CHAPTER 8. CONICS, PARAMETRIC CURVES, AND POLAR CURVES

Section 8.1 Conics (page 443)

1. The ellipse with foci $(0, \pm 2)$ has major axis along the y-axis and $c = 2$. If $a = 3$, then $b^2 = 9 - 4 = 5$. The ellipse has equation

$$\frac{x^2}{5} + \frac{y^2}{9} = 1.$$

2. The ellipse with foci $(0, 1)$ and $(4, 1)$ has $c = 2$, centre $(2, 1)$, and major axis along $y = 1$. If $\epsilon = 1/2$, then $a = c/\epsilon = 4$ and $b^2 = 16 - 4 = 12$. The ellipse has equation

$$\frac{(x-2)^2}{16} + \frac{(y-1)^2}{12} = 1.$$

3. A parabola with focus $(2, 3)$ and vertex $(2, 4)$ has $a = -1$ and principal axis $x = 2$. Its equation is $(x-2)^2 = -4(y-4) = 16 - 4y$.

4. A parabola with focus at $(0, -1)$ and principal axis along $y = -1$ will have vertex at a point of the form $(v, -1)$. Its equation will then be of the form $(y+1)^2 = \pm 4v(x - v)$. The origin lies on this curve if
$1 = \pm 4(-v^2)$. Only the $-$ sign is possible, and in this case $v = \pm 1/2$. The possible equations for the parabola are $(y+1)^2 = 1 \pm 2x$.

5. The hyperbola with semi-transverse axis $a = 1$ and foci $(0, \pm 2)$ has transverse axis along the y-axis, $c = 2$, and $b^2 = c^2 - a^2 = 3$. The equation is

$$y^2 - \frac{x^2}{3} = 1.$$

6. The hyperbola with foci at $(\pm 5, 1)$ and asymptotes $x = \pm(y - 1)$ is rectangular, has centre at $(0, 1)$ and has transverse axis along the line $y = 1$. Since $c = 5$ and $a = b$ (because the asymptotes are perpendicular to each other) we have $a^2 = b^2 = 25/2$. The equation of the hyperbola is

$$x^2 - (y-1)^2 = \frac{25}{2}.$$

7. If $x^2 + y^2 + 2x = -1$, then $(x+1)^2 + y^2 = 0$. This represents the single point $(-1, 0)$.

8. If $x^2 + 4y^2 - 4y = 0$, then

$$x^2 + 4\left(y^2 - y + \frac{1}{4}\right) = 1, \quad \text{or} \quad \frac{x^2}{1} + \frac{(y - \frac{1}{2})^2}{\frac{1}{4}} = 1.$$

This represents an ellipse with centre at $\left(0, \frac{1}{2}\right)$, semi-major axis 1, semi-minor axis $\frac{1}{2}$, and foci at $\left(\pm\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

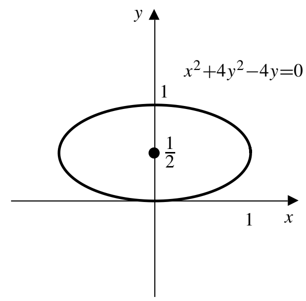

Fig. 8.1.8

9. If $4x^2 + y^2 - 4y = 0$, then

$$\begin{aligned} 4x^2 + y^2 - 4y + 4 &= 4 \\ 4x^2 + (y-2)^2 &= 4 \\ x^2 + \frac{(y-2)^2}{4} &= 1 \end{aligned}$$

This is an ellipse with semi-axes 1 and 2, centred at $(0, 2)$.

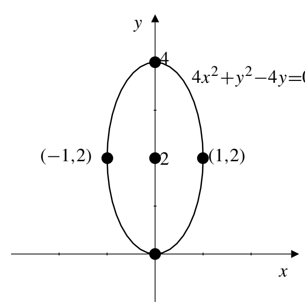

Fig. 8.1.9

10. If $4x^2 - y^2 - 4y = 0$, then

$$4x^2 - (y^2 + 4y + 4) = -4, \quad \text{or} \quad \frac{x^2}{1} - \frac{(y+2)^2}{4} = -1.$$

This represents a hyperbola with centre at $(0, -2)$, semi-transverse axis 2, semi-conjugate axis 1, and foci at $(0, -2 \pm \sqrt{5})$. The asymptotes are $y = \pm 2x - 2$.

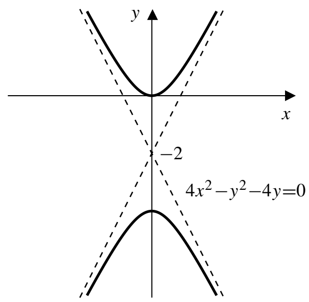

Fig. 8.1.10

11. If $x^2 + 2x - y = 3$, then $(x+1)^2 - y = 4$.
Thus $y = (x+1)^2 - 4$. This is a parabola with vertex $(-1, -4)$, opening upward.

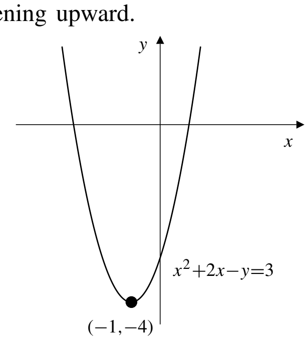

Fig. 8.1.11

12. If $x + 2y + 2y^2 = 1$, then

$$2\left(y^2 + y + \frac{1}{4}\right) = \frac{3}{2} - x$$
$$\Leftrightarrow \quad x = \frac{3}{2} - 2\left(y + \frac{1}{2}\right)^2.$$

This represents a parabola with vertex at $(\frac{3}{2}, -\frac{1}{2})$, focus at $(\frac{11}{8}, -\frac{1}{2})$ and directrix $x = \frac{13}{8}$.

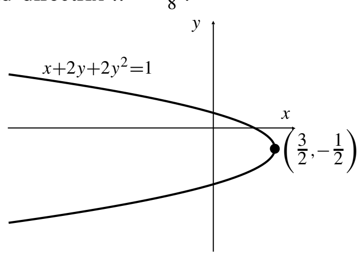

Fig. 8.1.12

13. If $x^2 - 2y^2 + 3x + 4y = 2$, then

$$\left(x + \frac{3}{2}\right)^2 - 2(y-1)^2 = \frac{9}{4}$$
$$\frac{\left(x+\frac{3}{2}\right)^2}{\frac{9}{4}} - \frac{(y-1)^2}{\frac{9}{8}} = 1$$

This is a hyperbola with centre $\left(-\frac{3}{2}, 1\right)$, and asymptotes the straight lines $2x + 3 = \pm 2\sqrt{2}(y-1)$.

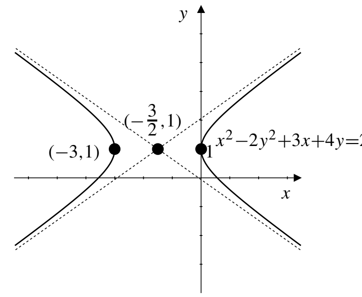

Fig. 8.1.13

14. If $9x^2 + 4y^2 - 18x + 8y = -13$, then

$$9(x^2 - 2x + 1) + 4(y^2 + 2y + 1) = 0$$
$$\Leftrightarrow 9(x-1)^2 + 4(y+1)^2 = 0.$$

This represents the single point $(1, -1)$.

15. If $9x^2 + 4y^2 - 18x + 8y = 23$, then

$$9(x^2 - 2x + 1) + 4(y^2 + 2y + 1) = 23 + 9 + 4 = 36$$
$$9(x-1)^2 + 4(y+1)^2 = 36$$
$$\frac{(x-1)^2}{4} + \frac{(y+1)^2}{9} = 1.$$

This is an ellipse with centre $(1, -1)$, and semi-axes 2 and 3.

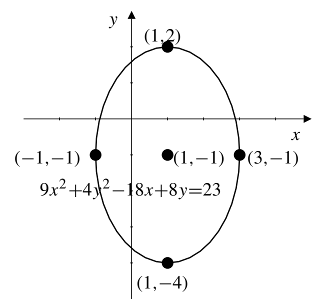

Fig. 8.1.15

16. The equation $(x-y)^2 - (x+y)^2 = 1$ simplifies to $4xy = -1$ and hence represents a rectangular hyperbola with centre at the origin, asymptotes along the coordinate axes, transverse axis along $y = -x$, conjugate axis along $y = x$, vertices at $\left(\frac{1}{2}, -\frac{1}{2}\right)$ and $\left(-\frac{1}{2}, \frac{1}{2}\right)$, semi-transverse and semi-conjugate axes equal to $1/\sqrt{2}$, semi-focal separation equal to $\sqrt{\frac{1}{2} + \frac{1}{2}} = 1$, and hence foci at the points $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ and $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. The eccentricity is $\sqrt{2}$.

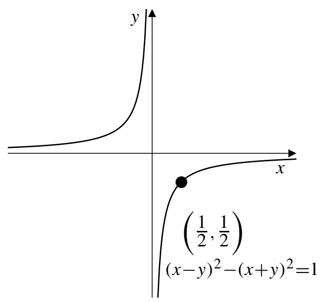

Fig. 8.1.16

17. The parabola has focus at $(3, 4)$ and principal axis along $y = 4$. The vertex must be at a point of the form $(v, 4)$, in which case $a = \pm(3 - v)$ and the equation of the parabola must be of the form

$$(y - 4)^2 = \pm 4(3 - v)(x - v).$$

This curve passes through the origin if $16 = \pm 4(v^2 - 3v)$. We have two possible equations for v: $v^2 - 3v - 4 = 0$ and $v^2 - 3v + 4 = 0$. The first of these has solutions $v = -1$ or $v = 4$. The second has no real solutions. The two possible equations for the parabola are

$$(y - 4)^2 = 4(4)(x + 1) \quad \text{or} \quad y^2 - 8y = 16x$$
$$(y - 4)^2 = 4(-1)(x - 4) \quad \text{or} \quad y^2 - 8y = -4x$$

18. The foci of the ellipse are $(0, 0)$ and $(3, 0)$, so the centre is $(3/2, 0)$ and $c = 3/2$. The semi-axes a and b must satisfy $a^2 - b^2 = 9/4$. Thus the possible equations of the ellipse are

$$\frac{(x - (3/2))^2}{(9/4) + b^2} + \frac{y^2}{b^2} = 1.$$

19. For $xy + x - y = 2$ we have $A = C = 0$, $B = 1$. We therefore rotate the coordinate axes (see text pages 407–408) through angle $\theta = \pi/4$.
(Thus $\cot 2\theta = 0 = (A - C)/B$.) The transformation is

$$x = \frac{1}{\sqrt{2}}(u - v), \qquad y = \frac{1}{\sqrt{2}}(u + v).$$

The given equation becomes

$$\frac{1}{2}(u^2 - v^2) + \frac{1}{\sqrt{2}}(u - v) - \frac{1}{\sqrt{2}}(u + v) = 2$$
$$u^2 - v^2 - 2\sqrt{2}v = 4$$
$$u^2 - \left(v + \sqrt{2}\right)^2 = 2$$
$$\frac{u^2}{2} - \frac{(v + \sqrt{2})^2}{2} = 1.$$

This is a rectangular hyperbola with centre $(0, -\sqrt{2})$, semi-axes $a = b = \sqrt{2}$, and eccentricity $\sqrt{2}$. The semi-focal separation is 2; the foci are at $(\pm 2, -\sqrt{2})$. The asymptotes are $u = \pm(v + \sqrt{2})$.
In terms of the original coordinates, the centre is $(1, -1)$, the foci are $(\pm\sqrt{2} + 1, \pm\sqrt{2} - 1)$, and the asymptotes are $x = 1$ and $y = -1$.

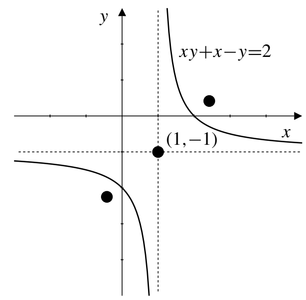

Fig. 8.1.19

20. We have $x^2 + 2xy + y^2 = 4x - 4y + 4$ and $A = 1$, $B = 2$, $C = 1$, $D = -4$, $E = 4$ and $F = -4$. We rotate the axes through angle θ satisfying $\tan 2\theta = B/(A - C) = \infty \Rightarrow \theta = \dfrac{\pi}{4}$. Then $A' = 2$, $B' = 0$, $C' = 0$, $D' = 0$, $E' = 4\sqrt{2}$ and the transformed equation is

$$2u^2 + 4\sqrt{2}v - 4 = 0 \quad \Rightarrow \quad u^2 = -2\sqrt{2}\left(v - \frac{1}{\sqrt{2}}\right)$$

which represents a parabola with vertex at $(u, v) = \left(0, \frac{1}{\sqrt{2}}\right)$ and principal axis along $u = 0$. The distance a from the focus to the vertex is given by $4a = 2\sqrt{2}$, so $a = 1/\sqrt{2}$ and the focus is at $(0, 0)$. The directrix is $v = \sqrt{2}$.
Since $x = \dfrac{1}{\sqrt{2}}(u - v)$ and $y = \dfrac{1}{\sqrt{2}}(u + v)$, the vertex of the parabola in terms of xy-coordinates is $(-\frac{1}{2}, \frac{1}{2})$, and the focus is $(0, 0)$. The directrix is $x - y = 2$. The principal axis is $y = -x$.

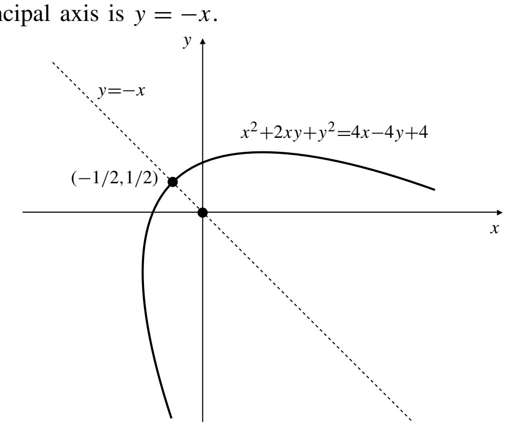

Fig. 8.1.20

21. For $8x^2 + 12xy + 17y^2 = 20$, we have $A = 8$, $B = 12$, $C = 17$, $F = -20$. Rotate the axes through angle θ where

$$\tan 2\theta = \frac{B}{A-C} = -\frac{12}{9} = -\frac{4}{3}.$$

Thus $\cos 2\theta = 3/5$, $\sin 2\theta = -4/5$, and

$$2\cos^2\theta - 1 = \cos 2\theta = \frac{3}{5} \quad\Rightarrow\quad \cos^2\theta = \frac{4}{5}.$$

We may therefore take $\cos\theta = \dfrac{2}{\sqrt{5}}$, and $\sin\theta = -\dfrac{1}{\sqrt{5}}$.
The transformation is therefore

$$x = \frac{2}{\sqrt{5}}u + \frac{1}{\sqrt{5}}v \qquad u = \frac{2}{\sqrt{5}}x - \frac{1}{\sqrt{5}}y$$
$$y = -\frac{1}{\sqrt{5}}u + \frac{2}{\sqrt{5}}v \qquad v = \frac{1}{\sqrt{5}}x + \frac{2}{\sqrt{5}}y$$

The coefficients of the transformed equation are

$$A' = 8\left(\frac{4}{5}\right) + 12\left(-\frac{2}{5}\right) + 17\left(\frac{1}{5}\right) = 5$$
$$B' = 0$$
$$C' = 8\left(\frac{1}{5}\right) - 12\left(-\frac{2}{5}\right) + 17\left(\frac{4}{5}\right) = 20.$$

The transformed equation is

$$5u^2 + 20v^2 = 20, \qquad \text{or} \qquad \frac{u^2}{4} + v^2 = 1.$$

This is an ellipse with centre $(0, 0)$, semi-axes $a = 2$ and $b = 1$, and foci at $u = \pm\sqrt{3}$, $v = 0$.
In terms of the original coordinates, the centre is $(0, 0)$, the foci are $\pm\left(\dfrac{2\sqrt{3}}{\sqrt{5}}, -\dfrac{\sqrt{3}}{\sqrt{5}}\right)$.

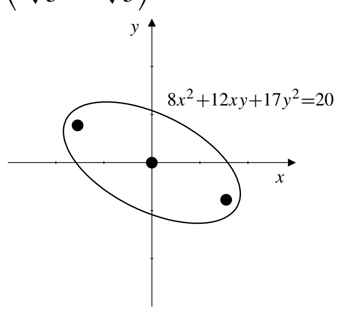

Fig. 8.1.21

22. We have $x^2 - 4xy + 4y^2 + 2x + y = 0$ and $A = 1$, $B = -4$, $C = 4$, $D = 2$, $E = 1$ and $F = 0$. We rotate the axes through angle θ satisfying $\tan 2\theta = B/(A-C) = \frac{4}{3}$. Then

$$\sec 2\theta = \sqrt{1 + \tan^2 2\theta} = \frac{5}{3} \quad\Rightarrow\quad \cos 2\theta = \frac{3}{5}$$
$$\Rightarrow \begin{cases} \cos\theta = \sqrt{\dfrac{1+\cos 2\theta}{2}} = \sqrt{\dfrac{4}{5}} = \dfrac{2}{\sqrt{5}}; \\ \sin\theta = \sqrt{\dfrac{1-\cos 2\theta}{2}} = \sqrt{\dfrac{1}{5}} = \dfrac{1}{\sqrt{5}}. \end{cases}$$

Then $A' = 0$, $B' = 0$, $C' = 5$, $D' = \sqrt{5}$, $E' = 0$ and the transformed equation is

$$5v^2 + \sqrt{5}u = 0 \quad\Rightarrow\quad v^2 = -\frac{1}{\sqrt{5}}u$$

which represents a parabola with vertex at $(u, v) = (0, 0)$, focus at $\left(-\dfrac{1}{4\sqrt{5}}, 0\right)$. The directrix is $u = \dfrac{1}{4\sqrt{5}}$ and the principal axis is $v = 0$. Since $x = \dfrac{2}{\sqrt{5}}u - \dfrac{1}{\sqrt{5}}v$ and $y = \dfrac{1}{\sqrt{5}}u + \dfrac{2}{\sqrt{5}}v$, in terms of the xy-coordinates, the vertex is at $(0, 0)$, the focus at $\left(-\dfrac{1}{10}, -\dfrac{1}{20}\right)$. The directrix is $2x + y = \frac{1}{4}$ and the principal axis is $2y - x = 0$.

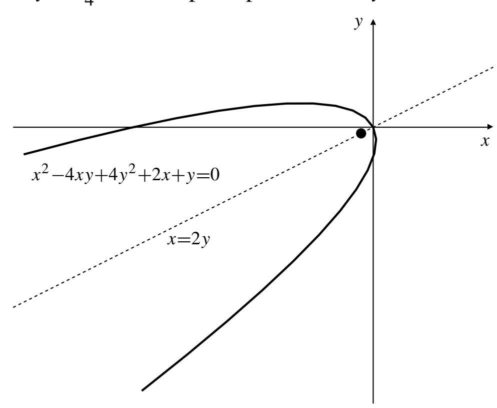

Fig. 8.1.22

23. The distance from P to F is $\sqrt{x^2 + y^2}$.
The distance from P to D is $x + p$. Thus

$$\frac{\sqrt{x^2+y^2}}{x+p} = \epsilon$$
$$x^2 + y^2 = \epsilon^2(x^2 + 2px + p^2)$$
$$(1-\epsilon^2)x^2 + y^2 - 2p\epsilon^2 x = \epsilon^2 p^2.$$

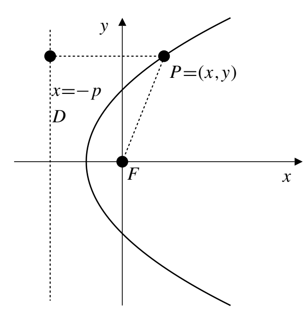

Fig. 8.1.23

24. Let the equation of the parabola be $y^2 = 4ax$. The focus F is at $(a, 0)$ and vertex at $(0, 0)$. Then the distance from the vertex to the focus is a. At $x = a$, $y = \sqrt{4a(a)} = \pm 2a$. Hence, $\ell = 2a$, which is twice the distance from the vertex to the focus.

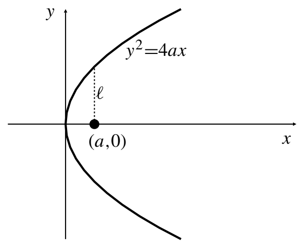

Fig. 8.1.24

25. We have $\dfrac{c^2}{a^2} + \dfrac{\ell^2}{b^2} = 1$. Thus

$$\begin{aligned}\ell^2 &= b^2\left(1 - \frac{c^2}{a^2}\right) \quad \text{but } c^2 = a^2 - b^2\\ &= b^2\left(1 - \frac{a^2 - b^2}{a^2}\right) = b^2\frac{b^2}{a^2}.\end{aligned}$$

Therefore $\ell = b^2/a$.

