CHAPTER 9. SEQUENCES, SERIES, AND POWER SERIES

Section 9.1 Sequences and Convergence (page 478)

1. $\left\{\frac{2n^2}{n^2+1}\right\} = \left\{2 - \frac{2}{n^2+1}\right\} = \left\{1, \frac{8}{5}, \frac{9}{5}, \ldots\right\}$ is bounded, positive, increasing, and converges to 2.

2. $\left\{\frac{2n}{n^2+1}\right\} = \left\{1, \frac{4}{5}, \frac{3}{5}, \frac{8}{17}, \ldots\right\}$ is bounded, positive, decreasing, and converges to 0.

3. $\left\{4 - \frac{(-1)^n}{n}\right\} = \left\{5, \frac{7}{2}, \frac{13}{3}, \ldots\right\}$ is bounded, positive, and converges to 4.

4. $\left\{\sin\frac{1}{n}\right\} = \left\{\sin 1, \sin\left(\frac{1}{2}\right), \sin\left(\frac{1}{3}\right), \ldots\right\}$ is bounded, positive, decreasing, and converges to 0.

5. $\left\{\frac{n^2-1}{n}\right\} = \left\{n - \frac{1}{n}\right\} = \left\{0, \frac{3}{2}, \frac{8}{3}, \frac{15}{4}, \ldots\right\}$ is bounded below, positive, increasing, and diverges to infinity.

6. $\left\{\frac{e^n}{\pi^n}\right\} = \left\{\frac{e}{\pi}, \left(\frac{e}{\pi}\right)^2, \left(\frac{e}{\pi}\right)^3, \ldots\right\}$ is bounded, positive, decreasing, and converges to 0, since $e < \pi$.

7. $\left\{\frac{e^n}{\pi^{n/2}}\right\} = \left\{\left(\frac{e}{\sqrt{\pi}}\right)^n\right\}$. Since $e/\sqrt{\pi} > 1$, the sequence is bounded below, positive, increasing, and diverges to infinity.

8. $\left\{\frac{(-1)^n n}{e^n}\right\} = \left\{\frac{-1}{e}, \frac{2}{e^2}, \frac{-3}{e^3}, \ldots\right\}$ is bounded, alternating, and converges to 0.

9. $\{2^n/n^n\}$ is bounded, positive, decreasing, and converges to 0.

10. $\frac{(n!)^2}{(2n)!} = \frac{1}{n+1}\,\frac{2}{n+2}\,\frac{3}{n+3}\cdots\frac{n}{2n} \le \left(\frac{1}{2}\right)^n.$

Also, $\frac{a_{n+1}}{a_n} = \frac{(n+1)^2}{(2n+2)(2n+1)} < \frac{1}{2}$. Thus the sequence $\left\{\frac{(n!)^2}{(2n)!}\right\}$ is positive, decreasing, bounded, and convergent to 0.

11. $\{n\cos(n\pi/2)\} = \{0, -2, 0, 4, 0, -6, \ldots\}$ is divergent.

12. $\left\{\frac{\sin n}{n}\right\} = \left\{\sin 1, \frac{\sin 2}{2}, \frac{\sin 3}{3}, \ldots\right\}$ is bounded and converges to 0.

13. $\{1, 1, -2, 3, 3, -4, 5, 5, -6, \ldots\}$ is divergent.

14. $\lim \frac{5-2n}{3n-7} = \lim \frac{\frac{5}{n} - 2}{3 - \frac{7}{n}} = -\frac{2}{3}.$

15. $\lim \frac{n^2-4}{n+5} = \lim \frac{n - \frac{4}{n}}{1 + \frac{5}{n}} = \infty.$

16. $\lim \frac{n^2}{n^3+1} = \lim \frac{\frac{1}{n}}{1 + \frac{1}{n^3}} = 0.$

17. $\lim(-1)^n \frac{n}{n^3+1} = 0.$

18. $\lim \frac{n^2 - 2\sqrt{n} + 1}{1 - n - 3n^2} = \lim \frac{1 - \frac{2}{n\sqrt{n}} + \frac{1}{n^2}}{\frac{1}{n^2} - \frac{1}{n} - 3} = -\frac{1}{3}.$

19. $\lim \frac{e^n - e^{-n}}{e^n + e^{-n}} = \lim \frac{1 - e^{-2n}}{1 + e^{-2n}} = 1.$

20. $\lim n \sin\frac{1}{n} = \lim_{x\to 0+} \frac{\sin x}{x} = \lim_{x\to 0+} \frac{\cos x}{1} = 1.$

21. $\lim \left(\frac{n-3}{n}\right)^n = \lim\left(1 + \frac{-3}{n}\right)^n = e^{-3}$ by l'Hôpital's Rule.

22.
$$\lim \frac{n}{\ln(n+1)} = \lim_{x\to\infty} \frac{x}{\ln(x+1)} = \lim_{x\to\infty} \frac{1}{\left(\frac{1}{x+1}\right)} = \lim_{x\to\infty} x + 1 = \infty.$$

23. $\lim(\sqrt{n+1} - \sqrt{n}) = \lim \frac{n+1-n}{\sqrt{n+1}+\sqrt{n}} = 0.$

24.
$$\lim\left(n - \sqrt{n^2-4n}\right) = \lim \frac{n^2 - (n^2-4n)}{n + \sqrt{n^2-4n}}$$
$$= \lim \frac{4n}{n + \sqrt{n^2-4n}} = \lim \frac{4}{1 + \sqrt{1 - \frac{4}{n}}} = 2.$$

25.
$$\lim(\sqrt{n^2+n} - \sqrt{n^2-1})$$
$$= \lim \frac{n^2+n-(n^2-1)}{\sqrt{n^2+n}+\sqrt{n^2-1}}$$
$$= \lim \frac{n+1}{n\left(\sqrt{1+\frac{1}{n}} + \sqrt{1 - \frac{1}{n^2}}\right)}$$
$$= \lim \frac{1 + \frac{1}{n}}{\sqrt{1+\frac{1}{n}} + \sqrt{1-\frac{1}{n^2}}} = \frac{1}{2}.$$

26. If $a_n = \left(\frac{n-1}{n+1}\right)^n$, then

$$\begin{aligned}\lim a_n &= \lim \left(\frac{n-1}{n}\right)^n \left(\frac{n}{n+1}\right)^n \\ &= \lim \left(1-\frac{1}{n}\right)^n \Big/ \lim \left(1+\frac{1}{n}\right)^n \\ &= \frac{e^{-1}}{e} = e^{-2} \quad \text{(by Theorem 6 of Section 3.4).}\end{aligned}$$

27. $$\begin{aligned}a_n = \frac{(n!)^2}{(2n)!} &= \frac{(1\cdot 2\cdot 3\cdots n)(1\cdot 2\cdot 3\cdots n)}{1\cdot 2\cdot 3\cdots n\cdot (n+1)\cdot (n+2)\cdots 2n} \\ &= \frac{1}{n+1}\cdot\frac{2}{n+2}\cdot\frac{3}{n+3}\cdots\frac{n}{n+n} \le \left(\frac{1}{2}\right)^n.\end{aligned}$$

Thus $\lim a_n = 0$.

