CHAPTER 7. APPLICATIONS OF INTEGRATION

Section 7.1 Volumes of Solids of Revolution (page 376)

1. By slicing:

$$V = \pi \int_0^1 x^4\, dx = \frac{\pi}{5} \text{ cu. units.}$$

By shells:

$$\begin{aligned} V &= 2\pi \int_0^1 y(1 - \sqrt{y})\, dy \\ &= 2\pi \left(\frac{y^2}{2} - \frac{2y^{5/2}}{5} \right)\Bigg|_0^1 = \frac{\pi}{5} \text{ cu. units.} \end{aligned}$$

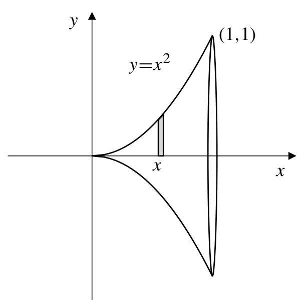

Fig. 7.1.1

2. Slicing:

$$\begin{aligned} V &= \pi \int_0^1 (1 - y)\, dy \\ &= \pi \left(y - \frac{1}{2} y^2 \right)\Bigg|_0^1 = \frac{\pi}{2} \text{ cu. units.} \end{aligned}$$

Shells:

$$\begin{aligned} V &= 2\pi \int_0^1 x^3\, dx \\ &= 2\pi \left(\frac{x^4}{4} \right)\Bigg|_0^1 = \frac{\pi}{2} \text{ cu. units.} \end{aligned}$$

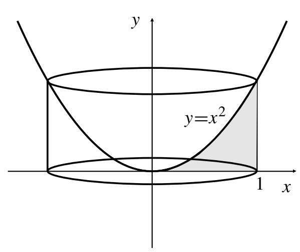

Fig. 7.1.2

3. By slicing:

$$\begin{aligned} V &= \pi \int_0^1 (x - x^4)\, dx \\ &= \pi \left(\frac{x^2}{2} - \frac{x^5}{5} \right)\Bigg|_0^1 = \frac{3\pi}{10} \text{ cu. units.} \end{aligned}$$

By shells:

$$\begin{aligned} V &= 2\pi \int_0^1 y(\sqrt{y} - y^2)\, dy \\ &= 2\pi \left(\frac{2y^{5/2}}{5} - \frac{y^4}{4} \right)\Bigg|_0^1 = \frac{3\pi}{10} \text{ cu. units.} \end{aligned}$$

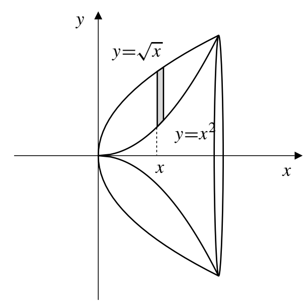

Fig. 7.1.3

4. Slicing:

$$\begin{aligned} V &= \pi \int_0^1 (y - y^4)\, dy \\ &= \pi \left(\frac{1}{2} y^2 - \frac{1}{5} y^5 \right)\Bigg|_0^1 = \frac{3\pi}{10} \text{ cu. units.} \end{aligned}$$

Shells:

$$\begin{aligned} V &= 2\pi \int_0^1 x(x^{1/2} - x^2)\, dx \\ &= 2\pi \left(\frac{2}{5} x^{5/2} - \frac{1}{4} x^4 \right)\Bigg|_0^1 = \frac{3\pi}{10} \text{ cu. units.} \end{aligned}$$

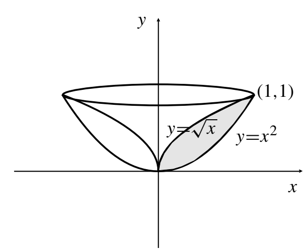

Fig. 7.1.4

5. a) About the x-axis:

$$\begin{aligned} V &= \pi \int_0^2 x^2(2-x)^2\,dx \\ &= \pi \int_0^2 (4x^2 - 4x^3 + x^4)\,dx \\ &= \pi \left(\frac{4x^3}{3} - x^4 + \frac{x^5}{5}\right)\Bigg|_0^2 = \frac{16\pi}{15} \text{ cu. units.} \end{aligned}$$

b) About the y-axis:

$$\begin{aligned} V &= 2\pi \int_0^2 x^2(2-x)\,dy \\ &= 2\pi \left(\frac{2x^3}{3} - \frac{x^4}{4}\right)\Bigg|_0^2 = \frac{8\pi}{3} \text{ cu. units.} \end{aligned}$$

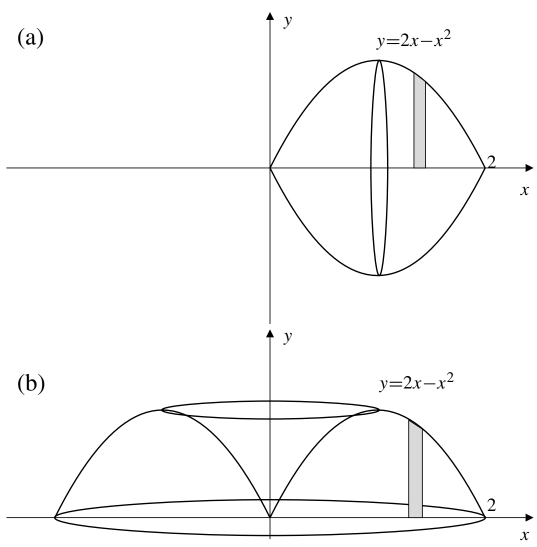

Fig. 7.1.5

6. Rotate about

a) the x-axis

$$\begin{aligned} V &= \pi \int_0^1 (x^2 - x^4)\,dx \\ &= \pi \left(\frac{1}{3}x^3 - \frac{1}{5}x^5\right)\Bigg|_0^1 = \frac{2\pi}{15} \text{ cu. units.} \end{aligned}$$

b) the y-axis

$$\begin{aligned} V &= 2\pi \int_0^1 x(x - x^2)\,dx \\ &= 2\pi \left(\frac{1}{3}x^3 - \frac{1}{4}x^4\right)\Bigg|_0^1 = \frac{\pi}{6} \text{ cu. units.} \end{aligned}$$

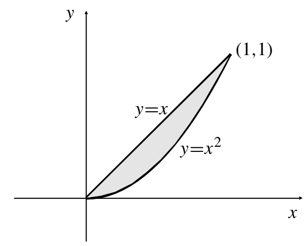

Fig. 7.1.6

7. a) About the x-axis:

$$\begin{aligned} V &= 2\pi \int_0^3 y(4y - y^2 - y)\,dy \\ &= 2\pi \left(y^3 - \frac{y^4}{4}\right)\Bigg|_0^3 = \frac{27\pi}{2} \text{ cu. units.} \end{aligned}$$

b) About the y-axis:

$$\begin{aligned} V &= \pi \int_0^3 \left[(4y - y^2)^2 - y^2\right] dy \\ &= \pi \int_0^3 (15y^2 - 8y^3 + y^4)\,dy \\ &= \pi \left(5y^3 - 2y^4 + \frac{y^5}{5}\right)\Bigg|_0^3 = \frac{108\pi}{5} \text{ cu. units.} \end{aligned}$$

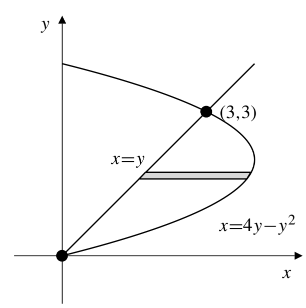

Fig. 7.1.7

8. Rotate about

a) the x-axis

$$\begin{aligned} V &= \pi \int_0^\pi [(1 + \sin x)^2 - 1]\,dx \\ &= \pi \int_0^\pi (2\sin x + \sin^2 x)\,dx \\ &= \left(-2\pi \cos x + \frac{\pi}{2}x - \frac{\pi}{4}\sin 2x\right)\Bigg|_0^\pi \\ &= 4\pi + \frac{1}{2}\pi^2 \text{ cu. units.} \end{aligned}$$

b) the y-axis

$$V = 2\pi \int_0^{\pi} x \sin x \, dx$$

$$U = x \qquad dV = \sin x \, dx$$
$$dU = dx \qquad V = -\cos x$$

$$= 2\pi \left[-x\cos x \Big|_0^{\pi} + \int_0^{\pi} \cos x \, dx \right]$$

$$= 2\pi^2 \text{ cu. units.}$$

9. a) About the x-axis:

$$V = \pi \int_0^1 \left(4 - \frac{1}{(1+x^2)^2}\right) dx \qquad \text{Let } x = \tan\theta, \quad dx = \sec^2\theta \, d\theta$$

$$= 4\pi - \pi \int_0^{\pi/4} \frac{\sec^2\theta}{\sec^4\theta} \, d\theta$$

$$= 4\pi - \pi \int_0^{\pi/4} \cos^2\theta \, d\theta$$

$$= 4\pi - \frac{\pi}{2}(\theta + \sin\theta\cos\theta)\Big|_0^{\pi/4}$$

$$= 4\pi - \frac{\pi^2}{8} - \frac{\pi}{4} = \frac{15\pi}{4} - \frac{\pi^2}{8} \text{ cu. units.}$$

b) About the y-axis:

$$V = 2\pi \int_0^1 x\left(2 - \frac{1}{1+x^2}\right) dx$$

$$= 2\pi \left(x^2 - \frac{1}{2}\ln(1+x^2)\right)\Big|_0^1$$

$$= 2\pi\left(1 - \frac{1}{2}\ln 2\right) = 2\pi - \pi \ln 2 \text{ cu. units.}$$

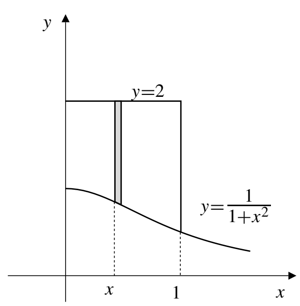

Fig. 7.1.9

10. By symmetry, rotation about the x-axis gives the same volume as rotation about the y-axis, namely

$$V = 2\pi \int_{1/3}^{3} x\left(\frac{10}{3} - x - \frac{1}{x}\right) dx$$

$$= 2\pi\left(\frac{5}{3}x^2 - \frac{1}{3}x^3 - x\right)\Big|_{1/3}^{3}$$

$$= \frac{512\pi}{81} \text{ cu. units.}$$

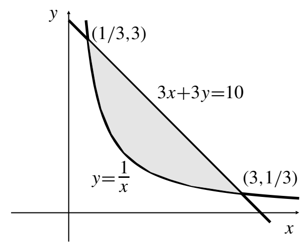

Fig. 7.1.10

11.

$$V = 2 \times 2\pi \int_0^1 (2-x)(1-x)\, dx$$

$$= 4\pi \int_0^1 (2 - 3x + x^2)\, dx$$

$$= 4\pi\left(2x - \frac{3x^2}{2} + \frac{x^3}{3}\right)\Big|_0^1 = \frac{10\pi}{3} \text{ cu. units.}$$

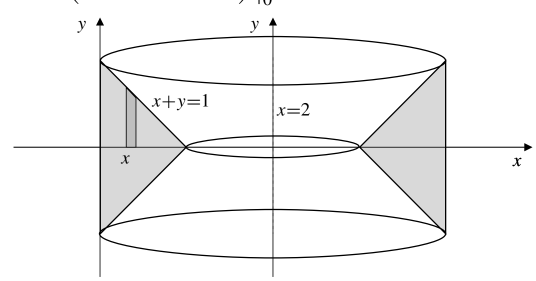

Fig. 7.1.11

12.

$$V = \pi \int_{-1}^{1} [(1)^2 - (x^2)^2]\, dx$$

$$= \pi\left(x - \frac{1}{5}x^5\right)\Big|_{-1}^{1}$$

$$= \frac{8\pi}{5} \text{ cu. units.}$$

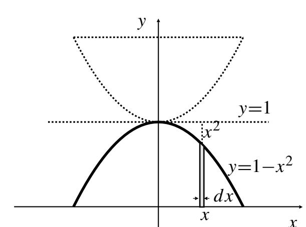

Fig. 7.1.12

13. The volume remaining is

$$V = 2 \times 2\pi \int_1^2 x\sqrt{4 - x^2}\, dx \qquad \text{Let } u = 4 - x^2, \quad du = -2x\, dx$$

$$= 2\pi \int_0^3 \sqrt{u}\, du = \frac{4\pi}{3} u^{3/2}\Big|_0^3 = 4\pi\sqrt{3} \text{ cu. units.}$$

Since the volume of the ball is $\frac{4}{3}\pi 2^3 = \frac{32\pi}{3}$ cu. units., therefore the volume removed is $\frac{32\pi}{3} - 4\pi\sqrt{3}$ cu. units. The percentage removed is

$$\frac{\frac{32\pi}{3} - 4\pi\sqrt{3}}{\frac{32\pi}{3}} \times 100 = 100\left(1 - \frac{3\sqrt{3}}{8}\right) \approx 35.$$

About 35% of the volume is removed.

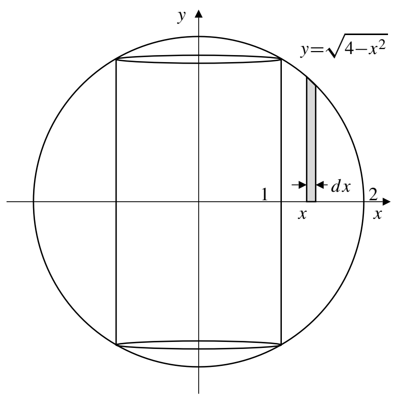

Fig. 7.1.13

14. The radius of the hole is $\sqrt{R^2 - \frac{1}{4}L^2}$. Thus, by slicing, the remaining volume is

$$\begin{aligned} V &= \pi \int_{-L/2}^{L/2} \left[\left(R^2 - x^2\right) - \left(R^2 - \frac{L^2}{4}\right)\right] dx \\ &= 2\pi\left(\frac{L^2}{4}x - \frac{1}{3}x^3\right)\Bigg|_0^{L/2} \\ &= \frac{\pi}{6}L^3 \text{ cu. units (independent of } R). \end{aligned}$$

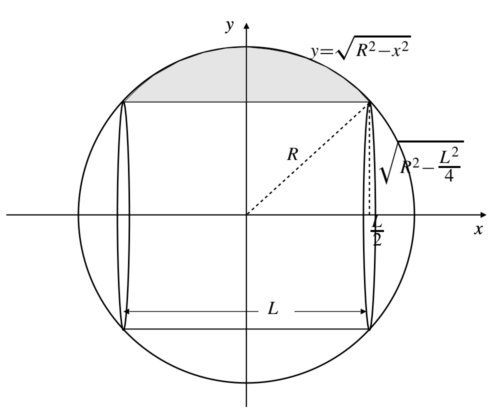

Fig. 7.1.14

15. The volume remaining is

$$\begin{aligned} V &= 2\pi \int_a^b xh\left(1 - \frac{x}{b}\right) dx \\ &= 2\pi h\left(\frac{x^2}{2} - \frac{x^3}{3b}\right)\Bigg|_a^b \\ &= \pi h(b^2 - a^2) - \frac{2}{3}\pi h\left(b^2 - \frac{a^3}{b}\right) \\ &= \frac{1}{3}\pi h\left(b^2 - 3a^2 + \frac{2a^3}{b}\right) \text{ cu. units.} \end{aligned}$$

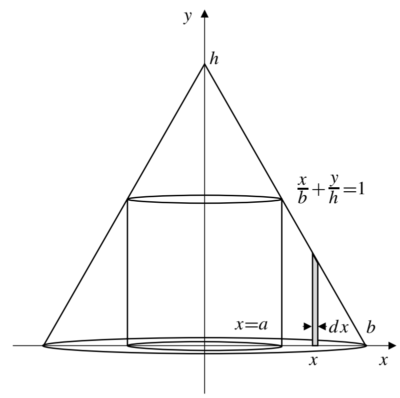

Fig. 7.1.15

16. Let a circular disk with radius a have centre at point $(a, 0)$. Then the disk is rotated about the y-axis which is one of its tangent lines. The volume is:

$$\begin{aligned} V &= 2 \times 2\pi \int_0^{2a} x\sqrt{a^2 - (x-a)^2}\, dx \quad \text{Let } u = x - a, \; du = dx \\ &= 4\pi \int_{-a}^{a} (u + a)\sqrt{a^2 - u^2}\, du \\ &= 4\pi \int_{-a}^{a} u\sqrt{a^2 - u^2}\, du + 4\pi a \int_{-a}^{a} \sqrt{a^2 - u^2}\, du \\ &= 0 + 4\pi a\left(\frac{1}{2}\pi a^2\right) = 2\pi^2 a^3 \text{ cu. units.} \end{aligned}$$

(Note that the first integral is zero because the integrand is odd and the interval is symmetric about zero; the second integral is the area of a semicircle.)

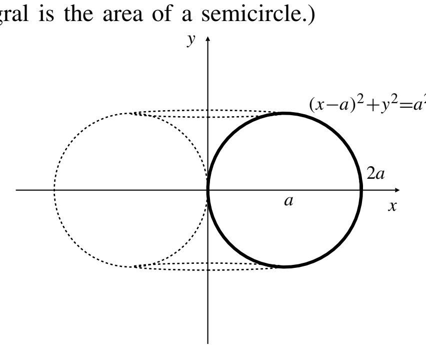

Fig. 7.1.16

17. Volume of the smaller piece:

$$\begin{aligned}
V &= \pi \int_b^a (a^2 - x^2)\,dx \\
&= \pi \left(a^2 x - \frac{x^3}{3} \right)\bigg|_b^a \\
&= \pi \left(a^2(a-b) - \frac{a^3 - b^3}{3} \right) \\
&= \frac{\pi}{3}(a-b)[3a^2 - (a^2 + ab + b^2)] \\
&= \frac{\pi}{3}(a-b)^2(2a+b) \text{ cu. units.}
\end{aligned}$$

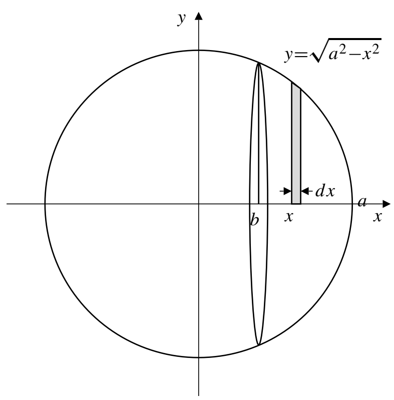

Fig. 7.1.17

18. Let the centre of the bowl be at (0, 30). Then the volume of the water in the bowl is

$$\begin{aligned}
V &= \pi \int_0^{20} \left[30^2 - (y-30)^2 \right] dy \\
&= \pi \int_0^{20} 60y - y^2\,dy \\
&= \pi \left[30y^2 - \frac{1}{3}y^3 \right]\bigg|_0^{20} \\
&\approx 29322 \text{ cm}^3.
\end{aligned}$$

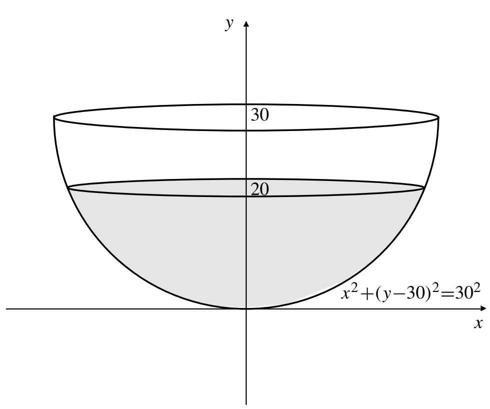

Fig. 7.1.18

19. The volume of the ellipsoid is

$$\begin{aligned}
V &= 2\pi \int_0^a b^2 \left(1 - \frac{x^2}{a^2} \right) dx \\
&= 2\pi b^2 \left(x - \frac{x^3}{3a^2} \right)\bigg|_0^a = \frac{4}{3}\pi a b^2 \text{ cu. units.}
\end{aligned}$$

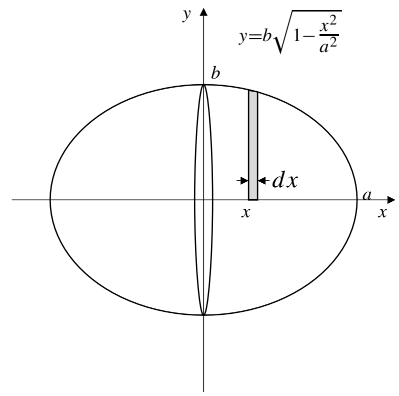

Fig. 7.1.19

20. The cross-section at height y is an annulus (ring) having inner radius $b - \sqrt{a^2 - y^2}$ and outer radius $b + \sqrt{a^2 - y^2}$. Thus the volume of the torus is

$$\begin{aligned}
V &= \pi \int_{-a}^a \left[(b + \sqrt{a^2 - y^2})^2 - (b - \sqrt{a^2 - y^2})^2 \right] dy \\
&= 2\pi \int_0^a 4b\sqrt{a^2 - y^2}\,dy \\
&= 8\pi b \frac{\pi a^2}{4} = 2\pi^2 a^2 b \text{ cu. units..}
\end{aligned}$$

We used the area of a quarter-circle of radius a to evaluate the last integral.

21. a) Volume of revolution about the x-axis is

$$\begin{aligned}
V &= \pi \int_0^\infty e^{-2x}\,dx \\
&= \pi \lim_{R\to\infty} \frac{e^{-2x}}{-2}\bigg|_0^R = \frac{\pi}{2} \text{ cu. units.}
\end{aligned}$$

b) Volume of revolution about the y-axis is

$$\begin{aligned}
V &= 2\pi \int_0^\infty x e^{-x}\,dx \\
&= 2\pi \lim_{R\to\infty} (-xe^{-x} - e^{-x})\bigg|_0^R = 2\pi \text{ cu. units.}
\end{aligned}$$

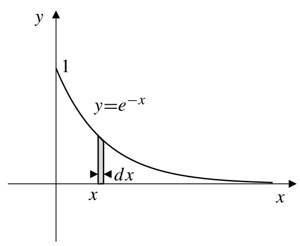

Fig. 7.1.21

22. The volume is

$$V = \pi \int_1^\infty x^{-2k}\,dx = \pi \lim_{R\to\infty} \frac{x^{1-2k}}{1-2k}\bigg|_1^R$$
$$= \pi \lim_{R\to\infty} \frac{R^{1-2k}}{1-2k} + \frac{\pi}{2k-1}.$$

In order for the solid to have finite volume we need

$$1 - 2k < 0, \quad \text{that is,} \quad k > \frac{1}{2}.$$

23. The volume is $V = 2\pi \int_1^\infty x^{1-k}\,dx$. This improper integral converges if $1 - k < -1$, i.e., if $k > 2$. The solid has finite volume only if $k > 2$.

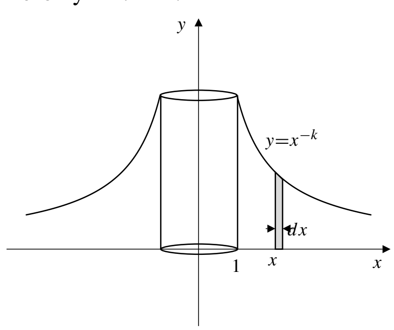

Fig. 7.1.23

24. A solid consisting of points on parallel line segments between parallel planes will certainly have congruent cross-sections in planes parallel to and lying between the two base planes, any solid satisfying the new definition will certainly satisfy the old one. But not vice versa; congruent cross-sections does not imply a family of parallel line segments giving all the points in a solid. For a counterexample, see the next exercise. Thus the earlier, incorrect definition defines a larger class of solids than does the current definition. However, the formula $V = Ah$ for the volume of such a solid is still valid, as all congruent cross-sections still have the same area, A, as the base region.

25. Since all isosceles right-angled triangles having leg length a cm are congruent, S does satisfy the condition for being a prism given in previous editions. It does not satisfy the condition in this edition because one of the line segments joining vertices of the triangular cross-sections, namely the x-axis, is not parallel to the line joining the vertices of the other end of the hypotenuses of the two bases.

The volume os S is still the constant cross-sectional area $a^2/2$ times the height b, that is, $V = a^2 b/2$ cm^3.

26. Using heights $f(x)$ estimated from the given graph, we obtain

$$V = \pi \int_1^9 \big(f(x)\big)^2\,dx$$
$$\approx \frac{\pi}{3}\Big[3^2 + 4(3.8)^2 + 2(5)^2 + 4(6.7)^2 + 2(8)^2 + 4(8)^2 + 2(7)^2 + 4(5.2)^2 + 3^2\Big] \approx 938 \text{ cu. units.}$$

27. Using heights $f(x)$ estimated from the given graph, we obtain

$$V = 2\pi \int_1^9 x f(x)\,dx$$
$$\approx \frac{2\pi}{3}\Big[1(3) + 4(2)(3.8) + 2(3)(5) + 4(4)(6.7) + 2(5)(8) + 4(6)(8) + 2(7)(7) + 4(8)(5.2) + 9(3)\Big] \approx 1537 \text{ cu. units.}$$

28. Using heights $f(x)$ estimated from the given graph, we obtain

$$V = 2\pi \int_1^9 (x+1) f(x)\,dx$$
$$\approx \frac{2\pi}{3}\Big[2(3) + 4(3)(3.8) + 2(4)(5) + 4(5)(6.7) + 2(6)(8) + 4(7)(8) + 2(8)(7) + 4(9)(5.2) + 10(3)\Big] \approx 1832 \text{ cu. units.}$$

29. The region is symmetric about $x = y$ so has the same volume of revolution about the two coordinate axes. The volume of revolution about the y-axis is

$$\begin{aligned} V &= 2\pi \int_0^8 x(4 - x^{2/3})^{3/2}\,dx \quad \text{Let } x = 8\sin^3 u \\ &\qquad\qquad\qquad\qquad\qquad dx = 24\sin^2 u \cos u\,du \\ &= 3072\pi \int_0^{\pi/2} \sin^5 u \cos^4 u\,du \\ &= 3072\pi \int_0^{\pi/2} (1 - \cos^2 u)^2 \cos^4 u \sin u\,du \quad \text{Let } v = \cos u \\ &\qquad\qquad\qquad\qquad\qquad dv = -\sin u\,du \\ &= 3072\pi \int_0^1 (1 - v^2)^2 v^4\,dv \\ &= 3072\pi \int_0^1 (v^4 - 2v^6 + v^8)\,dv \\ &= 3072\pi \left(\frac{1}{5} - \frac{2}{7} + \frac{1}{9}\right) = \frac{8192\pi}{105} \text{ cu. units.} \end{aligned}$$

30. The volume of the ball is $\frac{4}{3}\pi R^3$. Expressing this volume as the "sum" (i.e., integral) of volume elements that are concentric spherical shells of radius r and thickness dr, and therefore surface area kr^2 and volume $kr^2\,dr$, we obtain

$$\frac{4}{3}\pi R^3 = \int_0^R kr^2\,dr = \frac{k}{3}R^3.$$

Thus $k = 4\pi$.

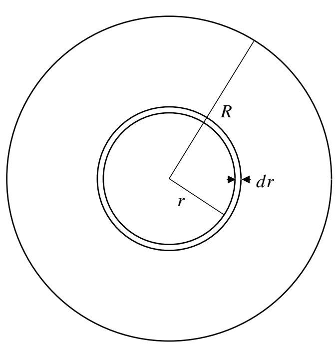

Fig. 7.1.30

31. Let the ball have radius R, and suppose its centre is x units above the top of the conical glass, as shown in the figure. (Clearly the ball which maximizes wine overflow from the glass must be tangent to the cone along some circle below the top of the cone — larger balls will have reduced displacement within the cone. Also, the ball will not be completely submerged.)

