some constant k and every $x \ge 2$. Thus, $\dfrac{1}{\sqrt{x}\ln x} \ge \dfrac{1}{kx^{3/4}}$ for $x \ge 2$ and $\displaystyle\int_2^\infty \frac{dx}{\sqrt{x}\ln x}$ diverges to infinity by comparison with $\displaystyle\frac{1}{k}\int_2^\infty \frac{dx}{x^{3/4}}$.

41. $\displaystyle\int_0^\infty \frac{dx}{xe^x} = \left(\int_0^1 + \int_1^\infty\right)\frac{dx}{xe^x}$. But

$$\int_0^1 \frac{dx}{xe^x} \ge \frac{1}{e}\int_0^1 \frac{dx}{x} = \infty.$$

Thus the given integral must diverge to infinity.

42. We are given that $\displaystyle\int_0^\infty e^{-x^2}\,dx = \tfrac{1}{2}\sqrt{\pi}$.

a) First we calculate

$$\begin{aligned}
\int_0^\infty x^2 e^{-x^2}\,dx &= \lim_{R\to\infty}\int_0^R x^2 e^{-x^2}\,dx \\
&\quad U = x \qquad dV = xe^{-x^2}\,dx \\
&\quad dU = dx \qquad V = -\tfrac{1}{2}e^{-x^2} \\
&= \lim_{R\to\infty}\left[-\frac{1}{2}xe^{-x^2}\Big|_0^R + \frac{1}{2}\int_0^R e^{-x^2}\,dx\right] \\
&= -\frac{1}{2}\lim_{R\to\infty} Re^{-R^2} + \frac{1}{2}\int_0^\infty e^{-x^2}\,dx \\
&= 0 + \frac{1}{4}\sqrt{\pi} = \frac{1}{4}\sqrt{\pi}.
\end{aligned}$$

b) Similarly,

$$\begin{aligned}
\int_0^\infty x^4 e^{-x^2}\,dx &= \lim_{R\to\infty}\int_0^R x^4 e^{-x^2}\,dx \\
&\quad U = x^3 \qquad dV = xe^{-x^2}\,dx \\
&\quad dU = 3x^2\,dx \qquad V = -\tfrac{1}{2}e^{-x^2} \\
&= \lim_{R\to\infty}\left[-\frac{1}{2}x^3e^{-x^2}\Big|_0^R + \frac{3}{2}\int_0^R x^2e^{-x^2}\,dx\right] \\
&= -\frac{1}{2}\lim_{R\to\infty} R^3e^{-R^2} + \frac{3}{2}\int_0^\infty x^2e^{-x^2}\,dx \\
&= 0 + \frac{3}{2}\left(\frac{1}{4}\sqrt{\pi}\right) = \frac{3}{8}\sqrt{\pi}.
\end{aligned}$$

43. Since f is continuous on $[a,b]$, there exists a positive constant K such that $|f(x)| \le K$ for $a \le x \le b$. If $a < c < b$, then

$$\begin{aligned}
&\left|\int_c^b f(x)\,dx - \int_a^b f(x)\,dx\right| \\
&= \left|\int_c^a f(x)\,dx\right| \le K(c-a) \to 0 \text{ as } c \to a+.
\end{aligned}$$

Thus $\displaystyle\lim_{c\to a+}\int_c^b f(x)\,dx = \int_a^b f(x)\,dx$.
Similarly

$$\begin{aligned}
&\left|\int_a^c f(x)\,dx - \int_a^b f(x)\,dx\right| \\
&= \left|\int_b^c f(x)\,dx\right| \le K(b-c) \to 0 \text{ as } c \to b-.
\end{aligned}$$

Thus $\displaystyle\lim_{c\to b-}\int_a^c f(x)\,dx = \int_a^b f(x)\,dx$.

44. $\displaystyle\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt$.

a) Since $\lim_{t\to\infty} t^{x-1}e^{-t/2} = 0$, there exists $T > 0$ such that $t^{x-1}e^{-t/2} \le 1$ if $t \ge T$. Thus

$$0 \le \int_T^\infty t^{x-1}e^{-t}\,dt \le \int_T^\infty e^{-t}\,dt = 2e^{-T/2}$$

and $\displaystyle\int_T^\infty t^{x-1}e^{-t}\,dt$ converges by the comparison theorem.

If $x > 0$, then

$$0 \le \int_0^T t^{x-1}e^{-t}\,dt < \int_0^T t^{x-1}\,dt$$

converges by Theorem 2(b). Thus the integral defining $\Gamma(x)$ converges.

b)
$$\begin{aligned}
\Gamma(x+1) &= \int_0^\infty t^x e^{-t}\,dt \\
&= \lim_{\substack{c\to 0+\\ R\to\infty}}\int_c^R t^x e^{-t}\,dt \\
&\quad U = t^x \qquad dV = e^{-t}\,dt \\
&\quad dU = xt^{x-1}\,dx \qquad V = -e^{-t} \\
&= \lim_{\substack{c\to 0+\\ R\to\infty}}\left(-t^xe^{-t}\Big|_c^R + x\int_c^R t^{x-1}e^{-t}\,dt\right) \\
&= 0 + x\int_0^\infty t^{x-1}e^{-t}\,dt = x\Gamma(x).
\end{aligned}$$

c) $\displaystyle\Gamma(1) = \int_0^\infty e^{-t}\,dt = 1 = 0!$.
By (b), $\Gamma(2) = 1\Gamma(1) = 1\times 1 = 1 = 1!$.
In general, if $\Gamma(k+1) = k!$ for some positive integer k, then
$\Gamma(k+2) = (k+1)\Gamma(k+1) = (k+1)k! = (k+1)!$.
Hence $\Gamma(n+1) = n!$ for all integers $n \ge 0$, by induction.

d)
$$\begin{aligned}
\Gamma\left(\frac{1}{2}\right) &= \int_0^\infty t^{-1/2}e^{-t}\,dt \quad \text{Let } t = x^2 \\
&\qquad\qquad\qquad\qquad\qquad dt = 2x\,dx \\
&= \int_0^\infty \frac{1}{x}e^{-x^2}2x\,dx = 2\int_0^\infty e^{-x^2}\,dx = \sqrt{\pi} \\
\Gamma\left(\frac{3}{2}\right) &= \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}.
\end{aligned}$$

Section 6.6 The Trapezoid and Midpoint Rules (page 354)

1. The exact value of I is

$$I = \int_0^2 (1+x^2)\,dx = \left(x + \frac{x^3}{3}\right)\Bigg|_0^2$$
$$= 2 + \frac{8}{3} \approx 4.6666667.$$

The approximations are

$$T_4 = \frac{1}{2}\left[\frac{1}{2} + \left(1+\frac{1}{4}\right) + (1+1) + \left(1+\frac{9}{4}\right) + \frac{5}{2}\right] = 4.75$$
$$M_4 = \frac{1}{2}\left[\left(1+\frac{1}{16}\right) + \left(1+\frac{9}{16}\right) + \left(1+\frac{25}{16}\right) + \left(1+\frac{49}{16}\right)\right]$$
$$= 4.625$$
$$T_8 = \frac{1}{2}(T_4 + M_4) = 4.6875$$
$$M_8 = \frac{1}{4}\left[\left(1+\frac{1}{64}\right) + \left(1+\frac{9}{64}\right) + \left(1+\frac{25}{64}\right)\right.$$
$$+ \left(1+\frac{49}{64}\right) + \left(1+\frac{81}{64}\right) + \left(1+\frac{121}{64}\right)$$
$$\left. + \left(1+\frac{169}{64}\right) + \left(1+\frac{225}{64}\right)\right] = 4.65625$$
$$T_{16} = \frac{1}{2}(T_8 + M_8) = 4.671875.$$

The exact errors are

$$I - T_4 = -0.0833333; \quad I - M_4 = 0.0416667;$$
$$I - T_8 = -0.0208333; \quad I - M_8 = 0.0104167;$$
$$I - T_{16} = -0.0052083.$$

If $f(x) = 1 + x^2$, then $f''(x) = 2 = K$, and $\dfrac{K(2-0)}{12} = \dfrac{1}{3}$. Therefore, the error bounds are

$$\text{Trapezoid}: |I - T_4| \le \frac{1}{3}\left(\frac{1}{2}\right)^2 \approx 0.0833333;$$
$$|I - T_8| \le \frac{1}{3}\left(\frac{1}{4}\right)^2 \approx 0.0208333;$$
$$|I - T_{16}| \le \frac{1}{3}\left(\frac{1}{8}\right)^2 \approx 0.0052083.$$
$$\text{Midpoint}: |I - M_4| \le \frac{1}{6}\left(\frac{1}{2}\right)^2 \approx 0.0416667;$$
$$|I - M_8| \le \frac{1}{6}\left(\frac{1}{4}\right)^2 \approx 0.0104167.$$

Note that the actual errors are equal to these estimates since f is a quadratic function.

2. The exact value of I is

$$I = \int_0^1 e^{-x}\,dx = -e^{-x}\Big|_0^1$$
$$= 1 - \frac{1}{e} \approx 0.6321206.$$

The approximations are

$$T_4 = \tfrac{1}{4}(\tfrac{1}{2}e^0 + e^{-1/4} + e^{-1/2} + e^{-3/4} + \tfrac{1}{2}e^{-1})$$
$$\approx 0.6354094$$
$$M_4 = \tfrac{1}{4}(e^{-1/8} + e^{-3/8} + e^{-5/8} + e^{-7/8})$$
$$\approx 0.6304774$$
$$T_8 = \tfrac{1}{2}(T_4 + M_4) \approx 0.6329434$$
$$M_8 = \tfrac{1}{8}(e^{-1/16} + e^{-3/16} + e^{-5/16} + e^{-7/16} +$$
$$e^{-9/16} + e^{-11/16} + e^{-13/16} + e^{-15/16})$$
$$\approx 0.6317092$$
$$T_{16} = \tfrac{1}{2}(T_8 + M_8) \approx 0.6323263.$$

The exact errors are

$$I - T_4 = -0.0032888; \quad I - M_4 = 0.0016432;$$
$$I - T_8 = -0.0008228; \quad I - M_8 = 0.0004114;$$
$$I - T_{16} = -0.0002057.$$

If $f(x) = e^{-x}$, then $f^{(2)}(x) = e^{-x}$. On [0,1], $|f^{(2)}(x)| \le 1$. Therefore, the error bounds are:

$$\text{Trapezoid}: |I - T_n| \le \frac{1}{12}\left(\frac{1}{n}\right)^2$$
$$|I - T_4| \le \frac{1}{12}\left(\frac{1}{16}\right) \approx 0.0052083;$$
$$|I - T_8| \le \frac{1}{12}\left(\frac{1}{64}\right) \approx 0.001302;$$
$$|I - T_{16}| \le \frac{1}{12}\left(\frac{1}{256}\right) \approx 0.0003255.$$
$$\text{Midpoint}: |I - M_n| \le \frac{1}{24}\left(\frac{1}{n}\right)^2$$
$$|I - M_4| \le \frac{1}{24}\left(\frac{1}{16}\right) \approx 0.0026041;$$
$$|I - M_8| \le \frac{1}{24}\left(\frac{1}{64}\right) \approx 0.000651.$$

Note that the actual errors satisfy these bounds.

3. The exact value of I is

$$I = \int_0^{\pi/2} \sin x\,dx = 1.$$

The approximations are

$$\begin{aligned}
T_4 &= \frac{\pi}{8}\left(0+\sin\frac{\pi}{8}+\sin\frac{\pi}{4}+\sin\frac{3\pi}{8}+\frac{1}{2}\right)\approx 0.9871158\\
M_4 &= \frac{\pi}{8}\left(\sin\frac{\pi}{16}+\sin\frac{3\pi}{16}+\sin\frac{5\pi}{16}+\sin\frac{7\pi}{16}\right)\approx 1.0064545\\
T_8 &= \frac{1}{2}(T_4+M_4)\approx 0.9967852\\
M_8 &= \frac{\pi}{16}\bigg(\sin\frac{\pi}{32}+\sin\frac{3\pi}{32}+\sin\frac{5\pi}{32}\\
&\quad +\sin\frac{7\pi}{32}+\sin\frac{9\pi}{32}+\sin\frac{11\pi}{32}\\
&\quad +\sin\frac{13\pi}{32}+\sin\frac{15\pi}{32}\bigg)\approx 1.0016082\\
T_{16} &= \frac{1}{2}(T_8+M_8)\approx 0.9991967.
\end{aligned}$$

The actual errors are

$$\begin{aligned}
I-T_4 &\approx 0.0128842; \quad I-M_4\approx -0.0064545;\\
I-T_8 &\approx 0.0032148; \quad I-M_8\approx -0.0016082;\\
I-T_{16} &\approx 0.0008033.
\end{aligned}$$

If $f(x)=\sin x$, then $f''(x)=-\sin x$, and $|f''(x)|\le 1=K$. Therefore, the error bounds are:

$$\begin{aligned}
\text{Trapezoid}: |I-T_4| &\le \frac{1}{12}\left(\frac{\pi}{2}-0\right)\left(\frac{\pi}{8}\right)^2\approx 0.020186;\\
|I-T_8| &\le \frac{1}{12}\left(\frac{\pi}{2}-0\right)\left(\frac{\pi}{16}\right)^2\approx 0.005047;\\
|I-T_{16}| &\le \frac{1}{12}\left(\frac{\pi}{2}-0\right)\left(\frac{\pi}{32}\right)^2\approx 0.001262.\\
\text{Midpoint}: |I-M_4| &\le \frac{1}{24}\left(\frac{\pi}{2}-0\right)\left(\frac{\pi}{8}\right)^2\approx 0.010093;\\
|I-M_8| &\le \frac{1}{24}\left(\frac{\pi}{2}-0\right)\left(\frac{\pi}{16}\right)^2\approx 0.002523.
\end{aligned}$$

Note that the actual errors satisfy these bounds.

4. The exact value of I is

$$I=\int_0^1\frac{dx}{1+x^2}=\tan^{-1}x\bigg|_0^1=\frac{\pi}{4}\approx 0.7853982.$$

The approximations are

$$\begin{aligned}
T_4 &= \frac{1}{4}\left[\frac{1}{2}(1)+\frac{16}{17}+\frac{4}{5}+\frac{16}{25}+\frac{1}{2}\left(\frac{1}{2}\right)\right]\\
&\approx 0.7827941\\
M_4 &= \frac{1}{4}\left[\frac{64}{65}+\frac{64}{73}+\frac{64}{89}+\frac{64}{113}\right]\\
&\approx 0.7867001\\
T_8 &= \frac{1}{2}(T_4+M_4)\approx 0.7847471\\
M_8 &= \frac{1}{8}\bigg[\frac{256}{257}+\frac{256}{265}+\frac{256}{281}+\frac{256}{305}+\\
&\quad \frac{256}{337}+\frac{256}{377}+\frac{256}{425}+\frac{256}{481}\bigg]\\
&\approx 0.7857237\\
T_{16} &= \frac{1}{2}(T_8+M_8)\approx 0.7852354.
\end{aligned}$$

The exact errors are

$$\begin{aligned}
I-T_4 &= 0.0026041; \; I-M_4=-0.0013019;\\
I-T_8 &= 0.0006511; \; I-M_8=-0.0003255;\\
I-T_{16} &= 0.0001628.
\end{aligned}$$

Since $f(x)=\dfrac{1}{1+x^2}$, then $f'(x)=\dfrac{-2x}{(1+x^2)^2}$ and $f''(x)=\dfrac{6x^2-2}{(1+x^2)^3}$. On [0,1], $|f''(x)|\le 4$. Therefore, the error bounds are

$$\begin{aligned}
\text{Trapezoid}: |I-T_n| &\le \frac{4}{12}\left(\frac{1}{n}\right)^2\\
|I-T_4| &\le \frac{4}{12}\left(\frac{1}{16}\right)\approx 0.0208333;\\
|I-T_8| &\le \frac{4}{12}\left(\frac{1}{64}\right)\approx 0.0052083;\\
|I-T_{16}| &\le \frac{4}{12}\left(\frac{1}{256}\right)\approx 0.001302.\\
\text{Midpoint}: |I-M_n| &\le \frac{4}{24}\left(\frac{1}{n}\right)^2\\
|I-M_4| &\le \frac{4}{24}\left(\frac{1}{16}\right)\approx 0.0104167;\\
|I-M_8| &\le \frac{4}{24}\left(\frac{1}{64}\right)\approx 0.0026042.
\end{aligned}$$

The exact errors are much smaller than these bounds. In part, this is due to very crude estimates made for $|f''(x)|$.