28. We have $\lim \frac{n^2}{2^n} = 0$ since 2^n grows much faster than n^2 and $\lim \frac{4^n}{n!} = 0$ by Theorem 3(b). Hence,

$$\lim \frac{n^2\,2^n}{n!} = \lim \frac{n^2}{2^n}\cdot\frac{2^{2n}}{n!} = \left(\lim \frac{n^2}{2^n}\right)\left(\lim \frac{4^n}{n!}\right) = 0.$$

29. $a_n = \frac{\pi^n}{1+2^{2n}} \quad \Rightarrow \quad 0 < a_n < (\pi/4)^n$. Since $\pi/4 < 1$, therefore $(\pi/4)^n \to 0$ as $n \to \infty$. Thus $\lim a_n = 0$.

30. Let $a_1 = 1$ and $a_{n+1} = \sqrt{1+2a_n}$ for $n = 1, 2, 3, \ldots$. Then we have $a_2 = \sqrt{3} > a_1$. If $a_{k+1} > a_k$ for some k, then

$$a_{k+2} = \sqrt{1+2a_{k+1}} > \sqrt{1+2a_k} = a_{k+1}.$$

Thus, $\{a_n\}$ is increasing by induction. Observe that $a_1 < 3$ and $a_2 < 3$. If $a_k < 3$ then

$$a_{k+1} = \sqrt{1+2a_k} < \sqrt{1+2(3)} = \sqrt{7} < \sqrt{9} = 3.$$

Therefore, $a_n < 3$ for all n, by induction. Since $\{a_n\}$ is increasing and bounded above, it converges. Let $\lim a_n = a$. Then

$$a = \sqrt{1+2a} \Rightarrow a^2 - 2a - 1 = 0 \Rightarrow a = 1 \pm \sqrt{2}.$$

Since $a = 1 - \sqrt{2} < 0$, it is not appropriate. Hence, we must have $\lim a_n = 1 + \sqrt{2}$.

31. Let $a_1 = 3$ and $a_{n+1} = \sqrt{15+2a_n}$ for $n = 1, 2, 3, \ldots$. Then we have $a_2 = \sqrt{21} > 3 = a_1$. If $a_{k+1} > a_k$ for some k, then

$$a_{k+2} = \sqrt{15+2a_{k+1}} > \sqrt{15+2a_k} = a_{k+1}.$$

Thus, $\{a_n\}$ is increasing by induction. Observe that $a_1 < 5$ and $a_2 < 5$. If $a_k < 5$ then

$$a_{k+1} = \sqrt{15+2a_k} < \sqrt{15+2(5)} = \sqrt{25} = 5.$$

Therefore, $a_n < 5$ for all n, by induction. Since $\{a_n\}$ is increasing and bounded above, it converges. Let $\lim a_n = a$. Then

$$a = \sqrt{15+2a} \Rightarrow a^2 - 2a - 15 = 0 \Rightarrow a = -3, \text{ or } a = 5.$$

Since $a > a_1$, we must have $\lim a_n = 5$.

32. Let $a_n = \left(1+\frac{1}{n}\right)^n$ so $\ln a_n = n\ln\left(1+\frac{1}{n}\right)$.

a) If $f(x) = x\ln\left(1+\frac{1}{x}\right) = x\ln(x+1) - x\ln x$, then

$$\begin{aligned}f'(x) &= \ln(x+1) + \frac{x}{x+1} - \ln x - 1 \\ &= \ln\left(\frac{x+1}{x}\right) - \frac{1}{x+1} \\ &= \int_x^{x+1} \frac{dt}{t} - \frac{1}{x+1} \\ &> \frac{1}{x+1}\int_x^{x+1} dt - \frac{1}{x+1} \\ &= \frac{1}{x+1} - \frac{1}{x+1} = 0.\end{aligned}$$

Since $f'(x) > 0$, $f(x)$ must be an increasing function. Thus, $\{a_n\} = \{e^{f(x_n)}\}$ is increasing.

b) Since $\ln x \le x - 1$,

$$\ln a_k = k\ln\left(1+\frac{1}{k}\right) \le k\left(1+\frac{1}{k}-1\right) = 1$$

which implies that $a_k \le e$ for all k. Since $\{a_n\}$ is increasing, e is an upper bound for $\{a_n\}$.

33. Suppose $\{a_n\}$ is ultimately increasing, say $a_{n+1} \ge a_n$ if $n \ge N$.
Case I. If there exists a real number K such that $a_n \le K$ for all n, then $\lim a_n = a$ exists by completeness.
Case II. Otherwise, for every integer K, there exists $n \ge N$ such that $a_n > K$, and hence $a_j > K$ for all $j \ge n$. Thus $\lim a_n = \infty$.

If $\{a_n\}$ is ultimately decreasing, then either it is bounded below, and therefore converges, or else it is unbounded below, and therefore diverges to negative infinity.

34. If $\{|a_n|\}$ is bounded then it is bounded above, and there exists a constant K such that $|a_n| \le K$ for all n. Therefore, $-K \le a_n \le K$ for all n, and so $\{a_n\}$ is bounded above and below, and is therefore bounded.

35. Suppose $\lim_{n\to\infty}|a_n| = 0$. Given any $\epsilon > 0$, there exists an integer $N = N(\epsilon)$ such that if $n > N$, then $||a_n| - 0| < \epsilon$. In this case $|a_n - 0| = |a_n| = ||a_n| - 0| < \epsilon$, so $\lim_{n\to\infty} a_n = 0$.

36. a) "If $\lim a_n = \infty$ and $\lim b_n = L > 0$, then $\lim a_n b_n = \infty$" is TRUE. Let R be an arbitrary, large positive number. Since $\lim a_n = \infty$, and $L > 0$, it must be true that $a_n \ge \dfrac{2R}{L}$ for n sufficiently large. Since $\lim b_n = L$, it must also be that $b_n \ge \dfrac{L}{2}$ for n sufficiently large. Therefore $a_n b_n \ge \dfrac{2R}{L}\dfrac{L}{2} = R$ for n sufficiently large. Since R is arbitrary, $\lim a_n b_n = \infty$.

b) "If $\lim a_n = \infty$ and $\lim b_n = -\infty$, then $\lim(a_n + b_n) = 0$" is FALSE. Let $a_n = 1 + n$ and $b_n = -n$; then $\lim a_n = \infty$ and $\lim b_n = -\infty$ but $\lim(a_n + b_n) = 1$.

c) "If $\lim a_n = \infty$ and $\lim b_n = -\infty$, then $\lim a_n b_n = -\infty$" is TRUE. Let R be an arbitrary, large positive number. Since $\lim a_n = \infty$ and $\lim b_n = -\infty$, we must have $a_n \ge \sqrt{R}$ and $b_n \le -\sqrt{R}$, for all sufficiently large n. Thus $a_n b_n \le -R$, and $\lim a_n b_n = -\infty$.

d) "If neither $\{a_n\}$ nor $\{b_n\}$ converges, then $\{a_n b_n\}$ does not converge" is FALSE. Let $a_n = b_n = (-1)^n$; then $\lim a_n$ and $\lim b_n$ both diverge. But $a_n b_n = (-1)^{2n} = 1$ and $\{a_n b_n\}$ does converge (to 1).

e) "If $\{|a_n|\}$ converges, then $\{a_n\}$ converges" is FALSE. Let $a_n = (-1)^n$. Then $\lim_{n\to\infty}|a_n| = \lim_{n\to\infty} 1 = 1$, but $\lim_{n\to\infty} a_n$ does not exist.