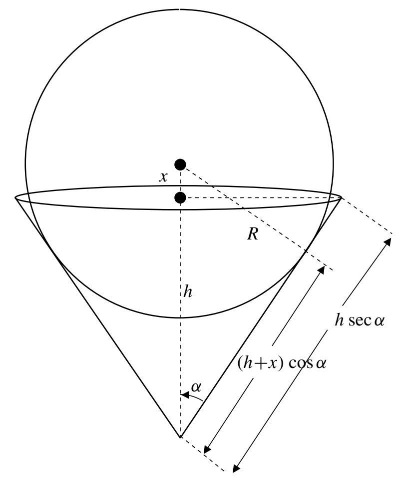

Fig. 7.1.31

Note that $\dfrac{R}{x+h} = \sin\alpha$, so $R = (x+h)\sin\alpha$.
Using the result of Exercise #17, the volume of wine displaced by the ball is

$$V = \frac{\pi}{3}(R - x)^2(2R + x).$$

We would like to consider V as a function of x for $-2R \le x \le R$ since $V = 0$ at each end of this interval, and $V > 0$ inside the interval. However, the actual interval of values of x for which the above formulation makes physical sense is smaller: x must satisfy $-R \le x \le h\tan^2\alpha$. (The left inequality signifies non-submersion of the ball; the right inequality signifies that the ball is tangent to the glass somewhere below the rim.) We look for a critical point of V, considered as a function of x. (As noted above, R is a function of x.) We have

$$\begin{aligned} 0 = \frac{dV}{dx} &= \frac{\pi}{3}\Bigg[2(R - x)\left(\frac{dR}{dx} - 1\right)(2R + x) \\ &\qquad + (R - x)^2\left(2\frac{dR}{dx} + 1\right)\Bigg] \end{aligned}$$

$$\frac{dR}{dx}(4R + 2x + 2R - 2x) = 4R + 2x - (R - x).$$

Thus

$$6R\sin\alpha = 3(R + x) = 3\left(R + \frac{R}{\sin\alpha} - h\right)$$

$$2R\sin^2\alpha = R\sin\alpha + R - h\sin\alpha$$

$$R = \frac{h\sin\alpha}{1 - 2\sin^2\alpha + \sin\alpha} = \frac{h\sin\alpha}{\cos 2\alpha + \sin\alpha}.$$

This value of R yields a positive value of V, and corresponds to $x = R(2\sin\alpha - 1)$. Since $\sin\alpha \ge \sin^2\alpha$,

$$-R \le x = \frac{h\sin\alpha(2\sin\alpha - 1)}{1 + \sin\alpha - 2\sin^2\alpha} \le \frac{h\sin^2\alpha}{\cos^2\alpha} = h\tan^2\alpha.$$

Therefore it gives the maximum volume of wine displaced.

32. Let P be the point $(t, \frac{5}{2} - t)$. The line through P perpendicular to AB has equation $y = x + \frac{5}{2} - 2t$, and meets the curve $xy = 1$ at point Q with x-coordinate s equal to the positive root of $s^2 + (\frac{5}{2} - 2t)s = 1$. Thus,

$$s = \frac{1}{2}\left[2t - \frac{5}{2} + \sqrt{\left(\frac{5}{2} - 2t\right)^2 + 4}\right].$$

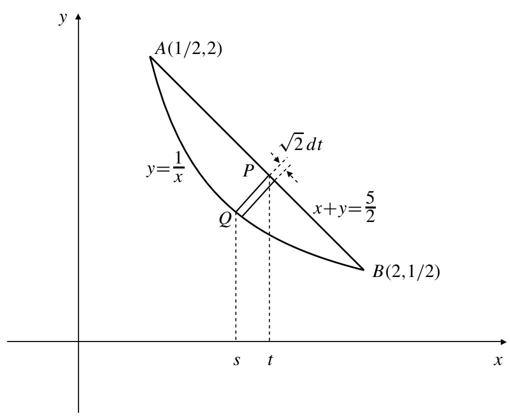

Fig. 7.1.32

The volume element at P has radius

$$\begin{aligned} PQ &= \sqrt{2}(t - s) \\ &= \sqrt{2}\left[\frac{5}{4} - \frac{1}{2}\sqrt{\left(\frac{5}{2} - 2t\right)^2 + 4}\right] \end{aligned}$$

and thickness $\sqrt{2}\,dt$. Hence, the volume of the solid is

$$\begin{aligned} V &= \pi \int_{1/2}^{2} \left[\sqrt{2}\left(\frac{5}{4} - \frac{1}{2}\sqrt{\left(\frac{5}{2} - 2t\right)^2 + 4}\right)\right]^2 \sqrt{2}\,dt \\ &= 2\sqrt{2}\pi \int_{1/2}^{2} \left[\frac{25}{16} - \frac{5}{4}\left(\sqrt{\left(\frac{5}{2} - 2t\right)^2 + 4}\right) + \right. \\ &\qquad \left. \frac{1}{4}\left[\left(\frac{5}{2} - 2t\right)^2 + 4\right]\right] dt \quad \begin{array}{l} \text{Let } u = 2t - \frac{5}{2} \\ du = 2\,dt \end{array} \\ &= \sqrt{2}\pi \int_{-3/2}^{3/2} \left(\frac{41}{16} - \frac{5}{4}\sqrt{u^2 + 4} + \frac{u^2}{4}\right) du \\ &= \sqrt{2}\pi \left(\frac{41}{16}u + \frac{1}{12}u^3\right)\Bigg|_{-3/2}^{3/2} - \\ &\qquad \frac{5\sqrt{2}\pi}{4} \int_{-3/2}^{3/2} \sqrt{u^2 + 4}\,du \quad \begin{array}{l} \text{Let } u = 2\tan v \\ du = 2\sec^2 v\,dv \end{array} \\ &= \frac{33\sqrt{2}\pi}{4} - 5\sqrt{2}\pi \int_{\tan^{-1}(-3/4)}^{\tan^{-1}(3/4)} \sec^3 v\,dv \\ &= \frac{33\sqrt{2}\pi}{4} - 10\sqrt{2}\pi \int_{0}^{\tan^{-1}(3/4)} \sec^3 v\,dv \\ &= \frac{33\sqrt{2}\pi}{4} - 5\sqrt{2}\pi \Big(\sec v \tan v + \\ &\qquad \ln|\sec v + \tan v|\Big)\Bigg|_{0}^{\tan^{-1}(3/4)} \\ &= \sqrt{2}\pi \left[\frac{33}{4} - 5\left(\frac{15}{16} + \ln 2 - 0 - \ln 1\right)\right] \\ &= \sqrt{2}\pi \left(\frac{57}{16} - 5\ln 2\right) \text{ cu. units.} \end{aligned}$$

Section 7.2 Other Volumes by Slicing (page 380)

1. $V = \displaystyle\int_0^2 3x\,dx = \frac{3}{2}x^2\Big|_0^2 = 6 \text{ m}^3$

2. A horizontal slice of thickness dz at height a has volume $dV = z(h - z)\,dz$. Thus the volume of the solid is

$$V = \int_0^h (z(h - z)\,dz = \left(\frac{hz^2}{2} - \frac{z^3}{3}\right)\Bigg|_0^h = \frac{h^3}{6} \text{ units}^3.$$

3. A horizontal slice of thickness dz at height a has volume $dV = \pi z\sqrt{1 - z^2}\,dz$. Thus the volume of the solid is

$$\begin{aligned} V &= \int_0^1 z\sqrt{1 - z^2}\,dz \qquad \text{let } u - 1 - z^2 \\ &= \frac{\pi}{2}\int_0^1 \sqrt{u}\,du = \frac{\pi}{2}\,\frac{2}{3}u^{3/2}\Bigg|_0^1 = \frac{\pi}{3} \text{ units}^3. \end{aligned}$$

4. $V = \displaystyle\int_1^3 x^2\,dx = \frac{x^3}{3}\Bigg|_1^3 = \frac{26}{3}$ cu. units

5. $$\begin{aligned} V &= \int_0^6 (2 + z)(8 - z)\,dz = \int_0^6 (16 + 6z - z^2)\,dz \\ &= \left(16z + 3z^2 - \frac{z^3}{3}\right)\Bigg|_0^6 = 132 \text{ ft}^3 \end{aligned}$$

6. The area of an equilateral triangle of edge $\sqrt{x}$ is $A(x) = \frac{1}{2}\sqrt{x}\left(\frac{\sqrt{3}}{2}\sqrt{x}\right) = \frac{\sqrt{3}}{4}x$ sq. units. The volume of the solid is

$$V = \int_1^4 \frac{\sqrt{3}}{4}x\,dx = \frac{\sqrt{3}}{8}x^2\Bigg|_1^4 = \frac{15\sqrt{3}}{8} \text{ cu. units.}$$

7. The area of cross-section at height y is

$$A(y) = \frac{2\pi(1 - (y/h))}{2\pi}(\pi a^2) = \pi a^2\left(1 - \frac{y}{h}\right) \text{ sq. units.}$$

The volume of the solid is

$$V = \int_0^h \pi a^2 \left(1 - \frac{y}{h}\right) dy = \frac{\pi a^2 h}{2} \text{ cu. units.}$$

8. Since $V = 4$, we have

$$4 = \int_0^2 kx^3\, dx = k\, \frac{x^4}{4}\bigg|_0^2 = 4k.$$

Thus $k = 1$.

9. The volume between height 0 and height z is z^3. Thus

$$z^3 = \int_0^z A(t)\, dt,$$

where $A(t)$ is the cross-sectional area at height t. Differentiating the above equation with respect to z, we get $3z^2 = A(z)$. The cross-sectional area at height z is $3z^2$ sq. units.

10. This is similar to Exercise 7. We have $4z = \displaystyle\int_0^z A(t)\, dt$, so $A(z) = 4$. Thus the square cross-section at height z has side 2 units.

11.
$$\begin{aligned} V &= 2\int_0^r \left(2\sqrt{r^2 - y^2}\right)^2 dy \\ &= 8\int_0^r (r^2 - y^2)\, dy = 8\left(r^2 y - \frac{y^3}{3}\right)\bigg|_0^r = \frac{16r^3}{3} \text{ cu. units.} \end{aligned}$$

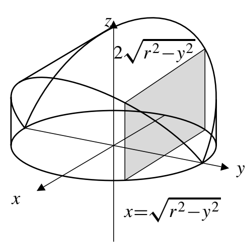

Fig. 7.2.11

12. The area of an equilateral triangle of base $2y$ is $\frac{1}{2}(2y)(\sqrt{3}y) = \sqrt{3}y^2$. Hence, the solid has volume

$$\begin{aligned} V &= 2\int_0^r \sqrt{3}(r^2 - x^2)\, dx \\ &= 2\sqrt{3}\left(r^2 x - \frac{1}{3}x^3\right)\bigg|_0^r \\ &= \frac{4}{\sqrt{3}} r^3 \text{ cu. units.} \end{aligned}$$

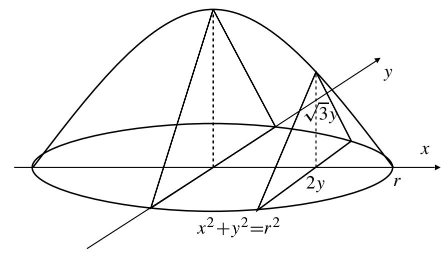

Fig. 7.2.12

13. The cross-section at distance y from the vertex of the partial cone is a semicircle of radius $y/2$ cm, and hence area $\pi y^2/8$ cm^2. The volume of the solid is

$$V = \int_0^{12} \frac{\pi}{8} y^2\, dy = \frac{\pi\, 12^3}{24} = 72\pi \text{ cm}^3.$$

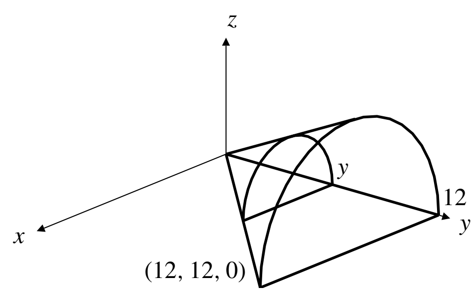

Fig. 7.2.13

14. The volume of a solid of given height h and given cross-sectional area $A(z)$ at height z above the base is given by

$$V = \int_0^h A(z)\, dz.$$

If two solids have the same height h and the same area function $A(z)$, then they must necessarily have the same volume.

15. Let the x-axis be along the diameter shown in the figure, with the origin at the centre of the base. The cross-section perpendicular to the x-axis at x is a rectangle having base $2\sqrt{r^2 - x^2}$ and height $h = \dfrac{a+b}{2} + \dfrac{a-b}{2}x$. Thus the volume of the truncated cylinder is

$$\begin{aligned} V &= \int_{-r}^r (2\sqrt{r^2 - x^2})\left(\frac{a+b}{2} + \frac{a-b}{2r}x\right) dx \\ &= \int_{-r}^r (a+b)\sqrt{r^2 - x^2}\, dx = \frac{\pi r^2(a+b)}{2} \text{ cu. units.} \end{aligned}$$

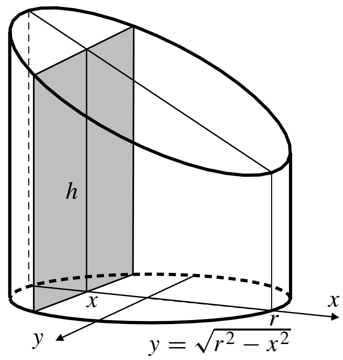

Fig. 7.2.15

16. The plane $z = k$ meets the ellipsoid in the ellipse

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 - \left(\frac{k}{c}\right)^2$$

$$\text{that is,} \quad \frac{x^2}{a^2\left[1-\left(\frac{k}{c}\right)^2\right]} + \frac{y^2}{b^2\left[1-\left(\frac{k}{c}\right)^2\right]} = 1$$

which has area

$$A(k) = \pi ab\left[1 - \left(\frac{k}{c}\right)^2\right].$$

The volume of the ellipsoid is found by summing volume elements of thickness dk:

$$\begin{aligned} V &= \int_{-c}^{c} \pi ab\left[1 - \left(\frac{k}{c}\right)^2\right] dk \\ &= \pi ab\left[k - \frac{1}{3c^2}k^3\right]\Bigg|_{-c}^{c} \\ &= \frac{4}{3}\pi abc \text{ cu. units.} \end{aligned}$$

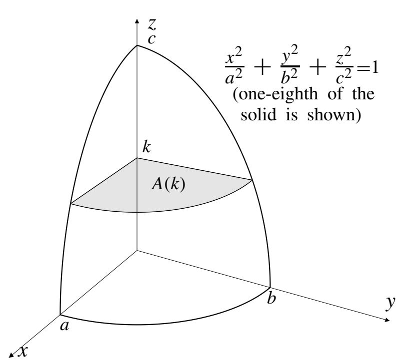

Fig. 7.2.16

17. Cross-sections of the wedge removed perpendicular to the x-axis are isosceles, right triangles. The volume of the wedge removed from the log is

$$\begin{aligned} V &= 2\int_0^{20} \frac{1}{2}(\sqrt{400 - x^2})^2\, dx \\ &= \left(400x - \frac{x^3}{3}\right)\Bigg|_0^{20} = \frac{16,000}{3} \text{ cm}^3. \end{aligned}$$

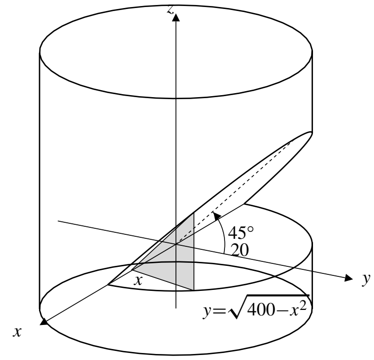

Fig. 7.2.17

18. The solution is similar to that of Exercise 15 except that the legs of the right-triangular cross-sections are $y - 10$ instead of y, and x goes from $-10\sqrt{3}$ to $10\sqrt{3}$ instead of -20 to 20. The volume of the notch is

$$\begin{aligned} V &= 2\int_0^{10\sqrt{3}} \frac{1}{2}(\sqrt{400 - x^2} - 10)^2\, dx \\ &= \int_0^{10\sqrt{3}} \left(500 - x^2 - 20\sqrt{400 - x^2}\right) dx \\ &= 3,000\sqrt{3} - \frac{4,000\pi}{3} \approx 1,007 \text{ cm}^3. \end{aligned}$$

19. The hole has the shape of two copies of the truncated cylinder of Exercise 13, placed base to base, with $a + b = 3\sqrt{2}$ in and $r = 2$ in. Thus the volume of wood removed (the volume of the hole) is $V = 2(\pi 2^2)(3\sqrt{2}/2) = 12\sqrt{2}\pi$ in^3.

20. One eighth of the region lying inside both cylinders is shown in the figure. If the region is sliced by a horizontal plane at height z, then the intersection is a rectangle with area

$$A(z) = \sqrt{b^2 - z^2}\sqrt{a^2 - z^2}.$$

The volume of the whole region is

$$V = 8\int_0^b \sqrt{b^2 - z^2}\sqrt{a^2 - z^2}\, dz.$$

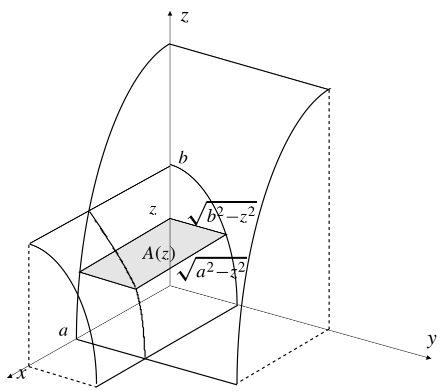

Fig. 7.2.20

21. By the result given in Exercise 18 with $a = 4$ cm and $b = 2$ cm, the volume of wood removed is

$$V = 8\int_0^2 \sqrt{4-z^2}\,\sqrt{16-z^2}\,dz \approx 97.28 \text{ cm}^3.$$

(We used the numerical integration routine in Maple to evaluate the integral.)

Section 7.3 Arc Length and Surface Area (page 387)

1. $y = 2x - 1, \quad y' = 2, \quad ds = \sqrt{1+2^2}\,dx$

$$L = \int_1^3 \sqrt{5}\,dx = 2\sqrt{5} \text{ units.}$$

2. $y = ax + b$, $A \le x \le B$, $y' = a$. The length is

$$L = \int_A^B \sqrt{1+a^2}\,dx = \sqrt{1+a^2}(B-A) \text{ units.}$$

3. $y = \frac{2}{3}x^{3/2}, \quad y' = \sqrt{x}, \quad ds = \sqrt{1+x}\,dx$

$$L = \int_0^8 \sqrt{1+x}\,dx = \frac{2}{3}(1+x)^{3/2}\Big|_0^8 = \frac{52}{3} \text{ units.}$$

4. $y^2 = (x-1)^3, \quad y = (x-1)^{3/2}, \quad y' = \frac{3}{2}\sqrt{x-1}$

$$L = \int_1^2 \sqrt{1+\frac{9}{4}(x-1)}\,dx = \frac{1}{2}\int_1^2 \sqrt{9x-5}\,dx$$
$$= \frac{1}{27}(9x-5)^{3/2}\Big|_1^2 = \frac{13^{3/2}-8}{27} \text{ units.}$$

5. $y = x^{2/3}, \quad y' = \frac{2}{3}x^{-1/3},$

$$ds = \sqrt{1+\frac{4}{9}x^{-2/3}}\,dx = \frac{\sqrt{9x^{2/3}+4}}{3|x|^{1/3}}\,dx$$
$$L = 2\int_0^1 \frac{\sqrt{9x^{2/3}+4}}{3x^{1/3}}\,dx \quad \text{Let } u = 9x^{2/3}+4, \quad du = 6x^{-1/3}\,dx$$
$$= \frac{1}{9}\int_4^{13} \sqrt{u}\,du = \frac{2(13^{3/2})-16}{27} \text{ units.}$$

6. $2(x+1)^3 = 3(y-1)^2, \quad y = 1 + \sqrt{\frac{2}{3}}(x+1)^{3/2}$

$$y' = \sqrt{\tfrac{3}{2}}(x+1)^{1/2},$$
$$ds = \sqrt{1+\frac{3x+3}{2}}\,dx = \sqrt{\frac{3x+5}{2}}\,dx$$
$$L = \frac{1}{\sqrt{2}}\int_{-1}^0 \sqrt{3x+5}\,dx = \frac{\sqrt{2}}{9}(3x+5)^{3/2}\Big|_{-1}^0$$
$$= \frac{\sqrt{2}}{9}\left(5^{3/2} - 2^{3/2}\right) \text{ units.}$$

7. $y = \frac{x^3}{12} + \frac{1}{x}, \quad y' = \frac{x^2}{4} - \frac{1}{x^2}$

$$ds = \sqrt{1+\left(\frac{x^2}{4}-\frac{1}{x^2}\right)^2}\,dx = \left(\frac{x^2}{4}+\frac{1}{x^2}\right)dx$$
$$L = \int_1^4 \left(\frac{x^2}{4}+\frac{1}{x^2}\right)dx = \left(\frac{x^3}{12}-\frac{1}{x}\right)\Big|_1^4 = 6 \text{ units.}$$

8. $y = \frac{x^3}{3} + \frac{1}{4x}, \quad y' = x^2 - \frac{1}{4x^2}$

$$ds = \sqrt{1+\left(x^2-\frac{1}{4x^2}\right)^2}\,dx = \left(x^2+\frac{1}{4x^2}\right)dx$$
$$L = \int_1^2 \left(x^2+\frac{1}{4x^2}\right)dx = \left(\frac{x^3}{3}-\frac{1}{4x}\right)\Big|_1^2 = \frac{59}{24} \text{ units.}$$

9. $y = \frac{\ln x}{2} - \frac{x^2}{4}, \quad y' = \frac{1}{2x} - \frac{x}{2}$

$$ds = \sqrt{1+\left(\frac{1}{2x}-\frac{x}{2}\right)^2}\,dx = \left(\frac{1}{2x}+\frac{x}{2}\right)dx$$
$$L = \int_1^e \left(\frac{1}{2x}+\frac{x}{2}\right)dx = \left(\frac{\ln x}{2}+\frac{x^2}{4}\right)\Big|_1^e$$
$$= \frac{1}{2} + \frac{e^2-1}{4} = \frac{e^2+1}{4} \text{ units.}$$

10. If $y = x^2 - \frac{\ln x}{8}$ then $y' = 2x - \frac{1}{8x}$ and

$$1 + (y')^2 = \left(2x + \frac{1}{8x}\right)^2.$$

Thus the arc length is given by

$$\begin{aligned} s &= \int_1^2 \sqrt{1+\left(2x-\frac{1}{8x}\right)^2}\,dx \\ &= \int_1^2 \left(2x+\frac{1}{8x}\right)dx \\ &= \left(x^2+\frac{1}{8}\ln x\right)\Big|_1^2 = 3+\frac{1}{8}\ln 2 \text{ units.} \end{aligned}$$

11. $$\begin{aligned} s &= \int_0^a \sqrt{1+\sinh^2 x}\,dx = \int_0^a \cosh x\,dx \\ &= \sinh x\Big|_0^a = \sinh a = \frac{e^a-e^{-a}}{2} \text{ units.} \end{aligned}$$

12. $$\begin{aligned} s &= \int_{\pi/6}^{\pi/4} \sqrt{1+\tan^2 x}\,dx \\ &= \int_{\pi/6}^{\pi/4} \sec x\,dx = \ln|\sec x+\tan x|\Big|_{\pi/6}^{\pi/4} \\ &= \ln(\sqrt{2}+1) - \ln\left(\frac{2}{\sqrt{3}}+\frac{1}{\sqrt{3}}\right) \\ &= \ln\frac{\sqrt{2}+1}{\sqrt{3}} \text{ units.} \end{aligned}$$

13. $y=x^2$, $0\le x\le 2$, $y'=2x$.

$$\begin{aligned} \text{length } &= \int_0^2 \sqrt{1+4x^2}\,dx \qquad \begin{array}{l}\text{Let } 2x=\tan\theta \\ 2\,dx=\sec^2\theta\,d\theta\end{array} \\ &= \frac{1}{2}\int_{x=0}^{x=2} \sec^3\theta \\ &= \frac{1}{4}\Big(\sec\theta\tan\theta+\ln|\sec\theta+\tan\theta|\Big)\Big|_{x=0}^{x=2} \\ &= \frac{1}{4}\Big(2x\sqrt{1+4x^2}+\ln(2x+\sqrt{1+4x^2})\Big)\Big|_0^2 \\ &= \frac{1}{4}\Big(4\sqrt{17}+\ln(4+\sqrt{17})\Big) \\ &= \sqrt{17}+\frac{1}{4}\ln(4+\sqrt{17}) \text{ units.} \end{aligned}$$

14. $$\begin{aligned} y &= \ln\frac{e^x-1}{e^x+1}, \qquad 2\le x\le 4 \\ y' &= \frac{e^x+1}{e^x-1}\,\frac{(e^x+1)e^x-(e^x-1)e^x}{(e^x+1)^2} \\ &= \frac{2e^x}{e^{2x}-1}. \end{aligned}$$

The length of the curve is

$$\begin{aligned} L &= \int_2^4 \sqrt{1+\frac{4e^{2x}}{(e^{2x}-1)^2}}\,dx \\ &= \int_2^4 \frac{e^{2x}+1}{e^{2x}-1}\,dx \\ &= \int_2^4 \frac{e^x+e^{-x}}{e^x-e^{-x}}\,dx = \ln\left|e^x-e^{-x}\right|\Big|_2^4 \\ &= \ln\left(e^4-\frac{1}{e^4}\right)-\ln\left(e^2-\frac{1}{e^2}\right) \\ &= \ln\left(\frac{e^8-1}{e^4}\,\frac{e^2}{e^4-1}\right) = \ln\frac{e^4+1}{e^2} \text{ units.} \end{aligned}$$

15. $x^{2/3}+y^{2/3}=x^{2/3}$. By symmetry, the curve has congruent arcs in the four quadrants. For the first quadrant arc we have

$$\begin{aligned} y &= \left(a^{2/3}-x^{2/3}\right)^{3/2} \\ y' &= \frac{3}{2}\left(a^{2/3}-x^{2/3}\right)^{1/2}\left(-\frac{2}{3}x^{-1/3}\right). \end{aligned}$$

Thus the length of the whole curve is

$$\begin{aligned} L &= 4\int_0^a \sqrt{1+\frac{a^{2/3}-x^{2/3}}{x^{2/3}}}\,dx \\ &= 4a^{1/3}\int_0^a x^{-1/3}\,dx \\ &= 4a^{1/3}\,\frac{3}{2}x^{2/3}\Big|_0^a = 6a \text{ units.} \end{aligned}$$

16. The required length is

$$L = \int_0^1 \sqrt{1+(4x^3)^2}\,dx = \int_0^1 \sqrt{1+16x^6}\,dx.$$

Using a calculator we calculate some Simpson's Rule approximations as described in Section 7.2:

$$\begin{array}{ll} S_2 \approx 1.59921 & S_4 \approx 1.60110 \\ S_8 \approx 1.60025 & S_{16} \approx 1.60023. \end{array}$$

To four decimal places the length is 1.6002 units.

17. $y = x^{1/3}$, $1 \le x \le 2$, $y' = \dfrac{1}{3}x^{-2/3}$.

Length $= \int_1^2 f(x)\,dx$, where $f(x) = \sqrt{1 + \dfrac{1}{9x^{4/3}}}$. We have

$$\begin{aligned} T_4 &= 1.03406 & M_4 &= 1.03363 \\ T_8 &= 1.03385 & M_8 &= 1.03374 \\ T_{16} &= 1.03378 & M_{16} &= 1.00376. \end{aligned}$$

Thus the length is approximately 1.0338 units.

18. For the ellipse $3x^2 + y^2 = 3$, we have $6x + 2yy' = 0$, so $y' = -3x/y$. Thus

$$ds = \sqrt{1 + \frac{9x^2}{3 - 3x^2}}\,dx = \sqrt{\frac{3 + 6x^2}{3 - 3x^2}}\,dx.$$

The circumference of the ellipse is

$$4\int_0^1 \sqrt{\frac{3 + 6x^2}{3 - 3x^2}}\,dx \approx 8.73775 \text{ units}$$

(with a little help from Maple's numerical integration routine.)

19. For the ellipse $x^2 + 2y^2 = 2$, we have $2x + 4yy' = 0$, so $y' = -x/(2y)$. Thus

$$ds = \sqrt{1 + \frac{x^2}{4 - 2x^2}}\,dx = \sqrt{\frac{4 - x^2}{4 - 2x^2}}\,dx$$

The length of the short arc from $(0, 1)$ to $(1, 1/\sqrt{2})$ is

$$\int_0^1 \sqrt{\frac{4 - x^2}{4 - 2x^2}}\,dx \approx 1.05810 \text{ units}$$

(with a little help from Maple's numerical integration routine).