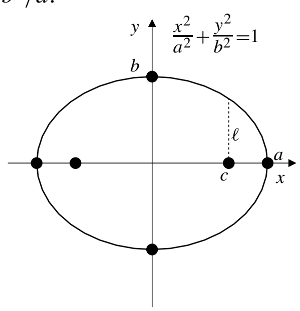

Fig. 8.1.25

26. Suppose the hyperbola has equation $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$. The vertices are at $(\pm a, 0)$ and the foci are at $(\pm c, 0)$ where $c = \sqrt{a^2 + b^2}$. At $x = \sqrt{a^2 + b^2}$,

$$\begin{aligned}\frac{a^2 + b^2}{a^2} - \frac{y^2}{b^2} &= 1\\ (a^2 + b^2)b^2 - a^2y^2 &= a^2b^2\\ y &= \pm\frac{b^2}{a}.\end{aligned}$$

Hence, $\ell = \dfrac{b^2}{a}$.

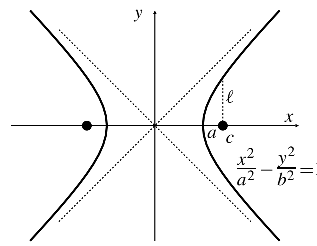

Fig. 8.1.26

27.

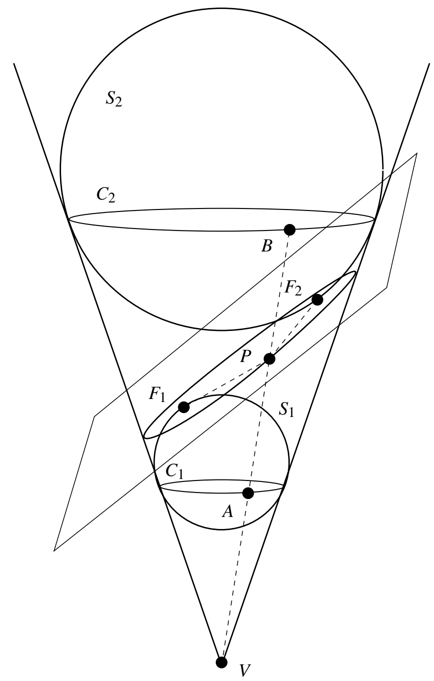

Fig. 8.1.27

Let the spheres S_1 and S_2 intersect the cone in the circles C_1 and C_2, and be tangent to the plane of the ellipse at the points F_1 and F_2, as shown in the figure.
Let P be any point on the ellipse, and let the straight line through P and the vertex of the cone meet C_1 and C_2 at A and B respectively. Then $PF_1 = PA$, since both segments are tangents to the sphere S_1 from P. Similarly, $PF_2 = PB$.
Thus $PF_1 + PF_2 = PA + PB = AB =$ constant (distance from C_1 to C_2 along all generators of the cone is the same.) Thus F_1 and F_2 are the foci of the ellipse.

28. Let F_1 and F_2 be the points where the plane is tangent to the spheres. Let P be an arbitrary point P on the hyperbola in which the plane intersects the cone. The spheres are tangent to the cone along two circles as shown in the figure. Let $PAVB$ be a generator of the cone (a straight line lying on the cone) intersecting these two circles at A and B as shown. (V is the vertex of the cone.) We have $PF_1 = PA$ because two tangents to a sphere from a point outside the sphere have equal lengths. Similarly, $PF_2 = PB$. Therefore

$$PF_2 - PF_1 = PB - PA = AB = \text{ constant,}$$

since the distance between the two circles in which the spheres intersect the cone, measured along the generators of the cone, is the same for all generators. Hence, F_1 and F_2 are the foci of the hyperbola.

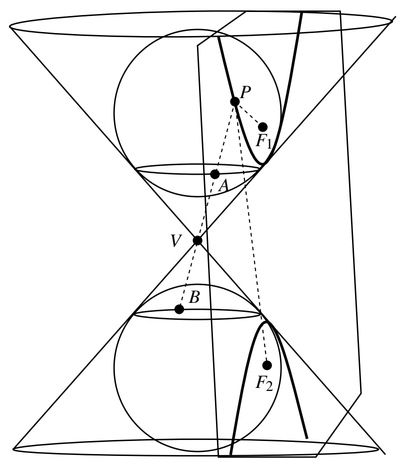

Fig. 8.1.28

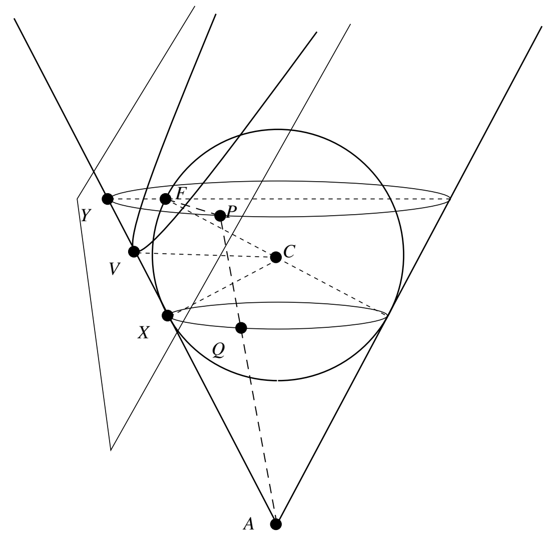

Fig. 8.1.29

29. Let the plane in which the sphere is tangent to the cone meet AV at X. Let the plane through F perpendicular to the axis of the cone meet AV at Y. Then $VF = VX$, and, if C is the centre of the sphere, $FC = XC$. Therefore VC is perpendicular to the axis of the cone. Hence YF is parallel to VC, and we have $YV = VX = VF$. If P is on the parabola, $FP \perp VF$, and the line from P to the vertex A of the cone meets the circle of tangency of the sphere and the cone at Q, then

$$FP = PQ = YX = 2VX = 2VF.$$

Since $FP = 2VF$, FP is the semi-latus rectum of the parabola. (See Exercise 18.) Therefore F is the focus of the parabola.

Section 8.2 Parametric Curves (page 449)

1. If $x = t$, $y = 1 - t$, $(0 \le t \le 1)$ then $x + y = 1$. This is a straight line segment.

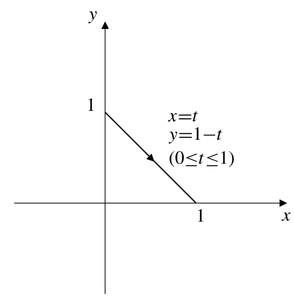

Fig. 8.2.1

2. If $x = 2 - t$ and $y = t + 1$ for $0 \le t < \infty$, then $y = 2 - x + 1 = 3 - x$ for $-\infty < x \le 2$, which is a half line.

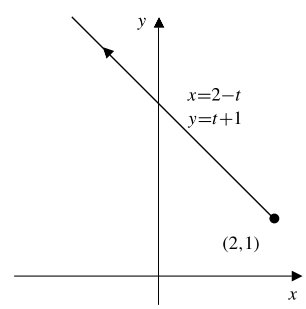

Fig. 8.2.2

3. If $x = 1/t$, $y = t - 1$, $(0 < t < 4)$, then $y = \dfrac{1}{x} - 1$. This is part of a hyperbola.

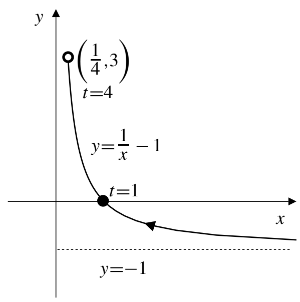

Fig. 8.2.3

4. If $x = \dfrac{1}{1+t^2}$ and $y = \dfrac{t}{1+t^2}$ for $-\infty < t < \infty$, then

$$x^2 + y^2 = \frac{1+t^2}{(1+t^2)^2} = \frac{1}{1+t^2} = x$$
$$\Leftrightarrow \quad \left(x - \frac{1}{2}\right)^2 + y^2 = \frac{1}{4}.$$

This curve consists of all points of the circle with centre at $(\frac{1}{2}, 0)$ and radius $\frac{1}{2}$ except the origin $(0, 0)$.

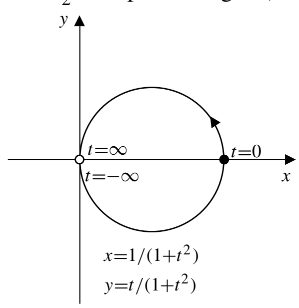

Fig. 8.2.4

5. If $x = 3\sin 2t$, $y = 3\cos 2t$, $(0 \le t \le \pi/3)$, then $x^2 + y^2 = 9$. This is part of a circle.

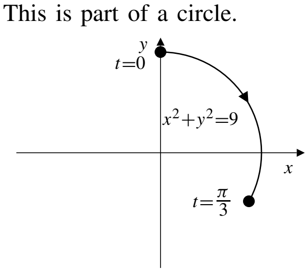

Fig. 8.2.5

6. If $x = a\sec t$ and $y = b\tan t$ for $-\dfrac{\pi}{2} < t < \dfrac{\pi}{2}$, then

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \sec^2 t - \tan^2 t = 1.$$

The curve is one arch of this hyperbola.

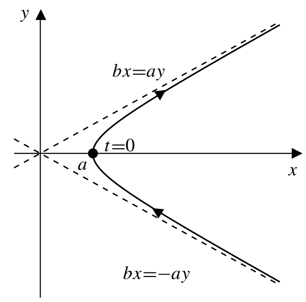

Fig. 8.2.6

7. If $x = 3\sin \pi t$, $y = 4\cos \pi t$, $(-1 \le t \le 1)$, then $\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$. This is an ellipse.

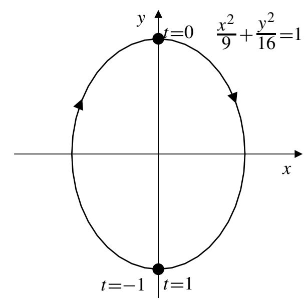

Fig. 8.2.7

8. If $x = \cos \sin s$ and $y = \sin \sin s$ for $-\infty < s < \infty$, then $x^2 + y^2 = 1$. The curve consists of the arc of this circle extending from $(a, -b)$ through $(1, 0)$ to (a, b) where $a = \cos(1)$ and $b = \sin(1)$, traversed infinitely often back and forth.

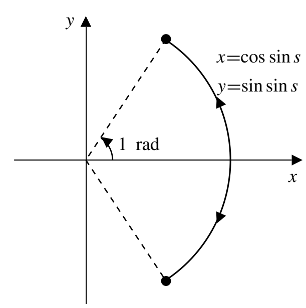

Fig. 8.2.8

9. If $x = \cos^3 t$, $y = \sin^3 t$, $(0 \le t \le 2\pi)$, then $x^{2/3} + y^{2/3} = 1$. This is an astroid.

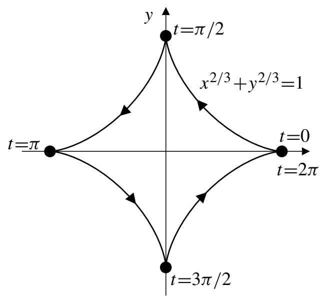

Fig. 8.2.9

10. If $x = 1 - \sqrt{4 - t^2}$ and $y = 2 + t$ for $-2 \le t \le 2$ then

$$(x-1)^2 = 4 - t^2 = 4 - (y-2)^2.$$

The parametric curve is the left half of the circle of radius 4 centred at $(1, 2)$, and is traced in the direction of increasing y.

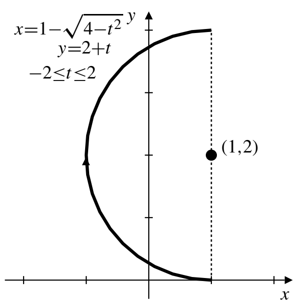

Fig. 8.2.10

11. $x = \cosh t$, $y = \sinh t$ represents the right half (branch) of the rectangular hyperbola $x^2 - y^2 = 1$.

12. $x = 2 - 3\cosh t$, $y = -1 + 2\sinh t$ represents the left half (branch) of the hyperbola

$$\frac{(x-2)^2}{9} - \frac{(y+1)^2}{4} = 1.$$

13. $x = t\cos t$, $y = t\sin t$, $(0 \le t \le 4\pi)$ represents two revolutions of a spiral curve winding outwards from the origin in a counterclockwise direction. The point on the curve corresponding to parameter value t is t units distant from the origin in a direction making angle t with the positive x-axis.

14. (i) If $x = \cos^4 t$ and $y = \sin^4 t$, then

$$\begin{aligned}(x-y)^2 &= (\cos^4 t - \sin^4 t)^2\\ &= \Big[(\cos^2 t + \sin^2 t)(\cos^2 t - \sin^2 t)\Big]^2\\ &= (\cos^2 t - \sin^2 t)^2\\ &= \cos^4 t + \sin^4 t - 2\cos^2 t\, \sin^2 t\end{aligned}$$

and

$$1 = (\cos^2 t + \sin^2 t)^2 = \cos^4 t + \sin^4 t + 2\cos^2 t\, \sin^2 t.$$

Hence,

$$1 + (x-y)^2 = 2(\cos^4 t + \sin^4 t) = 2(x+y).$$

(ii) If $x = \sec^4 t$ and $y = \tan^4 t$, then

$$\begin{aligned}(x-y)^2 &= (\sec^4 t - \tan^4 t)^2\\ &= (\sec^2 t + \tan^2 t)^2\\ &= \sec^4 t + \tan^4 t + 2\sec^2 t\, \tan^2 t\end{aligned}$$

and

$$1 = (\sec^2 t - \tan^2 t)^2 = \sec^4 t + \tan^4 t - 2\sec^2 t\, \tan^2 t.$$

Hence,

$$1 + (x-y)^2 = 2(\sec^4 t + \tan^4 t) = 2(x+y).$$

(iii) Similarly, if $x = \tan^4 t$ and $y = \sec^4 t$, then

$$\begin{aligned}1 + (x-y)^2 &= 1 + (y-x)^2\\ &= (\sec^2 t - \tan^2 t)^2 + (\sec^4 t - \tan^4 t)^2\\ &= 2(\tan^4 t + \sec^4 t)\\ &= 2(x+y).\end{aligned}$$

These three parametric curves above correspond to different parts of the parabola $1+(x-y)^2 = 2(x+y)$, as shown in the following diagram.

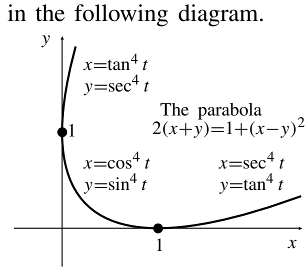

Fig. 8.2.14

15. The slope of $y = x^2$ at x is $m = 2x$. Hence the parabola can be parametrized $x = m/2$, $y = m^2/4$, $(-\infty < m < \infty)$.

16. If (x, y) is any point on the circle $x^2 + y^2 = R^2$ other than $(R, 0)$, then the line from (x, y) to $(R, 0)$ has slope $m = \dfrac{y}{x-R}$. Thus $y = m(x - R)$, and

$$\begin{aligned}&x^2 + m^2(x-R)^2 = R^2\\ &(m^2+1)x^2 - 2xRm^2 + (m^2-1)R^2 = 0\\ &\Big[(m^2+1)x - (m^2-1)R\Big](x-R) = 0\\ \Rightarrow\quad &x = \frac{(m^2-1)R}{m^2+1} \text{ or } x = R.\end{aligned}$$

The parametrization of the circle in terms of m is given by

$$x = \frac{(m^2-1)R}{m^2+1}$$
$$y = m\left[\frac{(m^2-1)R}{m^2+1} - R\right] = -\frac{2Rm}{m^2+1}$$

where $-\infty < m < \infty$. This parametrization gives every point on the circle except $(R, 0)$.

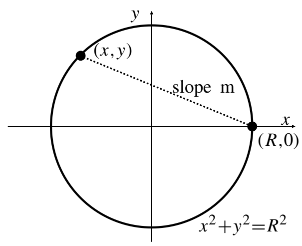

Fig. 8.2.16

17.

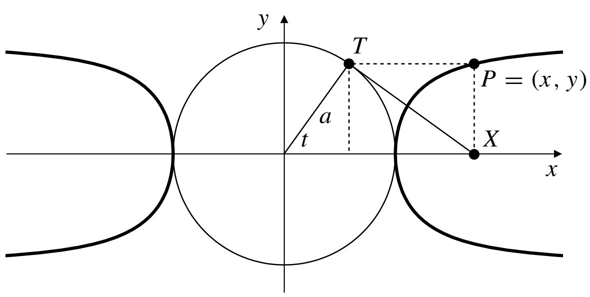

Fig. 8.2.17

Using triangles in the figure, we see that the coordinates of P satisfy

$$x = a\sec t, \quad y = a\sin t.$$

The Cartesian equation of the curve is

$$\frac{y^2}{a^2} + \frac{a^2}{x^2} = 1.$$

The curve has two branches extending to infinity to the left and right of the circle as shown in the figure.

18. The coordinates of P satisfy

$$x = a\sec t, \quad y = b\sin t.$$

The Cartesian equation is $\dfrac{y^2}{b^2} + \dfrac{a^2}{x^2} = 1.$

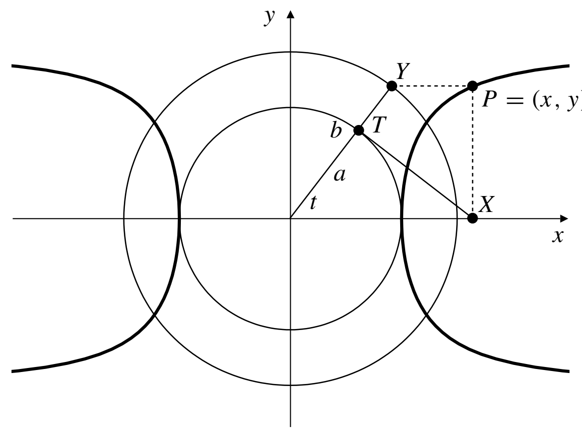

Fig. 8.2.18

19. If $x = \dfrac{3t}{1+t^3}$, $y = \dfrac{3t^2}{1+t^3}$, $(t \neq -1)$, then

$$x^3 + y^3 = \frac{27t^3}{(1+t^3)^3}(1+t^3) = \frac{27t^3}{(1+t^3)^2} = 3xy.$$

As $t \to -1$, we see that $|x| \to \infty$ and $|y| \to \infty$, but

$$x + y = \frac{3t(1+t)}{1+t^3} = \frac{3t}{1-t+t^2} \to -1.$$

Thus $x + y = -1$ is an asymptote of the curve.

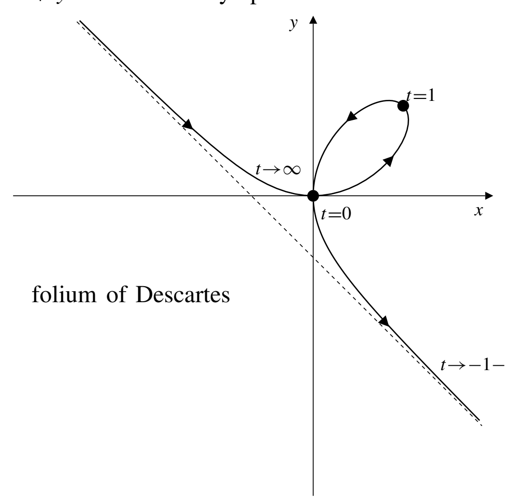

Fig. 8.2.19

20. Let C_0 and P_0 be the original positions of the centre of the wheel and a point at the bottom of the flange whose path is to be traced. The wheel is also shown in a subsequent position in which it makes contact with the rail at R. Since the wheel has been rotated by an angle θ,

$$OR = \text{arc } SR = a\theta.$$

Thus, the new position of the centre is $C = (a\theta, a)$. Let $P = (x, y)$ be the new position of the point; then

$$x = OR - PQ = a\theta - b\sin(\pi - \theta) = a\theta - b\sin\theta,$$
$$y = RC + CQ = a + b\cos(\pi - \theta) = a - b\cos\theta.$$

These are the parametric equations of the prolate cycloid.