Section 9.2 Infinite Series (page 484)

1.
$$\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots = \frac{1}{3}\left(1+\frac{1}{3}+\left(\frac{1}{3}\right)^2+\cdots\right) = \frac{1}{3}\cdot\frac{1}{1-\frac{1}{3}} = \frac{1}{2}.$$

2.
$$3-\frac{3}{4}+\frac{3}{16}-\frac{3}{64}+\cdots = \sum_{n=1}^{\infty} 3\left(-\frac{1}{4}\right)^{n-1} = \frac{3}{1+\frac{1}{4}} = \frac{12}{5}.$$

3.
$$\begin{aligned}\sum_{n=5}^{\infty}\frac{1}{(2+\pi)^{2n}} &= \frac{1}{(2+\pi)^{10}}+\frac{1}{(2+\pi)^{12}}+\frac{1}{(2+\pi)^{14}}+\cdots\\ &= \frac{1}{(2+\pi)^{10}}\left[1+\frac{1}{(2+\pi)^2}+\frac{1}{(2+\pi)^4}+\cdots\right]\\ &= \frac{1}{(2+\pi)^{10}}\cdot\frac{1}{1-\frac{1}{(2+\pi)^2}} = \frac{1}{(2+\pi)^8\left[(2+\pi)^2-1\right]}.\end{aligned}$$

4.
$$\sum_{n=0}^{\infty}\frac{5}{10^{3n}} = 5\left[1+\frac{1}{1000}+\left(\frac{1}{1000}\right)^2+\cdots\right] = \frac{5}{1-\frac{1}{1000}} = \frac{5000}{999}.$$

5.
$$\begin{aligned}\sum_{n=2}^{\infty}\frac{(-5)^n}{8^{2n}} &= \frac{(-5)^2}{8^4}+\frac{(-5)^3}{8^6}+\frac{(-5)^4}{8^8}+\cdots\\ &= \frac{25}{8^4}\left[1-\frac{5}{64}+\frac{5^2}{64^2}-\cdots\right]\\ &= \frac{25}{8^4}\cdot\frac{1}{1+\frac{5}{64}} = \frac{25}{64\times 69} = \frac{25}{4416}.\end{aligned}$$

6.
$$\sum_{n=0}^{\infty}\frac{1}{e^n} = 1+\frac{1}{e}+\left(\frac{1}{e}\right)^2+\cdots = \frac{1}{1-\frac{1}{e}} = \frac{e}{e-1}.$$

7.
$$\sum_{k=0}^{\infty}\frac{2^{k+3}}{e^{k-3}} = 8e^3\sum_{k=0}^{\infty}\left(\frac{2}{e}\right)^k = \frac{8e^3}{1-\frac{2}{e}} = \frac{8e^4}{e-2}.$$

8. $\sum_{j=1}^{\infty}\pi^{j/2}\cos(j\pi) = \sum_{j=2}^{\infty}(-1)^j\pi^{j/2}$ diverges because $\lim_{j\to\infty}(-1)^j\pi^{j/2}$ does not exist.

9. $\sum_{n=1}^{\infty}\dfrac{3+2^n}{2^{n+2}}$ diverges to ∞ because
$$\lim_{n\to\infty}\frac{3+2^n}{2^{n+2}} = \lim_{n\to\infty}\frac{\frac{3}{2^n}+1}{4} = \frac{1}{4} > 0.$$

10.
$$\begin{aligned}\sum_{n=0}^{\infty}\frac{3+2^n}{3^{n+2}} &= \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n+\frac{1}{9}\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n\\ &= \frac{1}{3}\cdot\frac{1}{1-\frac{1}{3}}+\frac{1}{9}\cdot\frac{1}{1-\frac{2}{3}} = \frac{1}{2}+\frac{1}{3} = \frac{5}{6}.\end{aligned}$$

11. Since $\dfrac{1}{n(n+2)} = \dfrac{1}{2}\left(\dfrac{1}{n} - \dfrac{1}{n+2}\right)$, therefore

$$\begin{aligned} s_n &= \frac{1}{1\times 3} + \frac{1}{2\times 4} + \frac{1}{3\times 5} + \cdots + \frac{1}{n(n+2)} \\ &= \frac{1}{2}\Big[\frac{1}{1} - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \frac{1}{4} - \frac{1}{6} + \cdots \\ &\qquad + \frac{1}{n-2} - \frac{1}{n} + \frac{1}{n-1} - \frac{1}{n+1} + \frac{1}{n} - \frac{1}{n+2}\Big] \\ &= \frac{1}{2}\left[1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right]. \end{aligned}$$

Thus $\lim s_n = \dfrac{3}{4}$, and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \frac{3}{4}$.

12. Let

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \frac{1}{1\times 3} + \frac{1}{3\times 5} + \frac{1}{5\times 7} + \cdots.$$

Since $\dfrac{1}{(2n-1)(2n+1)} = \dfrac{1}{2}\left(\dfrac{1}{2n-1} - \dfrac{1}{2n+1}\right)$, the partial sum is

$$\begin{aligned} s_n &= \frac{1}{2}\left(1 - \frac{1}{3}\right) + \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right) + \cdots \\ &\quad + \frac{1}{2}\left(\frac{1}{2n-3} - \frac{1}{2n-1}\right) + \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right) \\ &= \frac{1}{2}\left(1 - \frac{1}{2n+1}\right). \end{aligned}$$

Hence,

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \lim s_n = \frac{1}{2}.$$

13. Since $\dfrac{1}{(3n-2)(3n+1)} = \dfrac{1}{3}\left(\dfrac{1}{3n-2} - \dfrac{1}{3n+1}\right)$, therefore

$$\begin{aligned} s_n &= \frac{1}{1\times 4} + \frac{1}{4\times 7} + \frac{1}{7\times 10} + \cdots + \frac{1}{(3n-2)(3n+1)} \\ &= \frac{1}{3}\Big[\frac{1}{1} - \frac{1}{4} + \frac{1}{4} - \frac{1}{7} + \frac{1}{7} - \frac{1}{10} + \cdots \\ &\qquad + \frac{1}{3n-5} - \frac{1}{3n-2} + \frac{1}{3n-2} - \frac{1}{3n+1}\Big] \\ &= \frac{1}{3}\left(1 - \frac{1}{3n+1}\right) \quad \to \quad \frac{1}{3}. \end{aligned}$$

Thus $\sum_{n=1}^{\infty} \dfrac{1}{(3n-2)(3n+1)} = \dfrac{1}{3}$.