20.
$$\begin{aligned} S &= 2\pi \int_0^2 |x|\sqrt{1 + 4x^2}\,dx \quad \text{Let } u = 1 + 4x^2 \\ &\qquad\qquad\qquad\qquad\qquad\quad du = 8x\,dx \\ &= \frac{\pi}{4}\int_1^{17} \sqrt{u}\,du = \frac{\pi}{4}\left(\frac{2}{3}u^{3/2}\right)\Bigg|_1^{17} \\ &= \frac{\pi}{6}(17\sqrt{17} - 1) \text{ sq. units.} \end{aligned}$$

21. $y = x^3$, $0 \le x \le 1$. $ds = \sqrt{1 + 9x^4}\,dx$.
The area of the surface of rotation about the x-axis is

$$\begin{aligned} S &= 2\pi \int_0^1 x^3\sqrt{1 + 9x^4}\,dx \quad \text{Let } u = 1 + 9x^4 \\ &\qquad\qquad\qquad\qquad\qquad\quad du = 36x^3\,dx \\ &= \frac{\pi}{18}\int_1^{10} \sqrt{u}\,du = \frac{\pi}{27}(10^{3/2} - 1) \text{ sq. units.} \end{aligned}$$

22. $y = x^{3/2}$, $0 \le x \le 1$. $ds = \sqrt{1 + \frac{9}{4}x}\,dx$.
The area of the surface of rotation about the x-axis is

$$\begin{aligned} S &= 2\pi \int_0^1 x^{3/2}\sqrt{1 + \frac{9x}{4}}\,dx \quad \text{Let } 9x = 4u^2 \\ &\qquad\qquad\qquad\qquad\qquad\qquad\quad 9\,dx = 8u\,du \\ &= \frac{128\pi}{243}\int_0^{3/2} u^4\sqrt{1 + u^2}\,du \quad \text{Let } u = \tan v \\ &\qquad\qquad\qquad\qquad\qquad\qquad\quad du = \sec^2 v\,dv \\ &= \frac{128\pi}{243}\int_0^{\tan^{-1}(3/2)} \tan^4 v \sec^3 v\,dv \\ &= \frac{128\pi}{243}\int_0^{\tan^{-1}(3/2)} (\sec^7 v - 2\sec^5 v + \sec^3 v)\,dv. \end{aligned}$$

At this stage it is convenient to use the reduction formula

$$\int \sec^n v\,dv = \frac{1}{n-1}\sec^{n-2} v \tan v + \frac{n-2}{n-1}\int \sec^{n-2} v\,dv$$

(see Exercise 36 of Section 7.1) to reduce the powers of secant down to 3, and then use

$$\int_0^a \sec^3 v\,dv = \frac{1}{2}(\sec a \tan a + \ln|\sec a + \tan a|.$$

We have

$$\begin{aligned} I &= \int_0^a (\sec^7 v - 2\sec^5 v + \sec^3 v)\,dv \\ &= \frac{\sec^5 v \tan v}{6}\Bigg|_0^a + \left(\frac{5}{6} - 2\right)\int_0^a \sec^5 v\,dv + \int_0^a \sec^3 v\,dv \\ &= \frac{\sec^5 a \tan a}{6} - \frac{7}{6}\left[\frac{\sec^3 v \tan v}{4}\Bigg|_0^a + \frac{3}{4}\int_0^a \sec^3 v\,dv\right] \\ &\qquad + \int_0^a \sec^3 v\,dv \\ &= \frac{\sec^5 a \tan a}{6} - \frac{7\sec^3 a \tan a}{24} + \frac{1}{8}\int_0^a \sec^3 v\,dv \\ &= \frac{\sec^5 a \tan a}{6} - \frac{7\sec^3 a \tan a}{24} \\ &\qquad + \frac{\sec a \tan a + \ln|\sec a + \tan a|}{16}. \end{aligned}$$

Substituting $a = arctan(3/2)$ now gives the following value for the surface area:

$$S = \frac{28\sqrt{13}\pi}{81} + \frac{8\pi}{243}\ln\left(\frac{3 + \sqrt{13}}{2}\right) \text{ sq. units.}$$

23. If $y = x^{3/2}$, $0 \le x \le 1$, is rotated about the y-axis, the surface area generated is

$$\begin{aligned} S &= 2\pi \int_0^1 x\sqrt{1 + \frac{9x}{4}}\, dx \qquad \text{Let } u = 1 + \frac{9x}{4} \\ & \qquad\qquad\qquad\qquad\qquad du = \frac{9}{4}\, dx \\ &= \frac{32\pi}{81} \int_1^{13/4} (u-1)\sqrt{u}\, du \\ &= \frac{32\pi}{81} \left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} \right) \Bigg|_1^{13/4} \\ &= \frac{64\pi}{81} \left(\frac{(13/4)^{5/2} - 1}{5} - \frac{(13/4)^{3/2} - 1}{3} \right) \text{ sq. units.} \end{aligned}$$

24. We have

$$\begin{aligned} S &= 2\pi \int_0^1 e^x \sqrt{1 + e^{2x}}\, dx \quad \text{Let } e^x = \tan\theta \\ & \qquad\qquad\qquad\qquad\qquad e^x\, dx = \sec^2\theta\, d\theta \\ &= 2\pi \int_{x=0}^{x=1} \sqrt{1 + \tan^2\theta}\, \sec^2\theta\, d\theta = 2\pi \int_{x=0}^{x=1} \sec^3\theta\, d\theta \\ &= \pi \Big[\sec\theta\, \tan\theta + \ln|\sec\theta + \tan\theta| \Big] \Bigg|_{x=0}^{x=1}. \end{aligned}$$

Since

$$\begin{aligned} x = 1 &\Rightarrow \tan\theta = e,\ \sec\theta = \sqrt{1+e^2}, \\ x = 0 &\Rightarrow \tan\theta = 1,\ \sec\theta = \sqrt{2}, \end{aligned}$$

therefore

$$\begin{aligned} S &= \pi \Big[e\sqrt{1+e^2} + \ln|\sqrt{1+e^2} + e| - \sqrt{2} - \ln|\sqrt{2} + 1| \Big] \\ &= \pi \left[e\sqrt{1+e^2} - \sqrt{2} + \ln \frac{\sqrt{1+e^2} + e}{\sqrt{2} + 1} \right] \text{ sq. units.} \end{aligned}$$

25. If $y = \sin x$, $0 \le x \le \pi$, is rotated about the x-axis, the surface area generated is

$$\begin{aligned} S &= 2\pi \int_0^\pi \sin x \sqrt{1 + \cos^2 x}\, dx \qquad \text{Let } u = \cos x \\ & \qquad\qquad\qquad\qquad\qquad du = -\sin x\, dx \\ &= 2\pi \int_{-1}^1 \sqrt{1+u^2}\, du \qquad \text{Let } u = \tan\theta \\ & \qquad\qquad\qquad\qquad\qquad du = \sec^2\theta\, d\theta \\ &= 2\pi \int_{-\pi/4}^{\pi/4} \sec^3\theta\, d\theta = 4\pi \int_0^{\pi/4} \sec^3\theta\, d\theta \\ &= 2\pi \Big(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta| \Big) \Bigg|_0^{\pi/4} \\ &= 2\pi \Big(\sqrt{2} + \ln(1 + \sqrt{2}) \Big) \text{ sq. units.} \end{aligned}$$

26. $1 + (y')^2 = 1 + \left(\dfrac{x^2}{4} - \dfrac{1}{x^2} \right)^2 = \left(\dfrac{x^2}{4} + \dfrac{1}{x^2} \right)^2$

$$\begin{aligned} S &= 2\pi \int_1^4 \left(\frac{x^3}{12} + \frac{1}{x} \right) \left(\frac{x^2}{4} + \frac{1}{x^2} \right) dx \\ &= 2\pi \int_1^4 \left(\frac{x^5}{48} + \frac{x}{3} + \frac{1}{x^3} \right) dx \\ &= 2\pi \left(\frac{x^6}{288} + \frac{x^2}{6} - \frac{1}{2x^2} \right) \Bigg|_1^4 \\ &= \frac{275}{8}\pi \text{ sq. units.} \end{aligned}$$

27. For $y = \dfrac{x^3}{12} + \dfrac{1}{x}$, $1 \le x \le 4$, we have
$ds = \left(\dfrac{x^2}{4} + \dfrac{1}{x^2} \right) dx.$
The surface generated by rotating the curve about the y-axis has area

$$\begin{aligned} S &= 2\pi \int_1^4 x \left(\frac{x^2}{4} + \frac{1}{x^2} \right) dx \\ &= 2\pi \left(\frac{x^4}{16} + \ln|x| \right) \Bigg|_1^4 \\ &= 2\pi \left(\frac{255}{16} + \ln 4 \right) \text{ sq. units.} \end{aligned}$$

28. The area of the cone obtained by rotating the line $y = (h/r)x$, $0 \le x \le r$, about the y-axis is

$$\begin{aligned} S &= 2\pi \int_0^r x\sqrt{1 + (h/r)^2}\, dx = 2\pi \frac{\sqrt{r^2+h^2}}{r} \frac{x^2}{2} \Bigg|_0^r \\ &= \pi r \sqrt{r^2 + h^2} \text{ sq. units.} \end{aligned}$$

29. For the circle $(x-b)^2 + y^2 = a^2$ we have

$$2(x-b) + 2y\frac{dy}{dx} = 0 \qquad \Rightarrow \qquad \frac{dy}{dx} = -\frac{x-b}{y}.$$

Thus
$$ds = \sqrt{1 + \frac{(x-b)^2}{y^2}}\, dx = \frac{a}{y}\, dx = \frac{a}{\sqrt{a^2 - (x-b)^2}}\, dx$$
(if $y > 0$).
The surface area of the torus obtained by rotating the circle about the line $x = 0$ is

$$\begin{aligned} S &= 2 \times 2\pi \int_{b-a}^{b+a} x \frac{a}{\sqrt{a^2 - (x-b)^2}}\, dx \quad \text{Let } u = x - b \\ & \qquad\qquad\qquad\qquad\qquad\qquad du = dx \\ &= 4\pi a \int_{-a}^a \frac{u+b}{\sqrt{a^2 - u^2}}\, du \\ &= 8\pi ab \int_0^a \frac{du}{\sqrt{a^2-u^2}} \qquad \text{by symmetry} \\ &= 8\pi ab \sin^{-1}\frac{u}{a} \Bigg|_0^a = 4\pi^2 ab \text{ sq. units.} \end{aligned}$$

30. The top half of $x^2 + 4y^2 = 4$ is $y = \frac{1}{2}\sqrt{4 - x^2}$, so $\frac{dy}{dx} = \frac{-x}{2\sqrt{4 - x^2}}$, and

$$\begin{aligned}
S &= 2 \times 2\pi \int_0^2 \frac{\sqrt{4 - x^2}}{2} \sqrt{1 + \left(\frac{x}{2\sqrt{4 - x^2}}\right)^2}\, dx \\
&= \pi \int_0^2 \sqrt{16 - 3x^2}\, dx \quad \text{Let } x = \sqrt{\frac{16}{3}} \sin\theta \\
&\qquad\qquad\qquad\qquad dx = \sqrt{\frac{16}{3}} \cos\theta\, d\theta \\
&= \pi \int_0^{\pi/3} (4\cos\theta) \frac{4}{\sqrt{3}} \cos\theta\, d\theta \\
&= \frac{16\pi}{\sqrt{3}} \int_0^{\pi/3} \cos^2\theta\, d\theta \\
&= \frac{8\pi}{\sqrt{3}} \left(\theta + \sin\theta\, \cos\theta\right)\Big|_0^{\pi/3} \\
&= \frac{2\pi(4\pi + 3\sqrt{3})}{3\sqrt{3}} \text{ sq. units.}
\end{aligned}$$

31. For the ellipse $x^2 + 4y^2 = 4$ we have

$$2x\frac{dx}{dy} + 8y = 0 \qquad \Rightarrow \qquad \frac{dx}{dy} = -4\frac{y}{x}.$$

The arc length element on the ellipse is given by

$$\begin{aligned}
ds &= \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \\
&= \sqrt{1 + \frac{16y^2}{x^2}}\, dy = \frac{1}{x}\sqrt{4 + 12y^2}\, dy.
\end{aligned}$$

If the ellipse is rotated about the y-axis, the resulting surface has area

$$\begin{aligned}
S &= 2 \times 2\pi \int_0^1 x\, \frac{1}{x}\sqrt{4 + 12y^2}\, dy \\
&= 8\pi \int_0^1 \sqrt{1 + 3y^2}\, dy \qquad \text{Let } \sqrt{3}y = \tan\theta \\
&\qquad\qquad\qquad\qquad\quad \sqrt{3}dy = \sec^2\theta\, d\theta \\
&= \frac{8\pi}{\sqrt{3}} \int_0^{\pi/3} \sec^3\theta\, d\theta \\
&= \frac{8\pi}{2\sqrt{3}} \left(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right)\Big|_0^{\pi/3} \\
&= \frac{8\pi}{2\sqrt{3}} \left(2\sqrt{3} + \ln(2 + \sqrt{3})\right) \\
&= 8\pi \left(1 + \frac{\ln(2 + \sqrt{3})}{2\sqrt{3}}\right) \text{ sq. units.}
\end{aligned}$$

32. As in Example 4, the arc length element for the ellipse is

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \sqrt{\frac{a^2 - \frac{a^2 - b^2}{a^2}x^2}{a^2 - x^2}}\, dx.$$

To get the area of the ellipsoid, we must rotate both the upper and lower semi-ellipses (see the figure for Exercise 20 of Section 8.1):

$$\begin{aligned}
S &= 2 \times 2\pi \int_0^a \left[\left(c - b\sqrt{1 - \left(\frac{x}{a}\right)^2}\right) + \right. \\
&\qquad \left.\left(c + b\sqrt{1 - \left(\frac{x}{a}\right)^2}\right)\right] ds \\
&= 8\pi c \int_0^a \sqrt{\frac{a^2 - \frac{a^2 - b^2}{a^2}x^2}{a^2 - x^2}}\, dx \\
&= 8\pi c \left[\frac{1}{4} \text{ of the circumference of the ellipse}\right] \\
&= 8\pi c a E(\varepsilon)
\end{aligned}$$

where $\varepsilon = \frac{\sqrt{a^2 - b^2}}{a}$ and $E(\varepsilon) = \int_0^{\pi/2} \sqrt{1 - \varepsilon^2 \sin t}\, dt$ as defined in Example 4.

33. From Example 3, the length is

$$\begin{aligned}
s &= \frac{10}{\pi} \int_0^{\pi/2} \sqrt{1 + \frac{\pi^2}{4}\cos^2 t}\, dt \\
&= \frac{10}{\pi} \int_0^{\pi/2} \sqrt{1 + \frac{\pi^2}{4} - \frac{\pi^2}{4}\sin^2 t}\, dt \\
&= \frac{5}{\pi}\sqrt{4 + \pi^2} \int_0^{\pi/2} \sqrt{1 - \frac{\pi^2}{4 + \pi^2}\sin^2 t}\, dt \\
&= \frac{5}{\pi}\sqrt{4 + \pi^2}\, E\left(\frac{\pi}{\sqrt{4 + \pi^2}}\right).
\end{aligned}$$

34. Let the equation of the sphere be $x^2 + y^2 = R^2$. Then the surface area between planes $x = a$ and $x = b$ $(-R \le a < b \le R)$ is

$$\begin{aligned}
S &= 2\pi \int_a^b \sqrt{R^2 - x^2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\
&= 2\pi \int_a^b \sqrt{R^2 - x^2} \frac{R}{\sqrt{R^2 - x^2}}\, dx \\
&= 2\pi R \int_a^b dx = 2\pi R(b - a) \text{ sq. units.}
\end{aligned}$$

Thus, the surface area depends only on the radius R of the sphere, and the distance $(b-a)$ between the parellel planes.

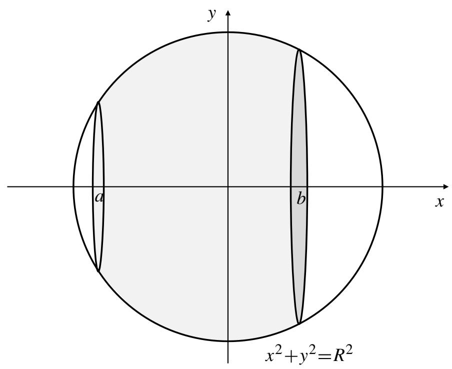

Fig. 7.3.34

35. If the curve $y = x^k$, $0 < x \le 1$, is rotated about the y-axis, it generates a surface of area

$$S = 2\pi \int_0^1 x\sqrt{1+k^2x^{2(k-1)}}\,dx$$
$$= 2\pi \int_0^1 \sqrt{x^2+k^2x^{2k}}\,dx.$$

If $k \le -1$, we have $S \ge 2\pi k \int_0^1 x^k\,dx$, which is infinite. If $k \ge 0$, the surface area S is finite, since x^k is bounded on $(0, 1]$ in that case.
Hence we need only consider the case $-1 < k < 0$. In this case $2 < 2-2k < 4$, and

$$S = 2\pi \int_0^1 x\sqrt{1+k^2x^{2(k-1)}}\,dx$$
$$= 2\pi \int_0^1 \sqrt{x^{2-2k}+k^2}\,x^k\,dx$$
$$< 2\pi\sqrt{1+k^2}\int_0^1 x^k\,dx < \infty.$$

Thus the area is finite if and only if $k > -1$.

36. $$S = 2\pi \int_0^1 |x|\sqrt{1+\frac{1}{x^2}}\,dx$$
$$= 2\pi \int_0^1 \sqrt{x^2+1}\,dx \quad \text{Let } x = \tan\theta, \quad dx = \sec^2\theta\,d\theta$$
$$= 2\pi \int_0^{\pi/4} \sec^3\theta\,d\theta$$
$$= \pi\Big(\sec\theta\,\tan\theta + \ln|\sec\theta+\tan\theta|\Big)\Big|_0^{\pi/4}$$
$$= \pi[\sqrt{2}+\ln(\sqrt{2}+1)] \text{ sq. units.}$$

37. a) Volume $V = \pi \int_1^\infty \frac{dx}{x^2} = \pi$ cu. units.

b) The surface area is

$$S = 2\pi \int_1^\infty \frac{1}{x}\sqrt{1+\frac{1}{x^4}}\,dx$$
$$> 2\pi \int_1^\infty \frac{dx}{x} = \infty.$$

c) Covering a surface with paint requires applying a layer of paint *of constant thickness* to the surface. Far to the right, the horn is thinner than any prescribed constant, so it can contain less paint than would be required to cover its surface.

Section 7.4 Mass, Moments, and Centre of Mass (page 394)

1. The mass of the wire is

$$m = \int_0^L \delta(s)\,ds = \int_0^L \sin\frac{\pi s}{L}\,ds$$
$$= -\frac{L}{\pi}\cos\frac{\pi s}{L}\Big|_0^L = \frac{2L}{\pi}.$$

Since $\delta(s)$ is symmetric about $s = L/2$ (that is, $\delta((L/2)-s) = \delta((L/2)+s)$), the centre of mass is at the midpoint of the wire: $\bar{s} = L/2$.

2. A slice of the wire of width dx at x has volume $dV = \pi(a+bx)^2\,dx$. Therefore the mass of the whole wire is

$$m = \int_0^L \delta_0\pi(a+bx)^2\,dx$$
$$= \delta_0\pi \int_0^L (a^2+2abx+b^2x^2)\,dx$$
$$= \delta_0\pi\left(a^2L+abL^2+\frac{1}{3}b^2L^3\right).$$

Its moment about $x = 0$ is

$$M_{x=0} = \int_0^L x\delta_0\pi(a+bx)^2\,dx$$
$$= \delta_0\pi \int_0^L (a^2x+2abx^2+b^2x^3)\,dx$$
$$= \delta_0\pi\left(\frac{1}{2}a^2L^2+\frac{2}{3}abL^3+\frac{1}{4}b^2L^4\right).$$

Thus, the centre of mass is

$$\bar{x} = \frac{\delta_0\pi\left(\frac{1}{2}a^2L^2+\frac{2}{3}abL^3+\frac{1}{4}b^2L^4\right)}{\delta_0\pi\left(a^2L+abL^2+\frac{1}{3}b^2L^3\right)}$$
$$= \frac{L\left(\frac{1}{2}a^2+\frac{2}{3}abL+\frac{1}{4}b^2L^2\right)}{a^2+abL+\frac{1}{3}b^2L^2}.$$

3. The mass of the plate is $m = \delta_0 \times \text{area} = \dfrac{\pi \delta_0 a^2}{4}$.
The moment about $x = 0$ is

$$\begin{aligned} M_{x=0} &= \int_0^a x\delta_0\sqrt{a^2 - x^2}\,dx \qquad \text{Let } u = a^2 - x^2 \\ &\qquad\qquad\qquad\qquad\qquad du = -2x\,dx \\ &= \frac{\delta_0}{2}\int_0^{a^2}\sqrt{u}\,du \\ &= \frac{\delta_0}{2}\,\frac{2}{3}\,u^{3/2}\bigg|_0^{a^2} = \frac{\delta_0 a^3}{3}. \end{aligned}$$

Thus $\bar{x} = \dfrac{M_{x=0}}{m} = \dfrac{\delta_0 a^3}{3}\,\dfrac{4}{\pi\delta_0 a^2} = \dfrac{4a}{3\pi}$. By symmetry, $\bar{y} = \bar{x}$. Thus the centre of mass of the plate is $\left(\dfrac{4a}{3\pi}, \dfrac{4a}{3\pi}\right)$.

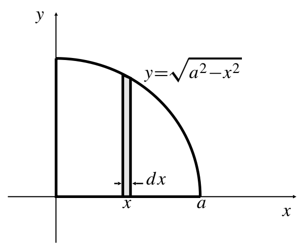

Fig. 7.4.3

4. A vertical strip has area $dA = \sqrt{a^2 - x^2}\,dx$. Therefore, the mass of the quarter-circular plate is

$$\begin{aligned} m &= \int_0^a (\delta_0 x)\sqrt{a^2 - x^2}\,dx \quad \text{Let } u = a^2 - x^2 \\ &\qquad\qquad\qquad\qquad\qquad du = -2x\,dx \\ &= \frac{1}{2}\delta_0\int_0^{a^2}\sqrt{u}\,du = \frac{1}{2}\delta_0\left(\frac{2}{3}u^{3/2}\right)\bigg|_0^{a^2} = \frac{1}{3}\delta_0 a^3. \end{aligned}$$

The moment about $x = 0$ is

$$\begin{aligned} M_{x=0} &= \int_0^a \delta_0 x^2\sqrt{a^2 - x^2}\,dx \quad \text{Let } x = a\sin\theta \\ &\qquad\qquad\qquad\qquad\qquad dx = a\cos\theta\,d\theta \\ &= \delta_0 a^4\int_0^{\pi/2}\sin^2\theta\cos^2\theta\,d\theta \\ &= \frac{\delta_0 a^4}{4}\int_0^{\pi/2}\sin^2 2\theta\,d\theta \\ &= \frac{\delta_0 a^4}{8}\int_0^{\pi/2}(1 - \cos 4\theta)\,d\theta = \frac{\pi\delta_0 a^4}{16}. \end{aligned}$$

The moment about $y = 0$ is

$$\begin{aligned} M_{y=0} &= \frac{1}{2}\delta_0\int_0^a x(a^2 - x^2)\,dx \\ &= \frac{1}{2}\delta_0\left(\frac{a^2x^2}{2} - \frac{x^4}{4}\right)\bigg|_0^a = \frac{1}{8}a^4\delta_0. \end{aligned}$$

Thus, $\bar{x} = \dfrac{3}{16}\pi a$ and $\bar{y} = \dfrac{3}{8}a$. Hence, the centre of mass is located at $(\dfrac{3}{16}\pi a, \dfrac{3}{8}a)$.

5. The mass of the plate is

$$\begin{aligned} m &= 2\int_0^4 ky\sqrt{4 - y}\,dy \qquad \text{Let } u = 4 - y \\ &\qquad\qquad\qquad\qquad\qquad du = -dy \\ &= 2k\int_0^4 (4 - u)u^{1/2}\,du \\ &= 2k\left(\frac{8}{3}u^{3/2} - \frac{2}{5}u^{5/2}\right)\bigg|_0^4 = \frac{256k}{15}. \end{aligned}$$

By symmetry, $M_{x=0} = 0$, so $\bar{x} = 0$.

$$\begin{aligned} M_{y=0} &= 2\int_0^4 ky^2\sqrt{4 - y}\,dy \qquad \text{Let } u = 4 - y \\ &\qquad\qquad\qquad\qquad\qquad du = -dy \\ &= 2k\int_0^4 (16u^{1/2} - 8u^{3/2} + u^{5/2})\,du \\ &= 2k\left(\frac{32}{3}u^{3/2} - \frac{16}{5}u^{5/2} + \frac{2}{7}u^{7/2}\right)\bigg|_0^4 = \frac{4096k}{105}. \end{aligned}$$

Thus $\bar{y} = \dfrac{4096k}{105}\cdot\dfrac{15}{256k} = \dfrac{16}{7}$. The centre of mass of the plate is $(0, 16/7)$.

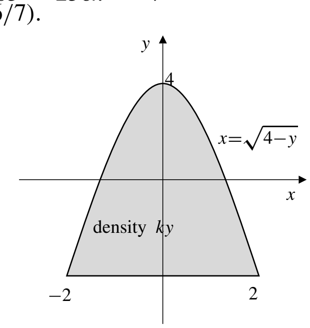

Fig. 7.4.5

6. A vertical strip at h has area $dA = (2 - \frac{2}{3}h)\,dh$. Thus, the mass of the plate is

$$\begin{aligned} m &= \int_0^3 (5h)\left(2 - \frac{2}{3}h\right)dh = 10\int_0^3\left(h - \frac{h^2}{3}\right)dh \\ &= 10\left(\frac{h^2}{2} - \frac{h^3}{9}\right)\bigg|_0^3 = 15 \text{ kg}. \end{aligned}$$

The moment about $x = 0$ is

$$\begin{aligned} M_{x=0} &= 10\int_0^3\left(h^2 - \frac{h^3}{3}\right)dh \\ &= 10\left(\frac{h^3}{3} - \frac{h^4}{12}\right)\bigg|_0^3 = \frac{45}{2} \text{ kg-m}. \end{aligned}$$

The moment about $y = 0$ is

$$M_{y=0} = 10\int_0^3 \frac{1}{2}\left(2 - \frac{2}{3}h\right)\left(h - \frac{1}{3}h^2\right) dh$$
$$= 10\int_0^3 \left(h - \frac{2}{3}h^2 + \frac{1}{9}h^3\right) dh$$
$$= 10\left(\frac{h^2}{2} - \frac{2h^3}{9} + \frac{h^4}{36}\right)\Bigg|_0^3 = \frac{15}{2} \text{ kg-m.}$$

Thus, $\bar{x} = \dfrac{\left(\frac{45}{2}\right)}{15} = \dfrac{3}{2}$ and $\bar{y} = \dfrac{\left(\frac{15}{2}\right)}{15} = \dfrac{1}{2}$. The centre of mass is located at $(\frac{3}{2}, \frac{1}{2})$.

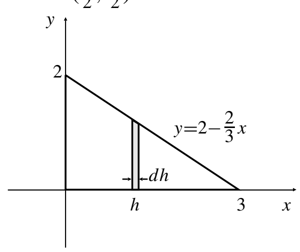

Fig. 7.4.6

7. The mass of the plate is

$$m = \int_0^a kx\,a\,dx = \frac{ka^3}{2}.$$

By symmetry, $\bar{y} = a/2$.

$$M_{x=0} = \int_0^a kx^2\,a\,dx = \frac{ka^4}{3}.$$

Thus $\bar{x} = \dfrac{ka^4}{3} \cdot \dfrac{2}{ka^3} = \dfrac{2a}{3}$. The centre of mass of the plate is $\left(\dfrac{2a}{3}, \dfrac{a}{2}\right)$.

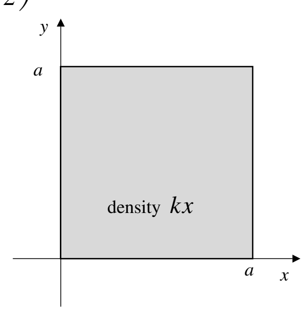

Fig. 7.4.7

8. A vertical strip has area $dA = 2\left(\dfrac{a}{\sqrt{2}} - r\right) dr$. Thus, the mass is

$$m = 2\int_0^{a/\sqrt{2}} kr\left[2\left(\frac{a}{\sqrt{2}} - r\right)\right] dr$$
$$= 4k\int_0^{a/\sqrt{2}} \left(\frac{a}{\sqrt{2}}r - r^2\right) dr = \frac{k}{3\sqrt{2}}a^3 \text{ g.}$$

Since the mass is symmetric about the y-axis, and the plate is symmetric about both the x- and y-axis, therefore the centre of mass must be located at the centre of the square.

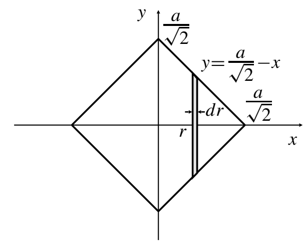

Fig. 7.4.8

9.
$$m = \int_a^b \delta(x)\Big(g(x) - f(x)\Big)\,dx$$
$$M_{x=0} = \int_a^b x\delta(x)\Big(g(x) - f(x)\Big)\,dx$$
$$M_{y=0} = \frac{1}{2}\int_a^b x\delta(x)\Big((g(x))^2 - (f(x))^2\Big)\,dx$$

Centre of mass: $\left(\dfrac{M_{x=0}}{m}, \dfrac{M_{y=0}}{m}\right)$.