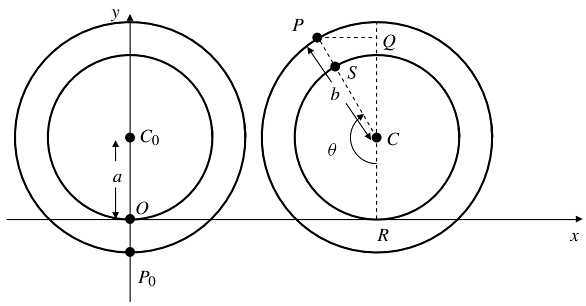

Fig. 8.2.20

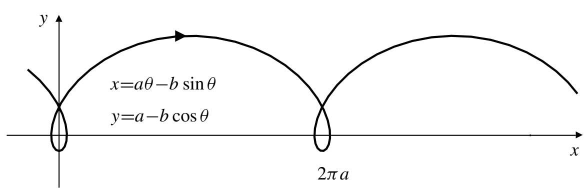

Fig. 8.2.20

21. Let t and θ_t be the angles shown in the figure below. Then arc AT_t = arc T_tP_t, that is, $at = b\theta_t$. The centre C_t of the rolling circle is $C_t = \big((a-b)\cos t, (a-b)\sin t\big)$. Thus

$$x - (a-b)\cos t = b\cos(\theta_t - t)$$
$$y - (a-b)\sin t = -b\sin(\theta_t - t).$$

Since $\theta_t - t = \dfrac{a}{b}t - t = \dfrac{a-b}{b}\,t$, therefore

$$x = (a-b)\cos t + b\cos\left(\frac{(a-b)t}{b}\right)$$
$$y = (a-b)\sin t - b\sin\left(\frac{(a-b)t}{b}\right).$$

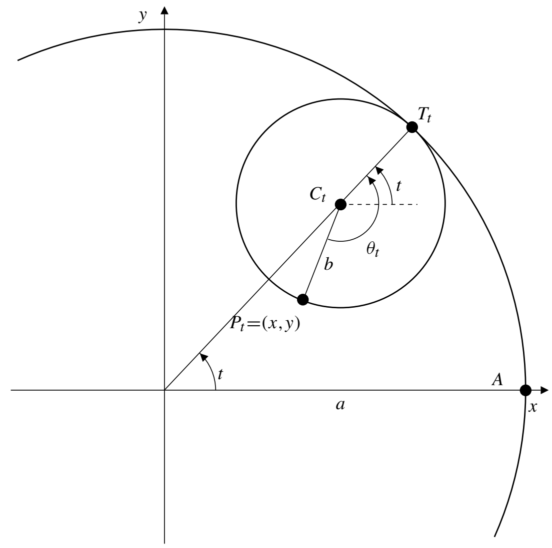

Fig. 8.2.21

If $a = 2$ and $b = 1$, then $x = 2\cos t$, $y = 0$. This is a straight line segment.
If $a = 4$ and $b = 1$, then

$$\begin{aligned}
x &= 3\cos t + \cos 3t \\
&= 3\cos t + (\cos 2t\cos t - \sin 2t\sin t) \\
&= 3\cos t + \Big((2\cos^2 t - 1)\cos t - 2\sin^2 t\cos t\Big) \\
&= 2\cos t + 2\cos^3 t - 2\cos t(1 - \sin^2 t) = 4\cos^3 t \\
y &= 3\sin t + \sin 3t \\
&= 3\sin t - \sin 2t\cos t - (\cos 2t\sin t) \\
&= 3\sin t - 2\sin t\cos^2 t - \Big((1 - 2\sin^2 t)\sin t\Big) \\
&= 2\sin t - 2\sin t + 2\sin^3 t + 2\sin^3 t = 4\sin^3 t
\end{aligned}$$

This is an astroid, similar to that of Exercise 11.

22. a) From triangles in the figure,

$$\begin{aligned}
x &= |TX| = |OT|\tan t = \tan t \\
y &= |OY| = sin\left(\tfrac{\pi}{2} - t\right) = |OY|\cos t \\
&= |OT|\cos t\cos t = \cos^2 t.
\end{aligned}$$

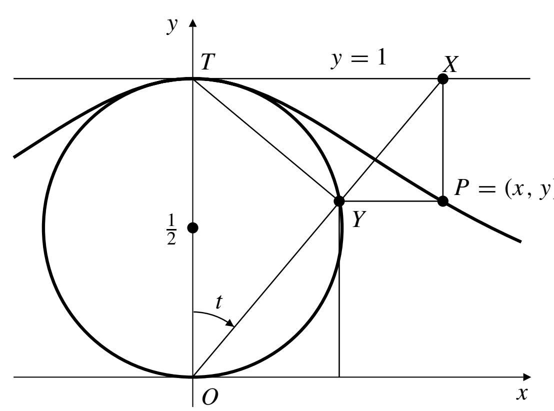

Fig. 8.2.22

b) $\frac{1}{y} = \sec^2 t = 1 + \tan^2 t = 1 + x^2$. Thus $y = \frac{1}{1+x^2}$.

23. $x = \sin t, \quad y = \sin(2t)$

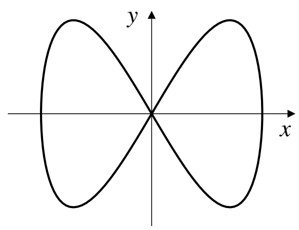

Fig. 8.2.23

24. $x = \sin t, \quad y = \sin(3t)$

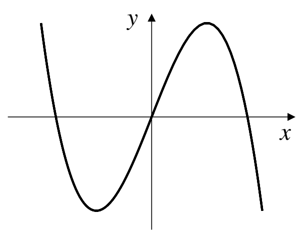

Fig. 8.2.24

25. $x = \sin(2t), \quad y = \sin(3t)$

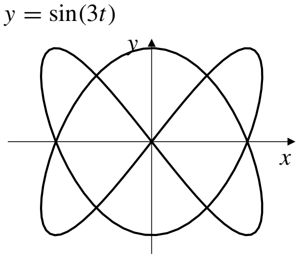

Fig. 8.2.25

26. $x = \sin(2t), \quad y = \sin(5t)$

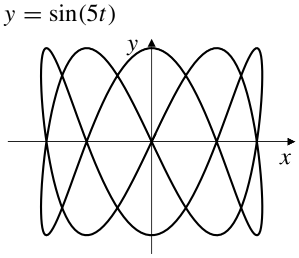

Fig. 8.2.26

27. $x = \left(1 + \frac{1}{n}\right)\cos t - \frac{1}{n}\cos(nt)$
$y = \left(1 + \frac{1}{n}\right)\sin t - \frac{1}{n}\sin(nt)$
represents a cycloid-like curve that is wound around the circle $x^2 + y^2 = 1$ instead of extending along the x-axis. If $n \geq 2$ is an integer, the curve closes after one revolution and has $n - 1$ cusps. The left figure below shows the curve for $n = 7$. If n is a rational number, the curve will wind around the circle more than once before it closes.

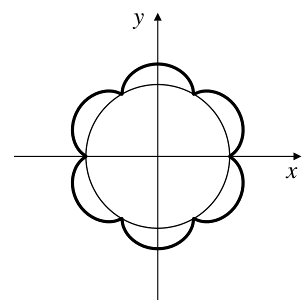

Fig. 8.2.27

28. $x = \left(1 + \frac{1}{n}\right)\cos t + \frac{1}{n}\cos((n-1)t)$
$y = \left(1 + \frac{1}{n}\right)\sin t - \frac{1}{n}\sin((n-1)t)$
represents a cycloid-like curve that is wound around the inside circle $x^2 + y^2 = \left(1 + (2/n)\right)^2$ and is externally tangent to $x^2 + y^2 = 1$. If $n \geq 2$ is an integer, the curve closes after one revolution and has n cusps. The figure shows the curve for $n = 7$. If n is a rational number but not an integer, the curve will wind around the circle more than once before it closes.

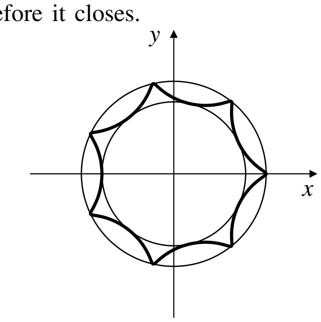

Fig. 8.2.28

Section 8.3 Smooth Parametric Curves and Their Slopes (page 453)

1. $x = t^2 + 1 \qquad y = 2t - 4$
$\frac{dx}{dt} = 2t \qquad \frac{dy}{dt} = 2$
No horizontal tangents. Vertical tangent at $t = 0$, i.e., at $(1, -4)$.

2. $x = t^2 - 2t \qquad y = t^2 + 2t$

$\frac{dx}{dt} = 2t - 2 \qquad \frac{dy}{dt} = 2t + 2$

Horizontal tangent at $t = -1$, i.e., at $(3, -1)$.
Vertical tangent at $t = 1$, i.e., at $(-1, 3)$.

3. $x = t^2 - 2t \qquad y = t^3 - 12t$

$\frac{dx}{dt} = 2(t-1) \qquad \frac{dy}{dt} = 3(t^2 - 4)$

Horizontal tangent at $t = \pm 2$, i.e., at $(0, -16)$ and $(8, 16)$.
Vertical tangent at $t = 1$, i.e., at $(-1, -11)$.

4. $x = t^3 - 3t \qquad y = 2t^3 + 3t^2$

$\frac{dx}{dt} = 3(t^2 - 1) \qquad \frac{dy}{dt} = 6t(t+1)$

Horizontal tangent at $t = 0$, i.e., at $(0, 0)$.
Vertical tangent at $t = 1$, i.e., at $(-2, 5)$.
At $t = -1$ (i.e., at $(2, 1)$) both dx/dt and dy/dt change sign, so the curve is not smooth there. (It has a cusp.)

5. $x = te^{-t^2/2} \qquad y = e^{-t^2}$

$\frac{dx}{dt} = (1 - t^2)e^{-t^2/2} \qquad \frac{dy}{dt} = -2te^{-t^2}$

Horizontal tangent at $t = 0$, i.e., at $(0, 1)$.
Vertical tangent at $t = \pm 1$, i.e. at $(\pm e^{-1/2}, e^{-1})$.

6. $x = \sin t \qquad y = \sin t - t\cos t$

$\frac{dx}{dt} = \cos t \qquad \frac{dy}{dt} = t\sin t$

Horizontal tangent at $t = n\pi$, i.e., at $(0, -(-1)^n n\pi)$ (for integers n).
Vertical tangent at $t = (n + \frac{1}{2})\pi$, i.e. at $(1, 1)$ and $(-1, -1)$.

7. $x = \sin(2t) \qquad y = \sin t$

$\frac{dx}{dt} = 2\cos(2t) \qquad \frac{dy}{dt} = \cos t$

Horizontal tangent at $t = (n + \frac{1}{2})\pi$, i.e., at $(0, \pm 1)$.
Vertical tangent at $t = \frac{1}{2}(n + \frac{1}{2})\pi$, i.e., at $(\pm 1, 1/\sqrt{2})$ and $(\pm 1, -1/\sqrt{2})$.

8. $x = \frac{3t}{1 + t^3} \qquad y = \frac{3t^2}{1 + t^3}$

$\frac{dx}{dt} = \frac{3(1 - 2t^3)}{(1 + t^3)^2} \qquad \frac{dy}{dt} = \frac{3t(2 - t^3)}{(1 + t^3)^2}$

Horizontal tangent at $t = 0$ and $t = 2^{1/3}$, i.e., at $(0, 0)$ and $(2^{1/3}, 2^{2/3})$.
Vertical tangent at $t = 2^{-1/3}$, i.e., at $(2^{2/3}, 2^{1/3})$. The curve also approaches $(0, 0)$ vertically as $t \to \pm\infty$.

9. $x = t^3 + t \qquad y = 1 - t^3$

$\frac{dx}{dt} = 3t^2 + 1 \qquad \frac{dy}{dt} = -3t^2$

At $t = 1$; $\frac{dy}{dx} = \frac{-3(1)^2}{3(1)^2 + 1} = -\frac{3}{4}$.

10. $x = t^4 - t^2 \qquad y = t^3 + 2t$

$\frac{dx}{dt} = 4t^3 - 2t \qquad \frac{dy}{dt} = 3t^2 + 2$

At $t = -1$; $\frac{dy}{dx} = \frac{3(-1)^2 + 2}{4(-1)^3 - 2(-1)} = -\frac{5}{2}$.

11. $x = \cos(2t) \qquad y = \sin t$

$\frac{dx}{dt} = -2\sin(2t) \qquad \frac{dy}{dt} = \cos t$

At $t = \frac{\pi}{6}$; $\frac{dy}{dx} = \frac{\cos(\pi/6)}{-2\sin(\pi/3)} = -\frac{1}{2}$.

12. $x = e^{2t} \qquad y = te^{2t}$

$\frac{dx}{dt} = 2e^{2t} \qquad \frac{dy}{dt} = e^{2t}(1 + 2t)$

At $t = -2$; $\frac{dy}{dx} = \frac{e^{-4}(1 - 4)}{2e^{-4}} = -\frac{3}{2}$.

13. $x = t^3 - 2t = -1 \qquad y = t + t^3 = 2 \quad \text{at } t = 1$

$\frac{dx}{dt} = 3t^2 - 2 = 1 \qquad \frac{dy}{dt} = 1 + 3t^2 = 4 \quad \text{at } t = 1$

Tangent line: $x = -1 + t$, $y = 2 + 4t$. This line is at $(-1, 2)$ at $t = 0$. If you want to be at that point at $t = 1$ instead, use

$$x = -1 + (t - 1) = t - 2, \quad y = 2 + 4(t - 1) = 4t - 2.$$

14. $x = t - \cos t = \frac{\pi}{4} - \frac{1}{\sqrt{2}}$

$\frac{dx}{dt} = 1 + \sin t = 1 + \frac{1}{\sqrt{2}}$

$y = 1 - \sin t = 1 - \frac{1}{\sqrt{2}} \quad \text{at } t = \frac{\pi}{4}$

$\frac{dy}{dt} = -\cos t = -\frac{1}{\sqrt{2}} \quad \text{at } t = \frac{\pi}{4}$

Tangent line: $x = \frac{\pi}{4} - \frac{1}{\sqrt{2}} + \left(1 + \frac{1}{\sqrt{2}}\right)t$,
$y = 1 - \frac{1}{\sqrt{2}} - \frac{t}{\sqrt{2}}$.

15. $x = t^3 - t$, $y = t^2$ is at $(0, 1)$ at $t = -1$ and $t = 1$. Since

$$\frac{dy}{dx} = \frac{2t}{3t^2 - 1} = \frac{\pm 2}{2} = \pm 1,$$

the tangents at $(0, 1)$ at $t = \pm 1$ have slopes ± 1.

16. $x = \sin t$, $y = \sin(2t)$ is at $(0, 0)$ at $t = 0$ and $t = \pi$. Since

$$\frac{dy}{dx} = \frac{2\cos(2t)}{\cos t} = \begin{cases} 2 & \text{if } t = 0 \\ -2 & \text{if } t = \pi, \end{cases}$$

the tangents at $(0, 0)$ at $t = 0$ and $t = \pi$ have slopes 2 and -2, respectively.

17. $x = t^3 \qquad y = t^2$

$\dfrac{dx}{dt} = 3t^2 \qquad \dfrac{dy}{dt} = 2t$ both vanish at $t = 0$.

$\dfrac{dy}{dx} = \dfrac{2}{3t}$ has no limit as $t \to 0$. $\dfrac{dx}{dy} = \dfrac{3t}{2} \to 0$ as $t \to 0$, but dy/dt changes sign at $t = 0$. Thus the curve is not smooth at $t = 0$. (In this solution, and in the next five, we are using the Remark following Example 2 in the text.)

18. $x = (t-1)^4$

$\dfrac{dx}{dt} = 4(t-1)^3$

$y = (t-1)^3$

$\dfrac{dy}{dt} = 3(t-1)^2$ both vanish at $t = 1$.

Since $\dfrac{dx}{dy} = \dfrac{4(t-1)}{3} \to 0$ as $t \to 1$, and dy/dt does not change sign at $t = 1$, the curve is smooth at $t = 1$ and therefore everywhere.

19. $x = t\sin t \qquad y = t^3$

$\dfrac{dx}{dt} = \sin t + t\cos t \qquad \dfrac{dy}{dt} = 3t^2$ both vanish at $t = 0$.

$\lim\limits_{t\to0} \dfrac{dy}{dx} = \lim\limits_{t\to0} \dfrac{3t^2}{\sin t + t\cos t} = \lim\limits_{t\to0} \dfrac{6t}{2\cos t - t\sin t} = 0,$
but dx/dt changes sign at $t = 0$. dx/dy has no limit at $t = 0$. Thus the curve is not smooth at $t = 0$.

20. $x = t^3 \qquad y = t - \sin t$

$\dfrac{dx}{dt} = 3t^2 \qquad \dfrac{dy}{dt} = 1 - \cos t$ both vanish at $t = 0$.

$\lim\limits_{t\to0} \dfrac{dx}{dy} = \lim\limits_{t\to0} \dfrac{3t^2}{1-\cos t} = \lim\limits_{t\to0} \dfrac{6t}{\sin t} = 6$ and dy/dt does not change sign at $t = 0$. Thus the curve is smooth at $t = 0$, and hence everywhere.

21. If $x = t^2 - 2t$ and $y = t^2 - 4t$, then

$$\frac{dx}{dt} = 2(t-1), \qquad \frac{dy}{dt} = 2(t-2)$$
$$\frac{d^2x}{dt^2} = \frac{d^2y}{dt^2} = 2$$
$$\frac{d^2y}{dx^2} = \frac{1}{dx/dt}\frac{d}{dt}\frac{dy}{dx} = \frac{1}{2(t-1)}\frac{d}{dt}\frac{t-2}{t-1} = \frac{1}{2(t-1)^3}.$$

Directional information is as follows:

	$t<1$	$1<t<2$	$t>2$
dx/dt	−	+	+
dy/dt	−	−	+
x	←	→	→
y	↓	↓	↑
curve	↙	↘	↗

The tangent is horizontal at $t = 2$, (i.e., $(0, -4)$), and is vertical at $t = 1$ (i.e., at $(-1, -3)$). Observe that $d^2y/dx^2 > 0$, and the curve is concave up, if $t > 1$. Similarly, $d^2y/dx^2 < 0$ and the curve is concave down if $t < 1$.

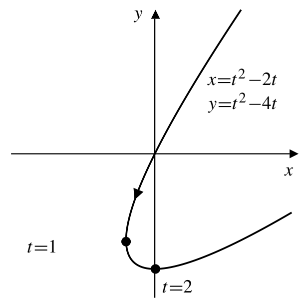

Fig. 8.3.21

22. If $x = f(t) = t^3$ and $y = g(t) = 3t^2 - 1$, then

$$f'(t) = 3t^2,\ f''(t) = 6t;$$
$$g'(t) = 6t, \quad g''(t) = 6.$$

Both $f'(t)$ and $g'(t)$ vanish at $t = 0$. Observe that

$$\frac{dy}{dx} = \frac{6t}{3t^2} = \frac{2}{t}.$$

Thus,

$$\lim_{t\to0+} \frac{dy}{dx} = \infty, \qquad \lim_{t\to0-} \frac{dy}{dx} = -\infty$$

and the curve has a cusp at $t = 0$, i.e., at $(0, -1)$. Since

$$\frac{d^2y}{dx^2} = \frac{(3t^2)(6) - (6t)(6t)}{(3t^2)^3} = -\frac{2}{3t^4} < 0$$

for all t, the curve is concave down everywhere.