14. Since

$$\frac{1}{n(n+1)(n+2)} = \frac{1}{2}\left[\frac{1}{n} - \frac{2}{n+1} + \frac{1}{n+2}\right],$$

the partial sum is

$$\begin{aligned} s_n &= \frac{1}{2}\left(1 - \frac{2}{2} + \frac{1}{3}\right) + \frac{1}{2}\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) + \cdots \\ &\quad + \frac{1}{2}\left(\frac{1}{n-1} - \frac{2}{n} + \frac{1}{n+1}\right) + \frac{1}{2}\left(\frac{1}{n} - \frac{2}{n+1} + \frac{1}{n+2}\right) \\ &= \frac{1}{2}\left(\frac{1}{2} - \frac{1}{n+1} + \frac{1}{n+2}\right). \end{aligned}$$

Hence,

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} = \lim s_n = \frac{1}{4}.$$

15. Since $\dfrac{1}{2n-1} > \dfrac{1}{2n} = \dfrac{1}{2}\cdot\dfrac{1}{n}$, therefore the partial sums of the given series exceed half those of the divergent harmonic series $\sum(1/2n)$. Hence the given series diverges to infinity.

16. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+2}$ diverges to infinity since $\lim \dfrac{n}{n+2} = 1 > 0$.

17. Since $n^{-1/2} = \dfrac{1}{\sqrt{n}} \ge \dfrac{1}{n}$ for $n \ge 1$, we have

$$\sum_{k=1}^{n} k^{-1/2} \ge \sum_{k=1}^{n} \frac{1}{k} \to \infty,$$

as $n \to \infty$ (harmonic series). Thus $\sum n^{-1/2}$ diverges to infinity.

18. $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n+1} = 2\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots\right)$ diverges to infinity since it is just twice the harmonic series with the first term omitted.

19. $s_n = -1 + 1 - 1 + \cdots + (-1)^n = \begin{cases} -1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$. Thus $\lim s_n$ does not exist, and $\sum(-1)^n$ diverges.

20. Since $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$, the given series is $\sum_{n=1}^{\infty} \dfrac{2}{n(n+1)}$ which converges to 2 by the result of Example 3 of this section.

21. The total distance is

$$\begin{aligned} &2 + 2\left[2\times\frac{3}{4} + 2\times\left(\frac{3}{4}\right)^2 + \cdots\right] \\ &= 2 + 2\times\frac{3}{2}\left[1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \cdots\right] \\ &= 2 + \frac{3}{1 - \frac{3}{4}} = 14 \text{ metres.} \end{aligned}$$

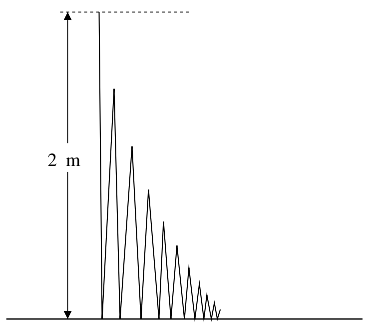

Fig. 9.2.21

22. The balance at the end of 8 years is

$$\begin{aligned} s_n &= 1000\Big[(1.1)^8 + (1.1)^7 + \cdots + (1.1)^2 + (1.1)\Big] \\ &= 1000(1.1)\left(\frac{(1.1)^8 - 1}{1.1 - 1}\right) \approx \$12,579.48. \end{aligned}$$

23. For $n > N$ let $s_n = \sum_{j=1}^{n} a_j$, and $S_n = \sum_{j=N}^{n} a_j$.
Then $s_n = S_n + C$, where $C = \sum_{j=1}^{N-1} a_j$. We have

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} S_n + C :$$

either both sides exist or neither does. Hence $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=N}^{\infty}$ both converge or neither does.

24. If $\{a_n\}$ is ultimately positive, then the sequence $\{s_n\}$ of partial sums of the series must be ultimately increasing. By Theorem 2, if $\{s_n\}$ is ultimately increasing, then either it is bounded above, and therefore convergent, or else it is not bounded above and diverges to infinity. Since $\sum a_n = \lim s_n$, $\sum a_n$ must either converge when $\{s_n\}$ converges and $\lim s_n = s$ exists, or diverge to infinity when $\{s_n\}$ diverges to infinity.

25. If $\{a_n\}$ is ultimately negative, then the series $\sum a_n$ must either converge (if its partial sums are bounded below), or diverge to $-\infty$ (if its partial sums are not bounded below).

26. "If $a_n = 0$ for every n, then $\sum a_n$ converge" is TRUE because $s_n = \sum_{k=0}^{n} 0 = 0$, for every n, and so $\sum a_n = \lim s_n = 0$.

27. "If $\sum a_n$ converges, then $\sum 1/a_n$ diverges to infinity" is FALSE. A counterexample is $\sum (-1)^n/2^n$.

28. "If $\sum a_n$ and $\sum b_n$ both diverge, then so does $\sum(a_n + b_n)$" is FALSE. Let $a_n = \dfrac{1}{n}$ and $b_n = -\dfrac{1}{n}$, then $\sum a_n = \infty$ and $\sum b_n = -\infty$ but $\sum(a_n + b_n) = \sum(0) = 0$.

29. "If $a_n \ge c > 0$ for all n, then $\sum a_n$ diverges to infinity" is TRUE. We have

$$s_n = a_1 + a_2 + a_3 + \cdots + a_n \ge c + c + c + \cdots + c = nc,$$

and $nc \to \infty$ as $n \to \infty$.

30. "If $\sum a_n$ diverges and $\{b_n\}$ is bounded, then $\sum a_n b_n$ diverges" is FALSE. Let $a_n = \dfrac{1}{n}$ and $b_n = \dfrac{1}{n+1}$. Then $\sum a_n = \infty$ and $0 \le b_n \le 1/2$. But $\sum a_n b_n = \sum \dfrac{1}{n(n+1)}$ which converges by Example 3.

31. "If $a_n > 0$ and $\sum a_n$ converges, then $\sum a_n^2$ converges" is TRUE.
Since $\sum a_n$ converges, therefore $\lim a_n = 0$.
Thus there exists N such that $0 < a_n \le 1$ for $n \ge N$.
Thus $0 < a_n^2 \le a_n$ for $n \ge N$.
If $S_n = \sum_{k=N}^{n} a_k^2$ and $s_n = \sum_{k=N}^{n} a_k$, then $\{S_n\}$ is increasing and bounded above:

$$S_n \le s_n \le \sum_{k=1}^{\infty} a_k < \infty.$$

Thus $\sum_{k=N}^{\infty} a_k^2$ converges, and so $\sum_{k=1}^{\infty} a_k^2$ converges.

Section 9.3 Convergence Tests for Positive Series (page 494)

1. $\sum \dfrac{1}{n^2+1}$ converges by comparison with $\sum \dfrac{1}{n^2}$ since $0 < \dfrac{1}{n^2+1} < \dfrac{1}{n^2}$.

2. $\sum_{n=1}^{\infty} \dfrac{n}{n^4-2}$ converges by comparison with $\sum_{n=1}^{\infty} \dfrac{1}{n^3}$ since

$$\lim \frac{\left(\dfrac{n}{n^4-2}\right)}{\left(\dfrac{1}{n^3}\right)} = 1, \quad \text{and} \quad 0 < 1 < \infty.$$

3. $\sum \dfrac{n^2+1}{n^3+1}$ diverges to infinity by comparison with $\sum \dfrac{1}{n}$, since $\dfrac{n^2+1}{n^3+1} > \dfrac{1}{n}$.

4. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+n+1}$ converges by comparison with $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ since

$$\lim \frac{\left(\frac{\sqrt{n}}{n^2+n+1}\right)}{\left(\frac{1}{n^{3/2}}\right)} = 1, \quad \text{and} \quad 0 < 1 < \infty.$$

5. Since $\sin x \le x$ for $x \ge 0$, we have

$$\left|\sin \frac{1}{n^2}\right| = \sin \frac{1}{n^2} \le \frac{1}{n^2},$$

so $\sum \left|\sin \frac{1}{n^2}\right|$ converges by comparison with $\sum \frac{1}{n^2}$.

6. $\sum_{n=8}^{\infty} \frac{1}{\pi^n + 5}$ converges by comparison with the geometric series $\sum_{n=8}^{\infty} \left(\frac{1}{\pi}\right)^n$ since $0 < \frac{1}{\pi^n+5} < \frac{1}{\pi^n}$.

7. Since $(\ln n)^3 < n$ for large n, $\sum \frac{1}{(\ln n)^3}$ diverges to infinity by comparison with $\sum \frac{1}{n}$.

8. $\sum_{n=1}^{\infty} \frac{1}{\ln(3n)}$ diverges to infinity by comparison with the harmonic series $\sum_{n=1}^{\infty} \frac{1}{3n}$ since $\frac{1}{\ln(3n)} > \frac{1}{3n}$ for $n \ge 1$.

9. Since $\lim_{n\to\infty} \frac{\pi^n}{\pi^n - n^\pi} = \lim \frac{1}{1 - \frac{n^\pi}{\pi^n}} = 1$, the series $\sum \frac{1}{\pi^n - n^\pi}$ converges by comparison with the geometric series $\sum \frac{1}{\pi^n}$.

10. $\sum_{n=0}^{\infty} \frac{1+n}{2+n}$ diverges to infinity since $\lim \frac{1+n}{2+n} = 1 > 0$.

11. $\sum \frac{1+n^{4/3}}{2+n^{5/3}}$ diverges to infinity by comparison with the divergent p-series $\sum \frac{1}{n^{1/3}}$, since

$$\lim_{n\to\infty} \frac{1+n^{4/3}}{2+n^{5/3}} \bigg/ \frac{1}{n^{1/3}} = \lim \frac{n^{1/3}+n^{5/3}}{2+n^{5/3}} = 1.$$

12. $\sum_{n=1}^{\infty} \frac{n^2}{1+n\sqrt{n}}$ diverges to infinity since

$$\lim \frac{n^2}{1+n\sqrt{n}} = \infty.$$

13. $\sum_{n=3}^{\infty} \frac{1}{n \ln n \sqrt{\ln \ln n}}$ diverges to infinity by the integral test, since

$$\int_3^\infty \frac{dt}{t \ln t \sqrt{\ln \ln t}} = \int_{\ln\ln 3}^\infty \frac{du}{\sqrt{u}} = \infty.$$

14. $\sum_{n=2}^{\infty} \frac{1}{n \ln n (\ln \ln n)^2}$ converges by the integral test:

$$\int_a^\infty \frac{dt}{t \ln t (\ln \ln t)^2} = \int_{\ln\ln a}^\infty \frac{du}{u^2} < \infty \quad \text{if } \ln\ln a > 0.$$

15. $\sum \frac{1-(-1)^n}{n^4}$ converges by comparison with $\sum \frac{1}{n^4}$, since $0 \le \frac{1-(-1)^n}{n^4} \le \frac{2}{n^4}$.

16. The series

$$\sum_{n=1}^{\infty} \frac{1+(-1)^n}{\sqrt{n}} = 0 + \frac{2}{\sqrt{2}} + 0 + \frac{2}{\sqrt{4}} + 0 + \frac{2}{\sqrt{6}} + \cdots$$
$$= 2\sum_{k=1}^{\infty} \frac{1}{\sqrt{2k}} = \sqrt{2}\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$$

diverges to infinity.

17. Since $\frac{1}{2^n(n+1)} < \frac{1}{2^n}$, the series $\sum \frac{1}{2^n(n+1)}$ converges by comparison with the geometric series $\sum \frac{1}{2^n}$.

18. $\sum_{n=1}^{\infty} \frac{n^4}{n!}$ converges by the ratio test since

$$\lim \frac{\frac{(n+1)^4}{(n+1)!}}{\frac{n^4}{n!}} = \lim \left(\frac{n+1}{n}\right)^4 \frac{1}{n+1} = 0.$$

19. $\sum \frac{n!}{n^2 e^n}$ diverges to infinity by the ratio test, since

$$\rho = \lim \frac{(n+1)!}{(n+1)^2 e^{n+1}} \cdot \frac{n^2 e^n}{n!} = \frac{1}{e} \lim \frac{n^2}{n+1} = \infty.$$

20. $\sum_{n=1}^{\infty} \frac{(2n)!6^n}{(3n)!}$ converges by the ratio test since

$$\lim \frac{(2n+2)!6^{n+1}}{(3n+3)!} \bigg/ \frac{(2n)!6^n}{(3n)!}$$
$$= \lim \frac{(2n+2)(2n+1)6}{(3n+3)(3n+2)(3n+1)} = 0.$$

21. $\displaystyle\sum_{n=2}^{\infty} \frac{\sqrt{n}}{3^n \ln n}$ converges by the ratio test, since

$$\begin{aligned}\rho &= \lim \frac{\sqrt{n+1}}{3^{n+1}\ln(n+1)} \cdot \frac{3^n \ln n}{\sqrt{n}} \\ &= \frac{1}{3}\lim\sqrt{\frac{n+1}{n}} \cdot \lim \frac{\ln n}{\ln(n+1)} = \frac{1}{3} < 1.\end{aligned}$$

22. $\displaystyle\sum_{n=0}^{\infty} \frac{n^{100}\,2^n}{\sqrt{n!}}$ converges by the ratio test since

$$\begin{aligned}&\lim \frac{(n+1)^{100}\,2^{n+1}}{\sqrt{(n+1)!}} \Big/ \frac{n^{100}\,2^n}{\sqrt{n!}} \\ &= \lim 2\left(\frac{n+1}{n}\right)^{100} \frac{1}{\sqrt{n+1}} = 0.\end{aligned}$$

23. $\displaystyle\sum \frac{(2n)!}{(n!)^3}$ converges by the ratio test, since

$$\rho = \lim \frac{(2n+2)!}{((n+1)!)^3} \cdot \frac{(n!)^3}{(2n)!} = \lim \frac{(2n+2)(2n+1)}{(n+1)^3} = 0 < 1.$$

24. $\displaystyle\sum_{n=1}^{\infty} \frac{1+n!}{(1+n)!}$ diverges by comparison with the harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+1}$ since $\displaystyle\frac{1+n!}{(1+n)!} > \frac{n!}{(1+n)!} = \frac{1}{n+1}$.