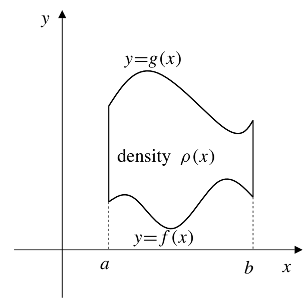

Fig. 7.4.9

10. The slice of the brick shown in the figure has volume $dV = 50\,dx$. Thus, the mass of the brick is

$$m = \int_0^{20} kx50\,dx = 25kx^2\Big|_0^{20} = 10000k \text{ g.}$$

The moment about $x = 0$, i.e., the yz-plane, is

$$M_{x=0} = 50k\int_0^{20} x^2\,dx = \frac{50}{3}kx^3\Big|_0^{20}$$
$$= \frac{50}{3}(8000)k \text{ g-cm.}$$

Thus, $\bar{x} = \dfrac{\frac{50}{3}(8000)k}{10000k} = \dfrac{40}{3}$. Since the density is independent of y and z, $\bar{y} = \dfrac{5}{2}$ and $\bar{z} = 5$. Hence, the centre of mass is located on the 20 cm long central axis of the brick, two-thirds of the way from the least dense 10×5 face to the most dense such face.

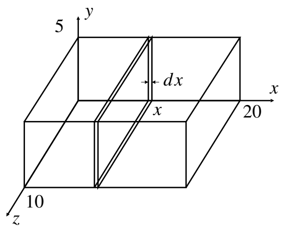

Fig. 7.4.10

11. Choose axes through the centre of the ball as shown in the following figure. The mass of the ball is

$$\begin{aligned} m &= \int_{-R}^{R} (y + 2R)\pi(R^2 - y^2)\,dy \\ &= 4\pi R\left(R^2 y - \frac{y^3}{3}\right)\Bigg|_0^R = \frac{8}{3}\pi R^4 \text{ kg.} \end{aligned}$$

By symmetry, the centre of mass lies along the y-axis; we need only calculate $\bar{y}$.

$$\begin{aligned} M_{y=0} &= \int_{-R}^{R} y(y + 2R)\pi(R^2 - y^2)\,dy \\ &= 2\pi \int_0^R y^2(R^2 - y^2)\,dy \\ &= 2\pi\left(R^2\frac{y^3}{3} - \frac{y^5}{5}\right)\Bigg|_0^R = \frac{4}{15}\pi R^5. \end{aligned}$$

Thus $\bar{y} = \dfrac{4\pi R^5}{15} \cdot \dfrac{3}{8\pi R^4} = \dfrac{R}{10}$. The centre of mass is on the line through the centre of the ball perpendicular to the plane mentioned in the problem, at a distance $R/10$ from the centre of the ball on the side opposite to the plane.

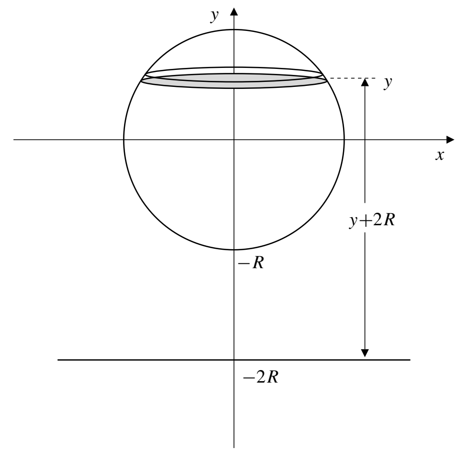

Fig. 7.4.11

12. A slice at height z has volume $dV = \pi y^2\,dz$ and density kz g/cm^3. Thus, the mass of the cone is

$$\begin{aligned} m &= \int_0^b kz\pi y^2\,dz \\ &= \pi k a^2 \int_0^b z\left(1 - \frac{z}{b}\right)^2 dz \\ &= \pi k a^2 \left(\frac{z^2}{2} - \frac{2z^3}{3b} + \frac{z^4}{4b^2}\right)\Bigg|_0^b \\ &= \frac{1}{12}\pi k a^2 b^2 \text{ g.} \end{aligned}$$

The moment about $z = 0$ is

$$M_{z=0} = \pi k a^2 \int_0^b z^2\left(1 - \frac{z}{b}\right)^2 dz = \frac{1}{30}\pi k a^2 b^3 \text{ g-cm.}$$

Thus, $\bar{z} = \dfrac{2b}{5}$. Hence, the centre of mass is on the axis of the cone at height $2b/5$ cm above the base.

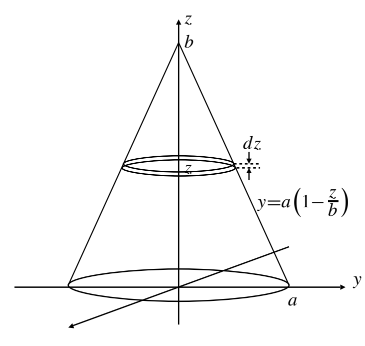

Fig. 7.4.12

13. By symmetry, $\bar{y} = 0$.

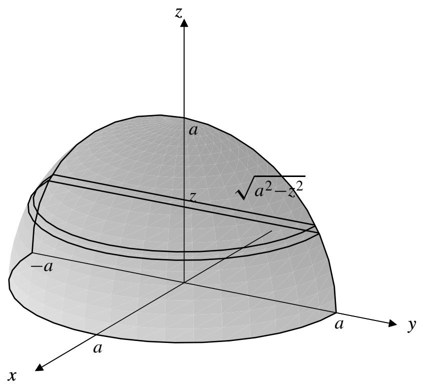

Fig. 7.4.13

A horizontal slice of the solid at height z with thickness dz is a half-disk of radius $\sqrt{a^2 - z^2}$ with centre of mass at $\bar{x} = \dfrac{4\sqrt{a^2 - z^2}}{3\pi}$, by Exercise 3 above. Its mass is

$$dm = \delta_0 z\, dz\, \frac{\pi}{2}(a^2 - z^2),$$

and its moment about $x = 0$ is

$$\begin{aligned} dM_{x=0} = dm\,\bar{x} &= \frac{\pi\delta_0}{2}\, z(a^2 - z^2)\frac{4\sqrt{a^2 - z^2}}{3\pi} \\ &= \frac{2\delta_0}{3}\, z(a^2 - z^2)^{3/2}. \end{aligned}$$

Thus the mass of the solid is

$$\begin{aligned} m &= \frac{\pi\delta_0}{2}\int_0^a (a^2 z - z^3)\, dz \\ &= \frac{\pi\delta_0}{2}\left(\frac{a^2 z^2}{2} - \frac{z^4}{4}\right)\Bigg|_0^a = \frac{\pi\delta_0 a^4}{8}. \end{aligned}$$

Also,

$$\begin{aligned} M_{z=0} &= \frac{\pi\delta_0}{2}\int_0^a (a^2 z^2 - z^4)\, dz \\ &= \frac{\pi\delta_0}{2}\left(\frac{a^2 z^3}{3} - \frac{z^5}{5}\right)\Bigg|_0^a = \frac{\pi\delta_0 a^5}{15}, \end{aligned}$$

and $\bar{z} = \dfrac{\pi\delta_0 a^5}{15} \cdot \dfrac{8}{\pi\delta_0 a^4} = \dfrac{8a}{15}$.

Finally,

$$\begin{aligned} M_{x=0} &= \frac{2\delta_0}{3}\int_0^a z(a^2 - z^2)^{3/2}\, dz \qquad \text{Let } u = a^2 - z^2,\ du = -2z\, dz \\ &= \frac{\delta_0}{3}\int_0^{a^2} u^{3/2}\, du \\ &= \frac{\delta_0}{3}\left(\frac{2}{5}u^{5/2}\right)\Bigg|_0^{a^2} = \frac{2\delta_0 a^5}{15}, \end{aligned}$$

so $\bar{x} = \dfrac{2\delta_0 a^5}{15} \cdot \dfrac{8}{\pi\delta_0 a^4} = \dfrac{16a}{15}$.

The centre of mass is $\left(\dfrac{16a}{15}, 0, \dfrac{8a}{15}\right)$.

14. Assume the cone has its base in the xy-plane and its vertex at height b on the z-axis. By symmetry, the centre of mass lies on the z-axis. A cylindrical shell of thickness dx and radius x about the z-axis has height $z = b(1 - (x/a))$. Since it's density is constant kx, its mass is

$$dm = 2\pi bkx^2\left(1 - \frac{x}{a}\right)dx.$$

Also its centre of mass is at half its height,

$$\bar{y}_{\text{shell}} = \frac{b}{2}\left(1 - \frac{x}{a}\right).$$

Thus its moment about $z = 0$ is

$$dM_{z=0} = \bar{y}_{\text{shell}}\, dm = \pi bkx^2\left(1 - \frac{x}{a}\right)^2 dx.$$

Hence

$$\begin{aligned} m &= \int_0^a 2\pi bkx^2\left(1 - \frac{x}{a}\right)dx = \frac{\pi kba^3}{6} \\ M_{z=0} &= \int_0^a \pi bkx^2\left(1 - \frac{x}{a}\right)^2 dx = \frac{\pi kb^2 a^3}{30} \end{aligned}$$

and $\bar{z} = M_{z=0}/m = b/5$. The centre of mass is on the axis of the cone at height $b/5$ cm above the base.

15.

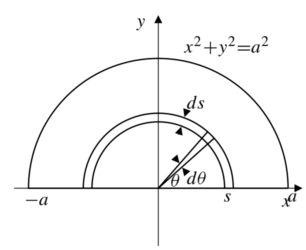

Fig. 7.4.15

Consider the area element which is the thin half-ring shown in the figure. We have

$$dm = ks\,\pi s\, ds = k\pi\, s^2\, ds.$$

Thus, $m = \dfrac{k\pi}{3}a^3$.

Regard this area element as itself composed of smaller elements at positions given by the angle θ as shown. Then

$$\begin{aligned} dM_{y=0} &= \left(\int_0^\pi (s\sin\theta)s\, d\theta\right) ks\, ds \\ &= 2ks^3\, ds, \\ M_{y=0} &= 2k\int_0^a s^3\, ds = \frac{ka^4}{2}. \end{aligned}$$

Therefore, $\bar{y} = \dfrac{ka^4}{2} \cdot \dfrac{3}{k\pi a^3} = \dfrac{3a}{2\pi}$. By symmetry, $\bar{x} = 0$. Thus, the centre of mass of the plate is $\left(0, \dfrac{3a}{2\pi}\right)$.

16.

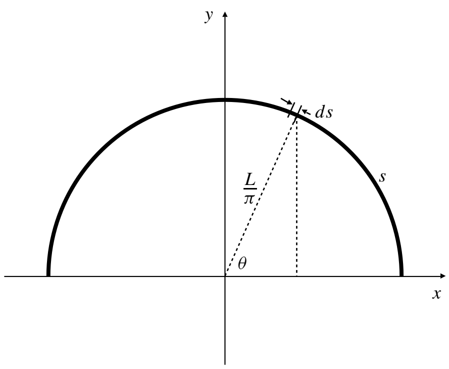

Fig. 7.4.16

The radius of the semicircle is $\dfrac{L}{\pi}$. Let s measure the distance along the wire from the point where it leaves the positive x-axis. Thus, the density at position s is $\delta\delta(s) = \sin\left(\dfrac{\pi s}{L}\right)$ g/cm. The mass of the wire is

$$m = \int_0^L \sin\frac{\pi s}{L}\,ds = -\frac{L}{\pi}\cos\frac{\pi s}{L}\bigg|_0^L = \frac{2L}{\pi}\ \text{g}.$$

Since an arc element ds at position s is at height $y = \dfrac{L}{\pi}\sin\theta = \dfrac{L}{\pi}\sin\dfrac{\pi s}{L}$, the moment of the wire about $y = 0$ is

$$\begin{aligned} M_{y=0} &= \int_0^L \frac{L}{\pi}\sin^2\frac{\pi s}{L}\,ds \qquad \text{Let } \theta = \pi s/L \\ &\qquad\qquad\qquad\qquad\qquad d\theta = \pi\,ds/L \\ &= \left(\frac{L}{\pi}\right)^2 \int_0^\pi \sin^2\theta\,d\theta \\ &= \frac{L^2}{2\pi^2}\Big(\theta - \sin\theta\cos\theta\Big)\bigg|_0^\pi = \frac{L^2}{2\pi}\ \text{g-cm}. \end{aligned}$$

Since the wire and the density function are both symmetric about the y-axis, we have $M_{x=0} = 0$.
Hence, the centre of mass is located at $\left(0, \dfrac{L}{4}\right)$.

17.
$$\begin{aligned} m &= \int_0^\infty Ce^{-kr^2}(4\pi r^2)\,dr \\ &= 4\pi C\int_0^\infty r^2 e^{-kr^2}\,dr \qquad \text{Let } u = \sqrt{k}\,r \\ &\qquad\qquad\qquad\qquad\qquad du = \sqrt{k}\,dr \\ &= \frac{4\pi C}{k^{3/2}}\int_0^\infty u^2 e^{-u^2}\,du \\ &\qquad U = u \qquad dV = ue^{-u^2}\,du \\ &\qquad dU = du \qquad V = -\tfrac{1}{2}e^{-u^2} \\ &= \frac{4\pi C}{k^{3/2}}\lim_{R\to\infty}\left(\frac{-ue^{-u^2}}{2}\bigg|_0^R + \frac{1}{2}\int_0^R e^{-u^2}\,du\right) \\ &= \frac{4\pi C}{k^{3/2}}\left(0 + \frac{1}{2}\int_0^\infty e^{-u^2}\,du\right) \\ &= \frac{4\pi C}{k^{3/2}}\frac{\sqrt{\pi}}{4} = C\left(\frac{\pi}{k}\right)^{3/2} \approx \frac{5.57C}{k^{3/2}}. \end{aligned}$$

18.
$$\begin{aligned} \bar{r} &= \frac{1}{m}\int_0^\infty rCe^{-kr^2}(4\pi r^2)\,dr \\ &= \frac{4\pi C}{C\pi^{3/2}k^{-3/2}}\int_0^\infty r^3 e^{-kr^2}\,dr \qquad \text{Let } u = kr^2 \\ &\qquad\qquad\qquad\qquad\qquad\qquad du = 2kr\,dr \\ &= \frac{4k^{3/2}}{\sqrt{\pi}}\frac{1}{2k^2}\int_0^\infty ue^{-u}\,du \\ &\qquad U = u \qquad dV = e^{-u}\,du \\ &\qquad dU = du \qquad V = -e^{-u} \\ &= \frac{2}{\sqrt{\pi k}}\lim_{R\to\infty}\left(-ue^{-u}\bigg|_0^R + \int_0^R e^{-u}\,du\right) \\ &= \frac{2}{\sqrt{\pi k}}\Big(0 + \lim_{R\to\infty}(e^0 - e^{-R}\Big) = \frac{2}{\sqrt{\pi k}}. \end{aligned}$$

Section 7.5 Centroids (page 399)

1.
$$\begin{aligned} A &= \frac{\pi r^2}{4} \\ M_{x=0} &= \int_0^r x\sqrt{r^2 - x^2}\,dx \qquad \text{Let } u = r^2 - x^2 \\ &\qquad\qquad\qquad\qquad\qquad du = -2x\,dx \\ &= \frac{1}{2}\int_0^{r^2} u^{1/2}\,du = \frac{u^{3/2}}{3}\bigg|_0^{r^2} = \frac{r^3}{3} \\ \bar{x} &= \frac{r^3}{3}\cdot\frac{4}{\pi r^2} = \frac{4r}{3\pi} = \bar{y} \text{ by symmetry.} \end{aligned}$$

The centroid is $\left(\dfrac{4r}{3\pi}, \dfrac{4r}{3\pi}\right)$.

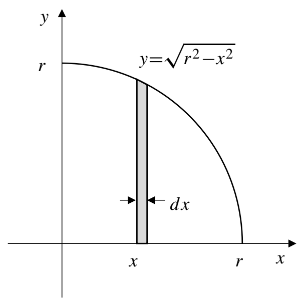

Fig. 7.5.1

2. By symmetry, $\bar{x} = 0$. A horizontal strip at y has mass $dm = 2\sqrt{9-y}\,dy$ and moment $dM_{y=0} = 2y\sqrt{9-y}\,dy$ about $y = 0$. Thus,

$$m = 2\int_0^9 \sqrt{9-y}\,dy = -2\left(\frac{2}{3}\right)(9-y)^{3/2}\Big|_0^9 = 36$$

and

$$\begin{aligned} M_{y=0} &= 2\int_0^9 y\sqrt{9-y}\,dy \quad \text{Let } u^2 = 9-y \\ &\qquad\qquad\qquad\qquad\quad 2u\,du = -dy \\ &= 4\int_0^3 (9u^2 - u^4)\,du = 4(3u^3 - \tfrac{1}{5}u^5)\Big|_0^3 = \frac{648}{5}. \end{aligned}$$

Thus, $\bar{y} = \dfrac{648}{5 \times 36} = \dfrac{18}{5}$. Hence, the centroid is at $\left(0, \dfrac{18}{5}\right)$.

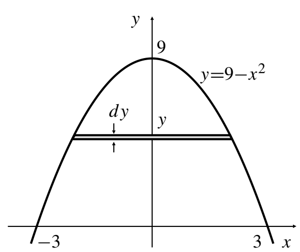

Fig. 7.5.2

3. The area and moments of the region are

$$\begin{aligned} A &= \int_0^1 \frac{dx}{\sqrt{1+x^2}} \qquad \text{Let } x = \tan\theta \\ &\qquad\qquad\qquad\qquad\quad dx = \sec^2\theta\,d\theta \\ &= \int_0^{\pi/4} \sec\theta\,d\theta \\ &= \ln|\sec\theta + \tan\theta|\Big|_0^{\pi/4} = \ln(1+\sqrt{2}) \\ M_{x=0} &= \int_0^1 \frac{x\,dx}{\sqrt{1+x^2}} = \sqrt{1+x^2}\Big|_0^1 = \sqrt{2} - 1 \\ M_{y=0} &= \frac{1}{2}\int_0^1 \frac{dx}{1+x^2} = \frac{1}{2}\tan^{-1}x\Big|_0^1 = \frac{\pi}{8}. \end{aligned}$$

Thus $\bar{x} = \dfrac{\sqrt{2}-1}{\ln(1+\sqrt{2})}$, and $\bar{y} = \dfrac{\pi}{8\ln(1+\sqrt{2})}$. The centroid is $\left(\dfrac{\sqrt{2}-1}{\ln(1+\sqrt{2})}, \dfrac{\pi}{8\ln(1+\sqrt{2})}\right)$.

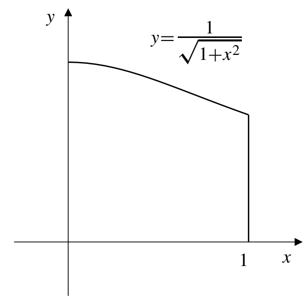

Fig. 7.5.3

4. The area of the sector is $A = \frac{1}{8}\pi r^2$. Its moment about $x = 0$ is

$$\begin{aligned} M_{x=0} &= \int_0^{r/\sqrt{2}} x^2\,dx + \int_{r/\sqrt{2}}^r x\sqrt{r^2-x^2}\,dx \\ &= \frac{r^3}{6\sqrt{2}} - \frac{1}{3}(r^2-x^2)^{3/2}\Big|_{r/\sqrt{2}}^r = \frac{r^3}{3\sqrt{2}}. \end{aligned}$$

Thus, $\bar{x} = \dfrac{r^3}{3\sqrt{2}} \times \dfrac{8}{\pi r^2} = \dfrac{8r}{3\sqrt{2}\pi}$. By symmetry, the centroid must lie on the line $y = x\left(\tan\dfrac{\pi}{8}\right) = x(\sqrt{2}-1)$. Thus, $\bar{y} = \dfrac{8r(\sqrt{2}-1)}{3\sqrt{2}\pi}$.

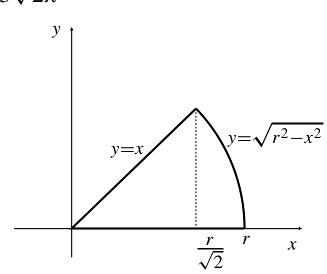

Fig. 7.5.4

5. By symmetry, $\bar{x} = 0$. We have

$$\begin{aligned} A &= 2\int_0^{\sqrt{3}} \left(\sqrt{4-x^2} - 1\right) dx \qquad \text{Let } x = 2\sin\theta, \; dx = 2\cos\theta\, d\theta \\ &= 2\left(4\int_0^{\pi/3} \cos^2\theta\, d\theta - \sqrt{3}\right) \\ &= 4(\theta + \sin\theta\cos\theta)\Big|_0^{\pi/3} - 2\sqrt{3} \\ &= 4\left(\frac{\pi}{3} + \frac{\sqrt{3}}{4}\right) - 2\sqrt{3} = \frac{4\pi}{3} - \sqrt{3} \\ M_{y=0} &= 2 \times \frac{1}{2}\int_0^{\sqrt{3}} \left(\sqrt{4-x^2} - 1\right)^2 dx \\ &= \int_0^{\sqrt{3}} \left(5 - x^2 - 2\sqrt{4-x^2}\right) dx \\ &= 5\sqrt{3} - \sqrt{3} - 2\int_0^{\sqrt{3}} \sqrt{4-x^2}\, dx \\ &= 4\sqrt{3} - 4\left(\frac{\pi}{3} + \frac{\sqrt{3}}{4}\right) = 3\sqrt{3} - \frac{4\pi}{3}. \end{aligned}$$

Thus $\bar{y} = \dfrac{9\sqrt{3} - 4\pi}{3} \cdot \dfrac{3}{4\pi - 3\sqrt{3}} = \dfrac{9\sqrt{3} - 4\pi}{4\pi - 3\sqrt{3}}$. The centroid is $\left(0, \dfrac{9\sqrt{3} - 4\pi}{4\pi - 3\sqrt{3}}\right)$.

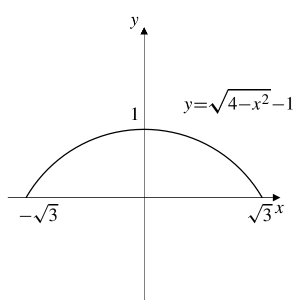

Fig. 7.5.5

6. By symmetry, $\bar{x} = 0$. The area is $A = \frac{1}{2}\pi ab$. The moment about $y = 0$ is

$$\begin{aligned} M_{y=0} &= \frac{1}{2}\int_{-a}^{a} b^2\left[1 - \left(\frac{x}{a}\right)^2\right] dx = b^2\int_0^a 1 - \frac{x^2}{a^2}\, dx \\ &= b^2\left(x - \frac{x^3}{3a^2}\right)\Big|_0^a = \frac{2}{3}ab^2. \end{aligned}$$

Thus, $\bar{y} = \dfrac{2ab^2}{3} \times \dfrac{2}{\pi ab} = \dfrac{4b}{3\pi}$.

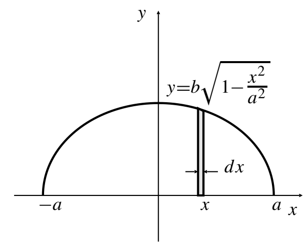

Fig. 7.5.6

7. The quadrilateral consists of two triangles, T_1 and T_2, as shown in the figure. The area and centroid of T_1 are given by

$$\begin{aligned} A_1 &= \frac{4 \times 1}{2} = 2, \\ \bar{x}_1 &= \frac{0+3+4}{3} = \frac{7}{3}, \quad \bar{y}_1 = \frac{0+1+0}{3} = \frac{1}{3}. \end{aligned}$$

The area and centroid of T_2 are given by

$$\begin{aligned} A_2 &= \frac{4 \times 2}{2} = 4, \\ \bar{x}_2 &= \frac{0+2+4}{3} = 2, \quad \bar{y}_2 = \frac{0-2+0}{3} = -\frac{2}{3}. \end{aligned}$$

It follows that

$$\begin{aligned} M_{1,x=0} &= \frac{7}{3} \times 2 = \frac{14}{3} \qquad & M_{2,x=0} &= 2 \times 4 = 8 \\ M_{1,y=0} &= \frac{1}{3} \times 2 = \frac{2}{3} \qquad & M_{2,y=0} &= -\frac{2}{3} \times 4 = -\frac{8}{3}. \end{aligned}$$

Since areas and moments are additive, we have for the whole quadrilateral

$$\begin{aligned} A &= 2 + 4 = 6, \\ M_{x=0} &= \frac{14}{3} + 8 = \frac{38}{3}, \quad M_{y=0} = \frac{2}{3} - \frac{8}{3} = -2. \end{aligned}$$

Thus $\bar{x} = \dfrac{38}{3 \times 6} = \dfrac{19}{9}$, and $\bar{y} = \dfrac{-2}{6} = -\dfrac{1}{3}$. The centroid of the quadrilateral is $\left(\dfrac{19}{9}, -\dfrac{1}{3}\right)$.

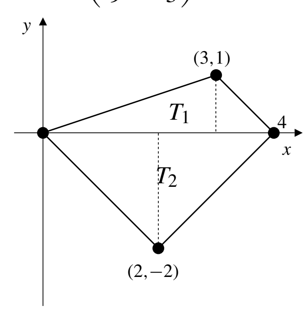

Fig. 7.5.7

8. The region is the union of a half-disk and a triangle. The centroid of the half-disk is known to be at $\left(1, \dfrac{4}{3\pi}\right)$ and that of the triangle is at $\left(\dfrac{2}{3}, -\dfrac{2}{3}\right)$. The area of the semi-circle is $\dfrac{\pi}{2}$ and the triangle is 2. Hence,

$$M_{x=0} = \left(\frac{\pi}{2}\right)(1) + (2)\left(\frac{2}{3}\right) = \frac{3\pi + 8}{6};$$
$$M_{y=0} = \left(\frac{\pi}{2}\right)\left(\frac{4}{3\pi}\right) + (2)\left(-\frac{2}{3}\right) = -\frac{2}{3}.$$

Since the area of the whole region is $\dfrac{\pi}{2} + 2$, then $\bar{x} = \dfrac{3\pi + 8}{3(\pi + 4)}$ and $\bar{y} = -\dfrac{4}{3(\pi + 4)}$.

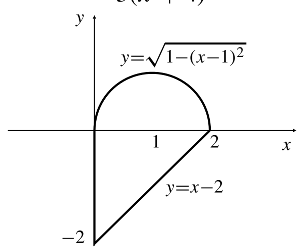

Fig. 7.5.8

9. A circular strip of the surface between heights y and $y + dy$ has area

$$dS = 2\pi x \frac{dy}{\cos\theta} = 2\pi x \frac{r}{x}\, dy = 2\pi r\, dy.$$

The total surface area is

$$S = 2\pi r \int_0^r dy = 2\pi r^2.$$

The moment about $y = 0$ is

$$M_{y=0} = 2\pi r \int_0^r y\, dy = \pi r (y^2)\Big|_0^r = \pi r^3.$$

Thus $\bar{y} = \dfrac{\pi r^3}{2\pi r^2} = \dfrac{r}{2}$. By symmetry, the centroid of the hemispherical surface is on the axis of symmetry of the hemisphere. It is halfway between the centre of the base circle and the vertex.

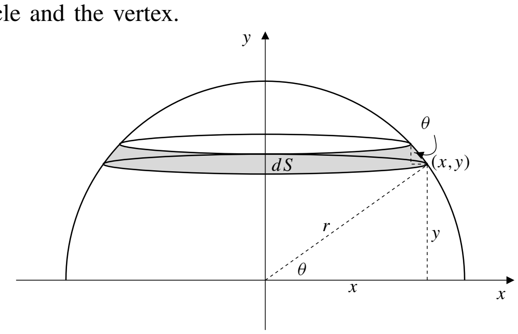

Fig. 7.5.9

10. By symmetry, $\bar{x} = \bar{y} = 0$. The volume is $V = \frac{2}{3}\pi r^3$. A thin slice of the solid at height z will have volume $dV = \pi y^2\, dz = \pi(r^2 - z^2)\, dz$. Thus, the moment about $z = 0$ is

$$\begin{aligned} M_{z=0} &= \int_0^r z\pi(r^2 - z^2)\, dz \\ &= \pi\left(\frac{r^2 z^2}{2} - \frac{z^4}{4}\right)\Bigg|_0^r = \frac{\pi r^4}{4}. \end{aligned}$$

Thus, $\bar{z} = \dfrac{\pi r^4}{4} \times \dfrac{3}{2\pi r^3} = \dfrac{3r}{8}$. Hence, the centroid is on the axis of the hemisphere at distance $3r/8$ from the base.