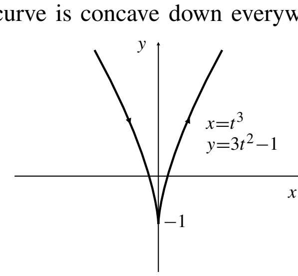

Fig. 8.3.22

23. $x = t^3 - 3t$, $y = 2/(1+t^2)$. Observe that $y \to 0$, $x \to \pm\infty$ as $t \to \pm\infty$.

$$\frac{dx}{dt} = 3(t^2-1), \qquad \frac{dy}{dt} = -\frac{4t}{(1+t^2)^2}$$
$$\frac{dy}{dx} = -\frac{4t}{3(t^2-1)(1+t^2)^2}$$
$$\frac{d^2x}{dt^2} = 6t, \qquad \frac{d^2y}{dt^2} = \frac{4(3t^2-1)}{(1+t^2)^3}$$
$$\frac{d^2y}{dx^2} = \frac{3(t^2-1)\dfrac{4(3t^2-1)}{(1+t^2)^3} - \dfrac{4t(6t)}{(1+t^2)^2}}{[3(t^2-1)]^3}$$
$$= \frac{60t^4+48t^2+12}{27(t^2-1)^3(1+t^2)^3}$$

Directional information:

t		-1		0		1	
dx/dt	$+$		$-$		$-$		$+$
dy/dt	$+$		$+$		$-$		$-$
x	$\rightarrow$		$\leftarrow$		$\leftarrow$		$\rightarrow$
y	$\uparrow$		$\uparrow$		$\downarrow$		$\downarrow$
curve	$\nearrow$		$\nwarrow$		$\swarrow$		$\searrow$

The tangent is horizontal at $t = 0$, i.e., $(0, 2)$, and vertical at $t = \pm 1$, i.e., $(\pm 2, 1)$.

t		-1		1	
$\frac{d^2y}{dx^2}$	$+$		$-$		$+$
curve	$\smile$		$\frown$		$\smile$

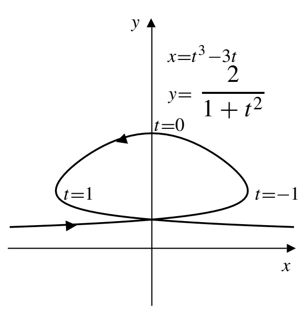

Fig. 8.3.23

24. If $x = f(t) = t^3 - 3t - 2$ and $y = g(t) = t^2 - t - 2$, then

$$f'(t) = 3t^2 - 3, \; f''(t) = 6t;$$
$$g'(t) = 2t - 1, \quad g''(t) = 2.$$

The tangent is horizontal at $t = \frac{1}{2}$, i.e., at $\left(-\frac{27}{8}, -\frac{9}{4}\right)$. The tangent is vertical at $t = \pm 1$, i.e., $(-4, -2)$ and $(0, 0)$. Directional information is as follows:

t		-1		$\frac{1}{2}$		1	
$f'(t)$	$+$		$-$		$-$		$+$
$g'(t)$	$-$		$-$		$+$		$+$
x	$\rightarrow$		$\leftarrow$		$\leftarrow$		$\rightarrow$
y	$\downarrow$		$\downarrow$		$\uparrow$		$\uparrow$
curve	$\searrow$		$\swarrow$		$\nwarrow$		$\nearrow$

For concavity,

$$\frac{d^2y}{dx^2} = \frac{3(t^2-1)(2) - (2t-1)(6t)}{[3(t^2-1)]^3} = -\frac{2(t^2-t+1)}{9(t^2-1)^3}$$

which is undefined at $t = \pm 1$, therefore

t		-1		1	
$\frac{d^2y}{dx^2}$	$-$		$+$		$-$
curve	$\frown$		$\smile$		$\frown$

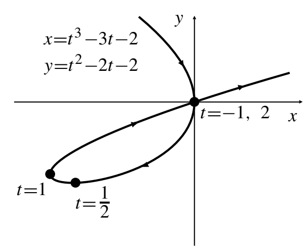

Fig. 8.3.24

25. $x = \cos t + t \sin t, \quad y = \sin t - t\cos t, \quad (t \ge 0).$

$$\frac{dx}{dt} = t\cos t, \quad \frac{dy}{dt} = t\sin t, \quad \frac{dy}{dt} = \tan t$$
$$\frac{d^2x}{dt^2} = \cos t - t\sin t$$
$$\frac{d^2y}{dt^2} = \sin t + t\cos t$$

$$\frac{d^2y}{dx^2} = \frac{\dfrac{dx}{dt}\dfrac{d^2y}{dt^2} - \dfrac{dy}{dt}\dfrac{d^2x}{dt^2}}{\left(\dfrac{dx}{dt}\right)^3}$$
$$= \frac{1}{t\cos^3 t}$$

Tangents are vertical at $t = \left(n + \frac{1}{2}\right)\pi$, and horizontal at $t = n\pi$ $(n = 0, 1, 2, \ldots)$.

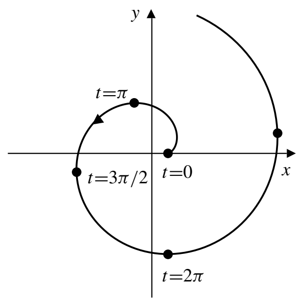

Fig. 8.3.25

Section 8.4 Arc Lengths and Areas for Parametric Curves (page 458)

1. $x = 3t^2 \qquad y = 2t^3 \quad (0 \le t \le 1)$

$\dfrac{dx}{dt} = 6t \qquad \dfrac{dy}{dt} = 6t^2$

$$\begin{aligned}
\text{Length } &= \int_0^1 \sqrt{(6t)^2 + (6t^2)^2}\,dt \\
&= 6\int_0^1 t\sqrt{1+t^2}\,dt \quad \text{Let } u = 1+t^2, \ du = 2t\,dt \\
&= 3\int_1^2 \sqrt{u}\,du = 2u^{3/2}\Big|_1^2 = 4\sqrt{2} - 2 \text{ units}
\end{aligned}$$

2. If $x = 1 + t^3$ and $y = 1 - t^2$ for $-1 \le t \le 2$, then the arc length is

$$\begin{aligned}
s &= \int_{-1}^2 \sqrt{(3t^2)^2 + (-2t)^2}\,dt \\
&= \int_{-1}^2 |t|\sqrt{9t^2+4}\,dt \\
&= \left(\int_0^1 + \int_0^2\right) t\sqrt{9t^2+4}\,dt \quad \text{Let } u = 9t^2+4, \ du = 18t\,dt \\
&= \frac{1}{18}\left(\int_4^{13} + \int_4^{40}\right)\sqrt{u}\,du \\
&= \frac{1}{27}\left(13\sqrt{13} + 40\sqrt{40} - 16\right) \text{ units.}
\end{aligned}$$

3. $x = a\cos^3 t$, $y = a\sin^3 t$, $(0 \le t \le 2\pi)$. The length is

$$\begin{aligned}
&\int_0^{2\pi} \sqrt{9a^2\cos^4 t\sin^2 t + 9a^2\sin^4 t\cos^2 t}\,dt \\
=&3a\int_0^{2\pi} |\sin t\cos t|\,dt \\
=&12a\int_0^{\pi/2} \frac{1}{2}\sin 2t\,dt \\
=&6a\left(-\frac{\cos 2t}{2}\right)\Big|_0^{\pi/2} = 6a \text{ units.}
\end{aligned}$$

4. If $x = \ln(1+t^2)$ and $y = 2\tan^{-1} t$ for $0 \le t \le 1$, then

$$\frac{dx}{dt} = \frac{2t}{1+t^2}; \qquad \frac{dy}{dt} = \frac{2}{1+t^2}.$$

The arc length is

$$\begin{aligned}
s &= \int_0^1 \sqrt{\frac{4t^2+4}{(1+t^2)^2}}\,dt \\
&= 2\int_0^1 \frac{dt}{\sqrt{1+t^2}} \quad \text{Let } t = \tan\theta, \ dt = \sec^2\theta\,d\theta \\
&= 2\int_0^{\pi/4} \sec\theta\,d\theta \\
&= 2\ln|\sec\theta + \tan\theta|\Big|_0^{\pi/4} = 2\ln(1+\sqrt{2}) \text{ units.}
\end{aligned}$$

5. $x = t^2\sin t$, $y = t^2\cos t$, $(0 \le t \le 2\pi)$.

$$\begin{aligned}
\frac{dx}{dt} &= 2t\sin t + t^2\cos t \\
\frac{dy}{dt} &= 2t\cos t - t^2\sin t \\
\left(\frac{ds}{dt}\right)^2 &= t^2\Big[4\sin^2 t + 4t\sin t\cos t + t^2\cos^2 t \\
&\qquad + 4\cos^2 t - 4t\sin t\cos t + t^2\sin^2 t\Big] \\
&= t^2(4+t^2).
\end{aligned}$$

The length of the curve is

$$\begin{aligned}
&\int_0^{2\pi} t\sqrt{4+t^2}\,dt \quad \text{Let } u = 4+t^2, \ du = 2t\,dt \\
=&\frac{1}{2}\int_4^{4+4\pi^2} u^{1/2}\,du = \frac{1}{3}u^{3/2}\Big|_4^{4+4\pi^2} \\
=&\frac{8}{3}\left((1+\pi^2)^{3/2} - 1\right) \text{ units.}
\end{aligned}$$

6. $x = \cos t + t\sin t \qquad y = \sin t - t\cos t \quad (0 \le t \le 2\pi)$

$\dfrac{dx}{dt} = t\cos t \qquad \dfrac{dy}{dt} = t\sin t$

$$\begin{aligned}
\text{Length } &= \int_0^{2\pi} \sqrt{t^2\cos^2 t + t^2\sin^2 t}\,dt \\
&= \int_0^{2\pi} t\,dt = \frac{t^2}{2}\Big|_0^{2\pi} = 2\pi^2 \text{ units.}
\end{aligned}$$

7. $x = t + \sin t \qquad y = \cos t \quad (0 \le t \le \pi)$

$\dfrac{dx}{dt} = 1 + \cos t \qquad \dfrac{dy}{dt} = -\sin t$

$$\begin{aligned}\text{Length} &= \int_0^\pi \sqrt{1+2\cos t+\cos^2 t+\sin^2 t}\,dt\\ &= \int_0^\pi \sqrt{4\cos^2(t/2)}\,dt = 2\int_0^\pi \cos\frac{t}{2}\,dt\\ &= 4\sin\frac{t}{2}\Big|_0^\pi = 4 \text{ units.}\end{aligned}$$

8. $x = \sin^2 t \qquad y = 2\cos t \quad (0 \le t \le \pi/2)$

$\dfrac{dx}{dt} = 2\sin t\cos t \qquad \dfrac{dy}{dt} = -2\sin t$

Length

$$\begin{aligned}&= \int_0^{\pi/2} \sqrt{4\sin^2 t\cos^2 t + 4\sin^2 t}\,dt\\ &= 2\int_0^{\pi/2} \sin t\sqrt{1+\cos^2 t}\,dt \quad \text{Let } \cos t = \tan u\\ &\qquad\qquad\qquad\qquad\qquad\quad -\sin t\,dt = \sec^2 u\,du\\ &= 2\int_0^{\pi/4} \sec^3 u\,du\\ &= \Big(\sec u\tan u + \ln(\sec u + \tan u)\Big)\Big|_0^{\pi/4}\\ &= \sqrt{2} + \ln(1+\sqrt{2}) \text{ units.}\end{aligned}$$

9. $x = a(t - \sin t) \qquad y = a(1-\cos t) \quad (0 \le t \le 2\pi)$

$\dfrac{dx}{dt} = a(1-\cos t) \qquad \dfrac{dy}{dt} = a\sin t$

$$\begin{aligned}\text{Length} &= \int_0^{2\pi} \sqrt{a^2(1-2\cos t+\cos^2 t+\sin^2 t)}\,dt\\ &= a\int_0^{2\pi} \sqrt{2-2\cos t}\,dt = a\int_0^{2\pi} \sqrt{\sin^2\frac{t}{2}}\,dt\\ &= 2a\int_0^{\pi} \sin\frac{t}{2}\,dt = -4a\cos\frac{t}{2}\Big|_0^\pi = 4a \text{ units.}\end{aligned}$$

10. If $x = at - a\sin t$ and $y = a - a\cos t$ for $0 \le t \le 2\pi$, then

$$\begin{aligned}&\frac{dx}{dt} = a - a\cos t, \qquad \frac{dy}{dt} = a\sin t;\\ &ds = \sqrt{(a-a\cos t)^2 + (a\sin t)^2}\,dt\\ &\quad = a\sqrt{2}\sqrt{1-\cos t}\,dt = a\sqrt{2}\sqrt{2\sin^2\left(\frac{t}{2}\right)}\,dt\\ &\quad = 2a\sin\left(\frac{t}{2}\right)\,dt.\end{aligned}$$

a) The surface area generated by rotating the arch about the x-axis is

$$\begin{aligned}S_x &= 2\pi\int_0^{2\pi} |y|\,ds\\ &= 4\pi\int_0^{\pi} (a - a\cos t)2a\sin\left(\frac{t}{2}\right)\,dt\\ &= 16\pi a^2\int_0^{\pi} \sin^3\left(\frac{t}{2}\right)\,dt\\ &= 16\pi a^2\int_0^{\pi} \left[1-\cos^2\left(\frac{t}{2}\right)\right]\sin\left(\frac{t}{2}\right)\,dt\\ &\qquad \text{Let } u = \cos\left(\frac{t}{2}\right)\\ &\qquad du = -\frac{1}{2}\sin\left(\frac{t}{2}\right)\,dt\\ &= -32\pi a^2\int_1^0 (1-u^2)\,du\\ &= 32\pi a^2\left[u - \frac{1}{3}u^3\right]\Big|_0^1\\ &= \frac{64}{3}\pi a^3 \text{ sq. units.}\end{aligned}$$

b) The surface area generated by rotating the arch about the y-axis is

$$\begin{aligned}S_y &= 2\pi\int_0^{2\pi} |x|\,ds\\ &= 2\pi\int_0^{2\pi} (at - a\sin t)2a\sin\left(\frac{t}{2}\right)\,dt\\ &= 4\pi a^2\int_0^{2\pi} \left[t - 2\sin\left(\frac{t}{2}\right)\cos\left(\frac{t}{2}\right)\right]\sin\left(\frac{t}{2}\right)\,dt\\ &= 4\pi a^2\int_0^{2\pi} t\sin\left(\frac{t}{2}\right)\,dt\\ &\quad - 8\pi a^2\int_0^{2\pi} \sin^2\left(\frac{t}{2}\right)\cos\left(\frac{t}{2}\right)\,dt\\ &= 4\pi a^2\left[-2t\cos\left(\frac{t}{2}\right)\Big|_0^{2\pi} + 2\int_0^{2\pi}\cos\left(\frac{t}{2}\right)\,dt\right] - 0\\ &= 4\pi a^2[4\pi + 0] = 16\pi^2 a^2 \text{ sq. units.}\end{aligned}$$

11. $x = e^t\cos t \qquad y = e^t\sin t \quad (0 \le t \le \pi/2)$

$\dfrac{dx}{dt} = e^t(\cos t - \sin t) \qquad \dfrac{dy}{dt} = e^t(\sin t + \cos t)$

Arc length element:

$$\begin{aligned}ds &= \sqrt{e^{2t}(\cos t - \sin t)^2 + e^{2t}(\sin t + \cos t)^2}\,dt\\ &= \sqrt{2}e^t\,dt.\end{aligned}$$

The area of revolution about the x-axis is

$$\begin{aligned}\int_{t=0}^{t=\pi/2} 2\pi y\,ds &= 2\sqrt{2}\pi \int_0^{\pi/2} e^{2t}\sin t\,dt\\ &= 2\sqrt{2}\pi\frac{e^{2t}}{5}(2\sin t-\cos t)\Big|_0^{\pi/2}\\ &= \frac{2\sqrt{2}\pi}{5}(2e^{\pi}+1)\ \text{sq. units.}\end{aligned}$$

12. The area of revolution of the curve in Exercise 11 about the y-axis is

$$\begin{aligned}\int_{t=0}^{t=\pi/2} 2\pi x\,ds &= 2\sqrt{2}\pi \int_0^{\pi/2} e^{2t}\cos t\,dt\\ &= 2\sqrt{2}\pi\frac{e^{2t}}{5}(2\cos t+\sin t)\Big|_0^{\pi/2}\\ &= \frac{2\sqrt{2}\pi}{5}(e^{\pi}-2)\ \text{sq. units.}\end{aligned}$$

13. $x = 3t^2 \qquad y = 2t^3 \quad (0 \le t \le 1)$

$\dfrac{dx}{dt} = 6t \qquad \dfrac{dy}{dt} = 6t^2$

Arc length element:

$ds = \sqrt{36(t^2+t^4)}\,dt = 6t\sqrt{1+t^2}\,dt.$

The area of revolution about the y-axis is

$$\begin{aligned}\int_{t=0}^{t=1} 2\pi x\,ds &= 36\pi\int_0^1 t^3\sqrt{1+t^2}\,dt \quad \text{Let } u=1+t^2,\ du=2t\,dt\\ &= 18\pi\int_1^2 (u-1)\sqrt{u}\,du\\ &= 18\pi\left(\frac{2}{5}u^{5/2}-\frac{2}{3}u^{3/2}\right)\Big|_1^2\\ &= \frac{72\pi}{15}(1+\sqrt{2})\ \text{sq. units.}\end{aligned}$$

14. The area of revolution of the curve of Exercise 13 about the x-axis is

$$\begin{aligned}\int_{t=0}^{t=1} 2\pi y\,ds &= 24\pi\int_0^1 t^4\sqrt{1+t^2}\,dt \quad \text{Let } t=\tan u,\ dt=\sec^2 u\,du\\ &= 24\pi\int_0^{\pi/4}\tan^4 u\sec^3 u\,du\\ &= 24\pi\int_0^{\pi/4}(\sec^7 u-2\sec^5 u+\sec^3 u)\,du\\ &= \frac{\pi}{2}\Big(7\sqrt{2}+3\ln(1+\sqrt{2})\Big)\ \text{sq. units.}\end{aligned}$$

We have omitted the details of evaluation of the final integral. See Exercise 24 of Section 8.3 for a similar evaluation.

15. $x = t^3-4t,\ y=t^2,\ (-2\le t\le 2).$

$$\begin{aligned}\text{Area } &= \int_{-2}^{2} t^2(3t^2-4)\,dt\\ &= 2\int_0^2 (3t^4-4t^2)\,dt\\ &= 2\left(\frac{3t^5}{5}-\frac{4t^3}{3}\right)\Big|_0^2 = \frac{256}{15}\ \text{sq. units.}\end{aligned}$$

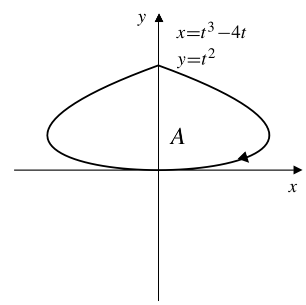

Fig. 8.4.15

16.

$$\begin{aligned}\text{Area of } R &= 4\times\int_{\pi/2}^{0}(a\sin^3 t)(-3a\sin t\cos^2 t)\,dt\\ &= -12a^2\int_{\pi/2}^{0}\sin^4 t\cos^2 t\,dt\\ &= 12a^2\left[\frac{t}{16}-\frac{\sin(4t)}{64}-\frac{\sin^3(2t)}{48}\right]\Big|_0^{\pi/2}\\ &\qquad\text{(See Exercise 34 of Section 6.4.)}\\ &= \frac{3}{8}\pi a^2\ \text{sq. units.}\end{aligned}$$

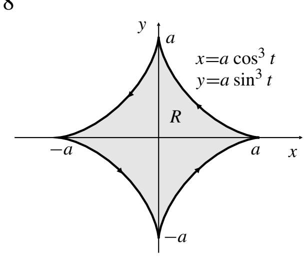

Fig. 8.4.16

17. $x=\sin^4 t,\ y=\cos^4 t,\ \left(0\le t\le\dfrac{\pi}{2}\right).$

$$\begin{aligned}\text{Area } &= \int_0^{\pi/2}(\cos^4 t)(4\sin^3 t\cos t)\,dt\\ &= 4\int_0^{\pi/2}\cos^5 t(1-\cos^2 t)\sin t\,dt \qquad \text{Let } u=\cos t,\ du=-\sin t\,dt\\ &= 4\int_0^1(u^5-u^7)\,du = 6\left(\frac{1}{6}-\frac{1}{8}\right)=\frac{1}{6}\ \text{sq. units.}\end{aligned}$$

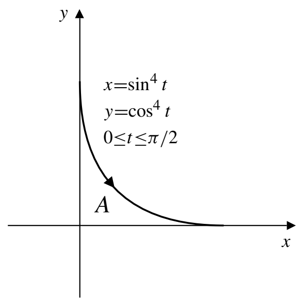

Fig. 8.4.17

18. If $x = \cos s \, \sin s = \frac{1}{2}\sin 2s$ and $y = \sin^2 s = \frac{1}{2} - \frac{1}{2}\cos 2s$ for $0 \le s \le \frac{1}{2}\pi$, then

$$x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}\sin^2 2s + \frac{1}{4}\cos^2 2s = \frac{1}{4}$$

which is the right half of the circle with radius $\frac{1}{2}$ and centre at $(0, \frac{1}{2})$. Hence, the area of R is

$$\frac{1}{2}\left[\pi\left(\frac{1}{2}\right)^2\right] = \frac{\pi}{8} \text{ sq. units.}$$

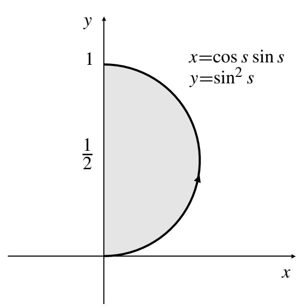

Fig. 8.4.18

19. $x = (2 + \sin t)\cos t$, $y = (2 + \sin t)\sin t$, $(0 \le t \le 2\pi)$. This is just the polar curve $r = 2 + \sin\theta$.

$$\begin{aligned}\text{Area } &= -\int_0^{2\pi} (2 + \sin t)\sin t \frac{d}{dt}\Big((2 + \sin t)\cos t\Big)\,dt \\ &= -\int_0^{2\pi} (2\sin t + \sin^2 t)(\cos^2 t - 2\sin t - \sin^2 t)\,dt \\ &= \int_0^{2\pi} \Big[4\sin^2 t + 4\sin^3 t + \sin^4 t \\ &\qquad - 2\sin t\cos^2 t - \sin^2 t\cos^2 t\Big]\,dt \\ &= \int_0^{2\pi} \Big[2(1 - \cos 2t) + \frac{1 - \cos 2t}{2}(-\cos 2t)\Big]\,dt \\ &\qquad + \int_0^{2\pi} \sin t\Big[4 - 6\cos^2 t\Big]\,dt \\ &= 4\pi + \frac{\pi}{2} + 0 = \frac{9\pi}{2} \text{ sq. units.}\end{aligned}$$