5. $T_4=\dfrac{2}{2}[3+2(5+8+7)+3]=46$

$T_8=\dfrac{1}{2}[3+2(3.8+5+6.7+8+8+7+5.2)+3]=46.7$

6. $M_4 = 2(3.8 + 6.7 + 8 + 5.2) = 47.4$

7. $$T_4 = 100 \times \frac{2}{2}[0 + 2(5.5 + 5 + 4.5) + 0] = 3,000 \text{ km}^2$$
$$T_8 = 100 \times \frac{1}{2}[0 + 2(4 + 5.5 + 5.5 + 5 + 5.5 + 4.5 + 4) + 0] = 3,400 \text{ km}^2$$

8. $M_4 = 100 \times 2(4 + 5.5 + 5.5 + 4) = 3,800 \text{ km}^2$

9. We have

$$\begin{aligned}
T_4 &= 0.4\Big(\tfrac{1}{2}(1.4142) + 1.3860 + 1.3026 + 1.1772 \\
&\qquad + \tfrac{1}{2}(0.9853)\Big) \approx 2.02622 \\
M_4 &= (0.4)(1.4071 + 1.3510 + 1.2411 + 1.0817) \approx 2.03236 \\
T_8 &= (T_4 + M_4)/2 \approx 2.02929 \\
M_8 &= (0.2)(1.4124 + 1.3983 + 1.3702 + 1.3285 \\
&\qquad + 1.2734 + 1.2057 + 1.1258 + 1.0348) \approx 2.02982 \\
T_{16} &= (T_8 + M_8)/2 \approx 2.029555.
\end{aligned}$$

10. The approximations for $I = \int_0^1 e^{-x^2}\,dx$ are

$$\begin{aligned}
M_8 &= \frac{1}{8}\Big(e^{-1/256} + e^{-9/256} + e^{-25/256} + e^{-49/256} + \\
&\qquad e^{-81/256} + e^{-121/256} + e^{-169/256} + e^{-225/256}\Big) \\
&\approx 0.7473 \\
T_{16} &= \frac{1}{16}\Big[\frac{1}{2}(1) + e^{-1/256} + e^{-1/64} + e^{-9/256} + e^{-1/16} + \\
&\qquad e^{-25/256} + e^{-9/64} + e^{-49/256} + e^{-1/4} + e^{-81/256} + \\
&\qquad e^{-25/64} + e^{-121/256} + e^{-9/16} + e^{-169/256} + e^{-49/64} + \\
&\qquad e^{-225/256} + \frac{1}{2}e^{-1}\Big] \\
&\approx 0.74658.
\end{aligned}$$

Since $f(x) = e^{-x^2}$, we have $f'(x) = -2xe^{-x^2}$, $f''(x) = 2(2x^2 - 1)e^{-x^2}$, and $f'''(x) = 4x(3 - 2x^2)e^{-x^2}$. Since $f'''(x) \neq 0$ on (0,1), therefore the maximum value of $|f''(x)|$ on [0, 1] must occur at an endpoint of that interval. We have $f''(0) = -2$ and $f''(1) = 2/e$, so $|f''(x)| \le 2$ on [0, 1]. The error bounds are

$$|I - M_n| \le \frac{2}{24}\left(\frac{1}{n}\right)^2 \Rightarrow |I - M_8| \le \frac{2}{24}\left(\frac{1}{64}\right) \approx 0.00130.$$
$$|I - T_n| \le \frac{2}{12}\left(\frac{1}{n}\right)^2 \Rightarrow |I - T_{16}| \le \frac{2}{12}\left(\frac{1}{256}\right) \approx 0.000651.$$

According to the error bounds,

$$\int_0^1 e^{-x^2}\,dx = 0.747,$$

accurate to two decimal places, with error no greater than 1 in the third decimal place.

11. $I = \int_0^{\pi/2} \frac{\sin x}{x}\,dx$. Note that $\lim_{x\to 0} \frac{\sin x}{x} = 1$.

$$\begin{aligned}
T_8 &= \frac{\pi}{16}\Big[\frac{1}{2} + \frac{16}{\pi}\sin\frac{\pi}{16} + \frac{8}{\pi}\sin\frac{\pi}{8} + \frac{16}{3\pi}\sin\frac{3\pi}{16} + \frac{4}{\pi}\sin\frac{\pi}{4} \\
&\qquad + \frac{16}{5\pi}\sin\frac{5\pi}{16} + \frac{8}{3\pi}\sin\frac{3\pi}{8} + \frac{16}{7\pi}\sin\frac{7\pi}{16} + \frac{1}{2}\left(\frac{2}{\pi}\right)\Big] \\
&\approx 1.3694596 \\
M_8 &= \frac{\pi}{16}\Big[\frac{32}{\pi}\sin\frac{\pi}{32} + \frac{32}{3\pi}\sin\frac{3\pi}{32} + \frac{32}{5\pi}\sin\frac{5\pi}{32} + \frac{32}{7\pi}\sin\frac{7\pi}{32} \\
&\qquad + \frac{32}{9\pi}\sin\frac{9\pi}{32} + \frac{32}{11\pi}\sin\frac{11\pi}{32} + \frac{32}{13\pi}\sin\frac{13\pi}{32} \\
&\qquad + \frac{32}{15\pi}\sin\frac{15\pi}{32}\Big] \approx 1.3714136 \\
T_{16} &= (T_8 + M_8)/2 \approx 1.3704366, \qquad I \approx 1.370.
\end{aligned}$$

12. The exact value of I is

$$I = \int_0^1 x^2\,dx = \frac{x^3}{3}\bigg|_0^1 = \frac{1}{3}.$$

The approximation is

$$T_1 = (1)\left[\frac{1}{2}(0)^2 + \frac{1}{2}(1)^2\right] = \frac{1}{2}.$$

The actual error is $I - T_1 = -\frac{1}{6}$. However, since $f(x) = x^2$, then $f''(x) = 2$ on [0,1], so the error estimate here gives

$$|I - T_1| \le \frac{2}{12}(1)^2 = \frac{1}{6}.$$

Since this is the actual size of the error in this case, the constant "12" in the error estimate cannot be improved (i.e., cannot be made larger).

13. $I = \int_0^1 x^2\,dx = \frac{1}{3}$. $M_1 = \left(\frac{1}{2}\right)^2(1) = \frac{1}{4}$. The actual error is $I - M_1 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$.
Since the second derivative of x^2 is 2, the error estimate is

$$|I - M_1| \le \frac{2}{24}(1 - 0)^2(1^2) = \frac{1}{12}.$$

Thus the constant in the error estimate for the Midpoint Rule cannot be improved; no smaller constant will work for $f(x) = x^2$.

14. Let $y = f(x)$. We are given that m_1 is the midpoint of $[x_0, x_1]$ where $x_1 - x_0 = h$. By tangent line approximate in the subinterval $[x_0, x_1]$,

$$f(x) \approx f(m_1) + f'(m_1)(x - m_1).$$

The error in this approximation is

$$E(x) = f(x) - f(m_1) - f'(m_1)(x - m_1).$$

If $f''(t)$ exists for all t in $[x_0, x_1]$ and $|f''(t)| \le K$ for some constant K, then by Theorem 4 of Section 3.5,

$$|E(x)| \le \frac{K}{2}(x - m_1)^2.$$

Hence,

$$|f(x) - f(m_1) - f'(m_1)(x - m_1)| \le \frac{K}{2}(x - m_1)^2.$$

We integrate both sides of this inequlity. Noting that $x_1 - m_1 = m_1 - x_0 = \frac{1}{2}h$, we obtain for the left side

$$\begin{aligned}&\left|\int_{x_0}^{x_1} f(x)\,dx - \int_{x_0}^{x_1} f(m_1)\,dx\right.\\ &\qquad \left.- \int_{x_0}^{x_1} f'(m_1)(x - m_1)\,dx\right|\\ &= \left|\int_{x_0}^{x_1} f(x)\,dx - f(m_1)h - f'(m_1)\frac{(x - m_1)^2}{2}\bigg|_{x_0}^{x_1}\right|\\ &= \left|\int_{x_0}^{x_1} f(x)\,dx - f(m_1)h\right|.\end{aligned}$$

Integrating the right-hand side, we get

$$\begin{aligned}\int_{x_0}^{x_1} \frac{K}{2}(x - m_1)^2\,dx &= \frac{K}{2}\frac{(x - m_1)^3}{3}\bigg|_{x_0}^{x_1}\\ &= \frac{K}{6}\left(\frac{h^3}{8} + \frac{h^3}{8}\right) = \frac{K}{24}h^3.\end{aligned}$$

Hence,

$$\begin{aligned}&\left|\int_{x_0}^{x_1} f(x)\,dx - f(m_1)h\right|\\ &= \left|\int_{x_0}^{x_1} [f(x) - f(m_1) - f'(m_1)(x - m_1)]\,dx\right|\\ &\le \frac{K}{24}h^3.\end{aligned}$$

A similar estimate holds on each subinterval $[x_{j-1}, x_j]$ for $1 \le j \le n$. Therefore,

$$\begin{aligned}\left|\int_a^b f(x)\,dx - M_n\right| &= \left|\sum_{j=1}^n\left(\int_{x_{j-1}}^{x_j} f(x)\,dx - f(m_j)h\right)\right|\\ &\le \sum_{j=1}^n\left|\int_{x_{j-1}}^{x_j} f(x)\,dx - f(m_j)h\right|\\ &\le \sum_{j=1}^n \frac{K}{24}h^3 = \frac{K}{24}nh^3 = \frac{K(b-a)}{24}h^2\end{aligned}$$

because $nh = b - a$.

Section 6.7 Simpson's Rule (page 359)

1. $I = \int_0^2 (1 + x^2)\,dx = \dfrac{14}{3} \approx 4.6666667$

$$\begin{aligned}S_4 &= \frac{1}{6}\left[1 + 4\left(1 + \frac{1}{4}\right) + 2(1+1) + 4\left(1 + \frac{9}{4}\right)\right.\\ &\qquad \left. + (1+4)\right] = \frac{14}{3}\\ S_8 &= \frac{1}{12}\left[1 + 4\left(1 + \frac{1}{16}\right) + 2\left(1 + \frac{1}{4}\right) + 4\left(1 + \frac{9}{16}\right)\right.\\ &\quad + 2\,(1+1) + 4\left(1 + \frac{25}{16}\right) + 2\left(1 + \frac{9}{4}\right)\\ &\quad \left. + 4\left(1 + \frac{49}{16}\right) + (1+4)\right] = \frac{14}{3}\end{aligned}$$

The errors are zero because Simpson approximations are exact for polynomials of degree up to three.

2. The exact value of I is

$$\begin{aligned}I &= \int_0^1 e^{-x}\,dx = -e^{-x}\bigg|_0^1\\ &= 1 - \frac{1}{e} \approx 0.6321206.\end{aligned}$$

The approximations are

$$\begin{aligned}S_4 &= \frac{1}{12}(e^0 + 4e^{-1/4} + 2e^{-1/2} + 4e^{-3/4} + e^{-1})\\ &\approx 0.6321342\\ S_8 &= \frac{1}{24}(e^0 + 4e^{-1/8} + 2e^{-1/4} + 4e^{-3/8} +\\ &\quad 2e^{-1/2} + 4e^{-5/8} + 2e^{-3/4} + 4e^{-7/8} + e^{-1})\\ &\approx 0.6321214.\end{aligned}$$

The actual errors are

$$I - S_4 = -0.0000136; \quad I - S_8 = -0.0000008.$$

These errors are evidently much smaller than the corresponding errors for the corresponding Trapezoid Rule approximations.

3. $I = \int_0^{\pi/2} \sin x\, dx = 1.$

$$\begin{aligned} S_4 &= \frac{\pi}{24}\left(0 + 4\sin\frac{\pi}{8} + 2\sin\frac{\pi}{4} + 4\sin\frac{3\pi}{8} + \sin\frac{\pi}{2}\right) \\ &\approx 1.0001346 \\ S_8 &= \frac{\pi}{48}\bigg(0 + 4\sin\frac{\pi}{16} + 2\sin\frac{\pi}{8} + 4\sin\frac{3\pi}{16} + 2\sin\frac{\pi}{4} \\ &\quad + 4\sin\frac{5\pi}{16} + 2\sin\frac{3\pi}{8} + 4\sin\frac{7\pi}{16} + \sin\frac{\pi}{2}\bigg) \\ &\approx 1.0000083. \end{aligned}$$

Errors: $I - S_4 \approx -0.0001346$; $I - S_8 \approx -0.0000083$.

4. The exact value of I is

$$I = \int_0^1 \frac{dx}{1+x^2} = \tan^{-1} x\bigg|_0^1 = \frac{\pi}{4} \approx 0.7853982.$$

The approximations are

$$\begin{aligned} S_4 &= \frac{1}{12}\left[1 + 4\left(\frac{16}{17}\right) + 2\left(\frac{4}{5}\right) + 4\left(\frac{16}{25}\right) + \frac{1}{2}\right] \\ &\approx 0.7853922 \\ S_8 &= \frac{1}{24}\bigg[1 + 4\left(\frac{64}{65}\right) + 2\left(\frac{16}{17}\right) + 4\left(\frac{64}{73}\right) + \\ &\quad 2\left(\frac{4}{5}\right) + 4\left(\frac{64}{89}\right) + 2\left(\frac{16}{25}\right) + 4\left(\frac{64}{113}\right) + \frac{1}{2}\bigg] \\ &\approx 0.7853981. \end{aligned}$$

The actual errors are

$$I - S_4 = 0.0000060; \quad I - S_8 = 0.0000001,$$

accurate to 7 decimal places. These errors are evidently much smaller than the corresponding errors for the corresponding Trapezoid Rule approximation.

5. $$\begin{aligned} S_8 &= \frac{1}{3}[3 + 4(3.8 + 6.7 + 8 + 5.2) + 2(5 + 8 + 7) + 3] \\ &\approx 46.93 \end{aligned}$$

6. $$\begin{aligned} S_8 &= 100 \times \frac{1}{3}[0 + 4(4 + 5.5 + 5.5 + 4) + 2(5.5 + 5 + 4.5) + 0] \\ &\approx 3,533 \text{ km}^2 \end{aligned}$$

7. If $f(x) = e^{-x}$, then $f^{(4)}(x) = e^{-x}$, and $|f^{(4)}(x)| \le 1$ on $[0, 1]$. Thus

$$\begin{aligned} |I - S_4| &\le \frac{1(1-0)}{180}\left(\frac{1}{4}\right)^4 \approx 0.000022 \\ |I - S_8| &\le \frac{1(1-0)}{180}\left(\frac{1}{8}\right)^4 \approx 0.0000014. \end{aligned}$$

If $f(x) = \sin x$, then $f^{(4)}(x) = \sin x$, and $|f^{(4)}(x)| \le 1$ on $[0, \pi/2]$. Thus

$$\begin{aligned} |I - S_4| &\le \frac{1((\pi/2) - 0)}{180}\left(\frac{\pi}{8}\right)^4 \approx 0.00021 \\ |I - S_8| &\le \frac{1((\pi/2) - 0)}{180}\left(\frac{\pi}{16}\right)^4 \approx 0.000013. \end{aligned}$$

8. Let $I = \int_a^b f(x)\, dx$, and the interval $[a, b]$ be subdivided into $2n$ subintervals of equal length $h = (b-a)/2n$. Let $y_j = f(x_j)$ and $x_j = a + jh$ for $0 \le j \le 2n$, then

$$\begin{aligned} S_{2n} &= \frac{1}{3}\left(\frac{b-a}{2n}\right)\Big[y_0 + 4y_1 + 2y_2 + \cdots \\ &\qquad + 2y_{2n-2} + 4y_{2n-1} + y_{2n}\Big] \\ &= \frac{1}{3}\left(\frac{b-a}{2n}\right)\left[y_0 + 4\sum_{j=1}^{2n-1} y_j - 2\sum_{j=1}^{n-1} y_{2j} + y_{2n}\right] \end{aligned}$$

and

$$\begin{aligned} T_{2n} &= \frac{1}{2}\left(\frac{b-a}{2n}\right)\left(y_0 + 2\sum_{j=1}^{2n-1} y_j + y_{2n}\right) \\ T_n &= \frac{1}{2}\left(\frac{b-a}{n}\right)\left(y_0 + 2\sum_{j=1}^{n-1} y_{2j} + y_{2n}\right). \end{aligned}$$

Since $T_{2n} = \frac{1}{2}(T_n + M_n) \Rightarrow M_n = 2T_{2n} - T_n$, then

$$\begin{aligned} \frac{T_n + 2M_n}{3} &= \frac{T_n + 2(2T_{2n} - T_n)}{3} = \frac{4T_{2n} - T_n}{3} \\ \frac{2T_{2n} + M_n}{3} &= \frac{2T_{2n} + 2T_{2n} - T_n}{3} = \frac{4T_{2n} - T_n}{3}. \end{aligned}$$

Hence,

$$\frac{T_n + 2M_n}{3} = \frac{2T_{2n} + M_n}{3} = \frac{4T_{2n} - T_n}{3}.$$

Using the formulas of T_{2n} and T_n obtained above,

$$\begin{aligned} &\frac{4T_{2n} - T_n}{3} \\ &= \frac{1}{3}\bigg[\frac{4}{2}\left(\frac{b-a}{2n}\right)\left(y_0 + 2\sum_{j=1}^{2n-1} y_j + y_{2n}\right) \\ &\qquad - \frac{1}{2}\left(\frac{b-a}{n}\right)\left(y_0 + 2\sum_{j=1}^{n-1} y_{2j} + y_{2n}\right)\bigg] \\ &= \frac{1}{3}\left(\frac{b-a}{2n}\right)\left[y_0 + 4\sum_{j=1}^{2n-1} y_j - 2\sum_{j=1}^{n-1} y_{2j} + y_{2n}\right] \\ &= S_{2n}. \end{aligned}$$

Hence,

$$S_{2n} = \frac{4T_{2n} - T_n}{3} = \frac{T_n + 2M_n}{3} = \frac{2T_{2n} + M_n}{3}.$$

9. We use the results of Exercise 9 of Section 7.6 and Exercise 8 of this section.

$$\begin{aligned} I &= \int_0^{1.6} f(x)\,dx \\ S_4 &= \frac{0.4}{3}(1.4142 + 4(1.3860) + 2(1.3026) + 4(1.1772) \\ &\quad + 0.9853) \approx 2.0343333 \\ S_8 &= (T_4 + 2M_4)/3 \approx 2.0303133 \\ S_{16} &= (T_8 + 2M_8)/3 \approx 2.0296433. \end{aligned}$$

10. The approximations for $I = \int_0^1 e^{-x^2}\,dx$ are

$$\begin{aligned} S_8 &= \frac{1}{3}\left(\frac{1}{8}\right)\Big[1 + 4\Big(e^{-1/64} + e^{-9/64} + e^{-25/64} + \\ &\quad e^{-49/64}\Big) + 2\Big(e^{-1/16} + e^{-1/4} + e^{-9/16}\Big) + e^{-1}\Big] \\ &\approx 0.7468261 \\ S_{16} &= \frac{1}{3}\left(\frac{1}{16}\right)\Big[1 + 4\Big(e^{-1/256} + e^{-9/256} + e^{-25/256} + \\ &\quad e^{-49/256} + e^{-81/256} + e^{-121/256} + e^{-169/256} + \\ &\quad e^{-225/256}\Big) + 2\Big(e^{-1/64} + e^{-1/16} + e^{-9/64} + e^{-1/4} + \\ &\quad e^{-25/64} + e^{-9/16} + e^{-49/64}\Big) + e^{-1}\Big] \\ &\approx 0.7468243. \end{aligned}$$

If $f(x) = e^{-x^2}$, then $f^{(4)}(x) = 4e^{-x^2}(4x^4 - 12x^2 + 3)$. On [0,1], $|f^{(4)}(x)| \le 12$, and the error bounds are

$$\begin{aligned} |I - S_n| &\le \frac{12(1)}{180}\left(\frac{1}{n}\right)^4 \\ |I - S_8| &\le \frac{12}{180}\left(\frac{1}{8}\right)^4 \approx 0.0000163 \\ |I - S_{16}| &\le \frac{12}{180}\left(\frac{1}{16}\right)^4 \approx 0.0000010. \end{aligned}$$

Comparing the two approximations,

$$I = \int_0^1 e^{-x^2}\,dx = 0.7468,$$

accurate to 4 decimal places.