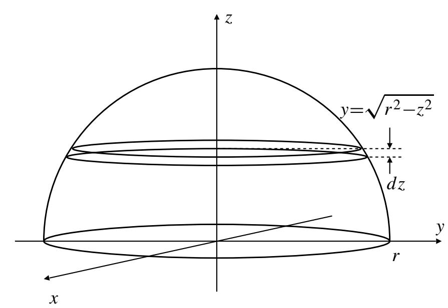

Fig. 7.5.10

11. The cone has volume $V = \frac{1}{3}\pi r^2 h$. (See the following figure.) The disk-shaped slice with vertical width dz has radius $y = r\left(1 - \dfrac{z}{h}\right)$, and therefore has volume

$$dV = \pi r^2 \left(1 - \frac{z}{h}\right)^2 dz = \pi \frac{r^2}{h^2}(h - z)^2\, dz.$$

We have

$$\begin{aligned} M_{z=0} &= \frac{\pi r^2}{h^2}\int_0^h z(h - z)^2\, dz \qquad \text{Let } u = h - z \\ &\qquad\qquad\qquad\qquad\qquad\quad\;\; du = -dz \\ &= \frac{\pi r^2}{h^2}\int_0^h (h - u)u^2\, du \\ &= \frac{\pi r^2}{h^2}\left(\frac{hu^3}{3} - \frac{u^4}{4}\right)\Bigg|_0^h = \frac{\pi r^2 h^2}{12}. \end{aligned}$$

Therefore $\bar{z} = \dfrac{\pi r^2 h^2}{12} \cdot \dfrac{3}{\pi r^2 h} = \dfrac{h}{4}$. The centroid of the solid cone is on the axis of the cone, at a distance above the base equal to one quarter of the height of the cone.

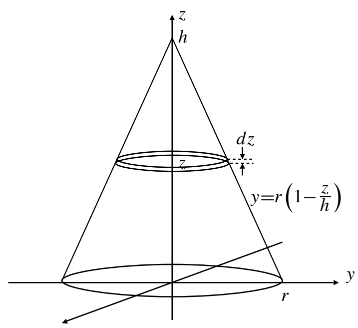

Fig. 7.5.11

12. A band at height z with vertical width dz has radius $y = r\left(1 - \dfrac{z}{h}\right)$, and has actual (slant) width

$$ds = \sqrt{1 + \left(\frac{dy}{dz}\right)^2}\,dz = \sqrt{1 + \frac{r^2}{h^2}}\,dz.$$

Its area is

$$dA = 2\pi r\left(1 - \frac{z}{h}\right)\sqrt{1 + \frac{r^2}{h^2}}\,dz.$$

Thus the area of the conical surface is

$$A = 2\pi r\sqrt{1 + \frac{r^2}{h^2}}\int_0^h \left(1 - \frac{z}{h}\right) dz = \pi r\sqrt{r^2 + h^2}.$$

The moment about $z = 0$ is

$$\begin{aligned}M_{z=0} &= 2\pi r\sqrt{1 + \frac{r^2}{h^2}}\int_0^h z\left(1 - \frac{z}{h}\right) dz\\ &= 2\pi r\sqrt{1 + \frac{r^2}{h^2}}\left(\frac{z^2}{2} - \frac{z^3}{3h}\right)\Bigg|_0^h = \frac{1}{3}\pi r h\sqrt{r^2 + h^2}.\end{aligned}$$

Thus, $\bar{z} = \dfrac{\pi r h\sqrt{r^2 + h^2}}{3} \times \dfrac{1}{\pi r\sqrt{r^2 + h^2}} = \dfrac{h}{3}$. By symmetry, $\bar{x} = \bar{y} = 0$. Hence, the centroid is on the axis of the conical surface, at distance $h/3$ from the base.

13. By symmetry, $\bar{x} = \dfrac{\pi}{2}$. The area and y-moment of the region are given by

$$\begin{aligned}A &= \int_0^\pi \sin x\,dx = 2\\ M_{y=0} &= \frac{1}{2}\int_0^\pi \sin^2 x\,dx\\ &= \frac{1}{4}(x - \sin x\cos x)\Bigg|_0^\pi = \frac{\pi}{4}.\end{aligned}$$

Thus $\bar{y} = \dfrac{\pi}{8}$, and the centroid is $\left(\dfrac{\pi}{2}, \dfrac{\pi}{8}\right)$.

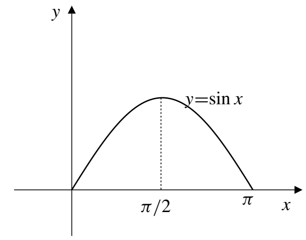

Fig. 7.5.13

14. The area of the region is

$$A = \int_0^{\pi/2} \cos x\,dx = \sin x\Bigg|_0^{\pi/2} = 1.$$

The moment about $x = 0$ is

$$\begin{aligned}M_{x=0} &= \int_0^{\pi/2} x\cos x\,dx\\ &\qquad U = x \qquad dV = \cos x\,dx\\ &\qquad dU = dx \qquad V = \sin x\\ &= x\sin x\Bigg|_0^{\pi/2} - \int_0^{\pi/2}\sin x\,dx = \frac{\pi}{2} - 1.\end{aligned}$$

Thus, $\bar{x} = \dfrac{\pi}{2} - 1$. The moment about $y = 0$ is

$$\begin{aligned}M_{y=0} &= \frac{1}{2}\int_0^{\pi/2} \cos^2 x\,dx\\ &= \frac{1}{4}\left(x + \frac{1}{2}\sin 2x\right)\Bigg|_0^{\pi/2} = \frac{\pi}{8}.\end{aligned}$$

Thus, $\bar{y} = \dfrac{\pi}{8}$. The centroid is $\left(\dfrac{\pi}{2} - 1, \dfrac{\pi}{8}\right)$.

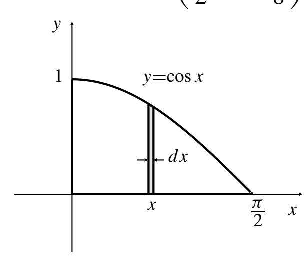

Fig. 7.5.14

15. The arc has length $L = \dfrac{\pi r}{2}$. By symmetry, $\bar{x} = \bar{y}$. An element of the arc between x and $x + dx$ has length

$$ds = \frac{dx}{\sin\theta} = \frac{r\,dx}{y} = \frac{r\,dx}{\sqrt{r^2 - x^2}}.$$

Thus

$$M_{x=0} = \int_0^r \frac{xr\,dx}{\sqrt{r^2-x^2}} = -r\sqrt{r^2-x^2}\Big|_0^r = r^2.$$

Hence $\bar{x} = r^2 \cdot \frac{2}{\pi r} = \frac{2r}{\pi}$, and the centroid is $\left(\frac{2r}{\pi}, \frac{2r}{\pi}\right)$.

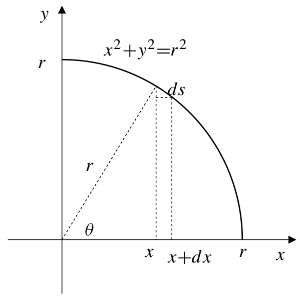

Fig. 7.5.15

16. The solid S in question consists of a solid cone $\mathcal{C}$ with vertex at the origin, height 1, and top a circular disk of radius 2, and a solid cylinder D of radius 2 and height 1 sitting on top of the cone. These solids have volumes $V_C = 4\pi/3$, $V_D = 4\pi$, and $V_S = V_C + V_D = 16\pi/3$.

By symmetry, the centroid of the solid lies on its vertical axis of symmetry; let us continue to call this the y-axis. We need only determine $\bar{y}_S$. Since D lies between $y = 1$ and $y = 2$, its centroid satisfies $\bar{y}_D = 3/2$. Also, by Exercise 11, the centroid of the solid cone satisfies $\bar{y}_C = 3/4$. Thus $\mathcal{C}$ and D have moments about $y = 0$:

$$M_{C,y=0} = \left(\frac{4\pi}{3}\right)\left(\frac{3}{4}\right) = \pi, \quad M_{D,y=0} = (4\pi)\left(\frac{3}{2}\right) = 6\pi.$$

Thus $M_{S,y=0} = \pi + 6\pi = 7\pi$, and $\bar{z}_S = 7\pi/(16\pi/3) = 21/16$. The centroid of the solid S is on its vertical axis of symmetry at height 21/16 above the vertex of the conical part.

17. The region in figure (a) is the union of a rectangle of area 2 and centroid $(1, 3/2)$ and a triangle of area 1 and centroid $(2/3, 2/3)$. Therefore its area is 3 and its centroid is $(\bar{x}, \bar{y})$, where

$$3\bar{x} = 2(1) + 1\left(\frac{2}{3}\right) = \frac{8}{3}$$
$$3\bar{y} = 2\left(\frac{3}{2}\right) + 1\left(\frac{2}{3}\right) = \frac{11}{3}.$$

Therefore, the centroid is $(8/9, 11/9)$.

18. The region in figure (b) is the union of a square of area $(\sqrt{2})^2 = 2$ and centroid $(0, 0)$ and a triangle of area 1/2 and centroid $(2/3, 2/3)$. Therefore its area is 5/2 and its centroid is $(\bar{x}, \bar{y})$, where

$$\frac{5}{2}\bar{x} = 2(0) + \frac{1}{2}\left(\frac{2}{3}\right) = \frac{1}{3}.$$

Therefore, $\bar{x} = \bar{y} = 2/15$, and the centroid is $(2/15, 2/15)$.

19. The region in figure (c) is the union of a half-disk of area $\pi/2$ and centroid $(0, 4/(3\pi))$ (by Example 1) and a triangle of area 1 and centroid $(0, -1/3)$. Therefore its area is $(\pi/2) + 1$ and its centroid is $(\bar{x}, \bar{y})$, where $\bar{x} = 0$ and

$$\frac{\pi+2}{2}\bar{y} = \frac{\pi}{2}\left(\frac{4}{3\pi}\right) + 1\left(\frac{-1}{3}\right) = \frac{1}{3}.$$

Therefore, the centroid is $(0, 2/[3(\pi+2)])$.

20. The region in figure (d) is the union of three half-disks, one with area $\pi/2$ and centroid $(0, 4/(3\pi))$, and two with areas $\pi/8$ and centroids $(-1/2, -2/(3\pi))$ and $(1/2, -2/(3\pi))$. Therefore its area is $3\pi/4$ and its centroid is $(\bar{x}, \bar{y})$, where

$$\frac{3\pi}{4}(\bar{x}) = \frac{\pi}{2}(0) + \frac{\pi}{8}\left(\frac{-1}{2}\right) + \frac{\pi}{8}\left(\frac{1}{2}\right) = 0$$
$$\frac{3\pi}{4}(\bar{y}) = \frac{\pi}{2}\left(\frac{4}{3\pi}\right) + \frac{\pi}{8}\left(\frac{-2}{3\pi}\right) + \frac{\pi}{8}\left(\frac{-2}{3\pi}\right) = \frac{1}{2}.$$

Therefore, the centroid is $(0, 2/(3\pi))$.

21. By symmetry the centroid is $(1, -2)$.

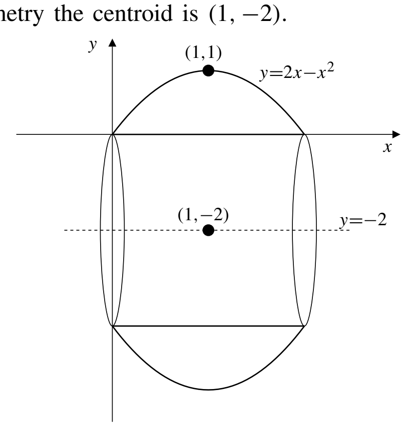

Fig. 7.5.21

22. The line segment from $(1, 0)$ to $(0, 1)$ has centroid $(\frac{1}{2}, \frac{1}{2})$ and length $\sqrt{2}$. By Pappus's Theorem, the surface area of revolution about $x = 2$ is

$$A = 2\pi\left(2 - \frac{1}{2}\right)\sqrt{2} = 3\pi\sqrt{2} \text{ sq. units.}$$

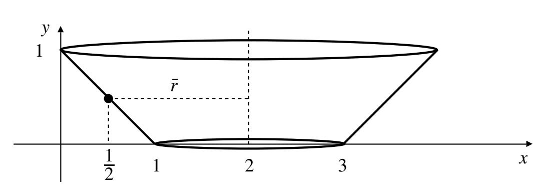

Fig. 7.5.22

23. The triangle T has centroid $\left(\frac{1}{3}, \frac{1}{3}\right)$ and area $\frac{1}{2}$. By Pappus's Theorem the volume of revolution about $x = 2$ is

$$V = \frac{1}{2} \times 2\pi \left(2 - \frac{1}{3}\right) = \frac{5\pi}{3} \text{ cu. units.}$$

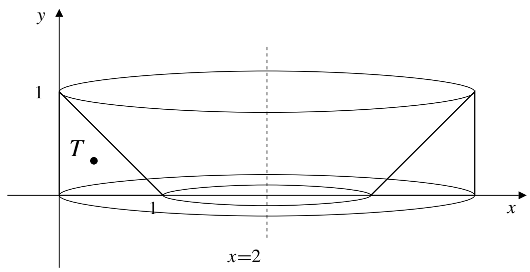

Fig. 7.5.23

24. The altitude h of the triangle is $\dfrac{s\sqrt{3}}{2}$. Its centroid is at height $\dfrac{h}{3} = \dfrac{s}{2\sqrt{3}}$ above the base side. Thus, by Pappus's Theorem, the volume of revolution is

$$V = 2\pi \left(\frac{s}{2\sqrt{3}}\right)\left(\frac{s}{2} \times \frac{\sqrt{3}s}{2}\right) = \frac{\pi s^3}{4} \text{ cu. units.}$$

The centroid of one side is $\dfrac{h}{2} = \dfrac{s\sqrt{3}}{4}$ above the base. Thus, the surface area of revolution is

$$S = 2 \times 2\pi \left(\frac{\sqrt{3}s}{4}\right)(s) = s^2\pi\sqrt{3} \text{ sq. units.}$$

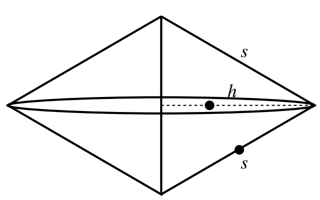

Fig. 7.5.24

25. For the purpose of evaluating the integrals in this problem and the next, the definite integral routine in the TI-85 calculator was used. For the region bounded by $y = 0$ and $y = \sqrt{x}\cos x$ between $x = 0$ and $x = \pi/2$, we have

$$\begin{aligned} A &= \int_0^{\pi/2} \sqrt{x}\cos x\, dx \approx 0.704038 \\ \bar{x} &= \frac{1}{A}\int_0^{\pi/2} x^{3/2}\cos x\, dx \approx 0.71377 \\ \bar{y} &= \frac{1}{2A}\int_0^{\pi/2} x\cos^2 x\, dx \approx 0.26053. \end{aligned}$$

26. The region bounded by $y = 0$ and $y = \ln(\sin x)$ between $x = 0$ and $x = \pi/2$ lies below the x-axis, so

$$\begin{aligned} A &= -\int_0^{\pi/2} \ln(\sin x)\, dx \approx 1.088793 \\ \bar{x} &= \frac{-1}{A}\int_0^{\pi/2} x\ln(\sin x)\, dx \approx 0.30239 \\ \bar{y} &= \frac{-1}{2A}\int_0^{\pi/2} \Big(\ln(\sin x)\Big)^2 dx \approx -0.93986. \end{aligned}$$

27. The area and moments of the region are

$$\begin{aligned} A &= \int_0^\infty \frac{dx}{(1+x)^3} = \lim_{R\to\infty} \frac{-1}{2(1+x)^2}\bigg|_0^R = \frac{1}{2} \\ M_{x=0} &= \int_0^\infty \frac{x\,dx}{(1+x)^3} \qquad \begin{array}{l}\text{Let } u = x+1 \\ du = dx\end{array} \\ &= \int_1^\infty \frac{u-1}{u^3}\, du \\ &= \lim_{R\to\infty}\left(-\frac{1}{u} + \frac{1}{2u^2}\right)\bigg|_1^R = 1 - \frac{1}{2} = \frac{1}{2} \\ M_{y=0} &= \frac{1}{2}\int_0^\infty \frac{dx}{(1+x)^6} = \lim_{R\to\infty} \frac{-1}{10(1+x)^5}\bigg|_0^R = \frac{1}{10}. \end{aligned}$$

The centroid is $\left(1, \frac{1}{5}\right)$.

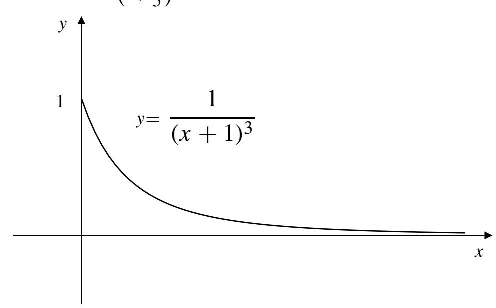

Fig. 7.5.27

28. The surface area is given by $S = 2\pi \displaystyle\int_{-\infty}^{\infty} e^{-x^2}\sqrt{1 + 4x^2e^{-2x^2}}\, dx$. Since $\displaystyle\lim_{x\to\pm\infty} 1 + 4x^2e^{-2x^2} = 1$, this expression must be bounded for all x, that is, $1 \le 1 + 4x^2e^{-2x^2} \le K^2$ for some constant K. Thus, $S \le 2\pi K \displaystyle\int_{-\infty}^{\infty} e^{-x^2}\, dx = 2K\pi\sqrt{\pi}$. The integral converges and the surface area is finite. Since the whole curve $y = e^{-x^2}$ lies above the x-axis, its centroid would have to satisfy $\bar{y} > 0$. However, Pappus's Theorem would then imply that the surface of revolution would have infinite area: $S = 2\pi\bar{y} \times (\text{length of curve}) = \infty$. The curve cannot, therefore, have any centroid.

29. By analogy with the formulas for the region $a \le x \le b$, $f(x) \le y \le g(y)$, the region $c \le y \le d$, $f(y) \le x \le g(y)$ will have centroid $(M_{x=0}/A, M_{y=0}/A)$, where

$$A = \int_c^d \Big(g(y) - f(y)\Big)\,dy$$
$$M_{x=0} = \frac{1}{2}\int_c^d \Big[\Big(g(y)\Big)^2 - \Big(f(y)\Big)^2\Big]\,dy$$
$$M_{y=0} = \int_c^d y\Big(g(y) - f(y)\Big)\,dy.$$

30. Let us take L to be the y-axis and suppose that a plane curve $\mathcal{C}$ lies between $x = a$ and $x = b$ where $0 < a < b$. Thus, $\bar{r} = \bar{x}$, the x-coordinate of the centroid of $\mathcal{C}$. Let ds denote an arc length element of $\mathcal{C}$ at position x. This arc length element generates, on rotation about L, a circular band of surface area $dS = 2\pi x\,ds$, so the surface area of the surface of revolution is

$$S = 2\pi\int_{x=a}^{x=b} x\,ds = 2\pi M_{x=0} = 2\pi\bar{r}s.$$

31.

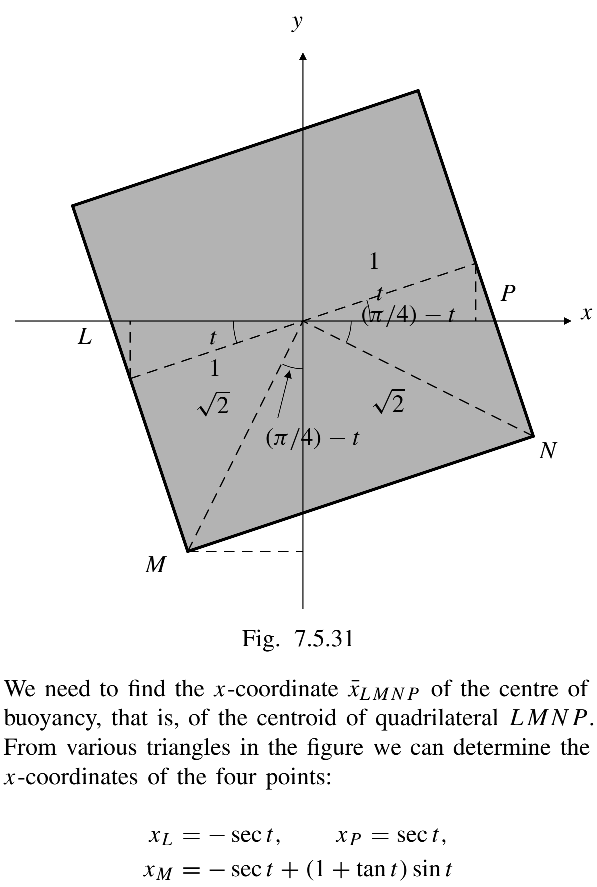

Fig. 7.5.31

We need to find the x-coordinate $\bar{x}_{LMNP}$ of the centre of buoyancy, that is, of the centroid of quadrilateral $LMNP$. From various triangles in the figure we can determine the x-coordinates of the four points:

$$x_L = -\sec t, \qquad x_P = \sec t,$$
$$x_M = -\sec t + (1 + \tan t)\sin t$$
$$x_N = \sec t + (1 - \tan t)\sin t$$

Triangle LMN has area $1 + \tan t$, and the x-coordinate of its centroid is

$$\bar{x}_{LMN} = \frac{-\sec t - \sec t + (1 + \tan t)\sin t + \sec t + (1 - \tan t)\sin t}{3} = \frac{2\sin t - \sec t}{3}.$$

Triangle LNP has area $1 - \tan t$, and the x-coordinate of its centroid is

$$\bar{x}_{LNP} = \frac{-\sec t + \sec t + \sec t + (1 - \tan t)\sin t}{3} = \frac{\sec t + (1 - \tan t)\sin t}{3}.$$

Therefore,

$$\begin{aligned}\bar{x}_{LMNP} &= \frac{1}{6}\Big[(2\sin t - \sec t)(1 + \tan t) \\ &\qquad + (\sec t + \sin t - \sin t\tan t)(1 - \tan t)\Big] \\ &= \frac{1}{6}\Big[3\sin t - 2\sec t\tan t + \sin t\tan^2 t\Big] \\ &= \frac{\sin t}{6}\Big[3 - \frac{2}{\cos^2 t} + \frac{\sin^2 t}{\cos^2 t}\Big] \\ &= \frac{\sin t}{6\cos^2 t}\Big[3\cos^2 t + \sin^2 t - 2\Big] \\ &= \frac{\sin t}{6\cos^2 t}\Big[2\cos^2 t - 1\Big] = \frac{\sin t}{6\cos^2 t}\Big[\cos(2t)\Big]\end{aligned}$$

which is positive provided $0 < t < \pi/4$. Thus the beam will rotate counterclockwise until an edge is on top.

Section 7.6 Other Physical Applications (page 406)

1. a) The pressure at the bottom is $p = 9,800 \times 6$ N/m^2. The force on the bottom is $4 \times p = 235,200$ N.

b) The pressure at depth h metres is $9,800h$ N/m^2. The force on a strip between depths h and $h + dh$ on one wall of the tank is

$$dF = 9,800h \times 2\,dh = 19,600\,h\,dh \text{ N}.$$

Thus, the total force on one wall is

$$F = 19,600\int_0^6 h\,dh = 19,600 \times 18 = 352,800 \text{ N}.$$

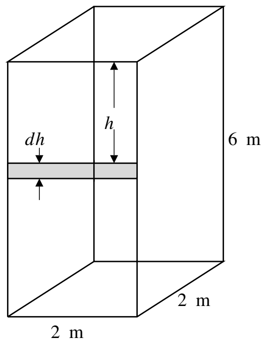

Fig. 7.6.1

2. A vertical slice of water at position y with thickness dy is in contact with the botttom over an area $8\sec\theta\,dy = \frac{4}{5}\sqrt{101}\,dy$ m^2, which is at depth $x = \frac{1}{10}y + 1$ m. The force exerted on this area is then $dF = \rho g(\frac{1}{10}y + 1)\frac{4}{5}\sqrt{101}\,dy$. Hence, the total force exerted on the bottom is

$$\begin{aligned} F &= \frac{4}{5}\sqrt{101}\,\rho g\int_0^{20}\left(\frac{1}{10}y+1\right)dy \\ &= \frac{4}{5}\sqrt{101}\,(1000)(9.8)\left(\frac{y^2}{20}+y\right)\bigg|_0^{20} \\ &\approx 3.1516\times 10^6 \text{ N.} \end{aligned}$$

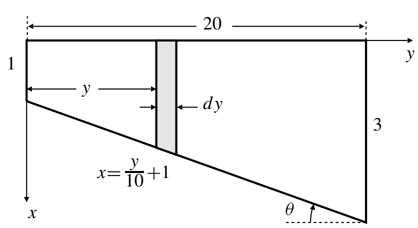

Fig. 7.6.2

3. A strip along the slant wall of the dam between depths h and $h + dh$ has area

$$dA = \frac{200\,dh}{\cos\theta} = 200\times\frac{26}{24}\,dh.$$

The force on this strip is

$$dF = 9,800\,h\,dA \approx 2.12\times 10^6\,h\,dh \text{ N.}$$

Thus the total force on the dam is

$$F = 2.12\times 10^6\int_0^{24} h\,dh \approx 6.12\times 10^8 \text{ N.}$$

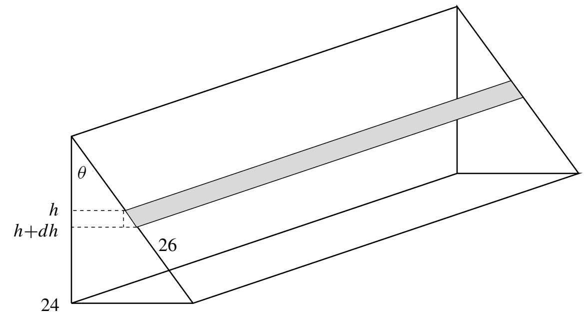

Fig. 7.6.3

4. The height of each triangular face is $2\sqrt{3}$ m and the height of the pyramid is $2\sqrt{2}$ m. Let the angle between the triangular face and the base be θ, then $\sin\theta = \sqrt{\frac{2}{3}}$ and $\cos\theta = \frac{1}{\sqrt{3}}$.