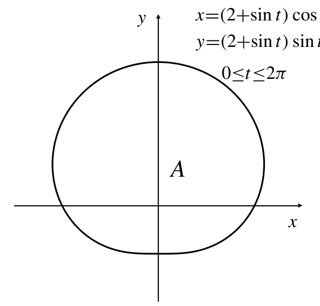

Fig. 8.4.19

20. To find the shaded area we subtract the area under the upper half of the hyperbola from that of a right triangle:

$$\begin{aligned}\text{Shaded area} &= \text{Area } \triangle ABC - \text{Area sector } ABC \\ &= \frac{1}{2}\sec t_0 \tan t_0 - \int_0^{t_0} \tan t(\sec t \tan t)\,dt \\ &= \frac{1}{2}\sec t_0 \tan t_0 - \int_0^{t_0} (\sec^3 t - \sec t)\,dt \\ &= \frac{1}{2}\sec t_0 \tan t_0 - \Bigg[\frac{1}{2}\sec t \tan t + \\ &\qquad \frac{1}{2}\ln|\sec t + \tan t| - \ln|\sec t + \tan t|\Bigg]\Bigg|_0^{t_0} \\ &= \frac{1}{2}\ln|\sec t_0 + \tan t_0| \text{ sq. units.}\end{aligned}$$

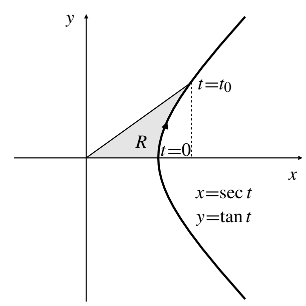

Fig. 8.4.20

21. See the figure below. The area is the area of a triangle less the area under the hyperbola:

$$\begin{aligned}A &= \frac{1}{2}\cosh t_0 \sinh t_0 - \int_0^{t_0} \sinh t \sinh t\,dt \\ &= \frac{1}{4}\sinh 2t_0 - \int_0^{t_0} \frac{\cosh 2t - 1}{2}\,dt \\ &= \frac{1}{4}\sinh 2t_0 - \frac{1}{4}\sinh 2t_0 + \frac{1}{2}t_0 \\ &= \frac{t_0}{2} \text{ sq. units.}\end{aligned}$$

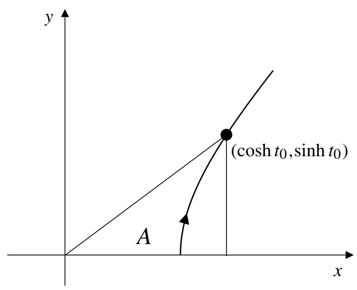

Fig. 8.4.21

22. If $x = f(t) = at - a\sin t$ and $y = g(t) = a - a\cos t$, then the volume of the solid obtained by rotating about the x-axis is

$$\begin{aligned} V &= \int_{t=0}^{t=2\pi} \pi y^2\,dx = \pi\int_{t=0}^{t=2\pi}[g(t)]^2 f'(t)\,dt \\ &= \pi\int_0^{2\pi}(a-a\cos t)^2(a-a\cos t)\,dt \\ &= \pi a^3\int_0^{2\pi}(1-\cos t)^3\,dt \\ &= \pi a^3\int_0^{2\pi}(1-3\cos t+3\cos^2 t-\cos^3 t)\,dt \\ &= \pi a^3\left[2\pi - 0 + \frac{3}{2}\int_0^{2\pi}(1+\cos 2t)\,dt - 0\right] \\ &= \pi a^3\left[2\pi + \frac{3}{2}(2\pi)\right] = 5\pi^2 a^3 \text{ cu. units.} \end{aligned}$$

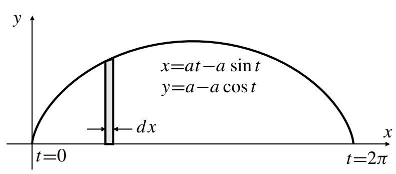

Fig. 8.4.22

23. Half of the volume corresponds to rotating $x = a\cos^3 t$, $y = a\sin^3 t$ $(0 \le t \le \pi/2)$ about the x-axis. The whole volume is

$$\begin{aligned} V &= 2\int_0^{\pi/2}\pi y^2\,(-dx) \\ &= 2\pi\int_0^{\pi/2} a^2\sin^6 t\,(3a\cos^2 t\sin t)\,dt \\ &= 6\pi a^3\int_0^{\pi/2}(1-\cos^2 t)^3\cos^2 t\sin t\,dt \quad \text{Let } u=\cos t,\ du = -\sin t\,dt \\ &= 6\pi a^3\int_0^1(1-3u^2+3u^4-u^6)u^2\,du \\ &= 6\pi a^3\left(\frac{1}{3}-\frac{3}{5}+\frac{3}{7}-\frac{1}{9}\right) = \frac{32\pi a^3}{105} \text{ cu. units.} \end{aligned}$$

Section 8.5 Polar Coordinates and Polar Curves (page 464)

1. $r = 3\sec\theta$
$r\cos\theta = 3$
$x = 3$ vertical straight line.

2. $r = -2\csc\theta \Rightarrow r\sin\theta = -2$
$\Leftrightarrow\ y = -2$ a horizontal line.

3. $r = 5/(3\sin\theta - 4\cos\theta)$
$3r\sin\theta - 4r\cos\theta = 5$
$3y - 4x = 5$ straight line.

4. $r = \sin\theta + \cos\theta$
$r^2 = r\sin\theta + r\cos\theta$
$x^2 + y^2 = y + x$
$\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{2}$
a circle with centre $\left(\frac{1}{2}, \frac{1}{2}\right)$ and radius $\frac{1}{\sqrt{2}}$.

5. $r^2 = \csc 2\theta$
$r^2\sin 2\theta = 1$
$2r^2\sin\theta\cos\theta = 1$
$2xy = 1$ a rectangular hyperbola.

6. $r = \sec\theta\,\tan\theta \Rightarrow r\cos\theta = \dfrac{r\sin\theta}{r\cos\theta}$
$x^2 = y$ a parabola.

7. $r = \sec\theta(1 + \tan\theta)$
$r\cos\theta = 1 + \tan\theta$
$x = 1 + \dfrac{y}{x}$
$x^2 - x - y = 0$ a parabola.

8. $r = \dfrac{2}{\sqrt{\cos^2\theta + 4\sin^2\theta}}$
$r^2\cos^2\theta + 4r^2\sin^2\theta = 4$
$x^2 + 4y^2 = 4$ an ellipse.

9. $r = \dfrac{1}{1 - \cos\theta}$
$r - x = 1$
$r^2 = (1 + x)^2$
$x^2 + y^2 = 1 + 2x + x^2$
$y^2 = 1 + 2x$ a parabola.

10. $r = \dfrac{2}{2 - \cos\theta}$

$2r - r\cos\theta = 2$

$4r^2 = (2 + x)^2$

$4x^2 + 4y^2 = 4 + 4x + x^2$

$3x^2 + 4y^2 - 4x = 4$ an ellipse.

11. $r = \dfrac{2}{1 - 2\sin\theta}$

$r - 2y = 2$

$x^2 + y^2 = r^2 = 4(1 + y)^2 = 4 + 8y + 4y^2$

$x^2 - 3y^2 - 8y = 4$ a hyperbola.

12. $r = \dfrac{2}{1 + \sin\theta}$

$r + r\sin\theta = 2$

$r^2 = (2 - y)^2$

$x^2 + y^2 = 4 - 4y + y^2$

$x^2 = 4 - 4y$ a parabola.

13. $r = 1 + \sin\theta$ (cardioid)

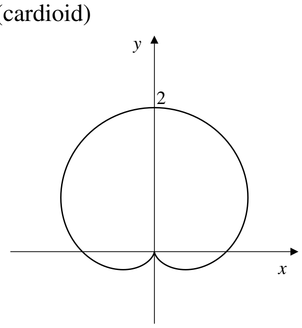

Fig. 8.5.13

14. If $r = 1 - \cos\left(\theta + \dfrac{\pi}{4}\right)$, then $r = 0$ at $\theta = -\dfrac{\pi}{4}$ and $\dfrac{7\pi}{4}$. This is a cardioid.

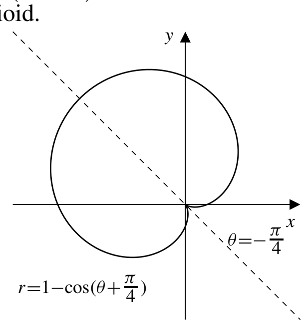

Fig. 8.5.14

15. $r = 1 + 2\cos\theta$

$r = 0$ if $\theta = \pm 2\pi/3$.

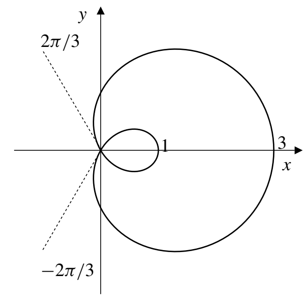

Fig. 8.5.15

16. If $r = 1 - 2\sin\theta$, then $r = 0$ at $\theta = \dfrac{\pi}{6}$ and $\dfrac{5\pi}{6}$.

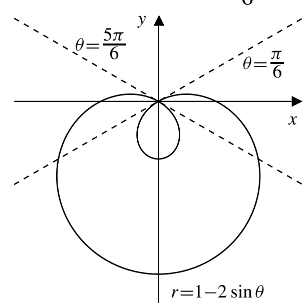

Fig. 8.5.16

17. $r = 2 + \cos\theta$

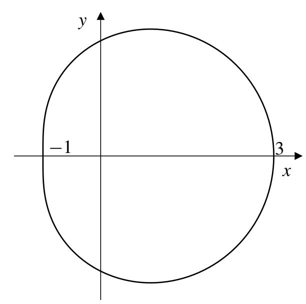

Fig. 8.5.17

18. If $r = 2\sin 2\theta$, then $r = 0$ at $\theta = 0,\ \pm\dfrac{\pi}{2}$ and π.

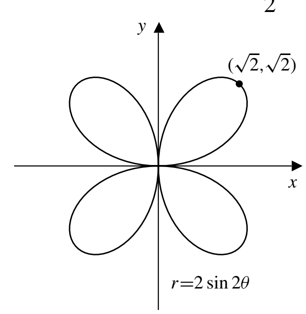

Fig. 8.5.18

19. $r = \cos 3\theta$ (three leaf rosette)

$r = 0$ at $\theta = \pm\pi/6,\ \pm\pi/2,\ \pm 5\pi/6$.

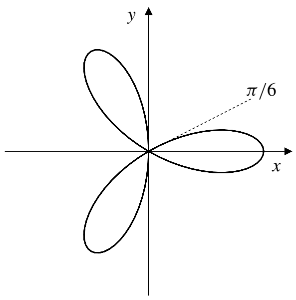

Fig. 8.5.19

20. If $r = 2\cos 4\theta$, then $r = 0$ at $\theta = \pm\dfrac{\pi}{8}$, $\pm\dfrac{3\pi}{8}$, $\pm\dfrac{5\pi}{8}$ and $\pm\dfrac{7\pi}{8}$. (an eight leaf rosette)

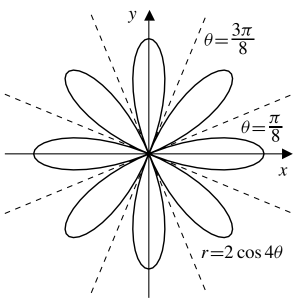

Fig. 8.5.20

21. $r^2 = 4\sin 2\theta$. Thus $r = \pm 2\sqrt{\sin 2\theta}$. This is a lemniscate. $r = 0$ at $\theta = 0$, $\theta = \pm\pi/2$, and $\theta = \pi$.

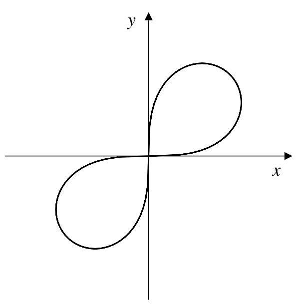

Fig. 8.5.21

22. If $r^2 = 4\cos 3\theta$, then $r = 0$ at $\theta = \pm\dfrac{\pi}{6}$, $\pm\dfrac{\pi}{2}$ and $\pm\dfrac{5\pi}{6}$. This equation defines two functions of r, namely $r = \pm 2\sqrt{\cos 3\theta}$. Each contributes 3 leaves to the graph.

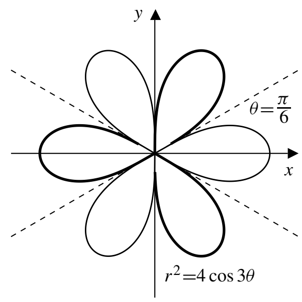

Fig. 8.5.22

23. $r^2 = \sin 3\theta$. Thus $r = \pm\sqrt{\sin 3\theta}$. This is a lemniscate. $r = 0$ at $\theta = 0$, $\pm\pi/3$, $\pm 2\pi/3$, π.

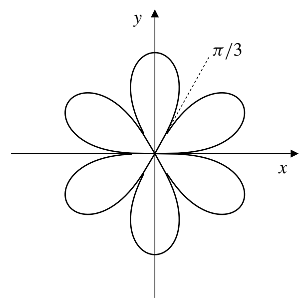

Fig. 8.5.23

24. If $r = \ln\theta$, then $r = 0$ at $\theta = 1$. Note that

$$y = r\sin\theta = \ln\theta\sin\theta = (\theta\ln\theta)\left(\frac{\sin\theta}{\theta}\right) \to 0$$

as $\theta \to 0+$. Therefore, the (negative) x-axis is an asymptote of the curve.

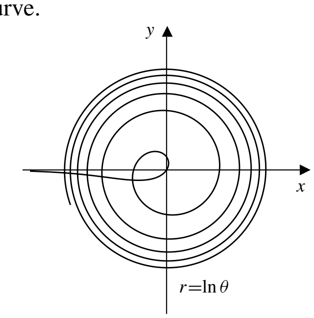

Fig. 8.5.24

25. $r = \sqrt{3}\cos\theta$, and $r = \sin\theta$ both pass through the origin, and so intersect there. Also
$\sin\theta = \sqrt{3}\cos\theta \quad\Rightarrow\quad \tan\theta = \sqrt{3} \quad\Rightarrow\quad \theta = \pi/3, \quad 4\pi/3.$
Both of these give the same point $[\sqrt{3}/2, \pi/3]$.
Intersections: the origin and $[\sqrt{3}/2, \pi/3]$.

26. $r^2 = 2\cos(2\theta)$, $r = 1$.
$\cos(2\theta) = 1/2 \quad\Rightarrow\quad \theta = \pm\pi/6$ or $\theta = \pm 5\pi/6$.
Intersections: $[1, \pm\pi/6]$ and $[1, \pm 5\pi/6]$.

27. $r = 1 + \cos\theta$, $r = 3\cos\theta$. Both curves pass through the origin, so intersect there. Also
$3\cos\theta = 1 + \cos\theta \quad\Rightarrow\quad \cos\theta = 1/2 \quad\Rightarrow\quad \theta = \pm\pi/3$.
Intersections: the origin and $[3/2, \pm\pi/3]$.

28. Let $r_1(\theta) = \theta$ and $r_2(\theta) = \theta + \pi$. Although the equation $r_1(\theta) = r_2(\theta)$ has no solutions, the curves $r = r_1(\theta)$ and $r = r_2(\theta)$ can still intersect if $r_1(\theta_1) = -r_2(\theta_2)$ for two angles θ_1 and θ_2 having the opposite directions in the polar plane. Observe that $\theta_1 = -n\pi$ and $\theta_2 = (n-1)\pi$ are two such angles provided n is any integer. Since

$$r_1(\theta_1) = -n\pi = -r_2((n-1)\pi),$$

the curves intersect at any point of the form $[n\pi, 0]$ or $[n\pi, \pi]$.

29. If $r = 1/\theta$ for $\theta > 0$, then

$$\lim_{\theta\to 0+} y = \lim_{\theta\to 0+} \frac{\sin\theta}{\theta} = 1.$$

Thus $y = 1$ is a horizontal asymptote.

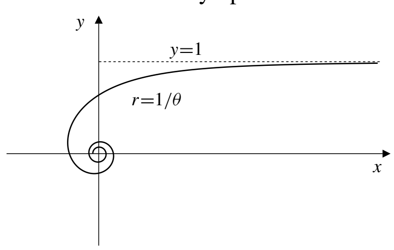

Fig. 8.5.29

30. The graph of $r = \cos n\theta$ has $2n$ leaves if n is an even integer and n leaves if n is an odd integer. The situation for $r^2 = \cos n\theta$ is reversed. The graph has $2n$ leaves if n is an odd integer (provided negative values of r are allowed), and it has n leaves if n is even.

31. If $r = f(\theta)$, then

$$x = r\cos\theta = f(\theta)\cos\theta$$
$$y = r\sin\theta = f(\theta)\sin\theta.$$

32. $r = \cos\theta\cos(m\theta)$
For odd m this flower has $2m$ petals, 2 large ones and 4 each of $(m-1)/2$ smaller sizes.
For even m the flower has $m+1$ petals, one large and 2 each of $m/2$ smaller sizes.

33. $r = 1 + \cos\theta\cos(m\theta)$
These are similar to the ones in Exercise 32, but the curve does not approach the origin except for $\theta = \pi$ in the case of even m. The petals are joined, and less distinct. The smaller ones cannot be distinguished.

34. $r = \sin(2\theta)\sin(m\theta)$
For odd m there are $m+1$ petals, 2 each of $(m+1)/2$ different sizes.
For even m there are always $2m$ petals. They are of n different sizes if $m = 4n-2$ or $m = 4n$.

35. $r = 1 + \sin(2\theta)\sin(m\theta)$
These are similar to the ones in Exercise 34, but the petals are joined, and less distinct. The smaller ones cannot be distinguished. There appear to be $m+2$ petals in both the even and odd cases.

36. $r = C + \cos\theta\cos(2\theta)$
The curve always has 3 bulges, one larger than the other two. For $C = 0$ these are 3 distinct petals. For $0 < C < 1$ there is a fourth supplementary petal inside the large one. For $C = 1$ the curve has a cusp at the origin. For $C > 1$ the curve does not approach the origin, and the petals become less distinct as C increases.

37. $r = C + \cos\theta\sin(3\theta)$
For $C < 1$ there appear to be 6 petals of 3 different sizes. For $C \ge 1$ there are only 4 of 2 sizes, and these coalesce as C increases.

38.

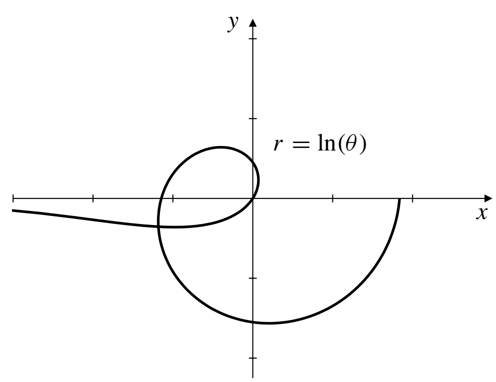

Fig. 8.5.38

We will have $[\ln\theta_1, \theta_1] = [\ln\theta_2, \theta_2]$ if

$$\theta_2 = \theta_1 + \pi \quad\text{and}\quad \ln\theta_1 = -\ln\theta_2,$$

that is, if $\ln\theta_1 + \ln(\theta_1 + \pi) = 0$. This equation has solution $\theta_1 \approx 0.29129956$. The corresponding intersection point has Cartesian coordinates $(\ln\theta_1\cos\theta_1, \ln\theta_1\sin\theta_1) \approx (-1.181442, -0.354230)$.

39.

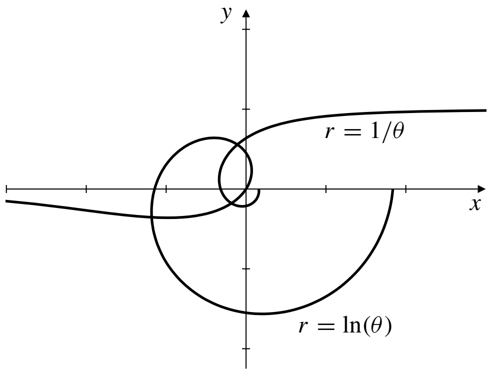

Fig. 8.5.39

The two intersections of $r = \ln\theta$ and $r = 1/\theta$ for $0 < \theta \le 2\pi$ correspond to solutions θ_1 and θ_2 of

$$\ln\theta_1 = \frac{1}{\theta_1}, \quad \ln\theta_2 = -\frac{1}{\theta_2 + \pi}.$$

The first equation has solution $\theta_1 \approx 1.7632228$, giving the point $(-0.108461, 0.556676)$, and the second equation has solution $\theta_2 \approx 0.7746477$, giving the point $(-0.182488, -0.178606)$.