25. $\displaystyle\sum \frac{2^n}{3^n - n^3}$ converges by the ratio test since

$$\begin{aligned}\rho &= \lim \frac{2^{n+1}}{3^{n+1}-(n+1)^3} \cdot \frac{3^n - n^3}{2^n} \\ &= \frac{2}{3}\lim \frac{3^n - n^3}{3^n - \dfrac{(n+1)^3}{3}} = \frac{2}{3}\lim \frac{1-\dfrac{n^3}{3^n}}{1-\dfrac{(n+1)^3}{3^{n+1}}} = \frac{2}{3} < 1.\end{aligned}$$

26. $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{\pi^n n!}$ converges by the ratio test since

$$\lim \frac{(n+1)^{n+1}}{\pi^{(n+1)}(n+1)!} \Big/ \frac{n^n}{\pi^n n!} = \frac{1}{\pi}\lim\left(1+\frac{1}{n}\right)^n = \frac{e}{\pi} < 1.$$

27. $f(x) = 1/x^4$ is positive, continuous, and decreasing on $[1,\infty)$. Let

$$A_n = \int_n^{\infty} \frac{dx}{x^4} = \lim_{R\to\infty}\left(-\frac{1}{3x^3}\right)\Big|_n^R = \frac{1}{3n^3}.$$

We use the approximation

$$s \approx s_n^* = s_n + \frac{1}{2}\left(\frac{1}{3(n+1)^3} + \frac{1}{3n^3}\right).$$

The error satisfies

$$\begin{aligned}|s - s_n^*| &\le \frac{1}{2}\left(\frac{1}{3n^3} - \frac{1}{3(n+1)^3}\right) \\ &= \frac{1}{6}\,\frac{(n+1)^3 - n^3}{n^3(n+1)^3} \\ &= \frac{1}{6}\,\frac{3n^2+3n+1}{n^3(n+1)^3} < \frac{7}{6n^4}.\end{aligned}$$

We have used $3n^2+3n+1 \le 7n^2$ and $n^3(n+1)^3 > n^6$ to obtain the last inequality. We will have $|s - s_n^*| < 0.001$ provided

$$\frac{7}{6n^4} < 0.001,$$

that is, if $n^4 > 7000/6$. Since $6^4 = 1296 > 7000/6$, $n = 6$ will do. Thus

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n^4} &\approx s_6^* = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \frac{1}{6^4} + \frac{1}{6}\left(\frac{1}{7^3} + \frac{1}{6^3}\right) \\ &\approx 1.082 \qquad \text{with error less than 0.001 in absolute value.}\end{aligned}$$

28. Since $f(x) = \dfrac{1}{x^3}$ is positive, continuous and decreasing on $[1,\infty)$, for any $n = 1, 2, 3, \ldots$, we have

$$s_n + A_{n+1} \le s \le s_n + A_n$$

where $s_n = \displaystyle\sum_{k=1}^{n}\frac{1}{k^3}$ and $A_n = \displaystyle\int_n^{\infty}\frac{dx}{x^3} = \frac{1}{2n^2}$. If $s_n^* = s_n + \dfrac{1}{2}(A_{n+1} + A_n)$, then

$$\begin{aligned}|s_n - s_n^*| &\le \frac{A_n - A_{n+1}}{2} = \frac{1}{4}\left[\frac{1}{n^2} - \frac{1}{(n+1)^2}\right] \\ &= \frac{1}{4}\,\frac{2n+1}{n^2(n+1)^2} < 0.001\end{aligned}$$

if $n = 8$. Thus, the error in the approximation $s \approx s_8^*$ is less than 0.001.

29. Since $f(x) = \dfrac{1}{x^{3/2}}$ is positive, continuous and decreasing on $[1,\infty)$, for any $n = 1, 2, 3, \ldots$, we have

$$s_n + A_{n+1} \le s \le s_n + A_n$$

where $s_n = \sum_{k=1}^{n} \frac{1}{k^{3/2}}$ and $A_n = \int_n^\infty \frac{dx}{x^{3/2}} = \frac{2}{\sqrt{n}}$. If $s_n^* = s_n + \frac{1}{2}(A_{n+1} + A_n) = s_n + \left(\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}}\right)$, then

$$\begin{aligned}|s_n - s_n^*| &\le \frac{A_n - A_{n+1}}{2}\\ &= \frac{1}{2}\left(\frac{2}{\sqrt{n}} - \frac{2}{\sqrt{n+1}}\right)\\ &= \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n}\sqrt{n+1}} = \frac{1}{\sqrt{n}\sqrt{n+1}(\sqrt{n}+\sqrt{n+1})}\\ &< \frac{1}{2n^{3/2}} < 0.001\end{aligned}$$

if $n \ge 63$. Thus, the error in the approximation $s \approx s_{63}^*$ is less than 0.001.

30. Again, we have $s_n + A_{n+1} \le s \le s_n + A_n$ where $s_n = \sum_{k=1}^{n} \frac{1}{k^2+4}$ and

$$A_n = \int_n^\infty \frac{dx}{x^2+4} = \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\Big|_n^\infty = \frac{\pi}{4} - \frac{1}{2}\tan^{-1}\left(\frac{n}{2}\right).$$

If $s_n^* = s_n + \frac{1}{2}(A_{n+1} + A_n)$, then

$$\begin{aligned}|s_n - s_n^*| &\le \frac{A_n - A_{n+1}}{2}\\ &= \frac{1}{2}\left[\frac{\pi}{4} - \frac{1}{2}\tan^{-1}\left(\frac{n}{2}\right) - \frac{\pi}{4} + \frac{1}{2}\tan^{-1}\left(\frac{n+1}{2}\right)\right]\\ &= \frac{1}{4}\left[\tan^{-1}\left(\frac{n+1}{2}\right) - \tan^{-1}\left(\frac{n}{2}\right)\right] = \frac{1}{4}(a-b),\end{aligned}$$

where $a = \tan^{-1}\left(\frac{n+1}{2}\right)$ and $b = \tan^{-1}\left(\frac{n}{2}\right)$. Now

$$\begin{aligned}\tan(a-b) &= \frac{\tan a - \tan b}{1 + \tan a \tan b}\\ &= \frac{\left(\frac{n+1}{2}\right) - \left(\frac{n}{2}\right)}{1 + \left(\frac{n+1}{2}\right)\left(\frac{n}{2}\right)}\\ &= \frac{2}{n^2+n+4}\\ \Leftrightarrow \quad a - b &= \tan^{-1}\left(\frac{2}{n^2+n+4}\right).\end{aligned}$$

We want error less than 0.001:

$$\begin{aligned}&\frac{1}{4}(a-b) = \frac{1}{4}\tan^{-1}\left(\frac{2}{n^2+n+4}\right) < 0.001\\ \Leftrightarrow \quad &\frac{2}{n^2+n+4} < \tan 0.004\\ \Leftrightarrow \quad &n^2 + n > 2\cot(0.004) - 4 \approx 496.\end{aligned}$$

$n = 22$ will do. The approximation $s \approx s_{22}^*$ has error less than 0.001.