11. $I = \int_0^1 x^4\,dx = \frac{1}{5}$. $S_2 = \frac{1}{6}\left[0^4 + 4\left(\frac{1}{2}\right)^4 + 1^4\right] = \frac{5}{24}$.
If $f(x) = x^4$, then $f^{(4)}(x) = 24$.
Error estimate: $|I - S_2| \le \frac{24(1-0)}{180}\left(\frac{1}{2}\right)^4 = \frac{1}{120}$.
Actual error: $|I - S_2| = \left|\frac{1}{5} - \frac{5}{24}\right| = \frac{1}{120}$.
Thus the error estimate cannot be improved.

12. The exact value of I is

$$I = \int_0^1 x^3\,dx = \frac{x^4}{4}\bigg|_0^1 = \frac{1}{4}.$$

The approximation is

$$S_2 = \frac{1}{3}\left(\frac{1}{2}\right)\left[0^3 + 4\left(\frac{1}{2}\right)^3 + 1^3\right] = \frac{1}{4}.$$

The actual error is zero. Hence, Simpson's Rule is exact for the cubic function $f(x) = x^3$. Since it is evidently exact for quadratic functions $f(x) = Bx^2 + Cx + D$, it must also be exact for arbitrary cubics
$f(x) = Ax^3 + Bx^2 + Cx + D$.

Section 6.8 Other Aspects of Approximate Integration (page 364)

1. $$\begin{aligned} &\int_0^1 \frac{dx}{x^{1/3}(1+x)} \qquad \text{Let } x = u^3 \\ &= 3\int_0^1 \frac{u^2\,du}{u(1+u^3)} = 3\int_0^1 \frac{u\,du}{1+u^3}. \end{aligned}$$

2. $$\begin{aligned} &\int_0^1 \frac{e^x}{\sqrt{1-x}}\,dx \quad \text{Let } t^2 = 1 - x \\ &\qquad\qquad\qquad\quad 2t\,dt = -dx \\ &= -\int_1^0 \frac{e^{1-t^2}}{t}2t\,dt = 2\int_0^1 e^{1-t^2}\,dt. \end{aligned}$$

3. One possibility: let $x = \sin\theta$ and get

$$I = \int_{-1}^1 \frac{e^x\,dx}{\sqrt{1-x^2}} = \int_{-\pi/2}^{\pi/2} e^{\sin\theta}\,d\theta.$$

Another possibility:

$$I = \int_{-1}^0 \frac{e^x\,dx}{\sqrt{1-x^2}} + \int_0^1 \frac{e^x\,dx}{\sqrt{1-x^2}} = I_1 + I_2.$$

In I_1 put $1 + x = u^2$; in I_2 put $1 - x = u^2$:

$$\begin{aligned} I_1 &= \int_0^1 \frac{2e^{u^2-1}u\,du}{u\sqrt{2-u^2}} = 2\int_0^1 \frac{e^{u^2-1}\,du}{\sqrt{2-u^2}} \\ I_2 &= \int_0^1 \frac{2e^{1-u^2}u\,du}{u\sqrt{2-u^2}} = 2\int_0^1 \frac{e^{1-u^2}\,du}{\sqrt{2-u^2}} \end{aligned}$$

so $I = 2\int_0^1 \frac{e^{u^2-1} + e^{1-u^2}}{\sqrt{2-u^2}}\,du.$

4. $$\int_1^\infty \frac{dx}{x^2+\sqrt{x}+1} \quad \text{Let } x = \frac{1}{t^2}, \quad dx = -\frac{2\,dt}{t^3}$$
$$= \int_1^0 \frac{1}{\left(\frac{1}{t^2}\right)^2 + \sqrt{\frac{1}{t^2}} + 1}\left(-\frac{2\,dt}{t^3}\right)$$
$$= 2\int_0^1 \frac{t\,dt}{t^4+t^3+1}.$$

5. $$\int_0^{\pi/2} \frac{dx}{\sqrt{\sin x}} \qquad \text{Let } \sin x = u^2, \quad 2u\,du = \cos x\,dx = \sqrt{1-u^4}\,dx$$
$$= 2\int_0^1 \frac{u\,du}{u\sqrt{1-u^4}}$$
$$= 2\int_0^1 \frac{du}{\sqrt{(1-u)(1+u)(1+y^2}} \qquad \text{Let } 1-u = v^2, \quad -du = 2v\,dv$$
$$= 4\int_0^1 \frac{v\,dv}{v\sqrt{(1+1-v^2)(1+(1-v^2)^2)}}$$
$$= 4\int_0^1 \frac{dv}{\sqrt{(2-v^2)(2-2v^2+v^4)}}.$$

6. Let

$$\int_0^\infty \frac{dx}{x^4+1} = \int_0^1 \frac{dx}{x^4+1} + \int_1^\infty \frac{dx}{x^4+1} = I_1 + I_2.$$

Let $x = \frac{1}{t}$ and $dx = -\frac{dt}{t^2}$ in I_2, then

$$I_2 = \int_1^0 \frac{1}{\left(\frac{1}{t}\right)^4 + 1}\left(-\frac{dt}{t^2}\right) = \int_0^1 \frac{t^2}{1+t^4}\,dt.$$

Hence,

$$\int_0^\infty \frac{dx}{x^4+1} = \int_0^1 \left(\frac{1}{x^4+1} + \frac{x^2}{1+x^4}\right)dx$$
$$= \int_0^1 \frac{x^2+1}{x^4+1}\,dx.$$

7. $I = \int_0^1 \sqrt{x}\,dx = \frac{2}{3} \approx 0.666667.$

$$T_2 = \frac{1}{2}\left(0 + \sqrt{\frac{1}{2}} + \frac{1}{2}\right) \approx 0.603553$$
$$T_4 = \frac{1}{4}\left(2T_2 + \sqrt{\frac{1}{4}} + \sqrt{\frac{3}{4}}\right) \approx 0.643283$$
$$T_8 = \frac{1}{8}\left(4T_4 + \sqrt{\frac{1}{8}} + \sqrt{\frac{3}{8}} + \sqrt{\frac{5}{8}} + \sqrt{\frac{7}{8}}\right) \approx 0.658130$$
$$T_{16} = \frac{1}{16}\left(8T_8 + \sqrt{\frac{1}{16}} + \sqrt{\frac{3}{16}} + \sqrt{\frac{5}{16}} + \sqrt{\frac{7}{16}} + \sqrt{\frac{9}{16}} + \sqrt{\frac{11}{16}} + \sqrt{\frac{13}{16}} + \sqrt{\frac{15}{16}}\right) \approx 0.663581.$$

The errors are

$$I - T_2 \approx 0.0631$$
$$I - T_4 \approx 0.0234$$
$$I - T_8 \approx 0.0085$$
$$I - T_{16} \approx 0.0031.$$

Observe that, although these errors are decreasing, they are not decreasing like $1/n^2$; that is,

$$|I - T_{2n}| >> \frac{1}{4}|I - T_n|.$$

This is because the second derivative of $f(x) = \sqrt{x}$ is $f''(x) = -1/(4x^{3/2})$, which is *not bounded* on [0, 1].

8. Let

$$I = \int_1^\infty e^{-x^2}\,dx \quad \text{Let } x = \frac{1}{t}, \quad dx = -\frac{dt}{t^2}$$
$$= \int_1^0 e^{-(1/t)^2}\left(-\frac{1}{t^2}\right)dt = \int_0^1 \frac{e^{-1/t^2}}{t^2}\,dt.$$

Observe that

$$\lim_{t\to0+} \frac{e^{-1/t^2}}{t^2} = \lim_{t\to0+} \frac{t^{-2}}{e^{1/t^2}} \quad \left[\frac{\infty}{\infty}\right]$$
$$= \lim_{t\to0+} \frac{-2t^{-3}}{e^{1/t^2}(-2t^{-3})}$$
$$= \lim_{t\to0+} \frac{1}{e^{1/t^2}} = 0.$$

Hence,

$$\begin{aligned}
S_2 &= \frac{1}{3}\left(\frac{1}{2}\right)\left[0 + 4(4e^{-4}) + e^{-1}\right]\\
&\approx 0.1101549\\
S_4 &= \frac{1}{3}\left(\frac{1}{4}\right)\left[0 + 4(16e^{-16}) + 2(4e^{-4})\right.\\
&\qquad \left. + 4\left(\frac{16}{9}e^{-16/9}\right) + e^{-1}\right]\\
&\approx 0.1430237\\
S_8 &= \frac{1}{3}\left(\frac{1}{8}\right)\left[0 + 4\left(64e^{-64} + \frac{64}{9}e^{-64/9} + \frac{64}{25}e^{-64/25} + \right.\right.\\
&\qquad \left.\left.\frac{64}{49}e^{-64/49}\right) + 2\left(16e^{-16} + 4e^{-4} + \frac{16}{9}e^{-16/9}\right) + e^{-1}\right]\\
&\approx 0.1393877.
\end{aligned}$$

Hence, $I \approx 0.14$, accurate to 2 decimal places. These approximations do not converge very quickly, because the fourth derivative of e^{-1/t^2} has very large values for some values of t near 0. In fact, higher and higher derivatives behave more and more badly near 0, so higher order methods cannot be expected to work well either.

9. Referring to Example 5, we have

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + R_n(f; 0, x),$$

where $R_n(f; 0, x) = \dfrac{e^X x^{n+1}}{(n+1)!}$, for some X between 0 and x. Now

$$|R_n(f; 0, -x^2)| \le \frac{x^{2n+2}}{(n+1)!}$$

if $0 \le x \le 1$ for any x, since $-x^2 \le X \le 0$. Therefore

$$\begin{aligned}
\left|\int_0^1 R_n(f; 0, -x^2)\,dx\right| &\le \frac{1}{(n+1)!}\int_0^1 x^{2n+2}\,dx\\
&= \frac{1}{(2n+3)(n+1)!}.
\end{aligned}$$

This error will be less than 10^{-4} if $(2n+3)(n+1)! > 10{,}000$. Since $15 \times 7! > 10{,}000$, $n = 6$ will do. Thus we use seven terms of the series $(0 \le n \le 6)$:

$$\begin{aligned}
&\int_0^1 e^{-x^2}\,dx\\
&\approx \int_0^1 \left(1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \frac{x^{10}}{5!} + \frac{x^{12}}{6!}\right) dx\\
&= 1 - \frac{1}{3} + \frac{1}{5\times 2!} - \frac{1}{7\times 3!} + \frac{1}{9\times 4!} - \frac{1}{11\times 5!} + \frac{1}{13\times 6!}\\
&\approx 0.74684 \quad \text{with error less than } 10^{-4}.
\end{aligned}$$

10. We are given that $\int_0^\infty e^{-x^2}\,dx = \frac{1}{2}\sqrt{\pi}$ and from the previous exercise $\int_0^1 e^{-x^2}\,dx = 0.74684$. Therefore,

$$\begin{aligned}
\int_1^\infty e^{-x^2}\,dx &= \int_0^\infty e^{-x^2}\,dx - \int_0^1 e^{-x^2}\,dx\\
&= \frac{1}{2}\sqrt{\pi} - 0.74684\\
&= 0.139 \qquad \text{(to 3 decimal places).}
\end{aligned}$$

11. If $f(x) = ax^3 + bx^2 + cx + d$, then, by symmetry,

$$\begin{aligned}
&\int_{-1}^1 f(x)\,dx = 2\int_0^1 (bx^2 + d)\,dx = 2\left(\frac{b}{3} + d\right)\\
&Af(-u) + Af(u) = 2A(bu^2 + d).
\end{aligned}$$

These two expressions are identical provided $A = 1$ and $u^2 = 1/3$, so $u = 1/\sqrt{3}$.

12. For any function f we use the approximation

$$\int_{-1}^1 f(x)\,dx \approx f(-1/\sqrt{3}) + f(1/\sqrt{3}).$$

We have

$$\begin{aligned}
&\int_{-1}^1 x^4\,dx \approx \left(-\frac{1}{\sqrt{3}}\right)^4 + \left(\frac{1}{\sqrt{3}}\right)^4 = \frac{2}{9}\\
&\quad \text{Error } = \int_{-1}^1 x^4\,dx - \frac{2}{9} = \frac{2}{5} - \frac{2}{9} \approx 0.17778\\
&\int_{-1}^1 \cos x\,dx \approx \cos\left(-\frac{1}{\sqrt{3}}\right) + \cos\left(\frac{1}{\sqrt{3}}\right) \approx 1.67582\\
&\quad \text{Error } = \int_{-1}^1 \cos x\,dx - 1.67582 \approx 0.00712\\
&\int_{-1}^1 e^x\,dx \approx e^{-1/\sqrt{3}} + e^{1/\sqrt{3}} \approx 2.34270\\
&\quad \text{Error } = \int_{-1}^1 e^x\,dx - 2.34270 \approx 0.00771.
\end{aligned}$$

13. If $F(x) = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$, then, by symmetry,

$$\begin{aligned}
&\int_{-1}^1 F(x)\,dx = 2\int_0^1 (bx^4 + dx^2 + f)\,dx = 2\left(\frac{b}{5} + \frac{d}{3} + f\right)\\
&AF(-u) + BF(0) + AF(u) = 2A(bu^4 + du^2 + f) + Bf.
\end{aligned}$$

These two expressions are identical provided

$$Au^4 = \frac{1}{5}, \quad Au^2 = \frac{1}{3}, \quad A + \frac{B}{2} = 1.$$

Dividing the first two equations gives $u^2 = 3/5$, so $u = \sqrt{3/5}$. Then $3A/5 = 1/3$, so $A = 5/9$, and finally, $B = 8/9$.

14. For any function f we use the approximation

$$\int_{-1}^{1} f(x)\,dx \approx \frac{5}{9}\Big[f(-\sqrt{3/5}) + f(\sqrt{3/5})\Big] + \frac{8}{9}f(0).$$

We have

$$\begin{aligned}\int_{-1}^{1} x^6\,dx &\approx \frac{5}{9}\left[\left(-\sqrt{\frac{3}{5}}\right)^6 + \left(\sqrt{\frac{3}{5}}\right)^6\right] + 0 = 0.24000\\ \text{Error } &= \int_{-1}^{1} x^6\,dx - 0.24000 \approx 0.04571\\ \int_{-1}^{1} \cos x\,dx &\approx \frac{5}{9}\left[\cos\left(-\sqrt{\frac{3}{5}}\right) + \cos\left(\sqrt{\frac{3}{5}}\right)\right] + \frac{8}{9}\\ &\approx 1.68300\\ \text{Error } &= \int_{-1}^{1} \cos x\,dx - 1.68300 \approx 0.00006\\ \int_{-1}^{1} e^x\,dx &\approx e^{-\sqrt{3/5}} + e^{\sqrt{3/5}} \approx 2.35034\\ \text{Error } &= \int_{-1}^{1} e^x\,dx - 2.35034 \approx 0.00006.\end{aligned}$$

15. $I = \displaystyle\int_0^1 e^{-x^2}\,dx$

$$\begin{aligned}T_0^0 &= T_1 = R_0 = (1)\left(\frac{1}{2}e^0 + \frac{1}{2}e^{-1}\right) \approx 0.6839397\\ T_1^0 &= T_2 = \frac{1}{2}\left(\frac{1}{2}e^0 + e^{-1/4} + \frac{1}{2}e^{-1}\right) \approx 0.7313703\\ T_2^0 &= T_4 = \frac{1}{4}\left(2T_2 + e^{-1/16} + e^{-9/16}\right) \approx 0.7429841\\ T_3^0 &= T_8 = \frac{1}{8}\left(4T_4 + e^{-1/64} + e^{-9/64} + e^{-25/64} + e^{-49/64}\right)\\ &\approx 0.7458656\\ T_1^1 &= S_2 = R_1 = \frac{4T_1^0 - T_0^0}{3} \approx 0.7471805\\ T_2^1 &= S_4 = \frac{4T_2^0 - T_1^0}{3} \approx 0.7468554\\ T_3^1 &= S_8 = \frac{4T_3^0 - T_2^0}{3} \approx 0.7468261\\ T_2^2 &= R_2 = \frac{16T_2^1 - T_1^1}{15} \approx 0.7468337\\ T_3^2 &= \frac{16T_3^1 - T_2^1}{15} \approx 0.7468242\\ T_3^3 &= R_3 = \frac{64T_3^2 - T_2^2}{63} \approx 0.7468241\\ I &\approx 0.746824 \quad \text{to 6 decimal places.}\end{aligned}$$