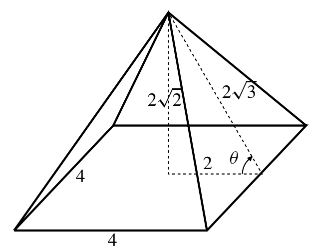

Fig. 7.6.4

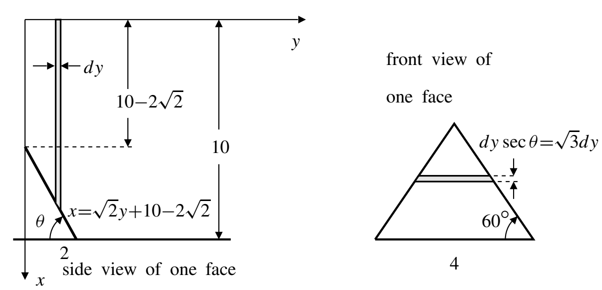

Fig. 7.6.4

A vertical slice of water with thickness dy at a distance y from the vertex of the pyramid exerts a force on the shaded strip shown in the front view, which has area $2\sqrt{3}y\,dy$ m^2 and which is at depth $\sqrt{2}y + 10 - 2\sqrt{2}$ m. Hence, the force exerted on the triangular face is

$$\begin{aligned} F &= \rho g\int_0^2(\sqrt{2}y+10-2\sqrt{2})2\sqrt{3}y\,dy \\ &= 2\sqrt{3}(9800)\left[\frac{\sqrt{2}}{3}y^3+(5-\sqrt{2})y^2\right]\bigg|_0^2 \\ &\approx 6.1495\times 10^5 \text{ N.} \end{aligned}$$

5. The unbalanced force is

$$F = 9,800 \times 5 \int_6^{20} h\,dh$$
$$= 9,800 \times 5 \left(\frac{h^2}{2}\right)\Bigg|_6^{20} \approx 8.92 \times 10^6 \text{ N.}$$

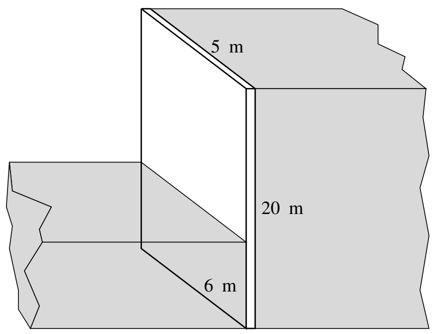

Fig. 7.6.5

6. The spring force is $F(x) = kx$, where x is the amount of compression. The work done to compress the spring 3 cm is

$$100 \text{ N·cm} = W = \int_0^3 kx\,dx = \frac{1}{2}kx^2\Big|_0^3 = \frac{9}{2}k.$$

Hence, $k = \dfrac{200}{9}$ N/cm. The work necessary to compress the spring a further 1 cm is

$$W = \int_3^4 kx\,dx = \left(\frac{200}{9}\right)\frac{1}{2}x^2\Big|_3^4 = \frac{700}{9} \text{ N·cm.}$$

7. A layer of water in the tank between depths h and $h+dh$ has weight $dF = \rho g\,dV = 4\rho g\,dh$. The work done to raise the water in this layer to the top of the tank is $dW = h\,dF = 4\rho g h\,dh$. Thus the total work done to pump all the water out over the top of the tank is

$$W = 4\rho g \int_0^6 h\,dh = 4 \times 9,800 \times 18 \approx 7.056 \times 10^5 \text{ N·m.}$$

8. The horizontal cross-sectional area of the pool at depth h is

$$A(h) = \begin{cases} 160, & \text{if } 0 \le h \le 1; \\ 240 - 80h, & \text{if } 1 < h \le 3. \end{cases}$$

The work done to empty the pool is

$$W = \rho g \int_0^3 hA(h)\,dh$$
$$= \rho g \left[\int_0^1 160h\,dh + \int_1^3 240h - 80h^2\,dh\right]$$
$$= 9800\left[80h^2\Big|_0^1 + \left(120h^2 - \frac{80}{3}h^3\right)\Bigg|_1^3\right]$$
$$= 3.3973 \times 10^6 \text{ N·m.}$$

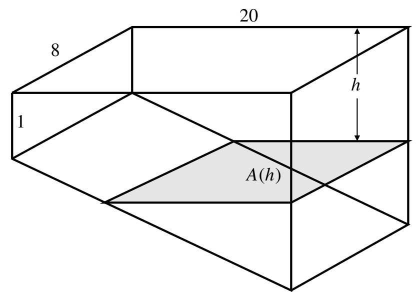

Fig. 7.6.8

9. A layer of water between depths y and $y + dy$ has volume $dV = \pi(a^2 - y^2)\,dy$ and weight $dF = 9,800\pi(a^2 - y^2)\,dy$ N. The work done to raise this water to height h m above the top of the bowl is

$$dW = (h + y)\,dF = 9,800\pi(h + y)(a^2 - y^2)\,dy \text{ N·m.}$$

Thus the total work done to pump all the water in the bowl to that height is

$$W = 9,800\pi \int_0^a (ha^2 + a^2y - hy^2 - y^3)\,dy$$
$$= 9,800\pi \left[ha^2y + \frac{a^2y^2}{2} - \frac{hy^3}{3} - \frac{y^4}{4}\right]\Bigg|_0^a$$
$$= 9,800\pi \left[\frac{2a^3h}{3} + \frac{a^4}{4}\right]$$
$$= 9,800\pi a^3 \frac{3a + 8h}{12} = 2450\pi a^3 \left(a + \frac{8h}{3}\right) \text{ N·m.}$$

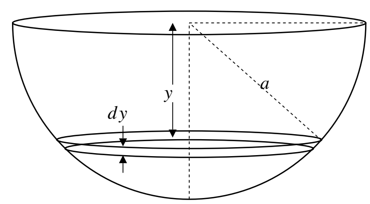

Fig. 7.6.9

10. Let the time required to raise the bucket to height h m be t minutes. Given that the velocity is 2 m/min, then $t = \frac{h}{2}$. The weight of the bucket at time t is $16 \text{ kg} - (1 \text{ kg/min})(t \text{ min}) = 16 - \frac{h}{2}$ kg. Therefore, the work done required to move the bucket to a height of 10 m is

$$\begin{aligned} W &= g\int_0^{10}\left(16 - \frac{h}{2}\right)dh \\ &= 9.8\left(16h - \frac{h^2}{4}\right)\bigg|_0^{10} = 1323 \text{ N·m.} \end{aligned}$$

Section 7.7 Applications in Business, Finance, and Ecology (page 409)

1. $$\begin{aligned} \text{Cost} &= \$4,000 + \int_0^{1,000}\left(6 - \frac{2x}{10^3} + \frac{6x^2}{10^6}\right)dx \\ &= \$11,000. \end{aligned}$$

2. The number of chips sold in the first year was

$$1,000\int_0^{52} te^{-t/10}\,dt = 100,000 - 620,000e^{-26/5}$$

that is, about 96,580.

3. The monthly charge is

$$\begin{aligned} &\int_0^x \frac{4}{1+\sqrt{t}}\,dt \qquad \text{let } t = u^2 \\ =&8\int_0^{\sqrt{x}} \frac{u}{1+u}\,du = 8\int_0^{\sqrt{x}}\left(1 - \frac{1}{1+u}\right)du \\ =&\$8\left(\sqrt{x} - \ln(1+\sqrt{x})\right). \end{aligned}$$

4. The price per kg at time t (years) is $\$10 + 5t$. Thus the revenue per year at time t is $400(10+5t)/(1+0.1t)$ \$/year. The total revenue over the year is

$$\int_0^1 \frac{400(10+5t)}{1+0.1t}\,dt \approx \$4,750.37.$$

5. The present value of continuous payments of \$1,000 per year for 10 years at a discount rate of 2% is

$$V = \int_0^{10} 1{,}000e^{-0.02t}\,dt = \frac{1{,}000}{-0.02}e^{-0.02t}\bigg|_0^{10} = \$9{,}063.46.$$

6. The present value of continuous payments of \$1,000 per year for 10 years at a discount rate of 5% is

$$V = \int_0^{10} 1{,}000e^{-0.05t}\,dt = \frac{1{,}000}{-0.05}e^{-0.05t}\bigg|_0^{10} = \$7{,}869.39.$$

7. The present value of continuous payments of \$1,000 per year for 10 years beginning 2 years from now at a discount rate of 8% is

$$V = \int_2^{12} 1{,}000e^{-0.08t}\,dt = \frac{1{,}000}{-0.08}e^{-0.08t}\bigg|_2^{12} = \$5{,}865.64.$$

8. The present value of continuous payments of \$1,000 per year for 25 years beginning 10 years from now at a discount rate of 5% is

$$V = \int_{10}^{35} 1{,}000e^{-0.05t}\,dt = \frac{1{,}000}{-0.05}e^{-0.05t}\bigg|_{10}^{35} = \$8{,}655.13.$$

9. The present value of continuous payments of \$1,000 per year for all future time at a discount rate of 2% is

$$V = \int_0^{\infty} 1{,}000e^{-0.02t}\,dt = \frac{1,000}{-0.02} = \$50,000.$$

10. The present value of continuous payments of \$1,000 per year beginning 10 years from now and continuing for all future time at a discount rate of 5% is

$$V = \int_{10}^{\infty} 1{,}000e^{-0.05t}\,dt = \frac{1{,}000}{-0.05}e^{-0.5} = \$12{,}130.61.$$

11. After t years, money is flowing at $\$(1,000 + 100t)$ per year. The present value of 10 years of payments discounted at 5% is

$$\begin{aligned} V &= 100\int_0^{10}(10+t)e^{-0.05t}\,dt \\ &\qquad U = 10 + t \quad dV = e^{-0.05t}\,dt \\ &\qquad dU = dt \qquad V = \frac{e^{-0.05t}}{-0.05} \\ &= 100(10+t)\frac{e^{-0.05t}}{-0.05}\bigg|_0^{10} + \frac{100}{0.05}\int_0^{10} e^{-0.05t}\,dt \\ &= -4261.23 + \frac{100}{-(0.05)^2}e^{-0.05t}\bigg|_0^{10} = \$11,477.54. \end{aligned}$$

12. After t years, money is flowing at \$1,000$(1.1)^t$ per year. The present value of 10 years of payments discounted at 5% is

$$\begin{aligned} V &= 1{,}000\int_0^{10} e^{t\ln(1.1)}e^{-0.05t}\,dt \\ &= \frac{1{,}000}{\ln(1.1)-0.05}e^{t(\ln(1.1)-0.05}\bigg|_0^{10} = \$12,650.23. \end{aligned}$$

13. The amount after 10 years is

$$A = 5,000\int_0^{10} e^{0.05t}\,dt = \frac{5,000}{0.05}e^{0.05t}\bigg|_0^{10} = \$64{,}872.13.$$

14. Let T be the time required for the account balance to reach \$1,000,000. The \$5, $000(1.1)^t\,dt$ deposited in the time interval $[t, t+dt]$ grows for $T-t$ years, so the balance after T years is

$$\begin{aligned} &\int_0^T 5,000(1.1)^t(1.06)^{T-t}\,dt = 1,000,000 \\ &(1.06)^T\int_0^T\left(\frac{1.1}{1.06}\right)^t dt = \frac{1,000,000}{5,000} = 200 \\ &\frac{(1.06)^T}{\ln(1.1/1.06)}\left[\left(\frac{1.1}{1.06}\right)^T - 1\right] = 200 \\ &(1.1)^T - (1.06)^T = 200\ln\frac{1.1}{1.06}. \end{aligned}$$

This equation can be solved by Newton's method or using a calculator "solve" routine. The solution is $T \approx 26.05$ years.

15. Let $P(\tau)$ be the value at time $\tau < t$ that will grow to $\$P = P(t)$ at time t. If the discount rate at time τ is $\delta(\tau)$, then

$$\frac{d}{d\tau}P(\tau) = \delta(\tau)P(\tau),$$

or, equivalently,

$$\frac{dP(\tau)}{P(\tau)} = \delta(\tau)\,d\tau.$$

Integrating this from 0 to t, we get

$$\ln P(t) - \ln P(0) = \int_0^t \delta(\tau)\,d\tau = \lambda(t),$$

and, taking exponentials of both sides and solving for $P(0)$, we get

$$P(0) = P(t)e^{-\lambda(t)} = Pe^{-\lambda(t)}.$$

The present value of a stream of payments due at a rate $P(t)$ at time t from $t=0$ to $t=T$ is

$$\int_0^T P(t)e^{-\lambda(t)}\,dt, \quad \text{where} \quad \lambda(t) = \int_0^t \delta(\tau)\,d\tau.$$

16. The analysis carried out in the text for the logistic growth model showed that the total present value of future harvests could be maximized by holding the population size x at a value that maximizes the quadratic expression

$$Q(x) = kx\left(1-\frac{x}{L}\right) - \delta x.$$

If the logistic model $dx/dt = kx(1-(x/L))$ is replaced with a more general growth model $dx/dt = F(x)$, exactly the same analysis leads us to maximize

$$Q(x) = F(x) - \delta x.$$

For realistic growth functions, the maximum will occur where $Q'(x) = 0$, that is, where $F'(x) = \delta$.

17. We are given $L = 80,000$, $k = 0.12$, and $\delta = 0.05$. According to the analysis in the text, the present value of future harvests will be maximized if the population level is maintained at

$$x = (k-\delta)\frac{L}{2k} = \frac{0.07}{0.24}(80,000) = 23,333.33$$

The annual revenue from harvesting to keep the population at this level (given a price of \$6 per fish) is

$$6(0.12)(23,333.33)\left(1-\frac{23,333.33}{80,000}\right) = \$11,900.$$

18. We are given that $k = 0.02$, $L = 150,000$, $p = \$10,000$. The growth rate at population level x is

$$\frac{dx}{dt} = 0.02x\left(1-\frac{x}{150,000}\right).$$

a) The maximum sustainable annual harvest is

$$\frac{dx}{dt}\bigg|_{x=L/2} = 0.02(75,000)(0.5) = 750 \text{ whales.}$$

b) The resulting annual revenue is $\$750p = \$7,500,000$.

c) If the whole population of 75,000 is harvested and the proceeds invested at 2%, the annual interest will be

$$75,000(\$10,000)(0.02) = \$15,000,000.$$

d) At 5%, the interest would be $(5/2)(\$15,000) = \$37,500,000$.

e) The total present value of all future harvesting revenue if the population level is maintained at 75,000 and $\delta = 0.05$ is

$$\int_0^\infty e^{-0.05t} 7,500,000\, dt = \frac{7,500,000}{0.05} = \$150,000,000.$$

19. If we assume that the cost of harvesting 1 unit of population is $\$C(x)$ when the population size is x, then the effective income from 1 unit harvested is $\$(p - C(x))$. Using this expression in place of the constant p in the analysis given in the text, we are led to choose x to maximize

$$Q(x) = \Big(p - C(x)\Big)\left[kx\left(1 - \frac{x}{L}\right) - \delta x\right].$$

A reasonable cost function $C(x)$ will increase as x decreases (the whales are harder to find), and will exceed p if $x \le x_0$, for some positive population level x_0. The value of x that maximizes $Q(x)$ must exceed x_0, so the model no longer predicts extinction, even for large discount rates δ. However, the optimizing population x may be so low that other factors not accounted for in the simple logistic growth model may still bring about extinction whether it is economically indicated or not.

Section 7.8 Probability (page 421)

1. The expected winnings on a toss of the coin are

$$\$1 \times 0.49 + \$2 \times 0.49 + \$50 \times 0.02 = \$2.47.$$

If you pay this much to play one game, in the long term you can expect to break even.

2. (a) We need $\sum_{n=1}^{6} Kn = 1$. Thus $21K = 1$, and $K = 1/21$.
(b) $\Pr(X \le 3) = (1/21)(1 + 2 + 3) = 2/7$.

3. From the second previous Exercise, the mean winings is $\mu = \$2.47$. Now

$$\begin{aligned}\sigma^2 &= 1 \times 0.49 + 4 \times 0.49 + 2{,}500 \times 0.02 - \mu^2\\ &\approx 52.45 - 6.10 = 46.35.\end{aligned}$$

The standard deviation is thus $\sigma \approx \$6.81$.

4. Since $\Pr(X = n) = n/21$, we have

$$\begin{aligned}\mu &= \sum_{n=1}^{6} n\Pr(X = n) = \frac{1 \times 1 + 2 \times 2 + \cdots + 6 \times 6}{21} = \frac{13}{3} \approx 4.33\\ \sigma^2 &= \sum_{n=1}^{6} n^2\Pr(X = n) - \mu^2 = \frac{1^2 + 2^3 + \cdots + 6^3}{21} - \mu^2\\ &= 21 - \frac{169}{9} = \frac{20}{9} \approx 2.22\\ \sigma &= \frac{\sqrt{20}}{3} \approx 1.49.\end{aligned}$$

5. The mean of X is

$$\mu = 1 \times \frac{9}{60} + (2 + 3 + 4 + 5) \times \frac{1}{6} + 6 \times 1160 \approx 3.5833.$$

The expectation of X^2 is

$$E(X^2) = 1^2 \times \frac{9}{60} + (2^2 + 3^2 + 4^2 + 5^2) \times \frac{1}{6} + 6^2 \times 1160 \approx 15.7500.$$

Hence the standard deviation of X is $\sqrt{15.75 - 3.5833^2} \approx 1.7059$.
Also $\Pr(X \le 3) = \dfrac{9}{60} + \dfrac{2}{6} = \dfrac{29}{60} \approx 0.4833$.

6. (a) Calculating as we did to construct the probability function in Example 2, but using the different values for the probabilities of "1" and "6", we obtain

$$\begin{aligned}f(2) &= \frac{9}{60} \times \frac{9}{60} \approx 0.0225\\ f(3) &= 2 \times \frac{9}{60} \times 16 = 0.0500\\ f(4) &= 2 \times \frac{9}{60} \times 16 + \frac{1}{36} = 0.0778\\ f(5) &= 2 \times \frac{9}{60} \times 16 + \frac{2}{36} = 0.1056\\ f(6) &= 2 \times \frac{9}{60} \times 16 + \frac{3}{36} = 0.1333\\ f(7) &= 2 \times \frac{9}{60} \times 1160 + \frac{4}{36} = 0.1661\\ f(8) &= 2 \times \frac{11}{60} \times 16 + \frac{3}{36} = 0.1444\\ f(9) &= 2 \times \frac{11}{60} \times 16 + \frac{2}{36} = 0.1167\\ f(10) &= 2 \times \frac{11}{60} \times 16 + \frac{1}{36} = 0.0889\\ f(11) &= 2 \times \frac{11}{60} \times 16 = 0.0611\\ f(12) &= \frac{11}{60} \times 1160 = 0.0336.\end{aligned}$$

(b) Multiplying each value $f(n)$ by n and summing, we get

$$\mu = \sum_{n=2}^{12} n f(n) \approx 7.1665.$$

Similarly,

$$E(X^2) = \sum_{n=2}^{12} n^2 f(n) \approx 57.1783,$$

so the standard deviation of X is

$$\sigma = \sqrt{E(X^2) - \mu^2} \approx 2.4124.$$

The mean is somewhat larger than the value (7) obtained for the unweighted dice, because the weighting favours more 6s than 1s showing if the roll is repeated many times. The standard deviation is just a tiny bit smaller than that found for the unweighted dice (2.4152); the distribution of probability is just slightly more concentrated around the mean here.

7. (a) The sample space consists of the eight triples (H, H, H), (H, H, T), (H, T, H), (T, H, H), (H, T, T), (T, H, T), (T, T, H), and (T, T, T).

(b) We have

$$\begin{aligned}
\Pr(H, H, H) &= (0.55)^3 = 0.166375 \\
\Pr(H, H, T) &= \Pr(H, T, H) = \Pr(T, H, H) = (0.55)^2(0.45) \\
&= 0.136125 \\
\Pr(H, T, T) &= \Pr(T, H, T) = \Pr(T, T, H) = (0.55)(0.45)^2 \\
&= 0.111375 \\
\Pr(T, T, T) &= (0.45)^3 = 0.091125.
\end{aligned}$$

(c) The probability function f for X is given by

$$\begin{aligned}
f(0) &= (0.45)^3 = 0.911125 \\
f(1) &= 3 \times (0.55)(0.45)^2 = 0.334125 \\
f(2) &= 3 \times (0.55)^2(0.45) = 0.408375 \\
f(3) &= (0.55)^3 = 0.166375.
\end{aligned}$$

(d) $\Pr(X \geq 1) = 1 - \Pr(X = 0) = 0.908875$.

(e) $E(X) = 0 \times f(0) + 1 \times f(1) + 2 \times f(2) + 3 \times f(3) = 1.6500$.

8. The number of red balls in the sack must be $0.6 \times 20 = 12$. Thus there are 8 blue balls.

(a) The probability of pulling out one blue ball is $8/20$. If you got a blue ball, then there would be only 7 blue balls left among the 19 balls remaining in the sack, so the probability of pulling out a second blue ball is $7/19$. Thus the probability of pulling out two blue balls is $\dfrac{8}{20} \times \dfrac{7}{19} = \dfrac{14}{95}$.

(b) The sample space for the three ball selection consists of all eight triples of the form (x, y, z), where each of x, y, z is either R(ed) or B(lue). Let X be the number of red balls among the three balls pulled out. Arguing in the same way as in (a), we calculate

$$\begin{aligned}
\Pr(X = 0) &= \Pr(B, B, B) = \frac{8}{20} \times \frac{7}{19} \times \frac{6}{18} = \frac{14}{285} \\
&\approx 0.0491 \\
\Pr(X = 1) &= \Pr(R, B, B) + \Pr(B, R, B) + \Pr(B, B, R) \\
&= 3 \times \frac{12}{20} \times \frac{8}{19} \times \frac{7}{18} = \frac{28}{95} \approx 0.2947 \\
\Pr(X = 2) &= \Pr(R, R, B) + \Pr(R, B, R) + \Pr(B, R, R) \\
&= 3 \times \frac{12}{20} \times \frac{11}{19} \times \frac{8}{18} = \frac{44}{95} \approx 0.4632 \\
\Pr(X = 3) &= \Pr(R, R, R) = \frac{12}{20} \times \frac{11}{19} \times \frac{10}{18} = \frac{11}{57} \\
&\approx 0.1930
\end{aligned}$$

Thus the expected value of X is

$$\begin{aligned}
E(X) &= 0 \times \frac{14}{285} + 1 \times \frac{28}{95} + 2 \times \frac{44}{95} + 3 \times \frac{11}{57} \\
&= \frac{9}{5} = 1.8.
\end{aligned}$$

9. We have $f(x) = Cx$ on $[0, 3]$.

a) C is given by

$$1 = \int_0^3 Cx\,dx = \frac{C}{2}x^2\bigg|_0^3 = \frac{9}{2}C.$$

Hence, $C = \dfrac{2}{9}$.

b) The mean is

$$\mu = E(X) = \frac{2}{9}\int_0^3 x^2\,dx = \frac{2}{27}x^3\bigg|_0^3 = 2.$$

Since $E(X^2) = \dfrac{2}{9}\displaystyle\int_0^3 x^3\,dx = \dfrac{2}{36}x^4\bigg|_0^3 = \dfrac{9}{2}$, the variance is

$$\sigma^2 = E(X^2) - \mu^2 = \frac{9}{2} - 4 = \frac{1}{2},$$

and the standard deviation is $\sigma = 1/\sqrt{2}$.

c) We have

$$\begin{aligned}
\Pr(\mu - \sigma \leq X \leq \mu + \sigma) &= \frac{2}{9}\int_{\mu-\sigma}^{\mu+\sigma} x\,dx \\
&= \frac{(\mu + \sigma)^2 - (\mu - \sigma)^2}{9} = \frac{4\mu\sigma}{9} \approx 0.6285.
\end{aligned}$$

10. We have $f(x) = Cx$ on $[1, 2]$.

a) To find C, we have

$$1 = \int_1^2 Cx\,dx = \frac{C}{2}x^2\bigg|_1^2 = \frac{3}{2}C.$$

Hence, $C = \frac{2}{3}$.

b) The mean is

$$\mu = E(X) = \frac{2}{3}\int_1^2 x^2\,dx = \frac{2}{9}x^3\bigg|_1^2 = \frac{14}{9} \approx 1.556.$$

Since $E(X^2) = \frac{2}{3}\int_1^2 x^3\,dx = \frac{1}{6}x^4\bigg|_1^2 = \frac{5}{2}$, the variance is

$$\sigma^2 = E(X^2) - \mu^2 = \frac{5}{2} - \frac{196}{81} = \frac{13}{162}$$

and the standard deviation is

$$\sigma = \sqrt{\frac{13}{162}} \approx 0.283.$$

c) We have

$$\begin{aligned}\Pr(\mu - \sigma \le X \le \mu + \sigma) &= \frac{2}{3}\int_{\mu-\sigma}^{\mu+\sigma} x\,dx \\ &= \frac{(\mu+\sigma)^2 - (\mu-\sigma)^2}{3} = \frac{4\mu\sigma}{3} \approx 0.5875.\end{aligned}$$

11. We have $f(x) = Cx^2$ on $[0, 1]$.

a) C is given by

$$1 = \int_0^1 Cx^2\,dx = \frac{C}{3}x^3\bigg|_0^1 = \frac{C}{3}.$$

Hence, $C = 3$.

b) The mean, variance, and standard deviation are

$$\mu = E(X) = 3\int_0^1 x^3\,dx = \frac{3}{4}$$
$$\sigma^2 = E(X^2) - \mu^2 = 3\int_0^1 x^4\,dx - \frac{9}{16} = \frac{3}{5} - \frac{9}{16} = \frac{3}{80}$$
$$\sigma = \sqrt{3/80}.$$

c) We have

$$\begin{aligned}\Pr(\mu - \sigma \le X \le \mu + \sigma) &= 3\int_{\mu-\sigma}^{\mu+\sigma} x^2\,dx \\ &= (\mu+\sigma)^3 - (\mu-\sigma)^3 \\ &= \left(\frac{3}{4} + \sqrt{\frac{3}{80}}\right)^3 - \left(\frac{3}{4} - \sqrt{\frac{3}{80}}\right)^3 \approx 0.668.\end{aligned}$$

12. We have $f(x) = C\sin x$ on $[0, \pi]$.

a) To find C, we calculate

$$1 = \int_0^\pi C\sin x\,dx = -C\cos x\bigg|_0^\pi = 2C.$$

Hence, $C = \frac{1}{2}$.

b) The mean is

$$\begin{aligned}\mu = E(X) &= \frac{1}{2}\int_0^\pi x\sin x\,dx \\ &\qquad U = x \quad dV = \sin x\,dx \\ &\qquad dU = dx \quad V = -\cos x \\ &= \frac{1}{2}\left[-x\cos x\bigg|_0^\pi + \int_0^\pi \cos x\,dx\right] \\ &= \frac{\pi}{2} = 1.571.\end{aligned}$$

Since

$$\begin{aligned}E(X^2) &= \frac{1}{2}\int_0^\pi x^2\sin x\,dx \\ &\qquad U = x^2 \quad dV = \sin x\,dx \\ &\qquad dU = 2x\,dx \quad V = -\cos x \\ &= \frac{1}{2}\left[-x^2\cos x\bigg|_0^\pi + 2\int_0^\pi x\cos x\,dx\right] \\ &\qquad U = x \quad dV = \cos x\,dx \\ &\qquad dU = dx \quad V = \sin x \\ &= \frac{1}{2}\left[\pi^2 + 2\left(x\sin x\bigg|_0^\pi - \int_0^\pi \sin x\,dx\right)\right] \\ &= \frac{1}{2}(\pi^2 - 4).\end{aligned}$$

Hence, the variance is

$$\sigma^2 = E(X^2) - \mu^2 = \frac{\pi^2 - 4}{2} - \frac{\pi^2}{4} = \frac{\pi^2 - 8}{4} \approx 0.467$$

and the standard deviation is

$$\sigma = \sqrt{\frac{\pi^2 - 8}{4}} \approx 0.684.$$

c) Then

$$\begin{aligned}\Pr(\mu-\sigma \le X \le \mu+\sigma) &= \frac{1}{2}\int_{\mu-\sigma}^{\mu+\sigma}\sin x\,dx\\ &= -\frac{1}{2}\Big[\cos(\mu+\sigma)-\cos(\mu-\sigma)\Big]\\ &= \sin\mu\sin\sigma = \sin\sigma \approx 0.632.\end{aligned}$$

13. We have $f(x) = C(x - x^2)$ on $[0, 1]$.

a) C is given by

$$1 = \int_0^1 C(x-x^2)\,dx = C\left(\frac{x^2}{2}-\frac{x^3}{3}\right)\bigg|_0^1 = \frac{C}{6}.$$

Hence, $C = 6$.

b) The mean, variance, and standard deviation are

$$\begin{aligned}\mu &= E(X) = 6\int_0^1 (x^2-x^3)\,dx = \frac{1}{2}\\ \sigma^2 &= E(X^2)-\mu^2 = 6\int_0^1 (x^3-x^4)\,dx - \frac{1}{4}\\ &= \frac{3}{10}-\frac{1}{4} = \frac{1}{20}\\ \sigma &= \sqrt{1/20}.\end{aligned}$$

c) We have

$$\begin{aligned}\Pr(\mu-\sigma \le X \le \mu+\sigma) &= 6\int_{(1/2)-\sigma}^{(1/2)+\sigma}(x-x^2)\,dx\\ &= 6\int_{(1/2)-\sigma}^{(1/2)+\sigma}\left[\frac{1}{4}-\left(x-\frac{1}{2}\right)^2\right]dx\\ &\qquad \text{Let } u = x-\tfrac{1}{2}\\ &\qquad du = dx\\ &= 12\int_0^{\sigma}\left[\frac{1}{4}-u^2\right]du = 12\left[\frac{\sigma}{4}-\frac{\sigma^3}{3}\right]\\ &= \frac{12}{\sqrt{20}}\left[\frac{1}{4}-\frac{1}{60}\right] \approx 0.626.\end{aligned}$$

14. It was shown in Section 6.1 (p. 349) that

$$\int x^n e^{-x}\,dx = -x^n e^{-x} + n\int x^{n-1}e^{-x}\,dx.$$

If $I_n = \displaystyle\int_0^\infty x^n e^{-x}\,dx$, then

$$I_n = \lim_{R\to\infty} -R^n e^{-R} + nI_{n-1} = nI_{n-1} \qquad \text{if } n \ge 1.$$

Since $I_0 = \displaystyle\int_0^\infty e^{-x}\,dx = 1$, therefore $I_n = n!$ for $n \ge 1$. Let $u = kx$; then

$$\int_0^\infty x^n e^{-kx}\,dx = \frac{1}{k^{n+1}}\int_0^\infty u^n e^{-u}\,du = \frac{1}{k^{n+1}}I_n = \frac{n!}{k^{n+1}}.$$

Now let $f(x) = Cxe^{-kx}$ on $[0,\infty)$.

a) To find C, observe that

$$1 = C\int_0^\infty xe^{-kx}\,dx = \frac{C}{k^2}.$$

Hence, $C = k^2$.

b) The mean is

$$\mu = E(X) = k^2\int_0^\infty x^2e^{-kx}\,dx = k^2\left(\frac{2}{k^3}\right) = \frac{2}{k}.$$

Since $E(X^2) = k^2\displaystyle\int_0^\infty x^3e^{-kx}\,dx = k^2\left(\frac{6}{k^4}\right) = \frac{6}{k^2}$, then the variance is

$$\sigma^2 = E(X^2)-\mu^2 = \frac{6}{k^2}-\frac{4}{k^2} = \frac{2}{k^2}$$

and the standard deviation is $\sigma = \dfrac{\sqrt{2}}{k}$.