Section 8.6 Slopes, Areas, and Arc Lengths for Polar Curves (page 468)

1. Area $= \frac{1}{2}\int_0^{2\pi} \theta\, d\theta = \frac{(2\pi)^2}{4} = \pi^2$.

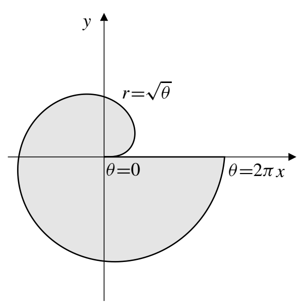

Fig. 8.6.1

2. Area $= \frac{1}{2}\int_0^{2\pi} \theta^2\, d\theta = \left.\frac{\theta^3}{6}\right|_0^{2\pi} = \frac{4}{3}\pi^3$ sq. units.

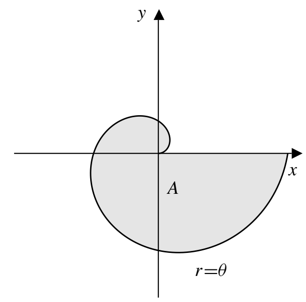

Fig. 8.6.2

3. Area $= 4 \times \frac{1}{2}\int_0^{\pi/4} a^2 \cos 2\theta\, d\theta$

$$= 2a^2 \left.\frac{\sin 2\theta}{2}\right|_0^{\pi/4} = a^2 \text{ sq. units.}$$

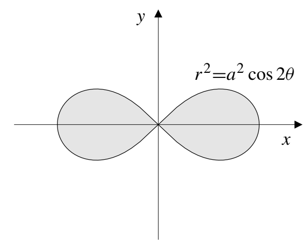

Fig. 8.6.3

4. Area $= \frac{1}{2}\int_0^{\pi/3} \sin^2 3\theta\, d\theta = \frac{1}{4}\int_0^{\pi/3} (1 - \cos 6\theta)\, d\theta$

$$= \frac{1}{4}\left.\left(\theta - \frac{1}{6}\sin 6\theta\right)\right|_0^{\pi/3} = \frac{\pi}{12} \text{ sq. units.}$$

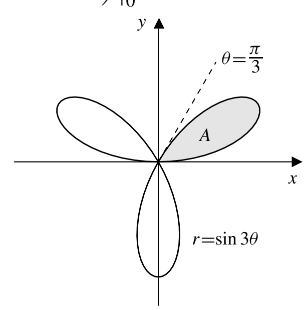

Fig. 8.6.4

5. Total area $= 16 \times \frac{1}{2}\int_0^{\pi/8} \cos^2 4\theta\, d\theta$

$$= 4\int_0^{\pi/8} (1 + \cos 8\theta)\, d\theta$$

$$= 4\left.\left(\theta + \frac{\sin 8\theta}{8}\right)\right|_0^{\pi/8} = \frac{\pi}{2} \text{ sq. units.}$$

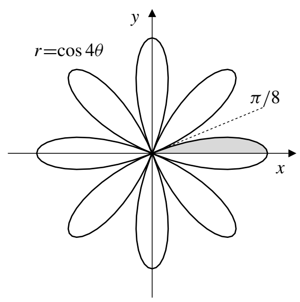

Fig. 8.6.5

6. The circles $r = a$ and $r = 2a\cos\theta$ intersect at $\theta = \pm\pi/3$. By symmetry, the common area is $4\times$(area of sector − area of right triangle) (see the figure), i.e.,

$$4\times\left[\left(\frac{1}{6}\pi a^2\right)-\left(\frac{1}{2}\frac{a}{2}\frac{\sqrt{3}a}{2}\right)\right]=\frac{4\pi-3\sqrt{3}}{6}a^2 \text{ sq. units.}$$

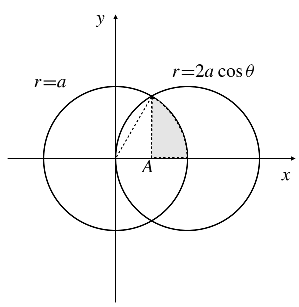

Fig. 8.6.6

7.
$$\begin{aligned}\text{Area} &= 2\times\frac{1}{2}\int_{\pi/2}^{\pi}(1-\cos\theta)^2\,d\theta-\frac{\pi}{2}\\ &=\int_{\pi/2}^{\pi}\left(1-2\cos\theta+\frac{1+\cos 2\theta}{2}\right)d\theta-\frac{\pi}{2}\\ &=\frac{3}{2}\left(\pi-\frac{\pi}{2}\right)-\left(2\sin\theta-\frac{\sin 2\theta}{4}\right)\Big|_{\pi/2}^{\pi}-\frac{\pi}{2}\\ &=\frac{\pi}{4}+2 \text{ sq. units.}\end{aligned}$$

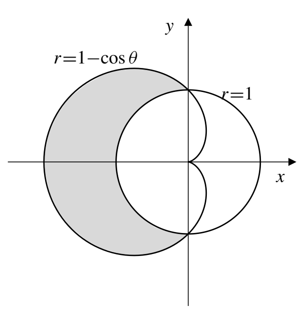

Fig. 8.6.7

8.
$$\begin{aligned}\text{Area} &= \frac{1}{2}\pi a^2+2\times\frac{1}{2}\int_0^{\pi/2}a^2(1-\sin\theta)^2\,d\theta\\ &=\frac{\pi a^2}{2}+a^2\int_0^{\pi/2}\left(1-2\sin\theta+\frac{1-\cos 2\theta}{2}\right)d\theta\\ &=\frac{\pi a^2}{2}+a^2\left(\frac{3}{2}\theta+2\cos\theta-\frac{1}{4}\sin 2\theta\right)\Big|_0^{\pi/2}\\ &=\left(\frac{5\pi}{4}-2\right)a^2 \text{ sq. units.}\end{aligned}$$

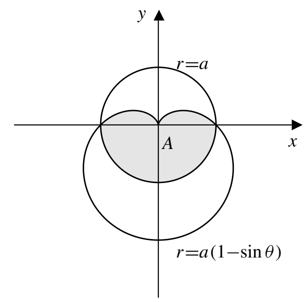

Fig. 8.6.8

9. For intersections: $1+\cos\theta = 3\cos\theta$. Thus $2\cos\theta = 1$ and $\theta = \pm\pi/3$. The shaded area is given by

$$\begin{aligned}&2\times\frac{1}{2}\left[\int_{\pi/3}^{\pi}(1+\cos\theta)^2\,d\theta-9\int_{\pi/3}^{\pi/2}\cos^2\theta\,d\theta\right]\\ &\quad=\int_{\pi/3}^{\pi}\left(1+2\cos\theta+\frac{1+\cos 2\theta}{2}\right)d\theta\\ &\qquad-\frac{9}{2}\int_{\pi/3}^{\pi/2}(1+\cos 2\theta)\,d\theta\\ &\quad=\frac{3}{2}\left(\frac{2\pi}{3}\right)+\left(2\sin\theta+\frac{\sin 2\theta}{4}\right)\Big|_{\pi/3}^{\pi}\\ &\qquad-\frac{9}{2}\left(\theta+\frac{\sin 2\theta}{2}\right)\Big|_{\pi/3}^{\pi/2}\\ &\quad=\frac{\pi}{4}-\sqrt{3}-\frac{\sqrt{3}}{8}+\frac{9}{4}\left(\frac{\sqrt{3}}{2}\right)=\frac{\pi}{4} \text{ sq. units.}\end{aligned}$$

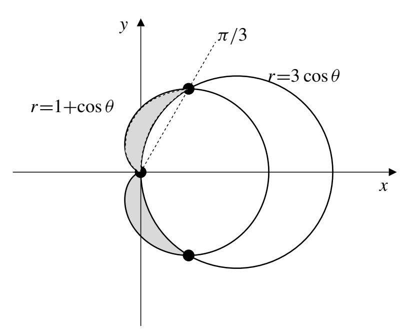

Fig. 8.6.9

10. Since $r^2 = 2\cos 2\theta$ meets $r = 1$ at $\theta = \pm\frac{\pi}{6}$ and $\pm\frac{5\pi}{6}$, the area inside the lemniscate and outside the circle is

$$4 \times \frac{1}{2}\int_0^{\pi/6}\Big[2\cos 2\theta - 1^2\Big]\,d\theta$$
$$= 2\sin 2\theta\Big|_0^{\pi/6} - \frac{\pi}{3} = \sqrt{3} - \frac{\pi}{3} \text{ sq. units.}$$

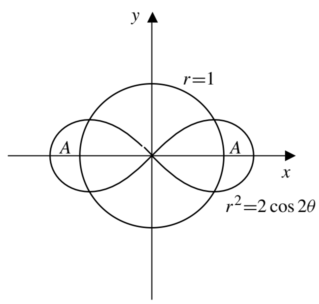

Fig. 8.6.10

11. $r = 0$ at $\theta = \pm 2\pi/3$. The shaded area is

$$2 \times \frac{1}{2}\int_{2\pi/3}^{\pi}(1 + 2\cos\theta)^2\,d\theta$$
$$= \int_{2\pi/3}^{\pi}\Big(1 + 4\cos\theta + 2(1 + \cos 2\theta)\Big)\,d\theta$$
$$= 3\left(\frac{\pi}{3}\right) + 4\sin\theta\Big|_{2\pi/3}^{\pi} + \sin 2\theta\Big|_{2\pi/3}^{\pi}$$
$$= \pi - 2\sqrt{3} + \frac{\sqrt{3}}{2} = \pi - \frac{3\sqrt{3}}{2} \text{ sq. units.}$$

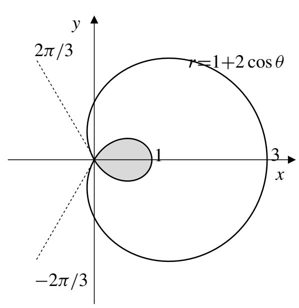

Fig. 8.6.11

12. $$s = \int_0^{\pi}\sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\,d\theta = \int_0^{\pi}\sqrt{4\theta^2 + \theta^4}\,d\theta$$
$$= \int_0^{\pi}\theta\sqrt{4 + \theta^2}\,d\theta \quad \text{Let } u = 4 + \theta^2,\ du = 2\theta\,d\theta$$
$$= \frac{1}{2}\int_4^{4+\pi^2}\sqrt{u}\,du = \frac{1}{3}u^{3/2}\Big|_4^{4+\pi^2}$$
$$= \frac{1}{3}\Big[(4 + \pi^2)^{3/2} - 8\Big] \text{ units.}$$

13. $r = e^{a\theta}$, $(-\pi \le \theta \le \pi)$. $\frac{dr}{d\theta} = ae^{a\theta}$.
$ds = \sqrt{e^{2a\theta} + a^2e^{2a\theta}}\,d\theta = \sqrt{1 + a^2}e^{a\theta}\,d\theta$. The length of the curve is

$$\int_{-\pi}^{\pi}\sqrt{1 + a^2}e^{a\theta}\,d\theta = \frac{\sqrt{1 + a^2}}{a}(e^{a\pi} - e^{-a\pi}) \text{ units.}$$

14. $$s = \int_0^{2\pi}\sqrt{a^2 + a^2\theta^2}\,d\theta$$
$$= a\int_0^{2\pi}\sqrt{1 + \theta^2}\,d\theta \quad \text{Let } \theta = \tan u,\ d\theta = \sec^2 u\,d\theta$$
$$= a\int_{\theta=0}^{\theta=2\pi}\sec^3 u\,du$$
$$= \frac{a}{2}\Big(\sec u\,\tan u + \ln|\sec u + \tan u|\Big)\Big|_{\theta=0}^{\theta=2\pi}$$
$$= \frac{a}{2}\Big[\theta\sqrt{1 + \theta^2} + \ln|\sqrt{1 + \theta^2} + \theta|\Big]\Big|_{\theta=0}^{\theta=2\pi}$$
$$= \frac{a}{2}\Big[2\pi\sqrt{1 + 4\pi^2} + \ln(2\pi + \sqrt{1 + 4\pi^2})\Big] \text{ units.}$$

15. $r^2 = \cos 2\theta$

$$2r\frac{dr}{d\theta} = -2\sin 2\theta \quad \Rightarrow \quad \frac{dr}{d\theta} = -\frac{\sin 2\theta}{r}$$
$$ds = \sqrt{\cos 2\theta + \frac{\sin^2 2\theta}{\cos 2\theta}}\,d\theta = \sqrt{\sec 2\theta}\,d\theta$$
$$\text{Length } = 4\int_0^{\pi/4}\sqrt{\sec 2\theta}\,d\theta.$$

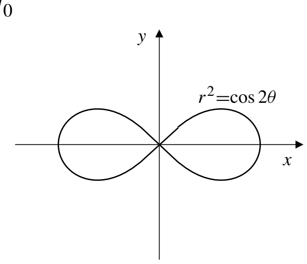

Fig. 8.6.15

16. If $r^2 = \cos 2\theta$, then

$$2r\frac{dr}{d\theta} = -2\sin 2\theta \Rightarrow \frac{dr}{d\theta} = -\frac{\sin 2\theta}{\sqrt{\cos 2\theta}}$$

and

$$ds = \sqrt{\cos 2\theta + \frac{\sin^2 2\theta}{\cos 2\theta}}\,d\theta = \frac{d\theta}{\sqrt{\cos 2\theta}}.$$

a) Area of the surface generated by rotation about the x- axis is

$$\begin{aligned} S_x &= 2\pi \int_0^{\pi/4} r\sin\theta\, ds \\ &= 2\pi \int_0^{\pi/4} \sqrt{\cos 2\theta}\, \sin\theta \frac{d\theta}{\sqrt{\cos 2\theta}} \\ &= -2\pi \cos\theta \Big|_0^{\pi/4} = (2-\sqrt{2})\pi \text{ sq. units.} \end{aligned}$$

b) Area of the surface generated by rotation about the y- axis is

$$\begin{aligned} S_y &= 2\pi \int_{-\pi/4}^{\pi/4} r\cos\theta\, ds \\ &= 4\pi \int_0^{\pi/4} \sqrt{\cos 2\theta}\, \cos\theta \frac{d\theta}{\sqrt{\cos 2\theta}} \\ &= 4\pi \sin\theta \Big|_0^{\pi/4} = 2\sqrt{2}\pi \text{ sq. units.} \end{aligned}$$

17. For $r = 1 + \sin\theta$,

$$\tan\psi = \frac{r}{dr/d\theta} = \frac{1+\sin\theta}{\cos\theta}.$$

If $\theta = \pi/4$, then $\tan\psi = \sqrt{2}+1$ and $\psi = 3\pi/8$.
If $\theta = 5\pi/4$, then $\tan\psi = 1-\sqrt{2}$ and $\psi = -\pi/8$.
The line $y = x$ meets the cardioid $r = 1+\sin\theta$ at the origin at an angle of 45°, and also at first and third quadrant points at angles of 67.5° and −22.5° as shown in the figure.

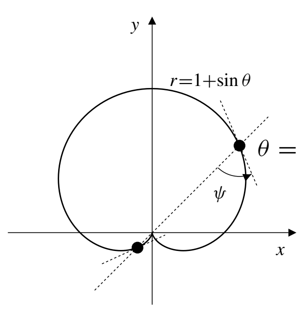

Fig. 8.6.17

18. The two curves $r^2 = 2\sin 2\theta$ and $r = 2\cos\theta$ intersect where

$$\begin{aligned} 2\sin 2\theta &= 4\cos^2\theta \\ 4\sin\theta\cos\theta &= 4\cos^2\theta \\ (\sin\theta - \cos\theta)\cos\theta &= 0 \\ \Leftrightarrow \quad \sin\theta = \cos\theta &\text{ or } \cos\theta = 0, \end{aligned}$$

i.e., at $P_1 = \left[\sqrt{2}, \frac{\pi}{4}\right]$ and $P_2 = (0, 0)$.

For $r^2 = 2\sin 2\theta$ we have $2r\dfrac{dr}{d\theta} = 4\cos 2\theta$. At P_1 we have $r = \sqrt{2}$ and $dr/d\theta = 0$. Thus the angle ψ between the curve and the radial line $\theta = \pi/4$ is $\psi = \pi/2$.
For $r = 2\cos\theta$ we have $dr/d\theta = -2\sin\theta$, so the angle between this curve and the radial line $\theta = \pi/4$ satisfies $\tan\psi = \left.\dfrac{r}{dr/d\theta}\right|_{\theta=\pi/4} = -1$, and $\psi = 3\pi/4$. The two curves intersect at P_1 at angle $\dfrac{3\pi}{4} - \dfrac{\pi}{2} = \dfrac{\pi}{4}$.
The Figure shows that at the origin, P_2, the circle meets the lemniscate twice, at angles 0 and $\pi/2$.

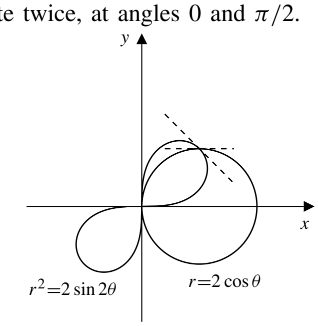

Fig. 8.6.18

19. The curves $r = 1-\cos\theta$ and $r = 1-\sin\theta$ intersect on the rays $\theta = \pi/4$ and $\theta = 5\pi/4$, as well as at the origin. At the origin their cusps clearly intersect at right angles.
For $r = 1-\cos\theta$, $\tan\psi_1 = (1-\cos\theta)/\sin\theta$.
At $\theta = \pi/4$, $\tan\psi_1 = \sqrt{2}-1$, so $\psi_1 = \pi/8$.
At $\theta = 5\pi/4$, $\tan\psi_1 = -(\sqrt{2}+1)$, so $\psi_1 = -3\pi/8$.
For $r = 1-\sin\theta$, $\tan\psi_2 = (1-\sin\theta)/(-\cos\theta)$.
At $\theta = \pi/4$, $\tan\psi_2 = 1-\sqrt{2}$, so $\psi_2 = -\pi/8$.
At $\theta = 5\pi/4$, $\tan\psi_2 = \sqrt{2}+1$, so $\psi_2 = 3\pi/8$.
At $\pi/4$ the curves intersect at angle $\pi/8-(-\pi/8) = \pi/4$.
At $5\pi/4$ the curves intersect at angle $3\pi/8 - (-3\pi/8)$ $= 3\pi/4$ (or $\pi/4$ if you use the supplementary angle).

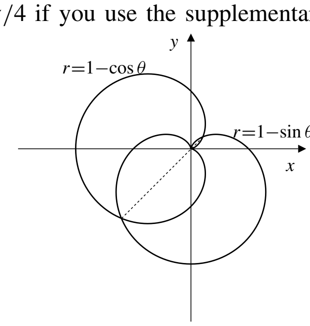

Fig. 8.6.19

20. We have $r = \cos\theta + \sin\theta$. For horizontal tangents:

$$\begin{aligned} 0 = \frac{dy}{d\theta} &= \frac{d}{d\theta}\left(\cos\theta\sin\theta + \sin^2\theta\right) \\ &= \cos^2\theta - \sin^2\theta + 2\sin\theta\cos\theta \\ \Leftrightarrow \quad \cos 2\theta = -\sin 2\theta &\quad \Leftrightarrow \quad \tan 2\theta = -1. \end{aligned}$$

Thus $\theta = -\frac{\pi}{8}$ or $\frac{3\pi}{8}$. The tangents are horizontal at $\left[\cos\left(\frac{\pi}{8}\right) - \sin\left(\frac{\pi}{8}\right), -\frac{\pi}{8}\right]$ and $\left[\cos\left(\frac{3\pi}{8}\right) + \sin\left(\frac{3\pi}{8}\right), \frac{3\pi}{8}\right]$.
For vertical tangent:

$$\begin{aligned} 0 = \frac{dx}{d\theta} &= \frac{d}{d\theta}\Big(\cos^2\theta + \cos\theta\sin\theta\Big) \\ &= -2\cos\theta\sin\theta + \cos^2\theta - \sin^2\theta \\ \Leftrightarrow \quad \sin 2\theta &= \cos 2\theta \quad \Leftrightarrow \quad \tan 2\theta = 1. \end{aligned}$$

Thus $\theta = \pi/8$ of $5\pi/8$. There are vertical tangents at $\left[\cos\left(\frac{\pi}{8}\right) + \sin\left(\frac{\pi}{8}\right), \frac{\pi}{8}\right]$ and $\left[\cos\left(\frac{5\pi}{8}\right) + \sin\left(\frac{5\pi}{8}\right), \frac{5\pi}{8}\right]$.