16. From Exercise 9 in Section 7.6, for $I = \displaystyle\int_0^{1.6} f(x)\,dx$,

$$\begin{aligned}T_0^0 &= T_1 = 1.9196\\ T_1^0 &= T_2 = 2.00188\\ T_2^0 &= T_4 = 2.02622\\ T_3^0 &= T_8 = 2.02929.\end{aligned}$$

Hence,

$$\begin{aligned}R_1 = T_1^1 &= \frac{4T_1^0 - T_0^0}{3} = 2.0346684\\ T_2^1 &= \frac{4T_2^0 - T_1^0}{3} = 2.0343333 = S_4\\ R_2 = T_2^2 &= \frac{16T_2^1 - T_1^1}{15} = 2.0346684\\ T_3^1 &= \frac{4T_3^0 - T_2^0}{3} = 2.0303133 = S_8\\ T_3^2 &= \frac{16T_3^1 - T_2^1}{15} = 2.0300453\\ R_3 = T_3^3 &= \frac{64T_3^2 - T_2^2}{63} = 2.0299719.\end{aligned}$$

17.
$$\begin{aligned}T_1^1 &= S_2 = \frac{2h}{3}\Big(y_0 + 4y_2 + y_4\Big)\\ T_2^1 &= S_4 = \frac{h}{3}\Big(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4\Big)\\ R_2 = T_2^2 &= \frac{16T_2^1 - T_1^1}{15}\\ &= \frac{\frac{16h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4) - \frac{2h}{3}(y_0 + 4y_2 + y_4)}{15}\\ &= \frac{h}{45}\Big(14y_0 + 64y_1 + 24y_2 + 64y_3 + 14y_4\Big)\\ &= \frac{2h}{45}\Big(7y_0 + 32y_1 + 12y_2 + 32y_3 + 7y_4\Big)\end{aligned}$$

18. Let

$$\begin{aligned}I &= \int_\pi^\infty \frac{\sin x}{1+x^2}\,dx \quad \text{Let } x = \frac{1}{t}\\ &\qquad\qquad\qquad\quad dx = -\frac{dt}{t^2}\\ &= \int_{1/\pi}^{0} \frac{\sin\left(\frac{1}{t}\right)}{1 + \left(\frac{1}{t^2}\right)}\left(-\frac{1}{t^2}\right)dt\\ &= \int_0^{1/\pi} \frac{\sin\left(\frac{1}{t}\right)}{1+t^2}\,dt.\end{aligned}$$

The transformation is not suitable because the derivative of $\sin\left(\frac{1}{t}\right)$ is $-\frac{1}{t^2}\cos\left(\frac{1}{t}\right)$, which has very large values at some points close to 0.

In order to approximate the integral I to an desired degree of accuracy, say with error less than ϵ in absolute value, we have to divide the integral into two parts:

$$\begin{aligned} I &= \int_\pi^\infty \frac{\sin x}{1+x^2}\,dx \\ &= \int_\pi^t \frac{\sin x}{1+x^2}\,dx + \int_t^\infty \frac{\sin x}{1+x^2}\,dx \\ &= I_1 + I_2. \end{aligned}$$

If $t \ge \tan\frac{\pi-\epsilon}{2}$, then

$$\begin{aligned} &\int_t^\infty \frac{\sin x}{1+x^2}\,dx < \int_t^\infty \frac{dx}{1+x^2} \\ &= \tan^{-1}(x)\bigg|_t^\infty = \frac{\pi}{2} - \tan^{-1}(t) \le \frac{\epsilon}{2}. \end{aligned}$$

Now let A be a numerical approximation to the proper integral $\int_\pi^t \frac{\sin x}{1+x^2}\,dx$, having error less than $\epsilon/2$ in absolute value. Then

$$\begin{aligned} |I - A| &= |I_1 + I_2 - A| \\ &\le |I_1 - A| + |I_2| \\ &\le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Hence, A is an approximation to the integral I with the desired accuracy.

19. $f(x) = \dfrac{\sin x}{x}, \qquad f'(x) = \dfrac{x\cos c - \sin x}{x^2},$

$$\begin{aligned} f''(x) &= \frac{x^2(\cos x - x\sin x - \cos x) - (x\cos x - \sin x)2x}{x^4} \\ &= \frac{-x^2\sin x - 2x\cos x + 2\sin x}{x^3}. \end{aligned}$$

Now use l'Hôpital's Rule to get

$$\begin{aligned} &\lim_{x\to 0} f''(x) \\ &= \lim_{x\to 0} \frac{-2x\sin x - x^2\cos x - 2\cos x + 2x\sin x + 2\cos x}{3x^2} \\ &= \lim_{x\to 0} -\frac{\cos x}{3} = -\frac{1}{3}. \end{aligned}$$

Review Exercises on Techniques of Integration (page 365)

1.
$$\begin{aligned} \frac{x}{2x^2+5x+2} &= \frac{A}{2x+1} + \frac{B}{x+2} \\ &= \frac{Ax + 2A + 2Bx + B}{2x^2+5x+2} \\ &\Rightarrow \begin{cases} A + 2B = 1 \\ 2A + B = 0 \end{cases} \end{aligned}$$

Thus $A = -1/3$ and $B = 2/3$. We have

$$\begin{aligned} &\int \frac{x\,dx}{2x^2+5x+2} = -\frac{1}{3}\int\frac{dx}{2x+1} + \frac{2}{3}\int\frac{dx}{x+2} \\ &= \frac{2}{3}\ln|x+2| - \frac{1}{6}\ln|2x+1| + C. \end{aligned}$$

2.
$$\begin{aligned} &\int \frac{x}{(x-1)^3}\,dx \quad \text{Let } u = x-1 \\ &\qquad\qquad\qquad\quad du = dx \\ &= \int \frac{u+1}{u^3}\,du = \int\left(\frac{1}{u^2} + \frac{1}{u^3}\right)du \\ &= -\frac{1}{u} - \frac{1}{2u^2} + C = -\frac{1}{x-1} - \frac{1}{2(x-1)^2} + C. \end{aligned}$$

3.
$$\begin{aligned} &\int \sin^3 x\cos^3 x\,dx \\ &= \int \sin^3 x(1-\sin^2 x)\cos x\,dx \quad \text{Let } u = \sin x \\ &\qquad\qquad\qquad\qquad\qquad\qquad du = \cos x\,dx \\ &= \int (u^3 - u^5)\,du = \frac{u^4}{4} - \frac{u^6}{6} + C \\ &= \frac{1}{4}\sin^4 x - \frac{1}{6}\sin^6 x + C. \end{aligned}$$

4.
$$\begin{aligned} &\int \frac{(1+\sqrt{x})^{1/3}}{\sqrt{x}}\,dx \quad \text{Let } u = 1+\sqrt{x} \\ &\qquad\qquad\qquad\qquad du = \frac{dx}{2\sqrt{x}} \\ &= 2\int u^{1/3}\,du = 2(\tfrac{3}{4})u^{4/3} + C \\ &= \tfrac{3}{2}(1+\sqrt{x})^{4/3} + C. \end{aligned}$$

5.
$$\begin{aligned} \frac{3}{4x^2-1} &= \frac{A}{2x-1} + \frac{B}{2x+1} \\ &= \frac{2Ax + A + 2Bx - B}{4x^2-1} \\ &\Rightarrow \begin{cases} 2A + 2B = 0 \\ A - B = 3 \end{cases} \Rightarrow A = -B = \frac{3}{2} \end{aligned}$$

$$\int \frac{3\,dx}{4x^2-1} = \frac{3}{2}\left(\int \frac{dx}{2x-1} - \int \frac{dx}{2x+1}\right)$$
$$= \frac{3}{4}\ln\left|\frac{2x-1}{2x+1}\right| + C.$$

6. $\int (x^2+x-2)\sin 3x\,dx$

$$U = x^2+x-2 \quad dV = \sin 3x$$
$$dU = (2x+1)\,dx \quad V = -\tfrac{1}{3}\cos 3x$$
$$= -\tfrac{1}{3}(x^2+x-2)\cos 3x + \tfrac{1}{3}\int (2x+1)\cos 3x\,dx$$

$$U = 2x+1 \quad dV = \cos 3x\,dx$$
$$dU = 2\,dx \quad V = \tfrac{1}{3}\sin 3x$$
$$= -\tfrac{1}{3}(x^2+x-2)\cos 3x + \tfrac{1}{9}(2x+1)\sin 3x - \tfrac{2}{9}\int \sin 3x\,dx$$
$$= -\tfrac{1}{3}(x^2+x-2)\cos 3x + \tfrac{1}{9}(2x+1)\sin 3x + \frac{2}{27}\cos 3x + C.$$

7. $\int \frac{\sqrt{1-x^2}}{x^4}\,dx$ Let $x = \sin\theta$, $dx = \cos\theta\,d\theta$

$$= \int \frac{\cos^2\theta}{\sin^4\theta}\,d\theta$$
$$= \int \csc^2\theta\cot^2\theta\,d\theta \quad \text{Let } v = \cot\theta,\ dv = -\csc^2\theta\,d\theta$$
$$= -\int v^2\,dv = -\frac{v^3}{3} + C$$
$$= -\frac{\cot^3\theta}{3} + C = -\frac{1}{3}\left(\frac{\sqrt{1-x^2}}{x}\right)^3 + C.$$

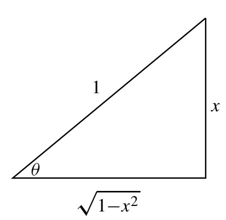

Fig. RT.7

8. $\int x^3\cos(x^2)\,dx$ Let $w = x^2$, $dw = 2x\,dx$

$$= \tfrac{1}{2}\int w\cos w\,dw$$
$$U = w \quad dV = \cos w\,dw$$
$$dU = dw \quad V = \sin w$$
$$= \tfrac{1}{2}w\sin w - \tfrac{1}{2}\int \sin w\,dw$$
$$= \tfrac{1}{2}x^2\sin(x^2) + \tfrac{1}{2}\cos(x^2) + C.$$

9. $\int \frac{x^2\,dx}{(5x^3-2)^{2/3}}$ Let $u = 5x^3-2$, $du = 15x^2\,dx$

$$= \frac{1}{15}\int u^{-2/3}\,du = \frac{1}{5}u^{1/3} + C$$
$$= \frac{1}{5}(5x^3-2)^{1/3} + C.$$

10. $\frac{1}{x^2+2x-15} = \frac{A}{x-3} + \frac{B}{x+5} = \frac{(A+B)x + (5A-3B)}{x^2+2x-15}$

$$\Rightarrow \begin{cases} A+B=0 \\ 5A-3B=1 \end{cases} \Rightarrow A = \frac{1}{8},\ B = -\frac{1}{8}.$$
$$\int \frac{dx}{x^2+2x-15} = \frac{1}{8}\int \frac{dx}{x-3} - \frac{1}{8}\int \frac{dx}{x+5}$$
$$= \frac{1}{8}\ln\left|\frac{x-3}{x+5}\right| + C.$$

11. $\int \frac{dx}{(4+x^2)^2}$ Let $x = 2\tan\theta$, $dx = 2\sec^2\theta\,d\theta$

$$= \int \frac{2\sec^2\theta\,d\theta}{16\sec^4\theta} = \frac{1}{8}\int \cos^2\theta\,d\theta$$
$$= \frac{1}{16}(\theta + \sin\theta\cos\theta) + C$$
$$= \frac{1}{16}\tan^{-1}\frac{x}{2} + \frac{1}{8}\left(\frac{x}{4+x^2}\right) + C.$$

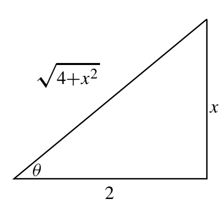

Fig. RT.11

12. $\int (\sin x + \cos x)^2\,dx = \int (1 + \sin 2x)\,dx$
$$= x - \tfrac{1}{2}\cos 2x + C.$$

13. $\displaystyle\int 2^x\sqrt{1+4^x}\,dx \qquad \begin{aligned}&\text{Let } 2^x = \tan\theta\\ &2^x \ln 2\,dx = \sec^2\theta\,d\theta\end{aligned}$

$$= \frac{1}{\ln 2}\int \sec^3\theta\,d\theta$$
$$= \frac{1}{2\ln 2}\Big(\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|\Big) + C$$
$$= \frac{1}{2\ln 2}\Big(2^x\sqrt{1+4^x} + \ln(2^x + \sqrt{1+4^x})\Big) + C.$$

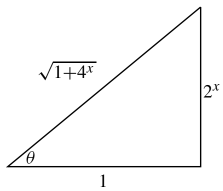

Fig. RT.13

14. $\displaystyle\int \frac{\cos x}{1+\sin^2 x}\,dx \qquad \begin{aligned}&\text{Let } u = \sin x\\ &du = \cos x\,dx\end{aligned}$

$$= \int \frac{du}{1+u^2} = \tan^{-1}u + C$$
$$= \tan^{-1}(\sin x) + C.$$

15. $\displaystyle\int \frac{\sin^3 x}{\cos^7 x}\,dx = \int \tan^3 x \sec^4 x\,dx$

$$= \int \tan^3 x(1+\tan^2 x)\sec^2 x\,dx \qquad \begin{aligned}&\text{Let } u = \tan x\\ &du = \sec^2 x\,dx\end{aligned}$$
$$= \int (u^3+u^5)\,du = \frac{u^4}{4} + \frac{u^6}{6} + C$$
$$= \frac{1}{4}\tan^4 x + \frac{1}{6}\tan^6 x + C.$$

16. We have

$$\int \frac{x^2\,dx}{(3+5x^2)^{3/2}} \qquad \begin{aligned}&\text{Let } x = \sqrt{\tfrac{3}{5}}\tan u\\ &dx = \sqrt{\tfrac{3}{5}}\sec^2 u\,du\end{aligned}$$
$$= \int \frac{(\frac{3}{5}\tan^2 u)(\sqrt{\frac{3}{5}}\sec^2 u)\,du}{(3)^{3/2}\sec^3 u}$$
$$= \frac{1}{5\sqrt{5}}\int (\sec u - \cos u)\,du$$
$$= \frac{1}{5\sqrt{5}}(\ln|\sec u + \tan u| - \sin u) + C$$
$$= \frac{1}{5\sqrt{5}}\left(\ln\left|\frac{\sqrt{5x^2+3}}{\sqrt{3}} + \frac{\sqrt{5}x}{\sqrt{3}}\right| - \frac{\sqrt{5}x}{\sqrt{5x^2+3}}\right) + C$$
$$= \frac{1}{5\sqrt{5}}\ln\Big(\sqrt{5x^2+3}+\sqrt{5}x\Big) - \frac{x}{5\sqrt{5x^2+3}} + C_0,$$

where $C_0 = C - \dfrac{1}{5\sqrt{5}}\ln\sqrt{3}$.

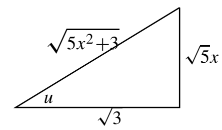

Fig. RT.16

17. $\displaystyle I = \int e^{-x}\sin 2x\,dx$

$$\begin{aligned}U &= e^{-x} & dV &= \sin 2x\,dx\\ dU &= -e^{-x}\,dx & V &= -\frac{1}{2}\cos 2x\end{aligned}$$
$$= -\frac{1}{2}e^{-x}\cos 2x - \frac{1}{2}\int e^{-x}\cos 2x\,dx$$

$$\begin{aligned}U &= e^{-x} & dV &= \cos 2x\,dx\\ dU &= -e^{-x}\,dx & V &= \frac{1}{2}\sin 2x\end{aligned}$$
$$= -\frac{1}{2}e^{-x}\cos 2x - \frac{1}{2}\left(\frac{1}{2}e^{-x}\sin 2x + \frac{1}{2}I\right)$$
$$= -\frac{1}{2}e^{-x}\cos 2x - \frac{1}{4}e^{-x}\sin 2x - \frac{1}{4}I$$
$$I = -e^{-x}\left(\frac{2}{5}\cos 2x + \frac{1}{5}\sin 2x\right) + C.$$

18. $\displaystyle I = \int \frac{2x^2+4x-3}{x^2+5x}\,dx = \int \frac{2x^2+10x-6x-3}{x^2+5x}\,dx$

$$= \int \left[2 - \frac{6x+3}{x(x+5)}\right]dx$$
$$\frac{6x+3}{x(x+5)} = \frac{A}{x} + \frac{B}{x+5} = \frac{(A+B)x+5A}{x(x+5)}$$
$$\Rightarrow \begin{cases}A+B=6\\ 5A=3\end{cases} \Rightarrow A = \frac{3}{5},\ B = \frac{27}{5}.$$
$$I = \int 2\,dx - \frac{3}{5}\int\frac{dx}{x} - \frac{27}{5}\int\frac{dx}{x+5}$$
$$= 2x - \frac{3}{5}\ln|x| - \frac{27}{5}\ln|x+5| + C.$$

19. $\displaystyle I = \int \cos(3\ln x)\,dx$

$$\begin{aligned}U &= \cos(3\ln x) & dV &= dx\\ dU &= -\frac{3\sin(3\ln x)\,dx}{x} & V &= x\end{aligned}$$
$$= x\cos(3\ln x) + 3\int \sin(3\ln x)\,dx$$

$$\begin{aligned}U &= \sin(3\ln x) & dV &= dx\\ dU &= \frac{3\cos(3\ln x)\,dx}{x} & V &= x\end{aligned}$$
$$= x\cos(3\ln x) + 3\Big(x\sin(3\ln x) - 3I\Big)$$
$$I = \frac{1}{10}x\cos(3\ln x) + \frac{3}{10}x\sin(3\ln x) + C.$$