c) Finally,

$$\begin{aligned}&\Pr(\mu-\sigma \le X \le \mu+\sigma)\\ &= k^2\int_{\mu-\sigma}^{\mu+\sigma} xe^{-kx}\,dx \quad \begin{array}{l}\text{Let } u = kx\\ du = k\,dx\end{array}\\ &= \int_{k(\mu-\sigma)}^{k(\mu+\sigma)} ue^{-u}\,du\\ &= -ue^{-u}\bigg|_{k(\mu-\sigma)}^{k(\mu+\sigma)} + \int_{k(\mu-\sigma)}^{k(\mu+\sigma)} e^{-u}\,du\\ &= -(2+\sqrt{2})e^{-(2+\sqrt{2})} + (2-\sqrt{2})e^{-(2-\sqrt{2})}\\ &\quad - e^{-(2+\sqrt{2})} + e^{-(2-\sqrt{2})}\\ &\approx 0.738.\end{aligned}$$

15. a) We have

$$1 = C\int_0^\infty e^{-x^2}\,dx = \frac{C}{2}\int_{-\infty}^\infty e^{-x^2}\,dx = \frac{C\sqrt{\pi}}{2}.$$

Thus $C = 2/\sqrt{\pi}$.

b) The mean, variance, and standard deviation are

$$\mu = \frac{2}{\sqrt{\pi}}\int_0^\infty xe^{-x^2}\,dx = -\frac{e^{-x^2}}{\sqrt{\pi}}\bigg|_0^\infty = \frac{1}{\sqrt{\pi}}$$

$$\sigma^2 = -\frac{1}{\pi} + \frac{2}{\sqrt{\pi}}\int_0^\infty x^2e^{-x^2}\,dx$$

$$\begin{aligned} U &= x \qquad dV = xe^{-x^2}\,dx \\ dU &= dx \qquad V = -\tfrac{1}{2}e^{-x^2}\end{aligned}$$

$$= -\frac{1}{\pi} + \frac{2}{\sqrt{\pi}}\left(-\frac{x}{2}e^{-x^2}\bigg|_0^\infty + \frac{1}{2}\int_0^\infty e^{-x^2}\,dx\right)$$

$$= -\frac{1}{\pi} + \frac{2}{\sqrt{\pi}}\left(0 + \frac{1}{2}\cdot\frac{\sqrt{\pi}}{2}\right) = \frac{1}{2} - \frac{1}{\pi}$$

$$\sigma = \sqrt{\frac{1}{2} - \frac{1}{\pi}} \approx 0.426.$$

c) We have

$$\Pr(\mu - \sigma \le X \le \mu + \sigma) = \frac{2}{\sqrt{\pi}}\int_{\mu-\sigma}^{\mu+\sigma} e^{-x^2}\,dx$$

$$\begin{aligned} &\text{Let } x = z/\sqrt{2} \\ &dx = dz/\sqrt{2}\end{aligned}$$

$$= \sqrt{\frac{2}{\pi}}\int_{\sqrt{2}(\mu-\sigma)}^{\sqrt{2}(\mu+\sigma)} e^{-z^2/2}\,dz.$$

But $\sqrt{2}(\mu - \sigma) \approx 0.195$ and $\sqrt{2}(\mu + \sigma) \approx 1.40$. Thus, if Z is a standard normal random variable, we obtain by interpolation in the table on page 386 in the text,

$$\begin{aligned}\Pr(\mu - \sigma \le X \le \mu + \sigma) &= 2\Pr(0.195 \le Z \le 1.400) \\ &\approx 2(0.919 - 0.577) \approx 0.68.\end{aligned}$$

16. No. The identity $\int_{-\infty}^\infty C\,dx = 1$ is not satisfied for any constant C.

17. $$f_{\mu,\sigma}(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/2\sigma^2}$$

$$\text{mean } = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^\infty xe^{-(x-\mu)^2/2\sigma^2}\,dx \qquad \text{Let } z = \frac{x-\mu}{\sigma}, \quad dz = \frac{1}{\sigma}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty (\mu + \sigma z)e^{-z^2/2}\,dz$$

$$= \frac{\mu}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-z^2/2}\,dz = \mu$$

$$\text{variance } = E\Big((x-\mu)^2\Big)$$

$$= \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^\infty (x-\mu)^2e^{-(x-\mu)^2/2\sigma^2}\,dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^\infty \sigma^2z^2e^{-z^2/2}\,dz = \sigma\,\mathrm{Var}(Z) = \sigma$$

18. Since $f(x) = \dfrac{2}{\pi(1+x^2)} > 0$ on $[0,\infty)$ and

$$\frac{2}{\pi}\int_0^\infty \frac{dx}{1+x^2} = \lim_{R\to\infty}\frac{2}{\pi}\tan^{-1}(R) = \frac{2}{\pi}\left(\frac{\pi}{2}\right) = 1,$$

therefore $f(x)$ is a probability density function on $[0,\infty)$. The expectation of X is

$$\begin{aligned}\mu = E(X) &= \frac{2}{\pi}\int_0^\infty \frac{x\,dx}{1+x^2} \\ &= \lim_{R\to\infty}\frac{1}{\pi}\ln(1+R^2) = \infty.\end{aligned}$$

No matter what the cost per game, you should be willing to play (if you have an adequate bankroll). Your expected winnings per game in the long term is infinite.

19. a) The density function for the uniform distribution on $[a,b]$ is given by $f(x) = 1/(b-a)$, for $a \le x \le b$. By Example 5, the mean and standard deviation are given by

$$\mu = \frac{b+a}{2}, \qquad \sigma = \frac{b-a}{2\sqrt{3}}.$$

Since $\mu + 2\sigma = \dfrac{b+a}{2} + \dfrac{b-a}{\sqrt{3}} > b$, and similarly, $\mu - 2\sigma < a$, therefore $\Pr(|X - \mu| \ge 2\sigma) = 0$.

b) For $f(x) = ke^{-kx}$ on $[0,\infty)$, we know that $\mu = \sigma = \dfrac{1}{k}$ (Example 6). Thus $\mu - 2\sigma < 0$ and $\mu + 2\sigma = \dfrac{3}{k}$. We have

$$\begin{aligned}\Pr(|X-\mu| \ge 2\sigma) &= \Pr\left(X \ge \frac{3}{k}\right) \\ &= k\int_{3/k}^\infty e^{-kx}\,dx \\ &= -e^{-kx}\bigg|_{3/k}^\infty = e^{-3} \approx 0.050.\end{aligned}$$

c) For $f_{\mu,\sigma}(x) = \dfrac{1}{\sigma\sqrt{2\pi}}e^{(x-\mu)^2/2\sigma^2}$, which has mean μ and standard deviation σ, we have

$$\Pr(|X-\mu| \ge 2\sigma) = 2\Pr(X \le \mu - 2\sigma)$$

$$= 2\int_{-\infty}^{\mu-2\sigma} \frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/2\sigma^2}\,dx$$

$$\text{Let } z = \frac{x-\mu}{\sigma}, \quad dz = \frac{1}{\sigma}\,dx$$

$$= \frac{2}{\sqrt{2\pi}}\int_{-\infty}^{-2} e^{-z^2}\,dz$$

$$= 2\Pr(Z \le -2) \approx 2 \times 0.023 = 0.046$$

from the table in this section.

20. The density function for T is $f(t) = ke^{-kt}$ on $[0, \infty)$, where $k = \frac{1}{\mu} = \frac{1}{20}$ (see Example 6). Then

$$\begin{aligned}\Pr(T \ge 12) &= \frac{1}{20}\int_{12}^{\infty} e^{-t/20}\,dt = 1 - \frac{1}{20}\int_0^{12} e^{-t/20}\,dt \\ &= 1 + e^{-t/20}\Big|_0^{12} = e^{-12/20} \approx 0.549.\end{aligned}$$

The probability that the system will last at least 12 hours is about 0.549.

21. If X is distributed normally, with mean $\mu = 5,000$, and standard deviation $\sigma = 200$, then

$$\begin{aligned}&\Pr(X \ge 5500)\\ &= \frac{1}{200\sqrt{2\pi}}\int_{5500}^{\infty} e^{-(x-5000)^2/(2\times 200^2)}\,dx \\ &\qquad \text{Let } z = \frac{x-5000}{200} \\ &\qquad dz = \frac{dx}{200} \\ &= \frac{1}{\sqrt{2\pi}}\int_{5/2}^{\infty} e^{-z^2/2}\,dz \\ &= \Pr(Z \ge 5/2) = \Pr(Z \le -5/2) \approx 0.006\end{aligned}$$

from the table in this section.

22. If X is the random variable giving the spinner's value, then $\Pr(X = 1/4) = 1/2$ and the density function for the other values of X is $f(x) = 1/2$. Thus the mean of X is

$$\mu = E(X) = \frac{1}{4}\Pr\left(X = \frac{1}{4}\right) + \int_0^1 x\,f(x)\,dx = \frac{1}{8} + \frac{1}{4} = \frac{3}{8}.$$

Also,

$$\begin{aligned}E(X^2) &= \frac{1}{16}\Pr\left(X = \frac{1}{4}\right) + \int_0^1 x^2\,f(x)\,dx = \frac{1}{32} + \frac{1}{6} = \frac{19}{96} \\ \sigma^2 &= E(X^2) - \mu^2 = \frac{19}{96} - \frac{9}{64} = \frac{11}{192}.\end{aligned}$$

Thus $\sigma = \sqrt{11/192}$.

Section 7.9 First-Order Differential Equations (page 429)

1.
$$\begin{aligned}&\frac{dy}{dx} = \frac{y}{2x} \\ &2\frac{dy}{y} = \frac{dx}{x} \\ &2\ln y = \ln x + C_1 \qquad \Rightarrow \qquad y^2 = Cx\end{aligned}$$

2.
$$\begin{aligned}&\frac{dy}{dx} = \frac{3y-1}{x} \\ &\int \frac{dy}{3y-1} = \int \frac{dx}{x} \\ &\frac{1}{3}\ln|3y-1| = \ln|x| + \frac{1}{3}\ln C \\ &\frac{3y-1}{x^3} = C \\ \Rightarrow\quad & y = \frac{1}{3}(1 + Cx^3).\end{aligned}$$

3.
$$\begin{aligned}&\frac{dy}{dx} = \frac{x^2}{y^2} \qquad \Rightarrow \qquad y^2\,dy = x^2\,dx \\ &\frac{y^3}{3} = \frac{x^3}{3} + C_1, \qquad \text{or} \quad x^3 - y^3 = C\end{aligned}$$

4.
$$\begin{aligned}&\frac{dy}{dx} = x^2y^2 \\ &\int \frac{dy}{y^2} = \int x^2\,dx \\ &-\frac{1}{y} = \frac{1}{3}x^3 + \frac{1}{3}C \\ \Rightarrow\quad & y = -\frac{3}{x^3 + C}.\end{aligned}$$

5.
$$\begin{aligned}&\frac{dY}{dt} = tY \qquad \Rightarrow \qquad \frac{dY}{Y} = t\,dt \\ &\ln Y = \frac{t^2}{2} + C_1, \qquad \text{or} \quad Y = Ce^{t^2/2}\end{aligned}$$

6.
$$\begin{aligned}&\frac{dx}{dt} = e^x \sin t \\ &\int e^{-x}\,dx = \int \sin t\,dt \\ &-e^{-x} = -\cos t - C \\ \Rightarrow\quad & x = -\ln(\cos t + C).\end{aligned}$$

7.
$$\begin{aligned}&\frac{dy}{dx} = 1 - y^2 \qquad \Rightarrow \qquad \frac{dy}{1-y^2} = dx \\ &\frac{1}{2}\left(\frac{1}{1+y} + \frac{1}{1-y}\right)dy = dx \\ &\frac{1}{2}\ln\left|\frac{1+y}{1-y}\right| = x + C_1 \\ &\frac{1+y}{1-y} = Ce^{2x} \qquad \text{or} \quad y = \frac{Ce^{2x}-1}{Ce^{2x}+1}\end{aligned}$$

8.
$$\begin{aligned}&\frac{dy}{dx} = 1 + y^2 \\ &\int \frac{dy}{1+y^2} = \int dx \\ &\tan^{-1} y = x + C \\ \Rightarrow\quad & y = \tan(x + C).\end{aligned}$$

9. $\frac{dy}{dt} = 2 + e^y \quad \Rightarrow \quad \frac{dy}{2+e^y} = dt$

$$\int \frac{e^{-y}\,dy}{2e^{-y}+1} = \int dt$$

$$-\frac{1}{2}\ln(2e^{-y}+1) = t + C_1$$

$$2e^{-y} + 1 = C_2 e^{-2t}, \qquad \text{or} \quad y = -\ln\left(Ce^{-2t} - \frac{1}{2}\right)$$

10. We have

$$\frac{dy}{dx} = y^2(1-y)$$

$$\int \frac{dy}{y^2(1-y)} = \int dx = x + K.$$

Expand the left side in partial fractions:

$$\frac{1}{y^2(1-y)} = \frac{A}{y} + \frac{B}{y^2} + \frac{C}{1-y}$$

$$= \frac{A(y-y^2) + B(1-y) + Cy^2}{y^2(1-y)}$$

$$\Rightarrow \begin{cases} -A + C = 0; \\ A - B = 0; \\ B = 1. \end{cases} \Rightarrow A = B = C = 1.$$

Hence,

$$\int \frac{dy}{y^2(1-y)} = \int \left(\frac{1}{y} + \frac{1}{y^2} + \frac{1}{1-y}\right) dy$$

$$= \ln|y| - \frac{1}{y} - \ln|1-y|.$$

Therefore,

$$\ln\left|\frac{y}{1-y}\right| - \frac{1}{y} = x + K.$$

11. $\frac{dy}{dx} - \frac{2}{x}y = x^2 \qquad$ (linear)

$$\mu = \exp\left(\int -\frac{2}{x}\,dx\right) = \frac{1}{x^2}$$

$$\frac{1}{x^2}\frac{dy}{dx} - \frac{2}{x^3}y = 1$$

$$\frac{d}{dx}\frac{y}{x^2} = 1$$

$$\frac{y}{x^2} = x + C, \qquad \text{so} \quad y = x^3 + Cx^2$$

12. We have $\frac{dy}{dx} + \frac{2y}{x} = \frac{1}{x^2}$. Let $\mu = \int \frac{2}{x}\,dx = 2\ln x = \ln x^2$, then $e^\mu = x^2$, and

$$\frac{d}{dx}(x^2y) = x^2\frac{dy}{dx} + 2xy$$

$$= x^2\left(\frac{dy}{dx} + \frac{2y}{x}\right) = x^2\left(\frac{1}{x^2}\right) = 1$$

$$\Rightarrow \quad x^2y = \int dx = x + C$$

$$\Rightarrow \quad y = \frac{1}{x} + \frac{C}{x^2}.$$

13. $\frac{dy}{dx} + 2y = 3 \qquad \mu = \exp\left(\int 2\,dx\right) = e^{2x}$

$$\frac{d}{dx}(e^{2x}y) = e^{2x}(y' + 2y) = 3e^{2x}$$

$$e^{2x}y = \frac{3}{2}e^{2x} + C \quad \Rightarrow \quad y = \frac{3}{2} + Ce^{-2x}$$

14. We have $\frac{dy}{dx} + y = e^x$. Let $\mu = \int dx = x$, then $e^\mu = e^x$, and

$$\frac{d}{dx}(e^xy) = e^x\frac{dy}{dx} + e^xy = e^x\left(\frac{dy}{dx} + y\right) = e^{2x}$$

$$\Rightarrow \quad e^xy = \int e^{2x}\,dx = \frac{1}{2}e^{2x} + C.$$

Hence, $y = \frac{1}{2}e^x + Ce^{-x}$.

15. $\frac{dy}{dx} + y = x \qquad \mu = \exp\left(\int 1\,dx\right) = e^x$

$$\frac{d}{dx}(e^xy) = e^x(y' + y) = xe^x$$

$$e^xy = \int xe^x\,dx = xe^x - e^x + C$$

$$y = x - 1 + Ce^{-x}$$

16. We have $\frac{dy}{dx} + 2e^xy = e^x$. Let $\mu = \int 2e^x\,dx = 2e^x$, then

$$\frac{d}{dx}\left(e^{2e^x}y\right) = e^{2e^x}\frac{dy}{dx} + 2e^xe^{2e^x}y$$

$$= e^{2e^x}\left(\frac{dy}{dx} + 2e^xy\right) = e^{2e^x}e^x.$$

Therefore,

$$e^{2e^x}y = \int e^{2e^x}e^x\,dx \quad \text{Let } u = 2e^x, \; du = 2e^x\,dx$$

$$= \frac{1}{2}\int e^u\,du = \frac{1}{2}e^{2e^x} + C.$$

Hence, $y = \frac{1}{2} + Ce^{-2e^x}$.

17. $\dfrac{dy}{dt} + 10y = 1, \qquad y\left(\frac{1}{10}\right) = \frac{2}{10}$

$$\mu = \int 10\,dt = 10t$$
$$\frac{d}{dt}(e^{10t}y) = e^{10t}\frac{dy}{dt} + 10e^{10t}y = e^{10t}$$
$$e^{10t}y(t) = \frac{1}{10}e^{10t} + C$$
$$y\left(\tfrac{1}{10}\right) = \tfrac{2}{10} \;\Rightarrow\; \frac{2e}{10} = \frac{e}{10} + C \;\Rightarrow\; C = \frac{e}{10}$$
$$y = \frac{1}{10} + \frac{1}{10}e^{1-10t}.$$

18. $\dfrac{dy}{dx} + 3x^2y = x^2, \qquad y(0) = 1$

$$\mu = \int 3x^2\,dx = x^3$$
$$\frac{d}{dx}(e^{x^3}y) = e^{x^3}\frac{dy}{dx} + 3x^2e^{x^3}y = x^2e^{x^3}$$
$$e^{x^3}y = \int x^2e^{x^3}\,dx = \frac{1}{3}e^{x^3} + C$$
$$y(0) = 1 \;\Rightarrow\; 1 = \frac{1}{3} + C \;\Rightarrow\; C = \frac{2}{3}$$
$$y = \frac{1}{3} + \frac{2}{3}e^{-x^3}.$$

19. $x^2y' + y = x^2e^{1/x}, \qquad y(1) = 3e$

$$y' + \frac{1}{x^2}y = e^{1/x}$$
$$\mu = \int \frac{1}{x^2}\,dx = -\frac{1}{x}$$
$$\frac{d}{dx}\left(e^{-1/x}y\right) = e^{-1/x}\left(y' + \frac{1}{x^2}y\right) = 1$$
$$e^{-1/x}y = \int 1\,dx = x + C$$
$$y(1) = 3e \;\Rightarrow\; 3 = 1 + C \;\Rightarrow\; C = 2$$
$$y = (x+2)e^{1/x}.$$

20. $y' + (\cos x)y = 2xe^{-\sin x}, \qquad y(\pi) = 0$

$$\mu = \int \cos x\,dx = \sin x$$
$$\frac{d}{dx}(e^{\sin x}y) = e^{\sin x}(y' + (\cos x)y) = 2x$$
$$e^{\sin x}y = \int 2x\,dx = x^2 + C$$
$$y(\pi) = 0 \;\Rightarrow\; 0 = \pi^2 + C \;\Rightarrow\; C = -\pi^2$$
$$y = (x^2 - \pi^2)e^{-\sin x}.$$

21. $y(x) = 2 + \displaystyle\int_0^x \frac{t}{y(t)}\,dt \quad\Longrightarrow\quad y(0) = 2$

$$\frac{dy}{dx} = \frac{x}{y}, \qquad \text{i.e. } y\,dy = x\,dx$$
$$y^2 = x^2 + C$$
$$2^2 = 0^2 + C \quad\Longrightarrow\quad C = 4$$
$$y = \sqrt{4 + x^2}.$$

22. $y(x) = 1 + \displaystyle\int_0^x \frac{(y(t))^2}{1+t^2}\,dt \quad\Longrightarrow\quad y(0) = 1$

$$\frac{dy}{dx} = \frac{y^2}{1+x^2}, \qquad \text{i.e. } dy/y^2 = dx/(1+x^2)$$
$$-\frac{1}{y} = \tan^{-1}x + C$$
$$-1 = 0 + C \quad\Longrightarrow\quad C = -1$$
$$y = 1/(1 - \tan^{-1}x).$$

23. $y(x) = 1 + \displaystyle\int_1^x \frac{y(t)}{t(t+1)}\,dt \quad\Longrightarrow\quad y(1) = 1$

$$\frac{dy}{dx} = \frac{y}{x(x+1)}, \qquad \text{for } x > 0$$
$$\frac{dy}{y} = \frac{dx}{x(x+1)} = \frac{dx}{x} - \frac{dx}{x+1}$$
$$\ln y = \ln\frac{x}{x+1} + \ln C$$
$$y = \frac{Cx}{x+1}, \quad\Longrightarrow\quad 1 = C/2$$
$$y = \frac{2x}{x+1}.$$

24. $y(x) = 3 + \displaystyle\int_0^x e^{-y}\,dt \quad\Longrightarrow\quad y(0) = 3$

$$\frac{dy}{dx} = e^{-y}, \qquad \text{i.e. } e^y\,dy = dx$$
$$e^y = x + C \quad\Longrightarrow\quad y = \ln(x + C)$$
$$3 = y(0) = \ln C \quad\Longrightarrow\quad C = e^3$$
$$y = \ln(x + e^3).$$

25. Since $a > b > 0$ and $k > 0$,

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} \frac{ab\left(e^{(b-a)kt} - 1\right)}{be^{(b-a)kt} - a} = \frac{ab(0-1)}{0-a} = b.$$

26. Since $b > a > 0$ and $k > 0$,

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} \frac{ab\left(e^{(b-a)kt} - 1\right)}{be^{(b-a)kt} - a} = \lim_{t\to\infty} \frac{ab\left(1 - e^{(a-b)kt}\right)}{b - ae^{(a-b)kt}} = \frac{ab(1-0)}{b-0} = a.$$

27. The solution given, namely

$$x = \frac{ab\left(e^{(b-a)kt} - 1\right)}{be^{(b-a)kt} - a},$$

is indeterminate (0/0) if $a = b$.
If $a = b$ the original differential equation becomes

$$\frac{dx}{dt} = k(a-x)^2,$$

which is separable and yields the solution

$$\frac{1}{a-x} = \int \frac{dx}{(a-x)^2} = k\int dt = kt + C.$$

Since $x(0) = 0$, we have $C = \frac{1}{a}$, so $\frac{1}{a-x} = kt + \frac{1}{a}$. Solving for x, we obtain

$$x = \frac{a^2kt}{1+akt}.$$

This solution also results from evaluating the limit of solution obtained for the case $a \neq b$ as b approaches a (using l'Hôpital's Rule, say).

28. Given that $m\frac{dv}{dt} = mg - kv$, then

$$\int \frac{dv}{g - \frac{k}{m}v} = \int dt$$
$$-\frac{m}{k}\ln\left|g - \frac{k}{m}v\right| = t + C.$$

Since $v(0) = 0$, therefore $C = -\frac{m}{k}\ln g$. Also, $g - \frac{k}{m}v$ remains positive for all $t > 0$, so

$$\frac{m}{k}\ln\frac{g}{g - \frac{k}{m}v} = t$$
$$\frac{g - \frac{k}{m}v}{g} = e^{-kt/m}$$
$$\Rightarrow \quad v = v(t) = \frac{mg}{k}\left(1 - e^{-kt/m}\right).$$

Note that $\lim_{t\to\infty} v(t) = \frac{mg}{k}$. This limiting velocity can be obtained directly from the differential equation by setting $\frac{dv}{dt} = 0$.

29. We proceed by separation of variables:

$$m\frac{dv}{dt} = mg - kv^2$$
$$\frac{dv}{dt} = g - \frac{k}{m}v^2$$
$$\frac{dv}{g - \frac{k}{m}v^2} = dt$$
$$\int \frac{dv}{\frac{mg}{k} - v^2} = \frac{k}{m}\int dt = \frac{kt}{m} + C.$$

Let $a^2 = mg/k$, where $a > 0$. Thus, we have

$$\int \frac{dv}{a^2 - v^2} = \frac{kt}{m} + C$$
$$\frac{1}{2a}\ln\left|\frac{a+v}{a-v}\right| = \frac{kt}{m} + C$$
$$\ln\left|\frac{a+v}{a-v}\right| = \frac{2akt}{m} + C_1 = 2\sqrt{\frac{kg}{m}}\,t + C_1$$
$$\frac{a+v}{a-v} = C_2e^{2t\sqrt{kg/m}}.$$

Assuming $v(0) = 0$, we get $C_2 = 1$. Thus

$$a + v = e^{2t\sqrt{kg/m}}(a - v)$$
$$v\left(1 + e^{2t\sqrt{kg/m}}\right) = a\left(e^{2t\sqrt{kg/m}} - 1\right)$$
$$= \sqrt{\frac{mg}{k}}\left(e^{2t\sqrt{kg/m}} - 1\right)$$
$$v = \sqrt{\frac{mg}{k}}\,\frac{e^{2t\sqrt{kg/m}} - 1}{e^{2t\sqrt{kg/m}} + 1}$$

Clearly $v \to \sqrt{\frac{mg}{k}}$ as $t \to \infty$. This also follows from setting $\frac{dv}{dt} = 0$ in the given differential equation.

30. The balance in the account after t years is $y(t)$ and $y(0) = 1000$. The balance must satisfy

$$\frac{dy}{dt} = 0.1y - \frac{y^2}{1,000,000}$$
$$\frac{dy}{dt} = \frac{10^5y - y^2}{10^6}$$
$$\int \frac{dy}{10^5y - y^2} = \int \frac{dt}{10^6}$$
$$\frac{1}{10^5}\int\left(\frac{1}{y} + \frac{1}{10^5 - y}\right)dy = \frac{t}{10^6} - \frac{C}{10^5}$$
$$\ln|y| - \ln|10^5 - y| = \frac{t}{10} - C$$
$$\frac{10^5 - y}{y} = e^{C-(t/10)}$$
$$y = \frac{10^5}{e^{C-(t/10)} + 1}.$$

Since $y(0) = 1000$, we have

$$1000 = y(0) = \frac{10^5}{e^C + 1} \quad \Rightarrow \quad C = \ln 99,$$

and

$$y = \frac{10^5}{99e^{-t/10} + 1}.$$

The balance after 1 year is

$$y = \frac{10^5}{99e^{-1/10} + 1} \approx \$1,104.01.$$

As $t \to \infty$, the balance can grow to

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} \frac{10^5}{e^{(4.60-0.1t)} + 1} = \frac{10^5}{0+1} = \$100,000.$$

For the account to grow to \$50,000, t must satisfy

$$\begin{aligned} 50,000 = y(t) &= \frac{100,000}{99e^{-t/10} + 1} \\ \Rightarrow \quad & 99e^{-t/10} + 1 = 2 \\ \Rightarrow \quad & t = 10 \ln 99 \approx 46 \text{ years.} \end{aligned}$$

31. The hyperbolas $xy = C$ satisfy the differential equation

$$y + x\frac{dy}{dx} = 0, \qquad \text{or} \quad \frac{dy}{dx} = -\frac{y}{x}.$$

Curves that intersect these hyperbolas at right angles must therefore satisfy $\frac{dy}{dx} = \frac{x}{y}$, or $x\,dx = y\,dy$, a separated equation with solutions $x^2 - y^2 = C$, which is also a family of rectangular hyperbolas. (Both families are degenerate at the origin for $C = 0$.)

32. Let $x(t)$ be the number of kg of salt in the solution in the tank after t minutes. Thus, $x(0) = 50$. Salt is coming into the tank at a rate of 10 g/L × 12 L/min = 0.12 kg/min. Since the contents flow out at a rate of 10 L/min, the volume of the solution is increasing at 2 L/min and thus, at any time t, the volume of the solution is $1000 + 2t$ L. Therefore the concentration of salt is $\frac{x(t)}{1000 + 2t}$ L. Hence, salt is being removed at a rate

$$\frac{x(t)}{1000+2t} \text{ kg/L} \times 10 \text{ L/min} = \frac{5x(t)}{500+t} \text{ kg/min}.$$

Therefore,

$$\begin{aligned} \frac{dx}{dt} &= 0.12 - \frac{5x}{500+t} \\ \frac{dx}{dt} &+ \frac{5}{500+t}x = 0.12. \end{aligned}$$

Let $\mu = \int \frac{5}{500+t}\,dt = 5\ln|500+t| = \ln(500+t)^5$ for $t > 0$. Then $e^\mu = (500+t)^5$, and

$$\begin{aligned} \frac{d}{dt}\Big[(500+t)^5 x\Big] &= (500+t)^5\frac{dx}{dy} + 5(500+t)^4 x \\ &= (500+t)^5\left(\frac{dx}{dy} + \frac{5x}{500+t}\right) \\ &= 0.12(500+t)^5. \end{aligned}$$

Hence,

$$\begin{aligned} (500+t)^5 x &= 0.12\int (500+t)^5\,dt = 0.02(500+t)^6 + C \\ \Rightarrow x &= 0.02(500+t) + C(500+t)^{-5}. \end{aligned}$$

Since $x(0) = 50$, we have $C = 1.25 \times 10^{15}$ and

$$x = 0.02(500+t) + (1.25 \times 10^{15})(500+t)^{-5}.$$

After 40 min, there will be

$$x = 0.02(540) + (1.25 \times 10^{15})(540)^{-5} = 38.023 \text{ kg}$$

of salt in the tank.