31. We have $s = \sum_{k=1}^{\infty} \frac{1}{2^k k!}$ and

$$s_n = \sum_{k=1}^{n} \frac{1}{2^k k!} = \frac{1}{2} + \frac{1}{2^2 2!} + \frac{1}{2^3 3!} + \cdots + \frac{1}{2^n n!}.$$

Then

$$\begin{aligned}0 &< s - s_n\\ &= \frac{1}{2^{n+1}(n+1)!} + \frac{1}{2^{n+2}(n+2)!} + \frac{1}{2^{n+3}(n+3)!} + \cdots\\ &= \frac{1}{2^{n+1}(n+1)!}\left[1 + \frac{1}{2(n+2)} + \frac{1}{2^2(n+2)(n+3)} + \cdots\right]\\ &< \frac{1}{2^{n+1}(n+1)!}\left[1 + \frac{1}{2(n+2)} + \left(\frac{1}{2(n+2)}\right)^2 + \cdots\right]\\ &= \frac{1}{2^{n+1}(n+1)!} \cdot \frac{1}{1 - \frac{1}{2(n+2)}}\\ &= \frac{n+2}{2^n(n+1)!(2n+3)} < 0.001\end{aligned}$$

if $n = 4$. Thus, $s \approx s_4 = \frac{1}{2} + \frac{1}{2^2 2!} + \frac{1}{2^3 3!} + \frac{1}{2^4 4!}$ with error less than 0.001.

32. We have $s = \sum_{k=1}^{\infty} \frac{1}{(2k-1)!}$ and

$$s_n = \sum_{k=1}^{n} \frac{1}{(2k-1)!} = \frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + \cdots + \frac{1}{(2n-1)!}.$$

Then

$$\begin{aligned}0 < s - s_n &= \frac{1}{(2n+1)!} + \frac{1}{(2n+3)!} + \frac{1}{(2n+5)!} + \cdots\\ &= \frac{1}{(2n+1)!}\left[1 + \frac{1}{(2n+2)(2n+3)} + \frac{1}{(2n+2)(2n+3)(2n+4)(2n+5)} + \cdots\right]\\ &< \frac{1}{(2n+1)!}\left[1 + \frac{1}{(2n+2)(2n+3)} + \frac{1}{[(2n+2)(2n+3)]^2} + \cdots\right]\\ &= \frac{1}{(2n+1)!}\left[\frac{1}{1 - \frac{1}{(2n+2)(2n+3)}}\right]\\ &= \frac{1}{(2n+1)!}\frac{4n^2+10n+6}{4n^2+10n+5} < 0.001\end{aligned}$$

if $n = 3$. Thus, $s \approx s_3 = 1 + \frac{1}{3!} + \frac{1}{5!} = 1.175$ with error less than 0.001.

33. We have $s = \sum_{k=0}^{\infty} \frac{2^k}{(2k)!}$ and $s_n = \sum_{k=0}^{n-1} \frac{2^k}{(2k)!}$. Thus

$$\begin{aligned}
0 &< s - s_n \\
&= \frac{2^n}{(2n)!} + \frac{2^{n+1}}{(2n+2)!} + \frac{2^{n+2}}{(2n+4)!} + \cdots \\
&= \frac{2^n}{(2n)!}\left[1 + \frac{2}{(2n+1)(2n+2)} + \frac{2^2}{(2n+1)(2n+2)(2n+3)(2n+4)} + \cdots\right] \\
&< \frac{2^n}{(2n)!}\left[1 + \frac{2}{(2n+1)(2n+2)} + \left(\frac{2}{(2n+1)(2n+2)}\right)^2 + \cdots\right] \\
&= \frac{2^n}{(2n)!} \cdot \frac{1}{1 - \frac{2}{(2n+1)(2n+2)}} \\
&= \frac{2^n}{(2n)!} \cdot \frac{4n^2 + 6n + 2}{4n^2 + 6n} < 0.001
\end{aligned}$$

if $n = 4$. Thus, $s \approx s_4$ with error less than 0.001.

34. We have $s = \sum_{k=1}^{\infty} \frac{1}{k^k}$ and

$$s_n = \sum_{k=1}^{n} \frac{1}{k^k} = \frac{1}{1} + \frac{1}{2^2} + \frac{1}{3^3} + \cdots + \frac{1}{n^n}.$$

Then

$$\begin{aligned}
0 < s - s_n &= \frac{1}{(n+1)^{n+1}} + \frac{1}{(n+2)^{n+2}} + \frac{1}{(n+3)^{n+3}} + \cdots \\
&< \frac{1}{(n+1)^{n+1}}\left[1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + \cdots\right] \\
&= \frac{1}{(n+1)^{n+1}}\left[\frac{1}{1 - \frac{1}{n+1}}\right] \\
&= \frac{1}{n(n+1)^n} < 0.001
\end{aligned}$$

if $n = 4$. Thus, $s \approx s_4 = 1 + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} = 1.291$ with error less than 0.001.

35. Let $f(x) = \frac{1}{1+x^2}$. Then f is decreasing on $[1, \infty)$. Since $\sum_{n=1}^{\infty} f(n)$ is a right Riemann sum for

$$\int_0^{\infty} f(x)\,dx = \lim_{R\to\infty} \tan^{-1} x \Big|_0^R = \frac{\pi}{2},$$

$\sum_{n=1}^{\infty} \frac{1}{1+n^2} = \sum_{n=1}^{\infty} f(n)$ converges by the integral test, and its sum is less than $\pi/2$.

36. Let $u = \ln\ln t$, $du = \frac{dt}{t \ln t}$ and $\ln\ln a > 0$; then

$$\int_a^{\infty} \frac{dt}{t \ln t (\ln\ln t)^p} = \int_{\ln\ln a}^{\infty} \frac{du}{u^p}$$

will converge if and only if $p > 1$. Thus, $\sum_{n=3}^{\infty} \frac{1}{n \ln n (\ln\ln n)^p}$ will converge if and only if $p > 1$. Similarly,

$$\sum_{n=N}^{\infty} \frac{1}{n(\ln n)(\ln\ln n)\cdots(\ln_j n)(\ln_{j+1} n)^p}$$

converges if and only if $p > 1$, where N is large enough that $\ln_j N > 1$.

37. Let $a_n > 0$ for all n. (Let's forget the "ultimately" part.) Let $\sigma = \lim(a_n)^{1/n}$.

CASE I. Suppose $\sigma < 1$. Pick λ such that $\sigma < \lambda < 1$. Then there exists N such that $(a_n)^{1/n} \le \lambda$ for all $n \ge N$. Therefore

$$a_N \le \lambda^N, \quad a_{N+1} \le \lambda^{N+1}, \quad a_{N+2} \le \lambda^{N+2}, \ldots.$$

Thus $\sum_{n=N}^{\infty} a_n$ converges by comparison with the geometric series $\sum_{n=N}^{\infty} \lambda^n$, and $\sum_{n=1}^{\infty} a_n$ also converges.

CASE II. Suppose $\sigma > 1$. Then $(a_n)^{1/n} \ge 1$, and $a_n \ge 1$, for all sufficiently large values of n. Therefore $\lim a_n \ne 0$ and $\sum a_n$ must diverge. Since $a_n > 0$ it diverges to infinity.

CASE III. Let $a_n = \frac{1}{n}$ and $b_n = \frac{1}{n^2}$. Since $\lim n^{1/n} = 1$ (because $\lim \frac{\ln n}{n} = 0$), we have $\lim(a_n)^{1/n} = 1$ and $\lim(b_n)^{1/n} = 1$. That is, $\sigma = 1$ for both series. But $\sum a_n$ diverges to infinity, while $\sum b_n$ converges. Thus the case $\sigma = 1$ provides no information on the convergence or divergence of a series.