20. $$\frac{1}{4x^3+x} = \frac{A}{x} + \frac{Bx+C}{4x^2+1}$$
$$= \frac{A(4x^2+1)+Bx^2+Cx}{4x^3+x}$$
$$\Rightarrow \begin{cases} 4A+B=0 \\ C=0, \quad A=1 \end{cases} \Rightarrow B=-4.$$
$$\int \frac{1}{4x^3+x}\,dx = \int \frac{dx}{x} - 4\int \frac{x\,dx}{4x^2+1}$$
$$= \ln|x| - \frac{1}{2}\ln(4x^2+1) + C.$$

21. $$\int \frac{x\ln(1+x^2)}{1+x^2}\,dx \quad \text{Let } u=\ln(1+x^2) \quad du = \frac{2x\,dx}{1+x^2}$$
$$= \frac{1}{2}\int u\,du = \frac{u^2}{4} + C$$
$$= \frac{1}{4}\Big(\ln(1+x^2)\Big)^2 + C.$$

22. $$\int \sin^2 x\,\cos^4 x\,dx$$
$$= \int \tfrac{1}{2}(1-\cos 2x)[\tfrac{1}{2}(1+\cos 2x)]^2\,dx$$
$$= \frac{1}{8}\int (1+\cos 2x - \cos^2 2x - \cos^3 2x)\,dx$$
$$= \frac{1}{8}x + \frac{1}{16}\sin 2x - \frac{1}{16}\int (1+\cos 4x)\,dx$$
$$- \frac{1}{8}\int (1-\sin^2 2x)\cos 2x\,dx$$
$$= \frac{x}{8} + \frac{1}{16}\sin 2x - \frac{x}{16} - \frac{1}{64}\sin 4x - \frac{1}{16}\sin 2x$$
$$+ \frac{1}{48}\sin^3 2x + C$$
$$= \frac{x}{16} - \frac{\sin 4x}{64} + \frac{\sin^3 2x}{48} + C.$$

23. $$\int \frac{x^2\,dx}{\sqrt{2-x^2}} \quad \text{Let } x=\sqrt{2}\sin\theta \quad dx=\sqrt{2}\cos\theta\,d\theta$$
$$= 2\int \sin^2\theta\,d\theta = \theta - \sin\theta\cos\theta + C$$
$$= \sin^{-1}\frac{x}{\sqrt{2}} - \frac{x\sqrt{2-x^2}}{2} + C.$$

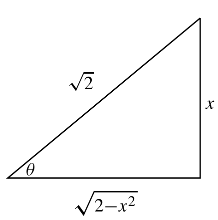

Fig. RT.23

24. We have
$$I = \int \tan^4 x\,\sec x\,dx$$
$$U=\tan^3 x \qquad dV = \tan x\,\sec x\,dx$$
$$dU = 3\tan^2 x\,\sec^2 x\,dx \qquad V=\sec x$$
$$= \tan^3 x\,\sec x - 3\int \tan^2 x\,\sec^3 x\,dx$$
$$= \tan^3 x\,\sec x - 3\int \tan^2 x(\tan^2 x+1)\sec x\,dx$$
$$= \tan^3 x\,\sec x - 3I - 3J \text{ where}$$
$$J = \int \tan^2 x\,\sec x\,dx$$
$$U=\tan x \qquad dV=\tan x\,\sec x\,dx$$
$$dU=\sec^2 x\,dx \qquad V=\sec x$$
$$= \tan x\,\sec x - \int \sec^3 x\,dx$$
$$= \tan x\,\sec x - \int (\tan^2 x+1)\sec x\,dx$$
$$= \tan x\,\sec x - J - \ln|\sec x+\tan x| + C$$
$$J = \tfrac{1}{2}\tan x\,\sec x - \tfrac{1}{2}\ln|\sec x+\tan x| + C.$$
$$I = \tfrac{1}{4}\tan^3 x\,\sec x - \tfrac{3}{8}\tan x\,\sec x$$
$$+ \tfrac{3}{8}\ln|\sec x+\tan x| + C.$$

25. $$\int \frac{x^2\,dx}{(4x+1)^{10}} \quad \text{Let } u=4x+1 \quad du=4\,dx$$
$$= \frac{1}{4}\int \left(\frac{u-1}{4}\right)^2 \frac{1}{u^{10}}\,du$$
$$= \frac{1}{64}\int (u^{-8} - 2u^{-9} + u^{-10})\,du$$
$$= -\frac{1}{448}u^{-7} + \frac{1}{256}u^{-8} - \frac{1}{576}u^{-9} + C$$
$$= \frac{1}{64}\left(-\frac{1}{7(4x+1)^7} + \frac{1}{4(4x+1)^8} - \frac{1}{9(4x+1)^9}\right) + C.$$

26. We have
$$\int x\sin^{-1}\left(\frac{x}{2}\right)dx$$
$$U=\sin^{-1}\left(\frac{x}{2}\right) \qquad dV = x\,dx$$
$$dU = \frac{dx}{\sqrt{4-x^2}} \qquad V=\frac{x^2}{2}$$
$$= \frac{x^2}{2}\sin^{-1}\left(\frac{x}{2}\right) - \frac{1}{2}\int \frac{x^2\,dx}{\sqrt{4-x^2}} \quad \text{Let } x=2\sin u \quad dx=2\cos u\,du$$
$$= \frac{x^2}{2}\sin^{-1}\left(\frac{x}{2}\right) - 2\int \sin^2 u\,du$$
$$= \frac{x^2}{2}\sin^{-1}\left(\frac{x}{2}\right) - \int (1-\cos 2u)\,du$$
$$= \frac{x^2}{2}\sin^{-1}\left(\frac{x}{2}\right) - u + \sin u\,\cos u + C$$

$$= \left(\frac{x^2}{2} - 1\right)\sin^{-1}\left(\frac{x}{2}\right) + \frac{1}{4}x\sqrt{4-x^2} + C.$$

27. $\int \sin^5(4x)\,dx$

$$= \int (1-\cos^2 4x)^2 \sin 4x\,dx \quad \text{Let } u = \cos 4x, \; du = -4\sin 4x\,dx$$
$$= -\frac{1}{4}\int (1-2u^2+u^4)\,du$$
$$= -\frac{1}{4}\left(u - \frac{2}{3}u^3 + \frac{1}{5}u^5\right) + C$$
$$= -\frac{1}{4}\cos 4x + \frac{1}{6}\cos^3 4x - \frac{1}{20}\cos^5 4x + C.$$

28. We have

$$I = \int \frac{dx}{x^5 - 2x^3 + x} = \int \frac{x\,dx}{x^6 - 2x^4 + x^2} \quad \text{Let } u = x^2, \; du = 2x\,dx$$
$$= \frac{1}{2}\int \frac{du}{u^3 - 2u^2 + u} = \frac{1}{2}\int \frac{du}{u(u-1)^2}$$
$$\frac{1}{u(u-1)^2} = \frac{A}{u} + \frac{B}{u-1} + \frac{C}{(u-1)^2}$$
$$= \frac{A(u^2-2u+1) + B(u^2-u) + Cu}{u^3 - 2u^2 + u}$$
$$\Rightarrow \begin{cases} A + B = 0 \\ -2A - B + C = 0 \\ A = 1 \end{cases} \Rightarrow A = 1,\ B = -1,\ C = 1.$$
$$\frac{1}{2}\int \frac{du}{u^3 - 2u^2 + u} = \frac{1}{2}\int \frac{du}{u} - \frac{1}{2}\int \frac{du}{u-1} + \frac{1}{2}\int \frac{du}{(u-1)^2}$$
$$= \frac{1}{2}\ln|u| - \frac{1}{2}\ln|u-1| - \frac{1}{2}\frac{1}{u-1} + K$$
$$= \frac{1}{2}\ln\frac{x^2}{|x^2-1|} - \frac{1}{2(x^2-1)} + K.$$

29. $\int \frac{dx}{2+e^x}$

$$= \int \frac{e^{-x}\,dx}{2e^{-x}+1} \quad \text{Let } u = 2e^{-x}+1, \; du = -2e^{-x}\,dx$$
$$= -\frac{1}{2}\int \frac{du}{u} = -\frac{1}{2}\ln(2e^{-x}+1) + C.$$

30. Let

$$I_n = \int x^n 3^x\,dx \qquad \begin{array}{ll} U = x^n & dV = 3^x\,dx \\ dU = nx^{n-1}\,dx & V = \dfrac{3^x}{\ln 3} \end{array}$$
$$= \frac{x^n 3^x}{\ln 3} - \frac{n}{\ln 3}I_{n-1}.$$
$$I_0 = \int 3^x\,dx = \frac{3^x}{\ln 3} + C.$$

Hence,

$$I_3 = \int x^3 3^x\,dx$$
$$= \frac{x^3 3^x}{\ln 3} - \frac{3}{\ln 3}\left[\frac{x^2 3^x}{\ln 3} - \frac{2}{\ln 3}\left(\frac{x3^x}{\ln 3} - \frac{1}{\ln 3}I_0\right)\right] + C_1$$
$$= 3^x\left[\frac{x^3}{\ln 3} - \frac{3x^2}{(\ln 3)^2} + \frac{6x}{(\ln 3)^3} - \frac{6}{(\ln 3)^4}\right] + C_1.$$

31. $\int \frac{\sin^2 x \cos x}{2 - \sin x}\,dx \quad \text{Let } u = \sin x, \; du = \cos x\,dx$

$$= \int \frac{u^2\,du}{2-u} \quad \text{Let } 2 - u = v, \; du = -dv$$
$$= -\int \frac{4 - 4v + v^2}{v}\,dv = \int \left(-\frac{4}{v} + 4 - v\right)dv$$
$$= -4\ln|v| + 4v - \frac{v^2}{2} + C$$
$$= -4\ln|2-u| + 4(2-u) - \frac{1}{2}(2-u)^2 + C$$
$$= -4\ln(2-\sin x) - 2\sin x - \frac{1}{2}\sin^2 x + C_1.$$

32. We have

$$\int \frac{x^2+1}{x^2+2x+2}\,dx = \int \left(1 - \frac{2x+1}{x^2+2x+2}\right)dx$$
$$= x - \int \frac{2x+1}{(x+1)^2+1}\,dx \quad \text{Let } u = x+1, \; du = dx$$
$$= x - \int \frac{2u-1}{u^2+1}\,du$$
$$= x - \ln|u^2+1| + \tan^{-1} u + C$$
$$= x - \ln(x^2+2x+2) + \tan^{-1}(x+1) + C.$$

33. $\int \frac{dx}{x^2\sqrt{1-x^2}} \quad \text{Let } x = \sin\theta, \; dx = \cos\theta\,d\theta$

$$= \int \frac{\cos\theta\,d\theta}{\sin^2\theta\cos\theta} = \int \csc^2\theta\,d\theta$$
$$= -\cot\theta + C = -\frac{\sqrt{1-x^2}}{x} + C.$$

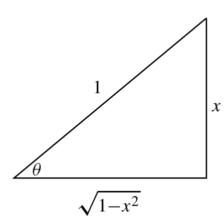

Fig. RT.33

34. We have

$$\int x^3(\ln x)^2\,dx \qquad \begin{array}{ll} U=(\ln x)^2 & dV=x^3\,dx \\ dU=\frac{2}{x}\ln x\,dx & V=\frac{1}{4}x^4 \end{array}$$
$$=\frac{1}{4}x^4(\ln x)^2-\frac{1}{2}\int x^3\ln x\,dx \qquad \begin{array}{ll} U=\ln x & dV=x^3\,dx \\ dU=\frac{1}{x}\,dx & V=\frac{1}{4}x^4 \end{array}$$
$$=\frac{1}{4}x^4(\ln x)^2-\frac{1}{8}x^4\ln x+\frac{1}{8}\int x^3\,dx$$
$$=\frac{x^4}{4}\left[(\ln x)^2-\frac{1}{2}\ln x+\frac{1}{8}\right]+C.$$

35. $$\int\frac{x^3\,dx}{\sqrt{1-4x^2}} \qquad \begin{array}{l}\text{Let } 2x=\sin\theta \\ 2\,dx=\cos\theta\,d\theta\end{array}$$
$$=\frac{1}{16}\int\frac{\sin^3\theta\cos\theta\,d\theta}{\cos\theta}=\frac{1}{16}\int(1-\cos^2\theta)\sin\theta\,d\theta$$
$$=\frac{1}{16}\left(-\cos\theta+\frac{1}{3}\cos^3\theta\right)+C$$
$$=\frac{1}{48}(1-4x^2)^{3/2}-\frac{1}{16}\sqrt{1-4x^2}+C.$$

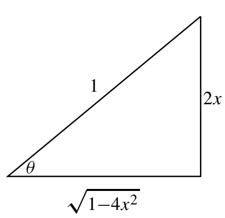

Fig. RT.35

36. $$\int\frac{e^{1/x}}{x^2}\,dx \qquad \begin{array}{l}\text{Let } u=\frac{1}{x} \\ du=-\frac{1}{x^2}\,dx\end{array}$$
$$=-\int e^u\,du=-e^u+C=-e^{1/x}+C.$$

37. $$\int\frac{x+1}{\sqrt{x^2+1}}\,dx$$
$$=\sqrt{x^2+1}+\int\frac{dx}{\sqrt{x^2+1}} \qquad \begin{array}{l}\text{Let } x=\tan\theta \\ dx=\sec^2\theta\,d\theta\end{array}$$
$$=\sqrt{x^2+1}+\int\sec\theta\,d\theta$$
$$=\sqrt{x^2+1}+\ln|\sec\theta+\tan\theta|+C$$
$$=\sqrt{x^2+1}+\ln(x+\sqrt{x^2+1})+C.$$

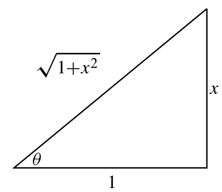

Fig. RT.37

38. $$\int e^{(x^{1/3})} \qquad \begin{array}{l}\text{Let } x=u^3 \\ dx=3u^2\,du\end{array}$$
$$=3\int u^2e^u\,du=3I_2$$
See solution to #16 of Section 6.6 for
$$I_n=\int u^ne^u\,dx=u^ne^u-nI_{n-1}.$$
$$=3[u^2e^u-2(ue^u-e^u)]+C$$
$$=e^{(x^{1/3})}(3x^{2/3}-6x^{1/3}+6)+C.$$

39. $$I=\int\frac{x^3-3}{x^3-9x}\,dx=\int\left(1+\frac{9x-3}{x^3-9x}\right)dx.$$

$$\frac{9x-3}{x^3-9x}=\frac{A}{x}+\frac{B}{x-3}+\frac{C}{x+3}$$
$$=\frac{Ax^2-9A+Bx^2+3Bx+Cx^2-3Cx}{x^3-9x}$$
$$\Rightarrow\begin{cases}A+B+C=0\\3B-3C=9\\-9A=-3\end{cases}\Rightarrow\begin{cases}A=1/3\\B=4/3\\C=-5/3.\end{cases}$$

Thus we have

$$I=x+\frac{1}{3}\int\frac{dx}{x}+\frac{4}{3}\int\frac{dx}{x-3}-\frac{5}{3}\int\frac{dx}{x+3}$$
$$=x+\frac{1}{3}\ln|x|+\frac{4}{3}\ln|x-3|-\frac{5}{3}\ln|x+3|+K.$$

40. $$\int\frac{10^{\sqrt{x+2}}\,dx}{\sqrt{x+2}} \qquad \begin{array}{l}\text{Let } u=\sqrt{x+2} \\ du=\frac{dx}{2\sqrt{x+2}}\end{array}$$
$$=2\int 10^u\,du=\frac{2}{\ln 10}10^u+C=\frac{2}{\ln 10}10^{\sqrt{x+2}}+C.$$

41. $$\int\sin^5x\cos^9x\,dx$$
$$=\int(1-\cos^2x)^2\cos^9x\sin x\,dx \qquad \begin{array}{l}\text{Let } u=\cos x \\ du=-\sin x\,dx\end{array}$$
$$=-\int(1-2u^2+u^4)u^9\,du$$
$$=-\frac{u^{10}}{10}+\frac{u^{12}}{6}-\frac{u^{14}}{14}+C$$
$$=\frac{\cos^{12}x}{6}-\frac{\cos^{10}x}{10}-\frac{\cos^{14}x}{14}+C.$$

42. Assume that $x \geq 1$ and let $x = \sec u$ and $dx = \sec u \tan u\,du$. Then

$$\begin{aligned}&\int \frac{x^2\,dx}{\sqrt{x^2-1}}\\&= \int \frac{\sec^3 u \tan u\,du}{\tan u} = \int \sec^3 u\,du\\&= \frac{1}{2}\sec u \tan u + \frac{1}{2}\ln|\sec u + \tan u| + C\\&= \frac{1}{2}x\sqrt{x^2-1} + \frac{1}{2}\ln|x + \sqrt{x^2-1}| + C.\end{aligned}$$

Differentiation shows that this solution is valid for $x \leq -1$ also.