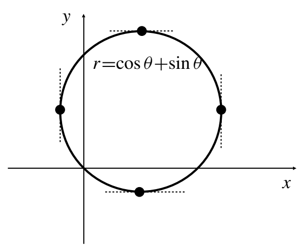

Fig. 8.6.20

21. $r = 2\cos\theta$. $\tan\psi = \frac{r}{dr/d\theta} = -\cot\theta$.
For horizontal tangents we want $\tan\psi = -\tan\theta$. Thus we want $-\tan\theta = -\cot\theta$, and so $\theta = \pm\pi/4$ or $\pm 3\pi/4$. The tangents are horizontal at $[\sqrt{2}, \pm\pi/4]$.
For vertical tangents we want $\tan\psi = \cot\theta$. Thus we want $-\cot\theta = \cot\theta$, and so $\theta = 0$, $\pm\pi/2$, or π. There are vertical tangents at the origin and at $[2, 0]$.

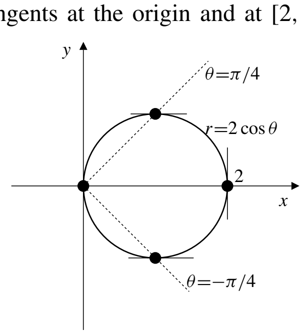

Fig. 8.6.21

22. We have $r^2 = \cos 2\theta$, and $2r\frac{dr}{d\theta} = -2\sin 2\theta$. For horizontal tangents:

$$\begin{aligned} 0 &= \frac{d}{d\theta} r\sin\theta = r\cos\theta + \sin\theta\left(-\frac{\sin 2\theta}{r}\right) \\ \Leftrightarrow \quad & \cos 2\theta\cos\theta = \sin 2\theta\sin\theta \\ \Leftrightarrow \quad & (\cos^2\theta - \sin^2\theta)\cos\theta = 2\sin^2\theta\cos\theta \\ \Leftrightarrow \quad & \cos\theta = 0 \quad \text{or} \quad \cos^2\theta = 3\sin^2\theta. \end{aligned}$$

There are no points on the curve where $\cos\theta = 0$. Therefore, horizontal tangents occur only where $\tan^2\theta = 1/3$. There are horizontal tangents at $\left[\frac{1}{\sqrt{2}}, \pm\frac{\pi}{6}\right]$ and $\left[\frac{1}{\sqrt{2}}, \pm\frac{5\pi}{6}\right]$.
For vertical tangents:

$$\begin{aligned} 0 &= \frac{d}{d\theta} r\cos\theta = -r\sin\theta + \cos\theta\left(-\frac{\sin 2\theta}{r}\right) \\ \Leftrightarrow \quad & \cos 2\theta\sin\theta = -\sin 2\theta\cos\theta \\ \Leftrightarrow \quad & (\cos^2\theta - \sin^2\theta)\sin\theta = -2\sin\theta\cos^2\theta \\ \Leftrightarrow \quad & \sin\theta = 0 \quad \text{or} \quad 3\cos^2\theta = \sin^2\theta. \end{aligned}$$

There are no points on the curve where $\tan^2\theta = 3$, so the only vertical tangents occur where $\sin\theta = 0$, that is, at the points with polar coordinates $[1, 0]$ and $[1, \pi]$.

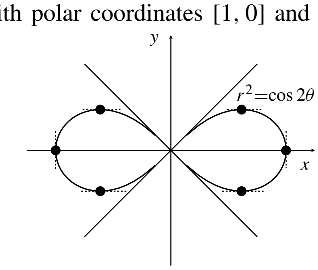

Fig. 8.6.22

23. $r = \sin 2\theta$. $\tan\psi = \frac{\sin 2\theta}{2\cos 2\theta} = \frac{1}{2}\tan 2\theta$.
For horizontal tangents:

$$\begin{aligned} &\tan 2\theta = -2\tan\theta \\ &\frac{2\tan\theta}{1-\tan^2\theta} = -2\tan\theta \\ &\tan\theta\Big(1 + (1 - \tan^2\theta)\Big) = 0 \\ &\tan\theta(2 - \tan^2\theta) = 0. \end{aligned}$$

Thus $\theta = 0$, π, $\pm\tan^{-1}\sqrt{2}$, $\pi \pm \tan^{-1}\sqrt{2}$.
There are horizontal tangents at the origin and the points

$$\left[\frac{2\sqrt{2}}{3}, \pm\tan^{-1}\sqrt{2}\right] \quad \text{and} \quad \left[\frac{2\sqrt{2}}{3}, \pi \pm \tan^{-1}\sqrt{2}\right].$$

Since the rosette $r = \sin 2\theta$ is symmetric about $x = y$, there must be vertical tangents at the origin and at the points

$$\left[\frac{2\sqrt{2}}{3}, \pm\tan^{-1}\frac{1}{\sqrt{2}}\right] \quad \text{and} \quad \left[\frac{2\sqrt{2}}{3}, \pi \pm \tan^{-1}\frac{1}{\sqrt{2}}\right].$$

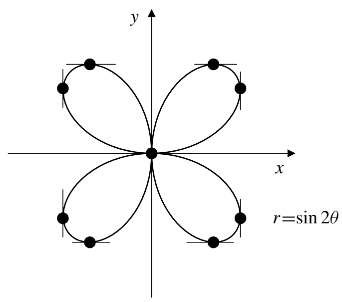

Fig. 8.6.23

24. We have $r = e^\theta$ and $\dfrac{dr}{d\theta} = e^\theta$. For horizontal tangents:

$$0 = \frac{d}{d\theta} r\sin\theta = e^\theta\cos\theta + e^\theta\sin\theta$$
$$\Leftrightarrow\quad \tan\theta = -1 \quad\Leftrightarrow\quad \theta = -\frac{\pi}{4} + k\pi,$$

where $k = 0, \pm1, \pm2, \ldots$. At the points $[e^{k\pi-\pi/4}, k\pi - \pi/4]$ the tangents are horizontal.
For vertical tangents:

$$0 = \frac{d}{d\theta} r\cos\theta = e^\theta\cos\theta - e^\theta\sin\theta$$
$$\Leftrightarrow\quad \tan\theta = 1 \quad\leftrightarrow\quad \theta = \frac{\pi}{4} + k\pi.$$

At the points $[e^{k\pi+\pi/4}, k\pi + \pi/4]$ the tangents are vertical.

25. $r = 2(1-\sin\theta)$, $\tan\psi = -\dfrac{1-\sin\theta}{\cos\theta}$.
For horizontal tangents $\tan\psi = -\cot\theta$, so

$$-\frac{1-\sin\theta}{\cos\theta} = -\frac{\sin\theta}{\cos\theta}$$
$$\cos\theta = 0, \quad\text{or}\quad 2\sin\theta = 1.$$

The solutions are $\theta = \pm\pi/2$, $\pm\pi/6$, and $\pm 5\pi/6$.
$\theta = \pi/2$ corresponds to the origin where the cardioid has a cusp, and therefore no tangent. There are horizontal tangents at $[4, -\pi/2]$, $[1, \pi/6]$, and $[1, 5\pi/6]$.
For vertical tangents $\tan\psi = \cot\theta$, so

$$-\frac{1-\sin\theta}{\cos\theta} = \frac{\cos\theta}{\sin\theta}$$
$$\sin^2\theta - \sin\theta = \cos^2\theta = 1 - \sin^2\theta$$
$$2\sin^2\theta - \sin\theta - 1 = 0$$
$$(\sin\theta - 1)(2\sin\theta + 1) = 0$$

The solutions here are $\theta = \pi/2$ (the origin again), $\theta = -\pi/6$ and $\theta = -5\pi/6$. There are vertical tangents at $[3, -\pi/6]$ and $[3, -5\pi/6]$.

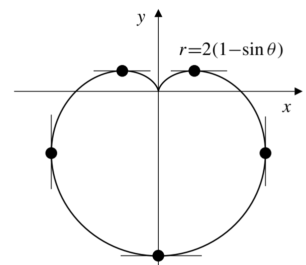

Fig. 8.6.25

26. $x = r\cos\theta = f(\theta)\cos\theta$, $y = r\sin\theta = f(\theta)\sin\theta$.

$$\frac{dx}{d\theta} = f'(\theta)\cos\theta - f(\theta)\sin\theta, \quad \frac{dy}{d\theta} = f'(\theta)\sin\theta + f(\theta)\cos\theta$$
$$ds = \sqrt{\Big(f'(\theta)\cos\theta - f(\theta)\sin\theta\Big)^2 + \Big(f'(\theta)\sin\theta + f(\theta)\cos\theta\Big)^2}\,d\theta$$
$$= \Big[\Big(f'(\theta)\Big)^2\cos^2\theta - 2f'(\theta)f(\theta)\cos\theta\sin\theta + \Big(f(\theta)\Big)^2\sin^2\theta$$
$$+\Big(f'(\theta)\Big)^2\sin^2\theta + 2f'(\theta)f(\theta)\sin\theta\cos\theta + \Big(f(\theta)\Big)^2\cos^2\theta\Big]^{1/2} d\theta$$
$$= \sqrt{\Big(f'(\theta)\Big)^2 + \Big(f(\theta)\Big)^2}\,d\theta.$$

Review Exercises 8 (page 469)

1. $x^2 + 2y^2 = 2 \quad\Leftrightarrow\quad \dfrac{x^2}{2} + y^2 = 1$
Ellipse, semi-major axis $a = \sqrt{2}$, along the x-axis. Semi-minor axis $b = 1$.
$c^2 = a^2 - b^2 = 1$. Foci: $(\pm1, 0)$.

2. $9x^2 - 4y^2 = 36 \quad\Leftrightarrow\quad \dfrac{x^2}{4} - \dfrac{y^2}{9} = 1$
Hyperbola, transverse axis along the x-axis.
Semi-transverse axis $a = 2$, semi-conjugate axis $b = 3$.
$c^2 = a^2 + b^2 = 13$. Foci: $(\pm\sqrt{13}, 0)$.
Asymptotes: $3x \pm 2y = 0$.

3. $x + y^2 = 2y + 3 \quad\Leftrightarrow\quad (y-1)^2 = 4 - x$
Parabola, vertex $(4, 1)$, opening to the left, principal axis $y = 1$.
$a = -1/4$. Focus: $(15/4, 1)$.

4. $2x^2 + 8y^2 = 4x - 48y$
$2(x^2 - 2x + 1) + 8(y^2 + 6y + 9) = 74$

$$\frac{(x-1)^2}{37} + \frac{(y+3)^2}{37/4} = 1.$$

Ellipse, centre $(1, -3)$, major axis along $y = -3$.
$a = \sqrt{37}$, $b = \sqrt{37}/2$, $c^2 = a^2 - b^2 = 111/4$.
Foci: $(1 \pm \sqrt{111}/2, -3)$.

5. $x = t$, $y = 2 - t$, $(0 \le t \le 2)$.
Straight line segment from $(0, 2)$ to $(2, 0)$.

6. $x = 2\sin(3t)$, $y = 2\cos(3t)$, $(0 \le t \le 2)$
Part of a circle of radius 2 centred at the origin from the point $(0, 2)$ clockwise to $(2\sin 6, 2\cos 6)$.

7. $x = \cosh t$, $y = \sinh^2 t$.
Parabola $x^2 - y = 1$, or $y = x^2 - 1$, traversed left to right.

8. $x = e^t$, $y = e^{-2t}$, $(-1 \le t \le 1)$.
Part of the curve $x^2 y = 1$ from $(1/e, e^2)$ to $(e, 1/e^2)$.

9. $x = \cos(t/2)$, $y = 4\sin(t/2)$, $(0 \le t \le \pi)$.
The first quadrant part of the ellipse $16x^2 + y^2 = 16$, traversed counterclockwise.

10. $x = \cos t + \sin t$, $y = \cos t - \sin t$, $(0 \le t \le 2\pi)$
The circle $x^2 + y^2 = 2$, traversed clockwise, starting and ending at $(1, 1)$.

11. $x = \dfrac{4}{1+t^2}$ $\quad y = t^3 - 3t$

$$\frac{dx}{dt} = -\frac{8t}{(1+t^2)^2} \qquad \frac{dy}{dt} = 3(t^2 - 1)$$

Horizontal tangent at $t = \pm 1$, i.e., at $(2, \pm 2)$.
Vertical tangent at $t = 0$, i.e., at $(4, 0)$.
Self-intersection at $t = \pm\sqrt{3}$, i.e., at $(1, 0)$.

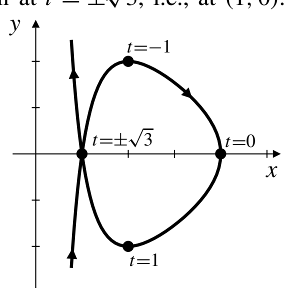

Fig. R-8.11

12. $x = t^3 - 3t$ $\quad y = t^3 + 3t$

$$\frac{dx}{dt} = 3(t^2 - 1) \qquad \frac{dy}{dt} = 3(t^2 + 1)$$

Horizontal tangent: none.
Vertical tangent at $t = \pm 1$, i.e., at $(2, -4)$ and $(-2, 4)$.

$$\text{Slope } \frac{dy}{dx} = \frac{t^2+1}{t^2-1} \quad \begin{cases} > 0 & \text{if } |t| > 1 \\ < 0 & \text{if } |t| < 1 \end{cases}$$

Slope $\to 1$ as $t \to \pm\infty$.

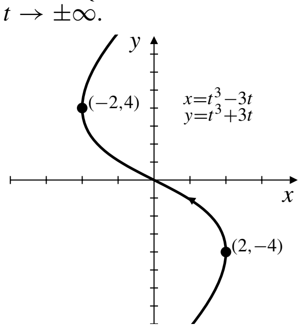

Fig. R-8.12

13. $x = t^3 - 3t$ $\quad y = t^3$

$$\frac{dx}{dt} = 3(t^2 - 1) \qquad \frac{dy}{dt} = 3t^2$$

Horizontal tangent at $t = 0$, i.e., at $(0, 0)$.
Vertical tangent at $t = \pm 1$, i.e., at $(2, -1)$ and $(-2, 1)$.

$$\text{Slope } \frac{dy}{dx} = \frac{t^2}{t^2-1} \quad \begin{cases} > 0 & \text{if } |t| > 1 \\ < 0 & \text{if } |t| < 1 \end{cases}$$

Slope $\to 1$ as $t \to \pm\infty$.

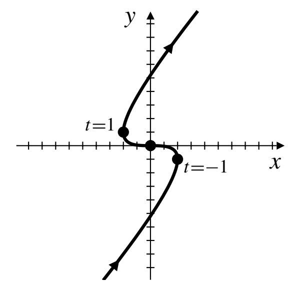

Fig. R-8.13

14. $x = t^3 - 3t$ $\quad y = t^3 - 12t$

$$\frac{dx}{dt} = 3(t^2 - 1) \qquad \frac{dy}{dt} = 3(t^2 - 4)$$

Horizontal tangent at $t = \pm 2$, i.e., at $(2, -16)$ and $(-2, 16)$.
Vertical tangent at $t = \pm 1$, i.e., at $(2, 11)$ and $(-2, -11)$.

$$\text{Slope } \frac{dy}{dx} = \frac{t^2-4}{t^2-1} \quad \begin{cases} > 0 & \text{if } |t| > 2 \text{ or } |t| < 1 \\ < 0 & \text{if } 1 < |t| < 2 \end{cases}$$

Slope $\to 1$ as $t \to \pm\infty$.

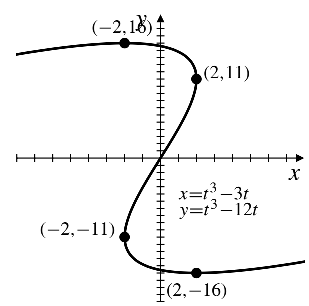

Fig. R-8.14

15. The curve $x = t^3 - t$, $y = |t^3|$ is symmetric about $x = 0$ since x is an odd function and y is an even function. Its self-intersection occurs at a nonzero value of t that makes $x = 0$, namely, $t = \pm 1$. The area of the loop is

$$A = 2\int_{t=0}^{t=1} (-x)\,dy = -2\int_0^1 (t^3 - t)3t^2\,dt$$
$$= \left(-t^6 + \frac{3}{2}t^4\right)\Bigg|_0^1 = \frac{1}{2} \text{ sq. units.}$$

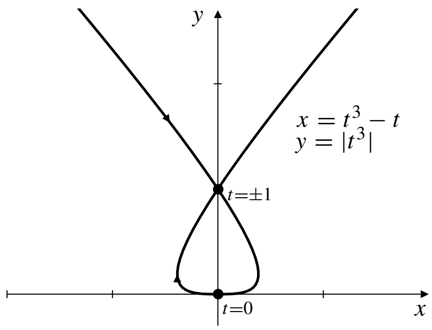

Fig. R-8.15

16. The volume of revolution about the y-axis is

$$\begin{aligned} V &= \pi \int_{t=0}^{t=1} x^2\,dy \\ &= \pi \int_0^1 (t^6 - 2t^4 + t^2)3t^2\,dt \\ &= 3\pi \int_0^1 (t^8 - 2t^6 + t^4)\,dt \\ &= 3\pi \left(\frac{1}{9} - \frac{2}{7} + \frac{1}{5}\right) = \frac{8\pi}{105} \text{ cu. units.} \end{aligned}$$

17. $x = e^t - t$, $y = 4e^{t/2}$, $(0 \le t \le 2)$. Length is

$$\begin{aligned} L &= \int_0^2 \sqrt{(e^t - 1)^2 + 4e^t}\,dt \\ &= \int_0^2 \sqrt{(e^t + 1)^2}\,dt = \int_0^2 (e^t + 1)\,dt \\ &= (e^t + t)\Big|_0^2 = e^2 + 1 \text{ units.} \end{aligned}$$

18. Area of revolution about the x-axis is

$$\begin{aligned} S &= 2\pi \int 4e^{t/2}(e^t + 1)\,dt \\ &= 8\pi \left(\frac{2}{3}e^{3t/2} + 2e^{t/2}\right)\Big|_0^2 \\ &= \frac{16\pi}{3}(e^3 + 3e - 4) \text{ sq. units.} \end{aligned}$$

19. $r = \theta$, $\left(\frac{-3\pi}{2} \le \theta \le \frac{3\pi}{2}\right)$

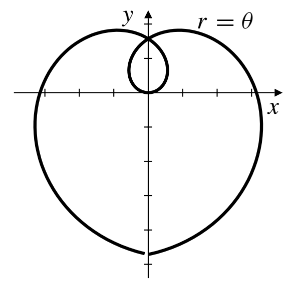

Fig. R-8.19

20. $r = |\theta|$, $(-2\pi \le \theta \le 2\pi)$

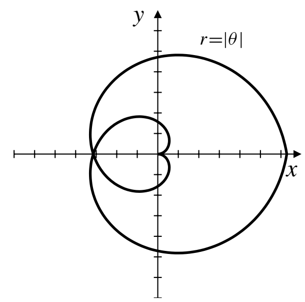

Fig. R-8.20

21. $r = 1 + \cos(2\theta)$

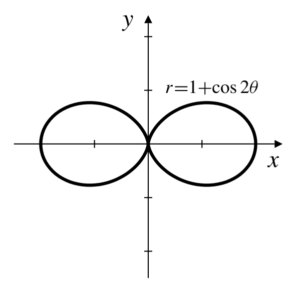

Fig. R-8.21

22. $r = 2 + \cos(2\theta)$

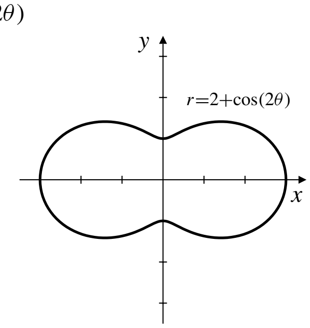

Fig. R-8.22

23. $r = 1 + 2\cos(2\theta)$

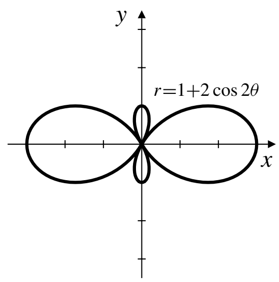

Fig. R-8.23

24. $r = 1 - \sin(3\theta)$

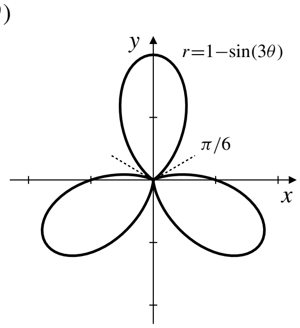

Fig. R-8.24

25. Area of a large loop:

$$\begin{aligned} A &= 2 \times \frac{1}{2} \int_0^{\pi/3} (1 + 2\cos(2\theta))^2 \, d\theta \\ &= \int_0^{\pi/3} [1 + 4\cos(2\theta) + 2(1 + \cos(4\theta))] \, d\theta \\ &= \left(3\theta + 2\sin(2\theta) + \frac{1}{2}\sin(4\theta) \right) \Big|_0^{\pi/3} \\ &= \pi + \frac{3\sqrt{3}}{4} \text{ sq. units.} \end{aligned}$$