Review Exercises 7 (page 430)

1.

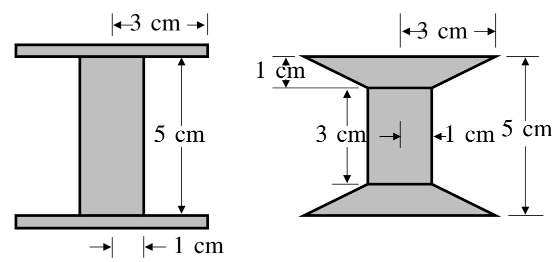

Fig. R-7.1

The volume of thread that can be wound on the left spool is $\pi(3^2 - 1^2)(5) = 40\pi$ cm^3.
The height of the winding region of the right spool at distance r from the central axis of the spool is of the form $h = A + Br$. Since $h = 3$ if $r = 1$, and $h = 5$ if $r = 3$, we have $A = 2$ and $B = 1$, so $h = 2 + r$. The volume of thread that can be wound on the right spool is

$$2\pi\int_1^3 r(2+r)\,dr = 2\pi\left(r^2 + \frac{r^3}{3}\right)\Bigg|_1^3 = \frac{100\pi}{3} \text{ cm}^3.$$

The right spool will hold $\frac{100}{3\times 40}(1,000) = 833.33$ m of thread.

2. Let $A(y)$ be the cross-sectional area of the bowl at height y above the bottom. When the depth of water in the bowl is Y, then the volume of water in the bowl is

$$V(Y) = \int_0^Y A(y)\,dy.$$

The water evaporates at a rate proportional to exposed surface area. Thus

$$\frac{dV}{dt} = kA(Y)$$
$$\frac{dV}{dY}\frac{dY}{dt} = kA(Y)$$
$$A(Y)\frac{dY}{dt} = kA(Y).$$

Hence $dY/dt = k$; the depth decreases at a constant rate.

3. The barrel is generated by revolving $x = a - by^2$, $(-2 \le y \le 2)$, about the y-axis. Since the top and bottom disks have radius 1 ft, we have $a - 4b = 1$. The volume of the barrel is

$$\begin{aligned} V &= 2\int_0^2 \pi(a-by^2)^2\,dy \\ &= 2\pi\left(a^2y - \frac{2aby^3}{3} + \frac{b^2y^5}{5}\right)\Bigg|_0^2 \\ &== 2\pi\left(2a^2 - \frac{16}{3}ab + \frac{32}{5}b^2\right). \end{aligned}$$

Since $V = 16$ and $a = 1 + 4b$, we have

$$2\pi\left(2(1+4b)^2 - \frac{16}{3}b(1+4b) + \frac{32}{5}b^2\right) = 16$$
$$128b^2 + 80b + 15 - \frac{60}{\pi} = 0.$$

Solving this quadratic gives two solutions, $b \approx 0.0476$ and $b \approx -0.6426$. Since the second of these leads to an unacceptable negative value for a, we must have $b \approx 0.0476$, and so $a = 1 + 4b \approx 1.1904$.

4. A vertical slice parallel to the top ridge of the solid at distance x to the right of the centre is a rectangle of base $2\sqrt{100-x^2}$ cm^and height $\sqrt{3}(10-x)$ cm. Thus the solid has volume

$$\begin{aligned} V &= 2\int_0^{10} \sqrt{3}(10-x)2\sqrt{100-x^2}\,dx \\ &= 40\sqrt{3}\int_0^{10}\sqrt{100-x^2}\,dx - 4\sqrt{3}\int_0^{10} x\sqrt{100-x^2}\,dx \\ &\qquad \text{Let } u = 100 - x^2 \\ &\qquad du = -2x\,dx \\ &= 40\sqrt{3}\frac{100\pi}{4} - 2\sqrt{3}\int_0^{100}\sqrt{u}\,du \\ &= 1,000\sqrt{3}\left(\pi - \frac{4}{3}\right) \text{ cm}^3. \end{aligned}$$

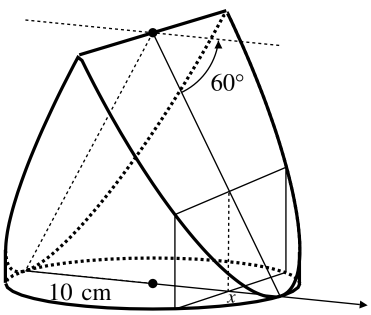

Fig. R-7.4

5. The arc length of $y = \frac{1}{a}\cosh(ax)$ from $x = 0$ to $x = 1$ is

$$\begin{aligned} s &= \int_0^1 \sqrt{1+\sinh^2(ax)}\,dx = \int_0^1 \cosh(ax)\,dx \\ &= \frac{1}{a}\sinh(ax)\Big|_0^1 = \frac{1}{a}\sinh a. \end{aligned}$$

We want $\frac{1}{a}\sinh a = 2$, that is, $\sinh a = 2a$. Solving this by Newton's Method or a calculator solve function, we get $a \approx 2.1773$.

6. The area of revolution of $y = \sqrt{x}$, $(0 \le x \le 6)$, about the x-axis is

$$\begin{aligned} S &= 2\pi\int_0^6 y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \\ &= 2\pi\int_0^6 \sqrt{x}\sqrt{1+\frac{1}{4x}}\,dx \\ &= 2\pi\int_0^6 \sqrt{x+\frac{1}{4}}\,dx \\ &= \frac{4\pi}{3}\left(x+\frac{1}{4}\right)^{3/2}\Bigg|_0^6 = \frac{4\pi}{3}\left[\frac{125}{8} - \frac{1}{8}\right] = \frac{62\pi}{3} \text{ sq. units.} \end{aligned}$$

7. The region is a quarter-elliptic disk with semi-axes $a = 2$ and $b = 1$. The area of the region is $A = \pi ab/4 = \pi/2$. The moments about the coordinate axes are

$$\begin{aligned} M_{x=0} &= \int_0^2 x\sqrt{1-\frac{x^2}{4}}\,dx \quad \text{Let } u = 1 - \frac{x^2}{4} \\ &\qquad\qquad\qquad\qquad du = -\frac{x}{2}\,dx \\ &= 2\int_0^1 \sqrt{u}\,du = \frac{4}{3} \\ M_{y=0} &= \frac{1}{2}\int_0^2 \left(1 - \frac{x^2}{4}\right)dx \\ &= \frac{1}{2}\left(x - \frac{x^3}{12}\right)\Bigg|_0^2 = \frac{2}{3}. \end{aligned}$$

Thus $\bar{x} = M_{x=0}/A = 8/(3\pi)$ and $\bar{y} = M_{y=0}/A = 4/(3\pi)$. The centroid is $\big(8/(3\pi), 4/(3\pi)\big)$.

8.

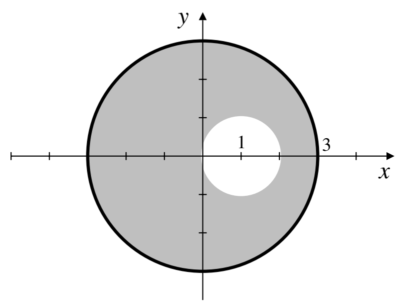

Fig. R-7.8

Let the disk have centre (and therefore centroid) at $(0, 0)$. Its area is 9π. Let the hole have centre (and therefore centroid) at $(1, 0)$. Its area is π. The remaining part has area 8π and centroid at $(\bar{x}, 0)$, where

$$(9\pi)(0) = (8\pi)\bar{x} + (\pi)(1).$$

Thus $\bar{x} = -1/8$. The centroid of the remaining part is $1/8$ ft from the centre of the disk on the side opposite the hole.

9. Let the area of cross-section of the cylinder be A. When the piston is y cm above the base, the volume of gas in the cylinder is $V = Ay$, and its pressure $P(y)$ satisfies $P(y)V = k$ (constant). The force exerted by the piston is

$$F(y) = P(y)A = \frac{kA}{Ay} = \frac{k}{y}.$$

We are told that $F = 1,000$ N when $y = 20$ cm. Thus $k = 20,000$ N·cm. The work done by the piston as it descends to 5 cm is

$$W = \int_5^{20} \frac{20,000}{y}\,dy = 20,000 \ln\frac{20}{5} \approx 27,726 \text{ N·cm}.$$

10. We are told that for any $a > 0$,

$$\pi \int_0^a \Big[\big(f(x)\big)^2 - \big(g(x)\big)^2\Big]\,dx = 2\pi \int_0^a x\Big[f(x) - g(x)\Big]\,dx.$$

Differentiating both sides of this equation with respect to a, we get

$$\big(f(a)\big)^2 - \big(g(a)\big)^2 = 2a\Big[f(a) - g(a)\Big],$$

or, equivalently, $f(a) + g(a) = 2a$. Thus f and g must satisfy

$$f(x) + g(x) = 2x \quad \text{for every } x > 0.$$

11. $$\frac{dy}{dx} = \frac{3y}{x-1} \Rightarrow \int \frac{dy}{y} = 3\frac{dx}{x-1}$$
$$\Rightarrow \ln|y| = \ln|x-1|^3 + \ln|C|$$
$$\Rightarrow y = C(x-1)^3.$$

Since $y = 4$ when $x = 2$, we have $4 = C(2-1)^3 = C$, so the equation of the curve is $y = 4(x-1)^3$.

12. The ellipses $3x^2 + 4y^2 = C$ all satisfy the differential equation

$$6x + 8y\frac{dy}{dx} = 0, \quad \text{or} \quad \frac{dy}{dx} = -\frac{3x}{4y}.$$

A family of curves that intersect these ellipses at right angles must therefore have slopes given by $\dfrac{dy}{dx} = \dfrac{4y}{3x}$. Thus

$$3\int \frac{dy}{y} = 4\int \frac{dx}{x}$$
$$3\ln|y| = 4\ln|x| + \ln|C|.$$

The family is given by $y^3 = Cx^4$.

13. The original \$8,000 grows to $\$8,000e^{0.08}$ in two years. Between t and $t + dt$, an amount $\$10,000\sin(2\pi t)\,dt$ comes in, and this grows to $\$10,000\sin(2\pi t)e^{0.04(2-t)}\,dt$ by the end of two years. Thus the amount in the account after 2 years is

$$8,000e^{0.08} + 10,000\int_0^2 \sin(2\pi t)e^{0.04(2-t)}\,dt \approx \$8,798.85.$$

(We omit the details of evaluation of the integral, which is done by the method of Example 4 of Section 7.1.)

Challenging Problems 7 (page 430)

1. a) The nth bead extends from $x = (n-1)\pi$ to $x = n\pi$, and has volume

$$\begin{aligned}
V_n &= \pi \int_{(n-1)\pi}^{n\pi} e^{-2kx}\sin^2 x\,dx \\
&= \frac{\pi}{2}\int_{(n-1)\pi}^{n\pi} e^{-2kx}(1 - \cos(2x))\,dx \\
&\qquad \text{Let } x = u + (n-1)\pi \\
&\qquad dx = du \\
&= \frac{\pi}{2}\int_0^{\pi} e^{-2ku}e^{-2k(n-1)\pi}\Big[1 - \cos(2u + 2(n-1)\pi)\Big]\,du \\
&= \frac{\pi}{2}e^{-2k(n-1)\pi}\int_0^{\pi} e^{-2ku}(1 - \cos(2u))\,du \\
&= e^{-2k(n-1)\pi}V_1.
\end{aligned}$$

Thus $\dfrac{V_{n+1}}{V_n} = \dfrac{e^{-2kn\pi}V_1}{e^{-2k(n-1)\pi}V_1} = e^{-2k\pi}$, which depends on k but not n.

b) $V_{n+1}/V_n = 1/2$ if $-2k\pi = \ln(1/2) = -\ln 2$, that is, if $k = (\ln 2)/(2\pi)$.

c) Using the result of Example 4 in Section 7.1, we calculate the volume of the first bead:

$$\begin{aligned}
V_1 &= \frac{\pi}{2}\int_0^\pi e^{-2kx}(1-\cos(2x))\,dx \\
&= \frac{\pi e^{-2kx}}{-4k}\bigg|_0^\pi - \frac{\pi}{2}\,\frac{e^{-2kx}(2\sin(2x)-2k\cos(2x))}{4(1+k^2)}\bigg|_0^\pi \\
&= \frac{\pi}{4k}(1-e^{-2k\pi}) - \frac{\pi}{4(1+k^2)}(k-ke^{-2k\pi}) \\
&= \frac{\pi}{4k(1+k^2)}(1-e^{-2k\pi}).
\end{aligned}$$

By part (a) and Theorem 1(d) of Section 6.1, the sum of the volumes of the first n beads is

$$\begin{aligned}
S_n &= \frac{\pi}{4k(1+k^2)}(1-e^{-2k\pi}) \\
&\quad \times \left[1 + e^{-2k\pi} + \left(e^{-2k\pi}\right)^2 + \cdots + \left(e^{-2k\pi}\right)^{n-1}\right] \\
&= \frac{\pi}{4k(1+k^2)}(1-e^{-2k\pi})\frac{1-e^{-2kn\pi}}{1-e^{-2k\pi}} \\
&= \frac{\pi}{4k(1+k^2)}(1-e^{-2kn\pi}).
\end{aligned}$$

Thus the total volume of all the beads is

$$V = \lim_{n\to\infty} S_n = \frac{\pi}{4k(1+k^2)} \text{ cu. units..}$$

2.

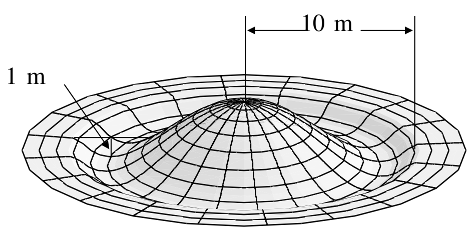

Fig. C-7.2

$$h(r) = a(r^2-100)(r^2-k^2), \quad \text{where } 0 < k < 10$$
$$h'(r) = 2ar(r^2-k^2) + 2ar(r^2-100) = 2ar(2r^2-100-k^2).$$

The deepest point occurs where $2r^2 = 100 + k^2$, i.e., $r^2 = 50 + (k^2/2)$. Since this depth must be 1 m, we require

$$a\left(\frac{k^2}{2}-50\right)\left(50-\frac{k^2}{2}\right) = -1,$$

or, equivalently, $a(100-k^2)^2 = 4$. The volume of the pool is

$$\begin{aligned}
V_P &= 2\pi a\int_k^{10} r(100-r^2)(r^2-k^2)\,dr \\
&= 2\pi a\left(\frac{250,000}{3} - 2,500k^2 + 25k^4 - \frac{1}{12}k^6\right).
\end{aligned}$$

The volume of the hill is

$$V_H = 2\pi a\int_0^k r(r^2-100)(r^2-k^2)\,dr = 2\pi a\left(25k^4 - \frac{1}{12}k^6\right).$$

These two volumes must be equal, so $k^2 = 100/3$ and $k \approx 5.77$ m. Thus $a = 4/(100-k^2)^2 = 0.0009$. The volume of earth to be moved is V_H with these values of a and k, namely

$$2\pi(0.0009)\left[25\left(\frac{100}{3}\right)^2 - \frac{1}{12}\left(\frac{100}{3}\right)^4\right] \approx 140 \text{ m}^3.$$

3.

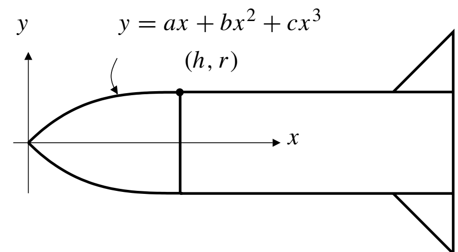

Fig. C-7.3

$f(x) = ax + bx^2 + cx^3$ must satisfy $f(h) = r$, $f'(h) = 0$, and $f'(x) > 0$ for $0 < x < h$. The first two conditions require that

$$\begin{aligned}
ah + bh^2 + ch^3 &= r \\
a + 2bh + 3ch^2 &= 0,
\end{aligned}$$

from which we obtain by solving for b and c,

$$b = \frac{3r-2ah}{h^2}, \quad c = \frac{ah-2r}{h^3}.$$

The volume of the nose cone is then

$$V(a) = \pi\int_0^h \left(f(x)\right)^2 dx = \frac{\pi h}{210}(13ahr + 78r^2 + 2a^2h^2).$$

Solving $dV/da = 0$ gives only one critical point, $a = -13r/(4h)$. This is unacceptable, because the condition $f'(x) > 0$ on $(0, h)$ forces us to require $a \ge 0$. In fact

$$f'(x) = a + \frac{2(3r-2ah)}{h^2}x + \frac{3(ah-2r)}{h^3}x^2$$

is clearly positive for small x if $a > 0$. Its two roots are $x_1 = h$ and $x_2 = h^2a/(3ah - 6r)$. a must be restricted so that x_2 is not in the interval $(0, h)$. If $a < 2r/h$, then $x_2 < 0$. If $2r/h < a < 3r/h$, then $x_2 > h$. If $a > 3r/h$, then $0 < x_2 < h$. Hence the interval of acceptable values of a is $0 \le a \le 3r/h$. We have

$$V(0) = \frac{13\pi r^2 h}{35}, \quad V\left(\frac{3r}{h}\right) = \frac{9\pi r^2 h}{14}.$$

The largest volume corresponds to $a = 3r/h$, which is the largest allowed value for a and so corresponds to the bluntest possible nose. The corresponding cubic $f(x)$ is

$$f(x) = \frac{r}{h^3}(3h^2x - 3hx^2 + x^3).$$

4. a) If $f(x) = \begin{cases} a + bx + cx^2 & \text{for } 0 \le x \le 1 \\ p + qx + rx^2 & \text{for } 1 \le x \le 3 \end{cases}$, then $f'(x) = \begin{cases} b + 2cx & \text{for } 0 < x < 1 \\ q + 2rx & \text{for } 1 < x < 3 \end{cases}$. We require that

$$\begin{aligned} &a = 1 && p + 3q + 9r = 0 \\ &a + b + c = 2 && p + q + r = 2 \\ &b + 2c = m && q + 2r = m. \end{aligned}$$

The solutions of these systems are $a = 1$, $b = 2 - m$, $c = m - 1$, $p = \frac{3}{2}(1 - m)$, $q = 2m + 1$, and $r = -\frac{1}{2}(1 + m)$. $f(x, m)$ is $f(x)$ with these values of the six constants.

b) The length of the spline is

$$L(m) = \int_0^1 \sqrt{1 + (b + 2cx)^2}\, dx + \int_1^3 \sqrt{1 + (q + 2rx)^2}\, dx$$

with the values of b, c, q, and r determined above. A plot of the graph of $L(m)$ reveals a minimum value in the neighbourhood of $m = -0.3$. The derivative of $L(m)$ is a horrible expression, but Mathematica determined its zero to be about $m = -0.281326$, and the corresponding minimum value of L is about 4.41748. The polygonal line ABC has length $3\sqrt{2} \approx 4.24264$, which is only slightly shorter.

5. Let $b = ka$ so that the cross-sectional curve is given by

$$y = f(x) = ax(1 - x)(x + k).$$

The requirement that $f(x) \ge 0$ for $0 \le x \le 1$ is satisfied provided either $a > 0$ and $k \ge 0$ or $a < 0$ and $k \le -1$. The volume of the wall is

$$V(a, k) = \int_0^1 2\pi(15 + x) f(x)\, dx = \frac{\pi a}{30}(78 + 155k).$$

To minimize this expression for $a > 0$ we should take $k = 0$. This gives $f(x) = ax^2(1 - x)$. To minimize $V(a, k)$ for $a < 0$ we should take $k = -1$. This gives $f(x) = -ax(1 - x)^2$. Since we want the maximum value of f to be 2 in either case, we calculate the critical points of these two possible functions. For $a > 0$ the CP is $x = 2/3$ and $f(2/3) = 2$ gives $a = 27/2$. The volume in this case is $V(27/2, 0) = (27\pi/60)(78 - 0)$. For $a < 0$ the CP is $x = 1/3$ and $f(1/3) = 2$ gives $a = -27/2$. The volume in this case is $V(-27/2, -1) = -(27\pi/60)(78 - 155) = (27\pi/60)(77)$. Thus the minimum volume occurs for $f(x) = (27/2)x(1 - x)^2$, i.e. $b = -a = 27/2$.

6. Starting with $V_1(r) = 2r$, and using repeatedly the formula

$$V_n(r) = \int_{-r}^{r} V_{n-1}(\sqrt{r^2 - x^2})\, dx,$$

Maple gave the following results:

$$\begin{aligned} V_1(r) &= 2r & V_2(r) &= \pi r^2 \\ V_3(r) &= \frac{4}{3}\pi r^3 & V_4(r) &= \frac{1}{2}\pi^2 r^4 \\ V_5(r) &= \frac{8}{15}\pi^2 r^5 & V_6(r) &= \frac{1}{6}\pi^3 r^6 \\ V_7(r) &= \frac{16}{105}\pi^3 r^7 & V_8(r) &= \frac{1}{24}\pi^4 r^8 \\ V_9(r) &= \frac{32}{945}\pi^4 r^9 & V_{10}(r) &= \frac{1}{120}\pi^5 r^{10} \end{aligned}$$

It appears that

$$\begin{aligned} V_{2n}(r) &= \frac{1}{n!}\pi^n r^{2n}, \qquad \text{and} \\ V_{2n-1}(r) &= \frac{2^n}{1 \cdot 3 \cdot 5 \cdots (2n-1)}\pi^{n-1} r^{2n-1} \\ &= \frac{2^{2n-1}(n-1)!}{(2n-1)!}\pi^{n-1} r^{2n-1}. \end{aligned}$$

These formulas predict that

$$V_{11}(r) = \frac{2^{11}5!}{11!}\pi^5 r^{11} \quad \text{and} \quad V_{12}(r) = \frac{1}{6!}\pi^6 r^{12},$$

both of which Maple is happy to confirm.

7. With y and θ as defined in the statement of the problem, we have

$$0 \le y \le 10 \quad \text{and} \quad 0 \le \theta < \pi.$$

The needle crosses a line if $y < 5\sin\theta$. The probability of this happening is the ratio of the area under the curve to the area of the rectangle in the figure, that is,

$$\Pr = \frac{1}{10\pi}\int_0^\pi 5\sin\theta\, d\theta = \frac{1}{\pi}.$$

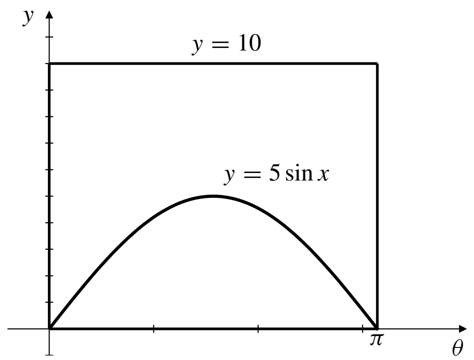

Fig. C-7.7

8.

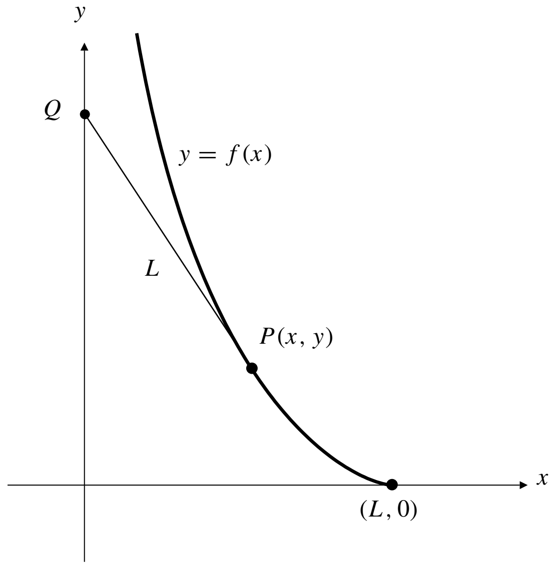

Fig. C-7.8

If $Q = (0, Y)$, then the slope of PQ is

$$\frac{y - Y}{x - 0} = f'(x) = \frac{dy}{dx}.$$

Since $|PQ| = L$, we have $(y - Y)^2 = L^2 - x^2$. Since the slope dy/dx is negative at P, $dy/dx = -\sqrt{L^2 - x^2}/x$. Thus

$$y = -\int \frac{\sqrt{L^2 - x^2}}{x}\,dx = L\ln\left(\frac{L + \sqrt{L^2 - x^2}}{x}\right) - \sqrt{L^2 - x^2} + C.$$

Since $y = 0$ when $x = L$, we have $C = 0$ and the equation of the tractrix is

$$y = L\ln\left(\frac{L + \sqrt{L^2 - x^2}}{x}\right) - \sqrt{L^2 - x^2}.$$

Note that the first term can be written in an alternate way:

$$y = L\ln\left(\frac{x}{L - \sqrt{L^2 - x^2}}\right) - \sqrt{L^2 - x^2}.$$

9. a) $S(a, a, c)$ is the area of the surface obtained by rotating the ellipse $(x^2/a^2) + (y^2/c^2) = 1$ (where $a > c$) about the y-axis. Since $y' = -cx/(a\sqrt{a^2 - x^2})$, we have

$$\begin{aligned}
S(a, a, c) &= 2 \times 2\pi \int_0^a x\sqrt{1 + \frac{c^2x^2}{a^2(a^2 - x^2)}}\,dx \\
&= \frac{4\pi}{a}\int_0^a x\frac{\sqrt{a^4 - (a^2 - c^2)x^2}}{\sqrt{a^2 - x^2}}\,dx \\
&\qquad \text{Let } x = a\sin u \\
&\qquad dx = a\cos u\,du \\
&= \frac{4\pi}{a}\int_0^{\pi/2} a\sin u\sqrt{a^4 - (a^2 - c^2)a^2\sin^2 u}\,du \\
&= 4\pi a\int_0^{\pi/2} \sin u\sqrt{a^2 - (a^2 - c^2)(1 - \cos^2 u)}\,du \\
&\qquad \text{Let } v = \cos u \\
&\qquad dv = -\sin u\,du \\
&= 4\pi a\int_0^1 \sqrt{c^2 + (a^2 - c^2)v^2}\,dv.
\end{aligned}$$

This integral can now be handled using tables or computer algebra. It evaluates to

$$S(a, a, c) = 2\pi a^2 + \frac{2\pi ac^2}{\sqrt{a^2 - c^2}}\ln\left(\frac{a + \sqrt{a^2 - c^2}}{c}\right).$$

b) $S(a, c, c)$ is the area of the surface obtained by rotating the ellipse of part (a) about the y-axis. Since $y' = -cx/(a\sqrt{a^2 - x^2})$, we have

$$\begin{aligned}
S(a, c, c) &= 2 \times 2\pi \int_0^a y\sqrt{1 + \frac{c^2x^2}{a^2(a^2 - x^2)}}\,dx \\
&= \frac{4\pi c}{a^2}\int_0^a \sqrt{a^2 - x^2}\frac{\sqrt{a^4 - (a^2 - c^2)x^2}}{\sqrt{a^2 - x^2}}\,dx \\
&= \frac{4\pi c}{a^2}\int_0^a \sqrt{a^4 - (a^2 - c^2)x^2}\,dx \\
&= 4\pi c\int_0^a \sqrt{1 - \frac{a^2 - c^2}{a^4}x^2}\,dx \\
&= 2\pi c^2 + \frac{2\pi a^2 c}{\sqrt{a^2 - c^2}}\cos^{-1}\frac{c}{a}.
\end{aligned}$$

c) Since $b = \left(\frac{b - c}{a - c}\right)a + \left(\frac{a - b}{a - c}\right)c$, we use

$$S(a, b, c) \approx \left(\frac{b - c}{a - c}\right)S(a, a, c) + \left(\frac{a - b}{a - c}\right)S(a, c, c).$$

d) We cannot evaluate $S(3, 2, 1)$ even numerically at this stage. The double integral necessary to calculate it is not treated until a later chapter. (The value is approximately 48.882 sq. units.) However, using the formulas obtained above,

$$S(3, 2, 1) \approx \frac{S(3, 3, 1) + S(3, 1, 1)}{2}$$
$$= \frac{1}{2}\left(18\pi + \frac{6\pi}{\sqrt{8}}\ln(3+\sqrt{8}) + 2\pi + \frac{18\pi}{\sqrt{8}}\cos^{-1}(1/3)\right)$$
$$\approx 49.595 \text{ sq. units.}$$