43. $I = \displaystyle\int \frac{x\,dx}{x^2+2x-1} = \int \frac{(x+1-1)\,dx}{(x+1)^2-2}$ Let $u = x+1$, $du = dx$

$$= \int \frac{u-1}{u^2-2}\,du = \frac{1}{2}\ln|u^2-2| - \int \frac{du}{u^2-2}.$$

$$\begin{aligned}\frac{1}{u^2-2} &= \frac{A}{u-\sqrt{2}} + \frac{B}{u+\sqrt{2}}\\&= \frac{Au+\sqrt{2}A+Bu-\sqrt{2}B}{u^2-2}\\&\Rightarrow \begin{cases}A+B=0\\ \sqrt{2}(A-B)=1\end{cases}\\&\Rightarrow A = -B = \frac{1}{2\sqrt{2}}.\end{aligned}$$

Thus we have

$$\begin{aligned}I &= \frac{1}{2}\ln|u^2-2| - \frac{1}{2\sqrt{2}}\ln\left|\frac{u-\sqrt{2}}{u+\sqrt{2}}\right| + K\\&= \frac{1}{2}\ln|x^2+2x-1| - \frac{1}{2\sqrt{2}}\ln\left|\frac{x+1-\sqrt{2}}{x+1+\sqrt{2}}\right| + K.\end{aligned}$$

44. $\displaystyle\int \frac{2x-3}{\sqrt{4-3x+x^2}}\,dx$ Let $u = 4-3x+x^2$, $du = (-3+2x)\,dx$

$$= \int \frac{du}{\sqrt{u}} = 2\sqrt{u} + C = 2\sqrt{4-3x+x^2} + C.$$

45. $\displaystyle\int x^2 \sin^{-1} 2x\,dx$

$U = \sin^{-1} 2x$ $\quad dV = x^2\,dx$

$dU = \dfrac{2\,dx}{\sqrt{1-4x^2}}$ $\quad V = \dfrac{x^3}{3}$

$$\begin{aligned}&= \frac{x^3}{3}\sin^{-1} 2x - \frac{2}{3}\int \frac{x^3\,dx}{\sqrt{1-4x^2}} \quad \text{Let } v = 1-4x^2,\ dv = -8x\,dx\\&= \frac{x^3}{3}\sin^{-1} 2x - \frac{2}{3}\int \frac{1-v}{4v^{1/2}}\left(-\frac{1}{8}\,dv\right)\\&= \frac{x^3}{3}\sin^{-1} 2x + \frac{1}{48}\int \left(v^{-1/2} - v^{1/2}\right)dv\\&= \frac{x^3}{3}\sin^{-1} 2x + \frac{1}{24}\sqrt{v} - \frac{1}{72}v^{3/2} + C\\&= \frac{x^3}{3}\sin^{-1} 2x + \frac{1}{24}\sqrt{1-4x^2} - \frac{1}{72}(1-4x^2)^{3/2} + C.\end{aligned}$$

46. Let $\sqrt{3}x = \sec u$ and $\sqrt{3}\,dx = \sec u \tan u\,du$. Then

$$\begin{aligned}&\int \frac{\sqrt{3x^2-1}}{x}\,dx\\&= \int \frac{\tan u \frac{1}{\sqrt{3}}\sec u \tan u\,du}{\frac{1}{\sqrt{3}}\sec u}\\&= \int \tan^2 u\,du = \int (\sec^2 u - 1)\,du\\&= \tan u - u + C = \sqrt{3x^2-1} - \sec^{-1}(\sqrt{3}x) + C\\&= \sqrt{3x^2-1} + \sin^{-1}\left(\frac{1}{\sqrt{3}\,x}\right) + C_1.\end{aligned}$$

47. $\displaystyle\int \cos^4 x \sin^4 x\,dx = \frac{1}{16}\int \sin^4 2x\,dx$

$$\begin{aligned}&= \frac{1}{64}\int (1-\cos 4x)^2\,dx\\&= \frac{1}{64}\int \left(1 - 2\cos 4x + \frac{1+\cos 8x}{2}\right)dx\\&= \frac{1}{64}\left(\frac{3x}{2} - \frac{\sin 4x}{2} + \frac{\sin 8x}{16}\right) + C\\&= \frac{1}{128}\left(3x - \sin 4x + \frac{\sin 8x}{8}\right) + C.\end{aligned}$$

48. $\displaystyle\int \sqrt{x-x^2}\,dx$

$$\begin{aligned}&= \int \sqrt{\tfrac{1}{4} - (x-\tfrac{1}{2})^2}\,dx \quad \text{Let } x - \tfrac{1}{2} = \tfrac{1}{2}\sin u,\ dx = \tfrac{1}{2}\cos u\,du\\&= \tfrac{1}{4}\int \cos^2 u\,du = \tfrac{1}{8}u + \tfrac{1}{8}\sin u \cos u + C\\&= \tfrac{1}{8}\sin^{-1}(2x-1) + \tfrac{1}{4}(2x-1)\sqrt{x-x^2} + C.\end{aligned}$$

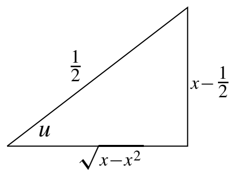

Fig. RT.48

49. $\displaystyle\int \frac{dx}{(4+x)\sqrt{x}}$ Let $x = u^2$, $dx = 2u\,du$

$$= \int \frac{2u\,du}{(4+u^2)u} = 2\int \frac{du}{4+u^2}$$
$$= \frac{2}{2}\tan^{-1}\frac{u}{2} + C = \tan^{-1}\frac{\sqrt{x}}{2} + C.$$

50. $\displaystyle\int x\tan^{-1}\left(\frac{x}{3}\right) dx$

$$U = \tan^{-1}\left(\frac{x}{3}\right) \qquad dV = x\,dx$$
$$dU = \frac{3\,dx}{9+x^2} \qquad V = \frac{x^2}{2}$$
$$= \frac{x^2}{2}\tan^{-1}\left(\frac{x}{3}\right) - \frac{3}{2}\int \frac{x^2}{9+x^2}\,dx$$
$$= \frac{x^2}{2}\tan^{-1}\left(\frac{x}{3}\right) - \frac{3}{2}\int \left(1 - \frac{9}{9+x^2}\right) dx$$
$$= \frac{x^2}{2}\tan^{-1}\left(\frac{x}{3}\right) - \frac{3x}{2} + \frac{9}{2}\tan^{-1}\left(\frac{x}{3}\right) + C.$$

51. $$I = \int \frac{x^4-1}{x^3+2x^2}\,dx$$
$$= \int \frac{x^4+2x^3-2x^3-4x^2+4x^2-1}{x^3+2x^2}\,dx$$
$$= \int \left(x - 2 + \frac{4x^2-1}{x^3+2x^2}\right) dx.$$

$$\frac{4x^2-1}{x^3+2x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+2}$$
$$= \frac{Ax^2+2Ax+Bx+2B+Cx^2}{x^3+2x^2}$$
$$\Rightarrow \begin{cases} A+C=4 \\ 2A+B=0 \\ 2B=-1 \end{cases} \Rightarrow \begin{cases} A=1/4 \\ B=-1/2 \\ C=15/4. \end{cases}$$

Thus

$$I = \frac{x^2}{2} - 2x + \frac{1}{4}\int \frac{dx}{x} - \frac{1}{2}\int \frac{dx}{x^2} + \frac{15}{4}\int \frac{dx}{x+2}$$
$$= \frac{x^2}{2} - 2x + \frac{1}{4}\ln|x| + \frac{1}{2x} + \frac{15}{4}\ln|x+2| + K.$$

52. Let $u = x^2$ and $du = 2x\,dx$; then we have

$$I = \int \frac{dx}{x(x^2+4)^2} = \int \frac{x\,dx}{x^2(x^2+4)^2} = \frac{1}{2}\int \frac{du}{u(u+4)^2}.$$

Since

$$\frac{1}{u(u+4)^2} = \frac{A}{u} + \frac{B}{u+4} + \frac{C}{(u+4)^2}$$
$$= \frac{A(u^2+8u+16)+B(u^2+4u)+Cu}{u(u+4)^2}$$
$$\Rightarrow \begin{cases} A+B=0 \\ 8A+4B+C=0 \\ 16A=1 \end{cases} \Rightarrow A=\frac{1}{16},\ B=-\frac{1}{16},\ C=-\frac{1}{4},$$

therefore

$$I = \frac{1}{32}\int \frac{du}{u} - \frac{1}{32}\int \frac{du}{u+4} - \frac{1}{8}\int \frac{du}{(u+4)^2}$$
$$= \frac{1}{32}\ln\left|\frac{u}{u+4}\right| + \frac{1}{8}\frac{1}{u+4} + C$$
$$= \frac{1}{32}\ln\left|\frac{x^2}{x^2+4}\right| + \frac{1}{8(x^2+4)} + C.$$

53. $\displaystyle\int \frac{\sin(2\ln x)}{x}\,dx$ Let $u = 2\ln x$, $du = \frac{2}{x}\,dx$

$$= \frac{1}{2}\int \sin u\,du = -\frac{1}{2}\cos u + C$$
$$= -\frac{1}{2}\cos(2\ln x) + C.$$

54. Since

$$I = \int \frac{\sin(\ln x)}{x^2}\,dx$$
$$U = \sin(\ln x) \qquad dV = \frac{dx}{x^2}$$
$$dU = \frac{\cos(\ln x)}{x}\,dx \qquad V = -\frac{1}{x}$$
$$= -\frac{\sin(\ln x)}{x} + \int \frac{\cos(\ln x)}{x^2}\,dx$$
$$U = \cos(\ln x) \qquad dV = \frac{dx}{x^2}$$
$$dU = -\frac{\sin(\ln x)}{x}\,dx \qquad V = \frac{-1}{x}$$
$$= -\frac{\sin(\ln x)}{x} - \frac{\cos(\ln x)}{x} - I,$$

therefore

$$I = -\frac{1}{2x}\Big[\sin(\ln x) + \cos(\ln x)\Big] + C.$$

55. $\displaystyle\int \frac{e^{2\tan^{-1}x}}{1+x^2}\,dx$ Let $u = 2\tan^{-1}x$, $du = \frac{2\,dx}{1+x^2}$

$$= \frac{1}{2}\int e^u\,du = \frac{1}{2}e^u + C = \frac{1}{2}e^{2\tan^{-1}x} + C.$$

56. We have

$$I = \int \frac{x^3 + x - 2}{x^2 - 7}\,dx = \int \frac{x^3 - 7x + 8x - 2}{x^2 - 7}\,dx$$
$$= \int \left(x + \frac{8x - 2}{x^2 - 7}\right) dx.$$

Since

$$\frac{8x - 2}{x^2 - 7} = \frac{A}{x + \sqrt{7}} + \frac{B}{x - \sqrt{7}} = \frac{(A + B)x + (B - A)\sqrt{7}}{x^2 - 7}$$
$$\Rightarrow \begin{cases} A + B = 8 \\ B - A = -\dfrac{2}{\sqrt{7}} \end{cases} \Rightarrow A = 4 + \frac{1}{\sqrt{7}},\ \ B = 4 - \frac{1}{\sqrt{7}},$$

therefore

$$I = \int \left(x + \frac{8x - 2}{x^2 - 7}\right) dx$$
$$= \frac{x^2}{2} + \left(4 + \frac{1}{\sqrt{7}}\right) \int \frac{dx}{x + \sqrt{7}} + \left(4 - \frac{1}{\sqrt{7}}\right) \int \frac{dx}{x - \sqrt{7}}$$
$$= \frac{x^2}{2} + \left(4 + \frac{1}{\sqrt{7}}\right) \ln|x + \sqrt{7}| + \left(4 - \frac{1}{\sqrt{7}}\right) \ln|x - \sqrt{7}| + C.$$

57. $$\int \frac{\ln(3 + x^2)}{3 + x^2}\, x\,dx \quad \text{Let } u = \ln(3 + x^2),\quad du = \frac{2x\,dx}{3 + x^2}$$
$$= \frac{1}{2}\int u\,du = \frac{u^2}{4} + C = \frac{1}{4}\Big(\ln(3 + x^2)\Big)^2 + C.$$

58. $$\int \cos^7 x\,dx = \int (1 - \sin^2 x)^3 \cos x\,dx \quad \text{Let } u = \sin x,\quad du = \cos x\,dx$$
$$= \int (1 - u^2)^3\,du = \int (1 - 3u^2 + 3u^4 - u^6)\,du$$
$$= u - u^3 + \tfrac{3}{5}u^5 - \tfrac{1}{7}u^7 + C$$
$$= \sin x - \sin^3 x + \tfrac{3}{5}\sin^5 x - \tfrac{1}{7}\sin^7 x + C.$$

59. $$\int \frac{\sin^{-1}(x/2)}{(4 - x^2)^{1/2}}\,dx \quad \text{Let } u = \sin^{-1}(x/2),\quad du = \frac{dx}{2\sqrt{1 - (x^2/4)}} = \frac{dx}{\sqrt{4 - x^2}}$$
$$= \int u\,du = \frac{u^2}{2} + C = \frac{1}{2}\Big(\sin^{-1}(x/2)\Big)^2 + C.$$

60. We have

$$\int \tan^4(\pi x)\,dx = \int \tan^2(\pi x)[\sec^2(\pi x) - 1]\,dx$$
$$= \int \tan^2(\pi x)\sec^2(\pi x)\,dx - \int [\sec^2(\pi x) - 1]\,dx$$
$$= \frac{1}{3\pi}\tan^3(\pi x) - \frac{1}{\pi}\tan(\pi x) + x + C.$$

61. $$\int \frac{(x + 1)\,dx}{\sqrt{x^2 + 6x + 10}}$$
$$= \int \frac{(x + 3 - 2)\,dx}{\sqrt{(x + 3)^2 + 1}} \quad \text{Let } u = x + 3,\quad du = dx$$
$$= \int \frac{(u - 2)\,du}{\sqrt{u^2 + 1}}$$
$$= \sqrt{u^2 + 1} - 2\int \frac{du}{\sqrt{u^2 + 1}} \quad \text{Let } u = \tan\theta,\quad du = \sec^2\theta\,d\theta$$
$$= \sqrt{x^2 + 6x + 10} - 2\int \sec\theta\,d\theta$$
$$= \sqrt{x^2 + 6x + 10} - 2\ln|\sec\theta + \tan\theta| + C$$
$$= \sqrt{x^2 + 6x + 10} - 2\ln\Big(x + 3 + \sqrt{x^2 + 6x + 10}\Big) + C.$$

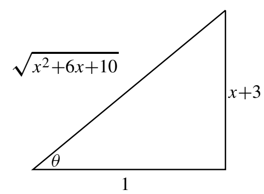

Fig. RT.61

62. $$\int e^x(1 - e^{2x})^{5/2}\,dx \quad \text{Let } e^x = \sin u,\quad e^x\,dx = \cos u\,du$$
$$= \int \cos^6 u\,du = \left(\frac{1}{2}\right)^3 \int (1 + \cos 2u)^3\,du$$
$$= \frac{1}{8}\int (1 + 3\cos 2u + 3\cos^2 2u + \cos^3 2u)\,du$$
$$= \frac{u}{8} + \frac{3}{16}\sin 2u + \frac{3}{16}\int (1 + \cos 4u)\,du +$$
$$\frac{1}{8}\int (1 - \sin^2 2u)\cos 2u\,du$$
$$= \frac{5u}{16} + \frac{3}{16}\sin 2u + \frac{3}{64}\sin 4u + \frac{\sin 2u}{16}$$
$$- \frac{1}{48}\sin^3 2u + C$$
$$= \frac{5}{16}\sin^{-1}(e^x) + \frac{1}{4}\sin[2\sin^{-1}(e^x)] +$$
$$\frac{3}{64}\sin[4\sin^{-1}(e^x)] - \frac{1}{48}\sin^3[2\sin^{-1}(e^x)] + C$$
$$= \frac{5}{16}\sin^{-1}(e^x) + \frac{1}{2}e^x\sqrt{1 - e^{2x}}$$
$$+ \frac{3}{16}e^x\sqrt{1 - e^{2x}}\Big(1 - 2e^{2x}\Big)$$
$$- \frac{1}{6}e^{3x}\Big(1 - e^{2x}\Big)^{3/2} + C.$$

63. $\displaystyle\int \frac{x^3\,dx}{(x^2+2)^{7/2}}$ Let $x = \sqrt{2}\tan\theta$, $dx = \sqrt{2}\sec^2\theta\,d\theta$

$$
\begin{aligned}
&= \int \frac{2\sqrt{2}\tan^3\theta\ \sqrt{2}\sec^2\theta\,d\theta}{8\sqrt{2}\sec^7\theta} \\
&= \frac{1}{2\sqrt{2}}\int \sin^3\theta\cos^2\theta\,d\theta \\
&= \frac{1}{2\sqrt{2}}\int (1-\cos^2\theta)\cos^2\theta\sin\theta\,d\theta \quad \text{Let } u=\cos\theta,\ du=-\sin\theta\,d\theta \\
&= \frac{1}{2\sqrt{2}}\int (u^4-u^2)\,du = \frac{1}{2\sqrt{2}}\left(\frac{u^5}{5}-\frac{u^3}{3}\right)+C \\
&= \frac{1}{2\sqrt{2}}\left(\frac{1}{5}\left(\frac{\sqrt{2}}{\sqrt{2+x^2}}\right)^5-\frac{1}{3}\left(\frac{\sqrt{2}}{\sqrt{2+x^2}}\right)^3\right)+C \\
&= \frac{2}{5(2+x^2)^{5/2}}-\frac{1}{3(2+x^2)^{3/2}}+C.
\end{aligned}
$$

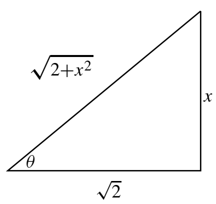

Fig. RT.63

64.

$$
\begin{aligned}
\int \frac{x^2}{2x^2-3}\,dx &= \frac{1}{2}\int\left(1+\frac{3}{2x^2-3}\right)dx \\
&= \frac{x}{2}+\frac{\sqrt{3}}{4}\int\left(\frac{1}{\sqrt{2}x-\sqrt{3}}-\frac{1}{\sqrt{2}x+\sqrt{3}}\right)dx \\
&= \frac{x}{2}+\frac{\sqrt{3}}{4\sqrt{2}}\ln\left|\frac{\sqrt{2}x-\sqrt{3}}{\sqrt{2}x+\sqrt{3}}\right|+C.
\end{aligned}
$$