26. Area of a small loop:

$$\begin{aligned} A &= 2 \times \frac{1}{2} \int_{\pi/3}^{\pi/2} (1 + 2\cos(2\theta))^2 \, d\theta \\ &= \int_{\pi/3}^{\pi/2} [1 + 4\cos(2\theta) + 2(1 + \cos(4\theta))] \, d\theta \\ &= \left(3\theta + 2\sin(2\theta) + \frac{1}{2}\sin(4\theta) \right) \Big|_{\pi/3}^{\pi/2} \\ &= \frac{\pi}{2} - \frac{3\sqrt{3}}{4} \text{ sq. units.} \end{aligned}$$

27. $r = 1 + \sqrt{2}\sin\theta$ approaches the origin in the directions for which $\sin\theta = -1/\sqrt{2}$, that is, $\theta = -3\pi/4$ and $\theta = -\pi/4$. The smaller loop corresponds to values of θ between these two values. By symmetry, the area of the loop is

$$\begin{aligned} A &= 2 \times \frac{1}{2} \int_{-\pi/2}^{-\pi/4} (1 + 2\sqrt{2}\sin\theta + 2\sin^2\theta) \, d\theta \\ &= \int_{-\pi/2}^{-\pi/4} (2 + 2\sqrt{2}\sin\theta - \cos(2\theta)) \, d\theta \\ &= \left(2\theta - 2\sqrt{2}\cos\theta - \frac{1}{2}\sin(2\theta) \right) \Big|_{-\pi/2}^{-\pi/4} \\ &= \frac{\pi}{2} - 2 + \frac{1}{2} = \frac{\pi - 3}{2} \text{ sq. units.} \end{aligned}$$

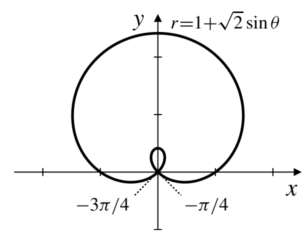

Fig. R-8.27

28. $r\cos\theta = x = 1/4$ and $r = 1 + \cos\theta$ intersect where

$$\begin{aligned} 1 + \cos\theta &= \frac{1}{4\cos\theta} \\ 4\cos^2\theta + 4\cos\theta - 1 &= 0 \\ \cos\theta &= \frac{-4 \pm \sqrt{16 + 16}}{8} = \frac{\pm\sqrt{2} - 1}{2}. \end{aligned}$$

Only $(\sqrt{2} - 1)/2$ is between -1 and 1, so is a possible value of $\cos\theta$. Let $\theta_0 = \cos^{-1}\dfrac{\sqrt{2} - 1}{2}$. Then

$$\sin\theta_0 = \sqrt{1 - \left(\frac{\sqrt{2} - 1}{2}\right)^2} = \frac{\sqrt{1 + 2\sqrt{2}}}{2}.$$

By symmetry, the area inside $r = 1 + \cos\theta$ to the left of the line $x = 1/4$ is

$$\begin{aligned} A &= 2 \times \frac{1}{2} \int_{\theta_0}^{\pi} \left(1 + 2\cos\theta + \frac{1 + \cos(2\theta)}{2} \right) d\theta + \cos\theta_0 \sin\theta_0 \\ &= \frac{3}{2}(\pi - \theta_0) + \left(2\sin\theta + \frac{1}{4}\sin(2\theta) \right) \Big|_{\theta_0}^{\pi} \\ &\quad + \frac{(\sqrt{2} - 1)\sqrt{1 + 2\sqrt{2}}}{4} \\ &= \frac{3}{2}\left(\pi - \cos^{-1}\frac{\sqrt{2} - 1}{2} \right) + \sqrt{1 + 2\sqrt{2}} \left(\frac{\sqrt{2} - 9}{8} \right) \text{ sq. units.} \end{aligned}$$

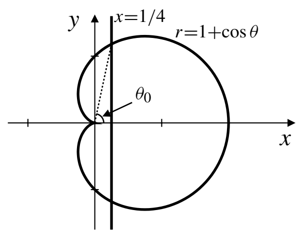

Fig. R-8.28

Challenging Problems 8 (page 469)

1. The surface of the water is elliptical (see Problem 2 below) whose semi-minor axis is 4 cm, the radius of the cylinder, and whose semi-major axis is $4\sec\theta$ cm because of the tilt of the glass. The surface area is that of the ellipse

$$x = 4\sec\theta\cos t,\quad y = 4\sin t,\quad (0 \le t \le 2\pi).$$

This area is

$$\begin{aligned} A &= 4\int_{t=0}^{t=\pi/2} x\,dy \\ &= 4\int_0^{\pi/2} (4\sec\theta\cos t)(4\cos t)\,dt \\ &= 32\sec\theta\int_0^{\pi/2}(1+\cos(2t))\,dt = 16\pi\sec\theta \text{ cm}^2. \end{aligned}$$

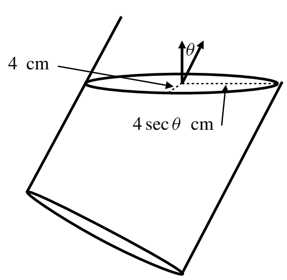

Fig. C-8.1

2. Let S_1 and S_2 be two spheres inscribed in the cylinder, one on each side of the plane that intersects the cylinder in the curve C that we are trying to show is an ellipse. Let the spheres be tangent to the cylinder around the circles C_1 and C_2, and suppose they are also tangent to the plane at the points F_1 and F_2, respectively, as shown in the figure.

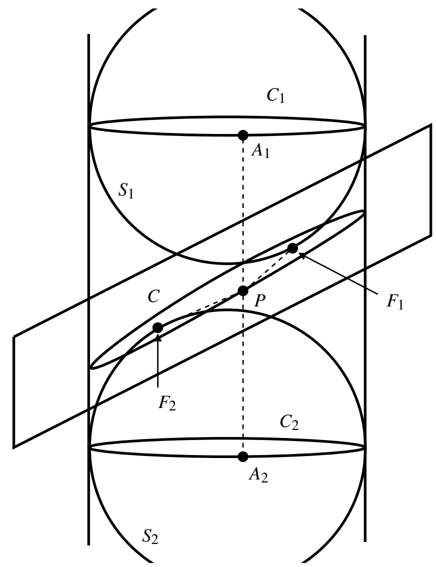

Fig. C-8.2

Let P be any point on C. Let A_1A_2 be the line through P that lies on the cylinder, with A_1 on C_1 and A_2 on C_2. Then $PF_1 = PA_1$ because both lengths are of tangents drawn to the sphere S_1 from the same exterior point P. Similarly, $PF_2 = PA_2$. Hence

$$PF_1 + PF_2 = PA_1 + PA_2 = A_1A_2,$$

which is constant, the distance between the centres of the two spheres. Thus C must be an ellipse, with foci at F_1 and F_2.

3. Given the foci F_1 and F_2, and the point P on the ellipse, construct N_1PN_2, the bisector of the angle F_1PF_2. Then construct T_1PT_2 perpendicular to N_1N_2 at P. By the reflection property of the ellipse, N_1N_2 is normal to the ellipse at P. Therefore T_1T_2 is tangent there.

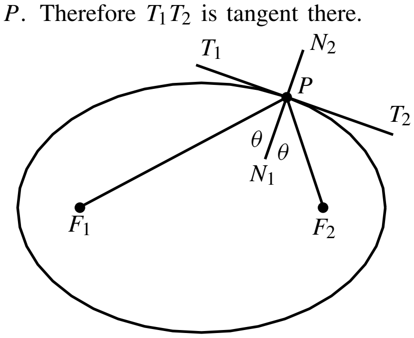

Fig. C-8.3

4. Without loss of generality, choose the axes and axis scales so that the parabola has equation $y = x^2$. If P is the point (x_0, x_0^2) on it, then the tangent to the parabola at P has equation

$$y = x_0^2 + 2x_0(x - x_0),$$

which intersects the principal axis $x = 0$ at $(0, -x_0^2)$. Thus $R = (0, -x_0^2)$ and $Q = (0, x_0^2)$. Evidently the vertex $V = (0, 0)$ bisects RQ.

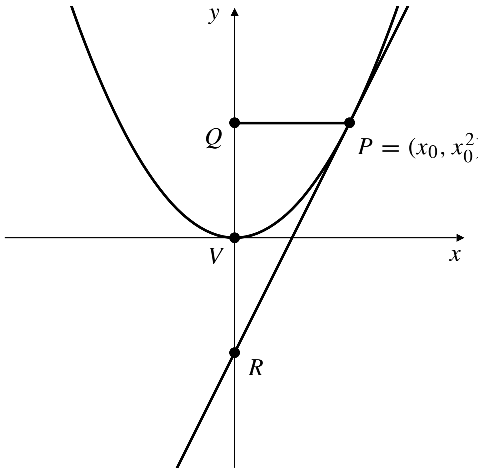

Fig. C-8.4

To construct the tangent at a given point P on a parabola with given vertex V and principal axis L, drop a perpendicular from P to L, meeting L at Q. Then find R on L on the side of V opposite Q and such that $QV = VR$. Then PR is the desired tangent.

5.

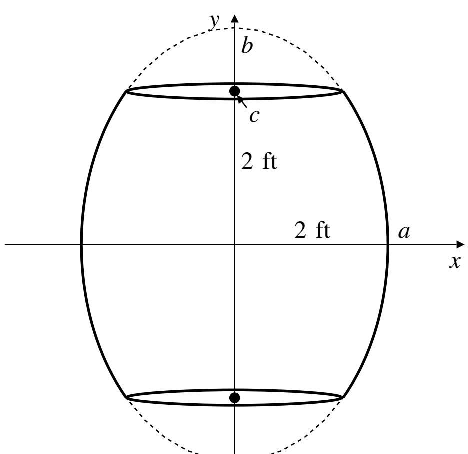

Fig. C-8.5

Let the ellipse be $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, with $a = 2$ and foci at $(0, \pm 2)$ so that $c = 2$ and $b^2 = a^2 + c^2 = 8$. The volume of the barrel is

$$V = 2\int_0^2 \pi x^2\,dy = 2\pi \int_0^2 4\left(1 - \frac{y^2}{8}\right) dy$$
$$= 8\pi \left(y - \frac{y^3}{24}\right)\bigg|_0^2 = \frac{40\pi}{3} \text{ ft}^3.$$

6.

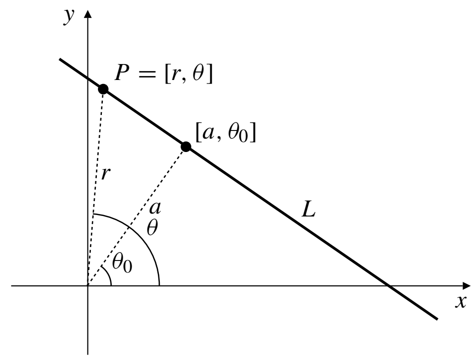

Fig. C-8.6

a) Let L be a line not passing through the origin, and let $[a, \theta_0]$ be the polar coordinates of the point on L that is closest to the origin. If $P = [r, \theta]$ is any point on the line, then, from the triangle in the figure,

$$\frac{a}{r} = \cos(\theta - \theta_0), \quad \text{or} \quad r = \frac{a}{\cos(\theta - \theta_0)}.$$

b) As shown in part (a), any line not passing through the origin has equation of the form

$$r = g(\theta) = \frac{a}{\cos(\theta - \theta_0)} = a\sec(\theta - \theta_0),$$

for some constants a and θ_0. We have

$$\begin{aligned}
g'(\theta) &= a\sec(\theta - \theta_0)\tan(\theta - \theta_0)\\
g''(\theta) &= a\sec(\theta - \theta_0)\tan^2(\theta - \theta_0)\\
&\quad + a\sec^3(\theta - \theta_0)\Big(g(\theta)\Big)^2 + 2\Big(g'(\theta)\Big)^2 - g(\theta)g''(\theta)\\
&= a^2\sec^2(\theta - \theta_0) + 2a^2\sec^2(\theta - \theta_0)\tan^2(\theta - \theta_0)\\
&\quad - a^2\sec^2(\theta - \theta_0)\tan^2(\theta - \theta_0) - a^2\sec^4(\theta - \theta_0)\\
&= a^2\Big[\sec^2(\theta - \theta_0)\Big(1 + \tan^2(\theta - \theta_0)\Big) - \sec^4(\theta - \theta_0)\Big]\\
&= 0.
\end{aligned}$$

c) If $r = g(\theta)$ is the polar equation of the tangent to $r = f(\theta)$ at $\theta = \alpha$, then $g(\alpha) = f(\alpha)$ and $g'(\alpha) = f'(\alpha)$. Suppose that

$$\Big(f(\alpha)\Big)^2 + 2\Big(f'(\alpha)\Big)^2 - f(\alpha)f''(\alpha) > 0.$$

By part (b) we have

$$\Big(g(\alpha)\Big)^2 + 2\Big(g'(\alpha)\Big)^2 - g(\alpha)g''(\alpha) = 0.$$

Subtracting, and using $g(\alpha) = f(\alpha)$ and $g'(\alpha) = f'(\alpha)$, we get $f''(\alpha) < g''(\alpha)$. It follows that $f(\theta) < g(\theta)$ for values of θ near α; that is, the graph of $r = f(\theta)$ is curving to the origin side of its tangent at α. Similarly, if

$$\Big(f(\alpha)\Big)^2 + 2\Big(f'(\alpha)\Big)^2 - f(\alpha)f''(\alpha) < 0,$$

then the graph is curving to the opposite side of the tangent, away from the origin.

7.

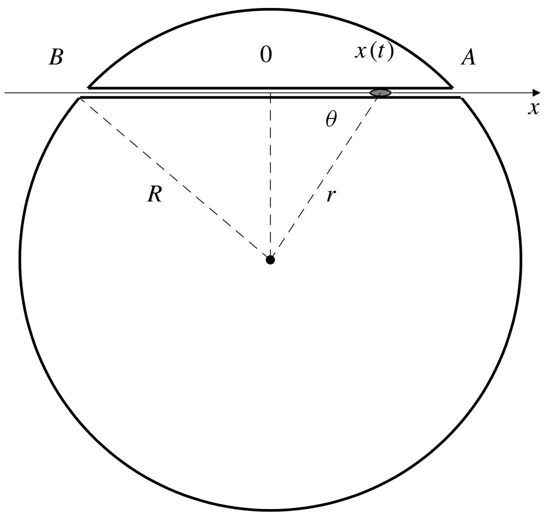

Fig. C-8.7

When the vehicle is at position x, as shown in the figure, the component of the gravitational force on it in the direction of the tunnel is

$$ma(r)\cos\theta = -\frac{mgr}{R}\cos\theta = -\frac{mg}{R}x.$$

By Newton's Law of Motion, this force produces an acceleration d^2x/dt^2 along the tunnel given by

$$m\frac{d^2x}{dt^2} = -\frac{mg}{R}x,$$

that is

$$\frac{d^2x}{dt^2} + \omega^2 x = 0, \quad \text{where} \quad \omega^2 = \frac{g}{R}.$$

This is the equation of simple harmonic motion, with period $T = 2\pi/\omega = 2\pi\sqrt{R/g}$.
For $R \approx 3960$ mi $\approx 2.09 \times 10^7$ ft, and $g \approx 32$ ft/s^2, we have $T \approx 5079$ s ≈ 84.6 minutes. This is a rather short time for a round trip between Atlanta and Baghdad, or any other two points on the surface of the earth.

8. Take the origin at station O as shown in the figure. Both of the lines L_1 and L_2 pass at distance $100\cos\epsilon$ from the origin. Therefore, by Problem 6(a), their equations are

$$L_1: \qquad r = \frac{100\cos\epsilon}{\cos\left[\theta - \left(\frac{\pi}{2} - \epsilon\right)\right]} = \frac{100\cos\epsilon}{\sin(\theta+\epsilon)}$$

$$L_2: \qquad r = \frac{100\cos\epsilon}{\cos\left[\theta - \left(\frac{\pi}{2} + \epsilon\right)\right]} = \frac{100\cos\epsilon}{\sin(\theta-\epsilon)}.$$

The search area $A(\epsilon)$ is, therefore,

$$\begin{aligned}
A(\epsilon) &= \frac{1}{2}\int_{\frac{\pi}{4}-\epsilon}^{\frac{\pi}{4}+\epsilon} \left(\frac{100^2\cos^2\epsilon}{\sin^2(\theta-\epsilon)} - \frac{100^2\cos^2\epsilon}{\sin^2(\theta+\epsilon)}\right) d\theta \\
&= 5,000\cos^2\epsilon \int_{\frac{\pi}{4}-\epsilon}^{\frac{\pi}{4}+\epsilon} \Big(\csc^2(\theta-\epsilon) - \csc^2(\theta+\epsilon)\Big) d\theta \\
&= 5,000\cos^2\epsilon\left[\cot\left(\tfrac{\pi}{4}+2\epsilon\right) - 2\cot\tfrac{\pi}{4} + \cot\left(\tfrac{\pi}{4}-2\epsilon\right)\right] \\
&= 5,000\cos^2\epsilon\left[\frac{\cos\left(\frac{\pi}{4}+2\epsilon\right)}{\sin\left(\frac{\pi}{4}+2\epsilon\right)} + \frac{\sin\left(\frac{\pi}{4}+2\epsilon\right)}{\cos\left(\frac{\pi}{4}+2\epsilon\right)} - 2\right] \\
&= 10,000\cos^2\epsilon\left[\csc\left(\tfrac{\pi}{2}+4\epsilon\right) - 1\right] \\
&= 10,000\cos^2\epsilon(\sec(4\epsilon) - 1)\ \text{mi}^2.
\end{aligned}$$

For $\epsilon = 3° = \pi/60$, we have $A(\epsilon) \approx 222.8$ square miles. Also

$$\begin{aligned}
A'(\epsilon) &= -20,000\cos\epsilon\sin\epsilon(\sec(4\epsilon) - 1) \\
&\quad + 40,000\cos^2\epsilon\sec(4\epsilon)\tan(4\epsilon) \\
A'(\pi/60) &\approx 8645.
\end{aligned}$$

When $\epsilon = 3°$, the search area increases at about $8645(\pi/180) \approx 151$ square miles per degree increase in ϵ.

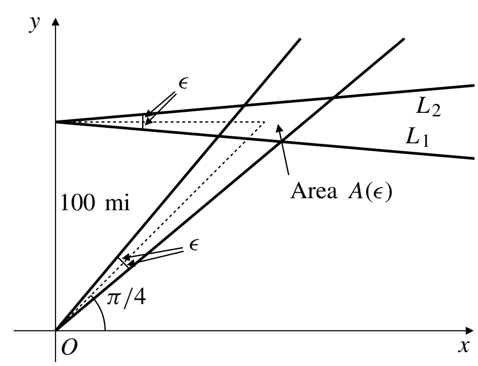

Fig. C-8.8

9. The easiest way to determine which curve is which is to calculate both their areas; the outer curve bounds the larger area.
The curve C_1 with parametric equations

$$x = \sin t, \quad y = \frac{1}{2}\sin(2t), \quad (0 \le t \le 2\pi)$$

has area

$$\begin{aligned} A_1 &= 4\int_{t=0}^{t=\pi/2} y\,dx \\ &= 4\int_0^{\pi/2} \frac{1}{2}\sin(2t)\cos t\,dt \\ &= 4\int_0^{\pi/2} \sin t\cos^2 t\,dt \end{aligned}$$

Let $u = \cos t$
$du = -\sin t\,dt$

$$= 4\int_0^1 u^2\,du = \frac{4}{3} \text{ sq. units.}$$

The curve C_2 with polar equation $r^2 = \cos(2\theta)$ has area

$$A_2 = \frac{4}{2}\int_0^{\pi/4} \cos(2\theta)\,d\theta = \sin(2\theta)\Big|_0^{\pi/4} = 1 \text{ sq. units.}$$

C_1 is the outer curve, and the area between the curves is 1/3 sq. units.

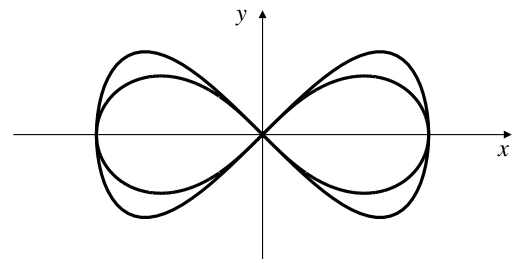

Fig. C-8.9