65. $\displaystyle\int \frac{x^{1/2}}{1+x^{1/3}}\,dx$ Let $x=u^6$, $dx = 6u^5\,du$

$$
\begin{aligned}
&= 6\int \frac{u^8}{u^2+1}\,du \\
&= 6\int \frac{u^8+u^6-u^6-u^4+u^4+u^2-u^2-1+1}{u^2+1}\,du \\
&= 6\int\left(u^6-u^4+u^2-1+\frac{1}{u^2+1}\right)du \\
&= 6\left(\frac{u^7}{7}-\frac{u^5}{5}+\frac{u^3}{3}-u+\tan^{-1}u\right)+C \\
&= \frac{6}{7}x^{7/6}-\frac{6}{5}x^{5/6}+2\sqrt{x}-6x^{1/6}+6\tan^{-1}x^{1/6}+C.
\end{aligned}
$$

66. We have

$$
\int \frac{dx}{x(x^2+x+1)^{1/2}}
$$

$$
= \int \frac{dx}{x[(x+\frac{1}{2})^2+\frac{3}{4}]^{1/2}} \quad \text{Let } x+\frac{1}{2}=\frac{\sqrt{3}}{2}\tan\theta,\ dx=\frac{\sqrt{3}}{2}\sec^2\theta\,d\theta
$$

$$
\begin{aligned}
&= \int \frac{\frac{\sqrt{3}}{2}\sec^2\theta\,d\theta}{\left(\frac{\sqrt{3}}{2}\tan\theta-\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\sec\theta\right)} \\
&= \int \frac{2\sec\theta\,d\theta}{\sqrt{3}\tan\theta-1} = 2\int\frac{d\theta}{\sqrt{3}\sin\theta-\cos\theta} \\
&= 2\int \frac{\sqrt{3}\sin\theta+\cos\theta}{3\sin^2\theta-\cos^2\theta}\,d\theta \\
&= 2\sqrt{3}\int\frac{\sin\theta\,d\theta}{3\sin^2\theta-\cos^2\theta}+2\int\frac{\cos\theta\,d\theta}{3\sin^2\theta-\cos^2\theta} \\
&= 2\sqrt{3}\int\frac{\sin\theta\,d\theta}{3-4\cos^2\theta}+2\int\frac{\cos\theta\,d\theta}{4\sin^2\theta-1}
\end{aligned}
$$

Let $u=\cos\theta$, $du=-\sin\theta\,d\theta$ in the first integral; let $v=\sin\theta$, $dv=\cos\theta\,d\theta$ in the second integral.

$$
\begin{aligned}
&= -2\sqrt{3}\int\frac{du}{3-4u^2}+2\int\frac{dv}{4v^2-1} \\
&= -\frac{\sqrt{3}}{2}\int\frac{du}{\frac{3}{4}-u^2}-\frac{1}{2}\int\frac{du}{\frac{1}{4}-v^2} \\
&= -\frac{\sqrt{3}}{2}\left(\frac{1}{2}\right)\left(\frac{2}{\sqrt{3}}\right)\ln\left|\frac{\cos\theta+\frac{\sqrt{3}}{2}}{\cos\theta-\frac{\sqrt{3}}{2}}\right| \\
&\qquad -\frac{1}{2}\left(\frac{1}{2}\right)(2)\ln\left|\frac{\sin\theta+\frac{1}{2}}{\sin\theta-\frac{1}{2}}\right|+C \\
&= \frac{1}{2}\ln\left|\frac{\left(\cos\theta-\frac{\sqrt{3}}{2}\right)\left(\sin\theta-\frac{1}{2}\right)}{\left(\cos\theta+\frac{\sqrt{3}}{2}\right)\left(\sin\theta+\frac{1}{2}\right)}\right|+C.
\end{aligned}
$$

Since $\sin\theta = \dfrac{2x+1}{2\sqrt{x^2+x+1}}$ and $\cos\theta = \dfrac{\sqrt{3}}{2\sqrt{x^2+x+1}}$, therefore

$$
\int\frac{dx}{x(x^2+x+1)^{1/2}} = \frac{1}{2}\ln\left|\frac{(x+2)-2\sqrt{x^2+x+1}}{(x+2)+2\sqrt{x^2+x+1}}\right|+C.
$$

67. $\displaystyle\int \frac{1+x}{1+\sqrt{x}}\,dx$ Let $x = u^2$, $dx = 2u\,du$

$$= 2\int \frac{u(1+u^2)}{1+u}\,du$$
$$= 2\int \frac{u^3+u^2-u^2-u+2u+2-2}{1+u}\,du$$
$$= 2\int \left(u^2-u+2-\frac{2}{1+u}\right)du$$
$$= 2\left(\frac{u^3}{3}-\frac{u^2}{2}+2u-2\ln|1+u|\right)+C$$
$$= \frac{2}{3}x^{3/2}-x+4\sqrt{x}-4\ln(1+\sqrt{x})+C.$$

68. $\displaystyle\int \frac{x\,dx}{4x^4+4x^2+5}$ Let $u = x^2$, $du = 2x\,dx$

$$= \frac{1}{2}\int \frac{du}{4u^2+4u+5}$$
$$= \frac{1}{2}\int \frac{du}{(2u+1)^2+4} \quad \text{Let } w = 2u+1,\ dw = 2du$$
$$= \frac{1}{4}\int \frac{dw}{w^2+4} = \frac{1}{8}\tan^{-1}\left(\frac{w}{2}\right)+C$$
$$= \frac{1}{8}\tan^{-1}\left(x^2+\frac{1}{2}\right)+C.$$

69. $\displaystyle\int \frac{x\,dx}{(x^2-4)^2}$ Let $u = x^2-4$, $du = 2x\,dx$

$$= \frac{1}{2}\int \frac{du}{u^2} = -\frac{1}{2u}+C$$
$$= -\frac{1}{2(x^2-4)}+C = -\frac{1}{2x^2-8}+C.$$

70. Use the partial fraction decomposition

$$\frac{1}{x^3+x^2+x} = \frac{A}{x}+\frac{Bx+C}{x^2+x+1}$$
$$= \frac{A(x^2+x+1)+Bx^2+Cx}{x^3+x^2+x}$$
$$\Rightarrow \begin{cases} A+B=0 \\ A+C=0 \\ A=1 \end{cases} \Rightarrow A=1,\ B=-1,\ C=-1.$$

Therefore,

$$\int \frac{dx}{x^3+x^2+x}$$
$$= \int \frac{dx}{x} - \int \frac{x+1}{x^2+x+1}\,dx \quad \text{Let } u = x+\tfrac{1}{2},\ du = dx$$
$$= \ln|x| - \int \frac{u+\frac{1}{2}}{u^2+\frac{3}{4}}\,du$$
$$= \ln|x| - \frac{1}{2}\ln\left(x^2+x+1\right) - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)+C.$$

71. $\displaystyle\int x^2\tan^{-1}x\,dx$

$U = \tan^{-1}x \quad dV = x^2\,dx$

$dU = \dfrac{dx}{1+x^2} \quad V = \dfrac{x^3}{3}$

$$= \frac{x^3}{3}\tan^{-1}x - \frac{1}{3}\int \frac{x^3\,dx}{1+x^2}$$
$$= \frac{x^3}{3}\tan^{-1}x - \frac{1}{3}\int \frac{x^3+x-x}{x^2+1}\,dx$$
$$= \frac{x^3}{3}\tan^{-1}x - \frac{1}{6}x^2 + \frac{1}{6}\ln(1+x^2)+C.$$

72. $\displaystyle\int e^x\sec(e^x)\,dx$ Let $u = e^x$, $du = e^x\,dx$

$$= \int \sec u\,du = \ln|\sec u+\tan u|+C$$
$$= \ln|\sec(e^x)+\tan(e^x)|+C.$$

73. $\displaystyle I = \int \frac{dx}{4\sin x-3\cos x}$ Let $z = \tan\dfrac{x}{2}$, $dx = \dfrac{2\,dz}{1+z^2}$

$\cos x = \dfrac{1-z^2}{1+z^2}$, $\sin x = \dfrac{2z}{1+z^2}$

$$= \int \frac{\dfrac{2\,dz}{1+z^2}}{\dfrac{8z}{1+z^2}-\dfrac{3-3z^2}{1+z^2}}$$
$$= 2\int \frac{dz}{3z^2+8z-3} = 2\int \frac{dz}{(3z-1)(z+3)}.$$

$$\frac{1}{(3z-1)(z+3)} = \frac{A}{3z-1}+\frac{B}{z+3}$$
$$= \frac{Az+3A+3Bz-B}{(3z-1)(z+3)}$$
$$\Rightarrow \begin{cases} A+3B=0 \\ 3A-B=1 \end{cases} \Rightarrow \begin{cases} A=3/10 \\ B=-1/10. \end{cases}$$

Thus

$$I = \frac{3}{5}\int \frac{dz}{3z-1} - \frac{1}{5}\int \frac{dz}{z+3}$$
$$= \frac{1}{5}\ln|3z-1| - \frac{1}{5}\ln|z+3|+C$$
$$= \frac{1}{5}\ln\left|\frac{3\tan^{-1}(x/2)-1}{\tan^{-1}(x/2)+3}\right|+C.$$

74. $\displaystyle\int \frac{dx}{x^{1/3}-1}$ Let $x = (u+1)^3$, $dx = 3(u+1)^2\,du$

$$= 3\int \frac{(u+1)^2}{u}\,du = 3\int \left(u+2+\frac{1}{u}\right)du$$
$$= 3\left(\frac{u^2}{2}+2u+\ln|u|\right)+C$$
$$= \frac{3}{2}(x^{1/3}-1)^2+6(x^{1/3}-1)+3\ln|x^{1/3}-1|+C.$$

75. $\displaystyle \int \frac{dx}{\tan x + \sin x}$

$\displaystyle = \int \frac{\cos x\,dx}{\sin x(1+\cos x)}$ Let $z = \tan(x/2)$, $\displaystyle dx = \frac{2\,dz}{1+z^2}$

$\displaystyle \cos x = \frac{1-z^2}{1+z^2}, \quad \sin x = \frac{2z}{1+z^2}$

$$= \int \frac{\frac{1-z^2}{1+z^2}\frac{2\,dz}{1+z^2}}{\frac{2z}{1+z^2}\left(1+\frac{1-z^2}{1+z^2}\right)}$$

$$= \int \frac{(1-z^2)\,dz}{z(1+z^2+1-z^2)} = \frac{1}{2}\int \frac{1-z^2}{z}\,dz$$

$$= \frac{1}{2}\ln|z| - \frac{z^2}{4} + C$$

$$= \frac{1}{2}\ln\left|\tan\frac{x}{2}\right| - \frac{1}{4}\left(\tan\frac{x}{2}\right)^2 + C.$$

Remark: Since

$$\tan^2\frac{x}{2} = \frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}} = \frac{1-\cos x}{1+\cos x},$$

the answer can also be written as

$$\frac{1}{4}\ln\left|\frac{1-\cos x}{1+\cos x}\right| - \frac{1}{4}\cdot\frac{1-\cos x}{1+\cos x} + C.$$

76. $\displaystyle \int \frac{x\,dx}{\sqrt{3-4x-4x^2}} = \int \frac{x\,dx}{\sqrt{4-(2x+1)^2}}$ Let $u = 2x+1$, $du = 2\,dx$

$$= \frac{1}{4}\int \frac{u-1}{\sqrt{4-u^2}}\,du$$

$$= -\frac{1}{4}\sqrt{4-u^2} - \frac{1}{4}\sin^{-1}\left(\frac{u}{2}\right) + C$$

$$= -\frac{1}{4}\sqrt{3-4x-4x^2} - \frac{1}{4}\sin^{-1}\left(x+\frac{1}{2}\right) + C.$$

77. $\displaystyle \int \frac{\sqrt{x}}{1+x}\,dx$ Let $x = u^2$, $dx = 2u\,du$

$$= 2\int \frac{u^2\,du}{1+u^2} = 2\int\left(1-\frac{1}{1+u^2}\right)du$$

$$= 2\left(u - \tan^{-1}u\right) + C = 2\sqrt{x} - 2\tan^{-1}\sqrt{x} + C.$$

78. $\displaystyle \int \sqrt{1+e^x}\,dx$ Let $u^2 = 1+e^x$, $2u\,du = e^x\,dx$

$$= \int \frac{2u^2\,du}{u^2-1} = \int\left(2+\frac{2}{u^2-1}\right)du$$

$$= \int\left(2+\frac{1}{u-1}-\frac{1}{u+1}\right)du$$

$$= 2u + \ln\left|\frac{u-1}{u+1}\right| + C$$

$$= 2\sqrt{1+e^x} + \ln\left|\frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1}\right| + C.$$

79. $\displaystyle I = \int \frac{x^4\,dx}{x^3-8} = \int\left(x+\frac{8x}{x^3-8}\right)dx.$

$$\frac{8x}{x^3-8} = \frac{A}{x-2} + \frac{Bx+C}{x^2+2x+4}$$

$$= \frac{Ax^2+2Ax+4A+Bx^2-2Bx+Cx-2C}{x^3-8}$$

$$\Rightarrow \begin{cases} A+B=0 \\ 2A-2B+C=8 \\ 4A-2C=0 \end{cases} \Rightarrow \begin{cases} B=-A \\ C=2A \\ 6A=8 \end{cases}$$

Thus $A = 4/3$, $B = -4/3$, $C = 8/3$. We have

$$I = \frac{x^2}{2} + \frac{4}{3}\int\frac{dx}{x-2} - \frac{4}{3}\int\frac{x-2}{x^2+2x+4}\,dx$$

$$= \frac{x^2}{2} + \frac{4}{3}\ln|x-2| - \frac{4}{3}\int\frac{x+1-3}{(x+1)^2+3}\,dx$$

$$= \frac{x^2}{2} + \frac{4}{3}\ln|x-2| - \frac{2}{3}\ln(x^2+2x+4)$$

$$+ \frac{4}{\sqrt{3}}\tan^{-1}\frac{x+1}{\sqrt{3}} + K.$$

80. By the procedure used in Example 4 of Section 7.1,

$$\int e^x\cos x\,dx = \tfrac{1}{2}e^x(\sin x+\cos x) + C,$$

$$\int e^x\sin x\,dx = \tfrac{1}{2}e^x(\sin x-\cos x) + C.$$

Now

$$\int xe^x\cos x\,dx$$

$U = x \quad dV = e^x\cos x\,dx$

$dU = dx \quad V = \frac{1}{2}e^x(\sin x+\cos x)$

$$= \tfrac{1}{2}xe^x(\sin+\cos x) - \tfrac{1}{2}\int e^x(\sin x+\cos x)\,dx$$

$$= \tfrac{1}{2}xe^x(\sin+\cos x)$$

$$- \tfrac{1}{4}e^x(\sin x-\cos x+\sin x+\cos x) + C$$

$$= \tfrac{1}{2}xe^x(\sin x+\cos x) - \tfrac{1}{2}e^x\sin x + C.$$

Other Review Exercises 6 (page 366)

1. $$\frac{d}{dx}e^x\Big[(ax+b)\cos x+(cx+d)\sin x\Big]$$
$$= e^x\Big[(ax+b)\cos x+(cx+d)\sin x+a\cos x+c\sin x$$
$$-(ax+b)\sin x+(cx+d)\cos x\Big]$$
$$= e^x\Big[\Big((a+c)x+b+a+d\Big)\cos x$$
$$+\Big((c-a)x+d+c-b\Big)\sin x\Big]$$

If $a+c=1$, $b+a+d=0$, $c-a=0$, and $d+c-b=0$, then $a=c=-d=1/2$ and $b=0$. Thus

$$I=\int xe^x\cos x\,dx=\frac{e^x}{2}\Big[x\cos x+(x-1)\sin x\Big]+C.$$

If $a+c=0$, $b+a+d=0$, $c-a=1$, and $d+c-b=0$, then $b=c=-a=1/2$ and $d=0$. Thus

$$J=\int xe^x\sin x\,dx=\frac{e^x}{2}\Big[x\sin x-(x-1)\cos x\Big]+C.$$

2. $$\int_0^\infty x^r e^{-x}\,dx$$
$$=\lim_{\substack{c\to0+\\R\to\infty}}\int_c^R x^re^{-x}\,dx$$
$$U=x^r \qquad dV=e^{-x}\,dx$$
$$dU=rx^{r-1}\,dr \qquad V=-e^{-x}$$
$$=\lim_{\substack{c\to0+\\R\to\infty}}-x^re^{-x}\Big|_c^R+r\int_0^\infty x^{r-1}e^{-x}\,dx$$
$$=\lim_{c\to0+}c^re^{-c}+r\int_0^\infty x^{r-1}e^{-x}\,dx$$

because $\lim_{R\to\infty}R^re^{-R}=0$ for any r. In order to ensure that $\lim_{c\to0+}c^re^{-c}=0$ we must have $\lim_{c\to0+}c^r=0$, so we need $r>0$.

3. $$\int_0^{\pi/2}\csc x\,dx=\lim_{c\to0+}-\ln|\csc x+\cot x|\Big|_c^{\pi/2}$$
$$=\lim_{c\to0+}\ln|\csc c+\cot c|=\infty \text{ (diverges)}$$

4. $$\int_1^\infty\frac{dx}{x+x^3}=\lim_{R\to\infty}\int_1^R\left(\frac{1}{x}-\frac{x}{1+x^2}\right)dx$$
$$=\lim_{R\to\infty}\left(\ln|x|-\frac12\ln(1+x^2)\right)\Bigg|_1^R$$
$$=\lim_{R\to\infty}\frac12\left(\ln\frac{R^2}{1+R^2}+\ln 2\right)=\frac{\ln 2}{2}$$

5. $$\int_0^1\sqrt{x}\ln x\,dx \quad \text{Let } x=u^2,\ dx=2u\,du$$
$$=\int_0^1 u(2\ln u)2u\,du$$
$$=4\int_0^1 u^2\ln u\,du$$
$$U=\ln u \qquad dV=u^2\,du$$
$$dU=\frac{du}{u} \qquad V=\frac{u^3}{3}$$
$$=4\lim_{c\to0+}\left(\frac{u^3}{3}\ln u\Big|_c^1-\frac13\int_c^1u^2\,du\right)$$
$$=-\frac43\lim_{c\to0+}c^3\ln c-\frac49(1-c^3)=-\frac49$$

6. $$\int_0^1\frac{dx}{x\sqrt{1-x^2}}>\int_0^1\frac{dx}{x}=\infty \text{ (diverges)}$$
Therefore $\displaystyle\int_{-1}^1\frac{dx}{x\sqrt{1-x^2}}$ diverges.

7. $$I=\int_0^\infty\frac{dx}{\sqrt{x}e^x}=\int_0^1+\int_1^\infty=I_1+I_2$$
$$I_1=\int_0^1\frac{dx}{\sqrt{x}e^x}<\int_0^1\frac{dx}{\sqrt{x}}=2$$
$$I_2=\int_1^\infty\frac{dx}{\sqrt{x}e^x}<\int_1^\infty e^{-x}\,dx=\frac1e$$
Thus I converges, and $I<2+(1/e)$.

8. Volume $=\int_0^{60}A(x)\,dx$. The approximation is

$$T_6=\frac{10}{2}\Big[10,200+2(9,200+8,000+7,100$$
$$+4,500+2,400)+100\Big]$$
$$\approx 364,000 \text{ m}^3.$$

9. $$S_6=\frac{10}{3}\Big[10,200+4(9,200+7,100+2,400)$$
$$+2(8,000+4,500)+100\Big]$$
$$\approx 367,000 \text{ m}^3$$

10. $$I=\int_0^1\sqrt{2+\sin(\pi x)}\,dx$$
$$T_4=\frac18\Big[\sqrt2+2(\sqrt{2+\sin(\pi/4)}+\sqrt{2+\sin(\pi/2)}$$
$$+\sqrt{2+\sin(3\pi/4)}+\sqrt2\Big]$$
$$\approx 1.609230$$
$$M_4=\frac14\Big[\sqrt{2+\sin(\pi/8)}+\sqrt{2+\sin(3\pi/8)}$$
$$\sqrt{2+\sin(5\pi/8)}+\sqrt{2+\sin(7\pi/8)}\Big]$$
$$\approx 1.626765$$
$$I\approx 1.6$$

11. $T_8 = \frac{1}{2}(T_4 + M_4) \approx 1.617996$
$S_8 = \frac{1}{3}(T_4 + 2M_4) \approx 1.62092$
$I \approx 1.62$

12. $I = \int_{1/2}^{\infty} \frac{x^2}{x^5 + x^3 + 1}\,dx$ Let $x = 1/t$, $dx = -(1/t^2)\,dt$

$$= \int_0^2 \frac{(1/t^4)\,dt}{(1/t^5) + (1/t^3) + 1} = \int_0^2 \frac{t\,dt}{t^5 + t^2 + 1}$$

$T_4 \approx 0.4444$ $M_4 \approx 0.4799$
$T_8 \approx 0.4622$ $M_8 \approx 0.4708$
$S_8 \approx 0.4681$ $S_{16} \approx 0.4680$
$I \approx 0.468$ to 3 decimal places

13. a) $T_4 = 1\left(\frac{0.730}{2} + 1.001 + 1.332 + 1.729 + \frac{2.198}{2}\right) = 5.526$

$S_4 = \frac{1}{3}\Big(0.730 + 2.198 + 4(1.001 + 1.729) + 2(1.332)\Big) = 5.504.$

b) If $T_8 = 5.5095$, then $S_8 = \frac{4T_8 - T_4}{3} = 5.504.$

c) Yes, $S_4 = S_8$ suggests that S_n may be independent of n, which is consistent with a polynomial of degree not exceeding 3.

Challenging Problems 6 (page 367)

1. a) Long division of $x^2 + 1$ into $x^4(1-x)^4 = x^8 - 4x^7 + 6x^6 - 4x^5 + x^4$ yields

$$\frac{x^4(1-x)^4}{x^2+1} = x^6 - 4x^5 + 5x^4 - 4x^2 + 4 - \frac{4}{x^2+1}.$$

Integrating both sides over [0, 1] leads at once to

$$\int_0^1 \frac{x^4(1-x)^4}{x^2+1}\,dx = \frac{22}{7} - 4\tan^{-1}1 = \frac{22}{7} - \pi.$$

Since $\frac{x^4(1-x)^4}{x^2+1} > 0$ on (0, 1), $\frac{22}{7} - \pi > 0$, and so $\pi < \frac{22}{7}$.

b) If $I = \int_0^1 x^4(1-x)^4\,dx$, then since $1 < x^2 + 1 < 2$ on (0, 1), we have

$$I > \int_0^1 \frac{x^4(1-x)^4}{x^2+1}\,dx > \frac{I}{2}.$$

Thus $I > (22/7) - \pi > I/2$, or

$$\frac{22}{7} - I < \pi < \frac{22}{7} - \frac{I}{2}.$$

c) $I = \int_0^1 (x^8 - 4x^7 + 6x^6 - 4x^5 + x^4)\,dx = \frac{1}{630}$. Thus

$$\frac{22}{7} - \frac{1}{630} < \pi < \frac{22}{7} - \frac{1}{1260}.$$

2. a) $I_n = \int (1-x^2)^n\,dx$

$U = (1-x^2)^n$ $dV = dx$
$dU = -2nx(1-x^2)^{n-1}\,dx$ $V = x$

$$\begin{aligned}
&= x(1-x^2)^n + 2n\int x^2(1-x^2)^{n-1}\,dx\\
&= x(1-x^2)^n - 2n\int (1-x^2-1)(1-x^2)^{n-1}\,dx\\
&= x(1-x^2)^n - 2nI_n + 2nI_{n-1}, \quad \text{so}
\end{aligned}$$

$$I_n = \frac{1}{2n+1}x(1-x^2)^n + \frac{2n}{2n+1}I_{n-1}.$$

b) Let $J_n = \int_0^1 (1-x^2)^n\,dx$. Observe that $J_0 = 1$. By (a), if $n > 0$, then we have

$$J_n = \frac{x(1-x^2)^n}{2n+1}\bigg|_0^1 + \frac{2n}{2n+1}J_{n-1} = \frac{2n}{2n+1}J_{n-1}.$$

Therefore,

$$\begin{aligned}
J_n &= \frac{2n}{2n+1}\cdot\frac{2n-2}{2n-1}\cdots\frac{4}{5}\cdot\frac{2}{3}J_0\\
&= \frac{[(2n)(2n-2)\cdots(4)(2)]^2}{(2n+1)!} = \frac{2^{2n}(n!)^2}{(2n+1)!}.
\end{aligned}$$

c) From (a):

$$I_{n-1} = \frac{2n+1}{2n}I_n - \frac{1}{2n}x(1-x^2)^n.$$

Thus

$$\begin{aligned}
&\int (1-x^2)^{-3/2}\,dx = I_{-3/2}\\
&= \frac{2(-1/2)+1}{-1}I_{-1/2} - \frac{1}{-1}x(1-x^2)^{-1/2}\\
&= \frac{x}{\sqrt{1-x^2}}.
\end{aligned}$$

3. a) $x^4+x^2+1 = (x^2+1)^2 - x^2 = (x^2-x+1)(x^2+x+1)$. Thus

$$\begin{aligned}\int \frac{x^2+1}{x^4+x^2+1} &= \int \frac{x^2+1}{(x^2-x+1)(x^2+x+1)}\,dx\\ &= \frac{1}{2}\int\left(\frac{1}{x^2-x+1}+\frac{1}{x^2+x+1}\right)dx\\ &= \frac{1}{2}\int\left(\frac{1}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}+\frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\right)dx\\ &= \frac{1}{\sqrt{3}}\left(\tan^{-1}\frac{2x-1}{\sqrt{3}}+\tan^{-1}\frac{2x+1}{\sqrt{3}}\right)+C.\end{aligned}$$

b) $x^4+1 = (x^2+1)^2 - 2x^2 = (x^2-\sqrt{2}x+1)(x^2+\sqrt{2}x+1)$. Thus

$$\begin{aligned}\int \frac{x^2+1}{x^4+1} &= \int \frac{x^2+1}{(x^2-\sqrt{2}x+1)(x^2+\sqrt{2}x+1)}\,dx\\ &= \frac{1}{2}\int\left(\frac{1}{x^2-\sqrt{2}x+1}+\frac{1}{x^2+\sqrt{2}x+1}\right)dx\\ &= \frac{1}{2}\int\left(\frac{1}{\left(x-\frac{1}{\sqrt{2}}\right)^2+\frac{3}{4}}+\frac{1}{\left(x+\frac{1}{\sqrt{2}}\right)^2+\frac{3}{4}}\right)dx\\ &= \frac{1}{\sqrt{3}}\left(\tan^{-1}\frac{2x-\sqrt{2}}{\sqrt{3}}+\tan^{-1}\frac{2x+\sqrt{2}}{\sqrt{3}}\right)+C.\end{aligned}$$

4. $$\begin{aligned}I_{m,n} &= \int_0^1 x^m(\ln x)^n\,dx \quad \text{Let } x = e^{-t}\\ &\qquad\qquad\qquad\qquad\quad dx = -e^{-t}\,dt\\ &= \int_0^\infty e^{-mt}(-t)^n e^{-t}\,dt\\ &= (-1)^n\int_0^\infty t^n e^{-(m+1)t}\,dt \quad \text{Let } u=(m+1)t\\ &\qquad\qquad\qquad\qquad\qquad\quad du = (m+1)\,dt\\ &= \frac{(-1)^n}{(m+1)^n}\int_0^\infty u^n e^{-u}\,du\\ &= \frac{(-1)^n}{(m+1)^n}\Gamma(n+1) \quad \text{(see \#50 in Section 7.5)}\\ &= \frac{(-1)^n n!}{(m+1)^n}.\end{aligned}$$

5. a) $0 < I_n = \int_0^1 x^n e^{-x}\,dx < \int_0^1 x^n\,dx = \frac{1}{n+1}$, because $0 < e^{-x} < 1$ on $(0,1)$. Thus $\lim_{n\to\infty} I_n = 0$ by the Squeeze Theorem.

b) $$\begin{aligned}I_0 &= \int_0^1 e^{-x}\,dx = -e^{-x}\Big|_0^1 = 1-\frac{1}{e}\\ I_n &= \int_0^1 x^n e^{-x}\,dx\\ &\qquad U = x^n \qquad\qquad dV = e^{-x}\,dx\\ &\qquad dU = nx^{n-1}\,dx \qquad V = -e^{-x}\\ &= -x^n e^{-x}\Big|_0^1 + n\int_0^1 x^{n-1}e^{-x}\,dx\\ &= nI_{n-1}-\frac{1}{e} \quad \text{if } n\ge 1\end{aligned}$$

c) The formula

$$I_n = n!\left(1-\frac{1}{e}\sum_{j=0}^{n}\frac{1}{j!}\right)$$

holds for $n=0$ by part (b). Assume that it holds for some integer $n = k \ge 0$. Then by (b),

$$\begin{aligned}I_{k+1} = (k+1)I_k - \frac{1}{e} &= (k+1)k!\left(1-\frac{1}{e}\sum_{j=0}^{k}\frac{1}{j!}\right)-\frac{1}{e}\\ &= (k+1)!\left(1-\frac{1}{e}\sum_{j=0}^{k}\frac{1}{j!}-\frac{1}{e(k+1)!}\right)\\ &= (k+1)!\left(1-\frac{1}{e}\sum_{j=0}^{k+1}\frac{1}{j!}\right).\end{aligned}$$

Thus the formula holds for all $n \ge 0$, by induction.

d) Since $\lim_{n\to\infty} I_n = 0$, we must have

$$\lim_{n\to\infty}\left(1-\frac{1}{e}\sum_{j=0}^{n}\frac{1}{j!}\right) = 0.$$

Thus $e = \lim_{n\to\infty}\sum_{j=0}^{n}\frac{1}{j!}$.

6. $I = \int_0^1 e^{-Kx}\,dx = \frac{e^{-Kx}}{-K}\Big|_0^1 = \frac{1}{K}\left(1-\frac{1}{e^K}\right)$.

For very large K, the value of I is very small ($I < 1/K$). However,

$$\begin{aligned}T_{100} &= \frac{1}{100}(1+\cdots) > \frac{1}{100}\\ S_{100} &= \frac{1}{300}(1+\cdots) > \frac{1}{300}\\ M_{100} &= \frac{1}{100}(e^{-K/200}+\cdots) < \frac{1}{100}.\end{aligned}$$

In each case the $\cdots$ represent terms much less than the first term (shown) in the sum. Evidently M_{100} is smallest if k is much greater than 100, and is therefore the best approximation. T_{100} appears to be the worst.

7. a) Let $f(x) = Ax^5 + Bx^4 + Cx^3 + Dx^2 + Ex + F$. Then

$$\int_{-h}^{h} f(x)\,dx = 2\left(\frac{Bh^5}{5} + \frac{Dh^3}{3} + Fh\right).$$

Also

$$\begin{aligned}
&2h\Big[af(-h) + bf(-h/2) + cf(0) + bf(h/2) + af(h)\Big]\\
&= 2\Big[a\left(2Bh^5 + 2Dh^3 + 2F\right)\\
&\quad + b\left(\frac{2Bh^5}{16} + \frac{2Dh^3}{4} + 2F\right) + cFh\Big].
\end{aligned}$$

These expressions will be identical if the coefficients of like powers of h on the two sides are identical. Thus

$$2a + \frac{2b}{16} = \frac{1}{5}, \quad 2a + \frac{2b}{4} = \frac{1}{3}, \quad 2a + 2b + c = 1.$$

Solving these equations, we get $a = 7/90$, $b = 16/45$, and $c = 2/15$. The approximation for the integral of any function f on $[m-h, m+h]$ is

$$\begin{aligned}
\int_{m-h}^{m+h} f(x)\,dx &\approx 2h\Big[\frac{7}{90}f(m-h) + \frac{16}{45}f(m-\tfrac{1}{2}h)\\
&\quad + \frac{2}{15}f(m) + \frac{16}{45}f(m+\tfrac{1}{2}h) + \frac{7}{90}f(m+h)\Big].
\end{aligned}$$

b) If $m = h = 1/2$, we obtain

$$\begin{aligned}
\int_0^1 e^{-x}\,dx &\approx 1\Big[\frac{7}{90}e^0 + \frac{16}{45}e^{-1/4} + \frac{2}{15}e^{-1/2}\\
&\quad + \frac{16}{45}e^{-3/4} + \frac{7}{90}e^{-1}\Big]\\
&\approx 0.63212087501.
\end{aligned}$$

With two intervals having $h = 1/4$ and $m = 1/4$ and $m = 3/4$, we get

$$\begin{aligned}
\int_0^1 e^{-x}\,dx &\approx \frac{1}{2}\Big[\frac{7}{90}e^0 + \frac{16}{45}e^{-1/8} + \frac{2}{15}e^{-1/4}\\
&\quad + \frac{16}{45}e^{-3/8} + \frac{7}{45}e^{-1/2}\\
&\quad + \frac{16}{45}e^{-5/8} + \frac{2}{15}e^{-3/4} + \frac{16}{45}e^{-7/8} + \frac{7}{90}e^{-1}\Big]\\
&\approx 0.63212055883.
\end{aligned}$$

8. a) $f'(x) < 0$ on $[1,\infty)$, and $\lim_{x\to\infty} f(x) = 0$. Therefore

$$\begin{aligned}
\int_1^\infty |f'(x)|\,dx &= -\int_1^\infty f'(x)\,dx\\
&= -\lim_{R\to\infty}\int_1^R f'(x)\,dx\\
&= \lim_{R\to\infty}(f(1) - f(R)) = f(1).
\end{aligned}$$

Thus

$$\left|\int_R^\infty f'(x)\cos x\,dx\right| \le \int_R^\infty |f'(x)|\,dx \to 0 \text{ as } R \to \infty.$$

Thus $\displaystyle\lim_{R\to\infty}\int_1^R f'(x)\cos x\,dx$ exists.

b) $\displaystyle\int_1^\infty f(x)\sin x\,dx$

$$\begin{array}{ll} U = f(x) & dV = \sin x\,dx\\ dU = f'(x)\,dx & V = -\cos x \end{array}$$

$$\begin{aligned}
&= \lim_{R\to\infty} f(x)\cos x\Big|_1^R + \int_1^\infty f'(x)\cos x\,dx\\
&= -f(1)\cos(1) + \int_1^\infty f'(x)\cos x\,dx;
\end{aligned}$$

the integral converges.

c) $f(x) = 1/x$ satisfies the conditions of part (a), so

$$\int_1^\infty \frac{\sin x}{x}\,dx \quad \text{converges}$$

by part (b). Similarly, it can be shown that

$$\int_1^\infty \frac{\cos(2x)}{x}\,dx \quad \text{converges.}$$

But since $|\sin x| \ge \sin^2 x = \frac{1}{2}(1 - \cos(2x))$, we have

$$\int_1^\infty \frac{|\sin x|}{x}\,dx \ge \int_1^\infty \frac{1-\cos(2x)}{2x}.$$

The latter integral diverges because $\int_1^\infty (1/x)\,dx$ diverges to infinity while $\int_1^\infty (\cos(2x))/(2x)\,dx$ converges. Therefore

$$\int_1^\infty \frac{|\sin x|}{x}\,dx \quad \text{diverges to infinity.}$$