38. Let $a_n = 2^{n+1}/n^n$. Then

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{2 \times 2^{1/n}}{n} = 0.$$

Since this limit is less than 1, $\sum_{n=1}^{\infty} a_n$ converges by the root test.

39. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^{n^2}$ converges by the root test of Exercise 31 since

$$\sigma=\lim_{n\to\infty}\left[\left(\frac{n}{n+1}\right)^{n^2}\right]^{1/n}=\lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n}=\frac{1}{e}<1.$$

40. Let $a_n=\dfrac{2^{n+1}}{n^n}$. Then

$$\begin{aligned}\frac{a_{n+1}}{a_n}&=\frac{2^{n+2}}{(n+1)^{n+1}}\cdot\frac{n^n}{2^{n+1}}\\&=\frac{2}{(n+1)\left(\frac{n}{n+1}\right)^n}=\frac{2}{n+1}\cdot\frac{1}{\left(1+\frac{1}{n}\right)^n}\\&\to 0\times\frac{1}{e}=0\text{ as }n\to\infty.\end{aligned}$$

Thus $\sum_{n=1}^{\infty}a_n$ converges by the ratio test.
(Remark: the question contained a typo. It was intended to ask that #33 be repeated, using the ratio test. That is a little harder.)

41. Trying to apply the ratio test to $\displaystyle\sum\frac{2^{2n}(n!)^2}{(2n)!}$, we obtain

$$\rho=\lim\frac{2^{2n+2}((n+1)!)^2}{(2n+2)!}\cdot\frac{(2n)!}{2^{2n}(n!)^2}=\lim\frac{4(n+1)^2}{(2n+2)(2n+1)}=1.$$

Thus the ratio test provides no information. However,

$$\begin{aligned}\frac{2^{2n}(n!)^2}{(2n)!}&=\frac{[2n(2n-2)\cdots6\cdot4\cdot2]^2}{2n(2n-1)(2n-2)\cdots3\cdot2\cdot1}\\&=\frac{2n}{2n-1}\cdot\frac{2n-2}{2n-3}\cdot\cdots\cdot\frac{4}{3}\cdot\frac{2}{1}>1.\end{aligned}$$

Since the terms exceed 1, the series diverges to infinity.

42. We have

$$\begin{aligned}a_n=\frac{(2n)!}{2^{2n}(n!)^2}&=\frac{1\times2\times3\times4\times\cdots\times2n}{(2\times4\times6\times8\times\cdots\times2n)^2}\\&=\frac{1\times3\times5\times\cdots\times(2n-1)}{2\times4\times6\times\cdots\times(2n-2)\times2n}\\&=1\times\frac{3}{2}\times\frac{5}{4}\times\frac{7}{6}\times\cdots\times\frac{2n-1}{2n-2}\times\frac{1}{2n}>\frac{1}{2n}.\end{aligned}$$

Therefore $\displaystyle\sum_{n=1}^{\infty}\frac{(2n)!}{2^{2n}(n!)^2}$ diverges to infinity by comparison with the harmonic series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2n}$.

43. a) If n is a positive integer and $k>0$, then $(1+k)^n\ge 1+nk>nk$, so $n<\dfrac{1}{k}(1+k)^n$.

b) Let $$\begin{aligned}s_N=\sum_{n=0}^{N}\frac{n}{2^n}&<\frac{1}{k}\sum_{n=0}^{N}\left(\frac{1+k}{2}\right)^n\\&=\frac{1}{k}\sum_{n=0}^{N}r^n=\frac{1}{k}\cdot\frac{1-r^{N+1}}{1-r},\end{aligned}$$
where $r=(1+k)/n$. Thus

$$\begin{aligned}s_n&<\frac{1}{k}\cdot\frac{1-\left(\frac{1+k}{2}\right)^{N+1}}{1-\frac{1+k}{2}}\\&=\frac{2}{k(1-k)}\left(1-\left(\frac{1+k}{2}\right)^{N+1}\right)\le\frac{2}{k(1-k)}.\end{aligned}$$

Therefore, $s=\displaystyle\sum_{n=0}^{\infty}\frac{n}{2^n}\le\frac{2}{k(1-k)}$.
Since the maximum value of $k(1-k)$ is $1/4$ (at $k=1/2$), the best upper bound we get for s by this method is $s\le 8$.

c) $$\begin{aligned}s-s_n=\sum_{j=n+1}^{\infty}\frac{j}{2^j}&<\frac{1}{k}\sum_{j=n+1}^{\infty}\left(\frac{1+k}{2}\right)^j\\&=\frac{1}{k}\left(\frac{1+k}{2}\right)^{n+1}\cdot\frac{1}{1-\frac{1+k}{2}}\\&=\frac{(1+k)^{n+1}}{k(1-k)2^n}=\frac{G(k)}{2^n},\end{aligned}$$
where $G(k)=\dfrac{(1+k)^{n+1}}{k(1-k)}$. For minimum $G(k)$, look for a critical point:

$$\frac{k(1-k)(n+1)(1+k)^n-(1+k)^{n+1}(1-2k)}{k^2(1-k)^2}=0$$
$$(k-k^2)(n+1)-(1+k)(1-2k)=0$$
$$k^2(n+1)-k(n+1)+1-k-2k^2=0$$
$$(n-1)k^2-(n+2)k+1=0$$
$$\begin{aligned}k&=\frac{(n+2)\pm\sqrt{(n+2)^2-4(n-1)}}{2(n-1)}\\&=\frac{n+2\pm\sqrt{n^2+8}}{2(n-1)}.\end{aligned}$$

For given n, the upper bound is minimal if $k=\dfrac{n+2-\sqrt{n^2+8}}{2(n-1)}$ (for $n\ge2$).

44. If $s=\displaystyle\sum_{k=1}^{\infty}c_k=\sum_{k=1}^{\infty}\frac{1}{k^2(k+1)}$, then we have

$$s_n+A_{n+1}\le s\le s_n+A_n$$

where $s_n = \sum_{k=1}^{n} \frac{1}{k^2(k+1)}$ and

$$\begin{aligned} A_n &= \int_n^\infty \frac{dx}{x^2(x+1)} = \int_n^\infty \left(\frac{-1}{x} + \frac{1}{x^2} + \frac{1}{x+1}\right) dx \\ &= -\ln x - \frac{1}{x} + \ln(x+1)\Big|_n^\infty \\ &= \ln\left(1+\frac{1}{x}\right) - \frac{1}{x}\Big|_n^\infty \\ &= \frac{1}{n} - \ln\left(1+\frac{1}{n}\right). \end{aligned}$$

If $s_n^* = s_n + \frac{1}{2}(A_{n+1} + A_n)$, then

$$\begin{aligned} |s_n - s_n^*| &\le \frac{A_n - A_{n+1}}{2} \\ &= \frac{1}{2}\left[\frac{1}{n} - \ln\left(1+\frac{1}{n}\right) - \frac{1}{n+1} + \ln\left(1+\frac{1}{n+1}\right)\right] \\ &= \frac{1}{2}\left[\frac{1}{n(n+1)} + \ln\left(\frac{n^2+2n}{n^2+2n+1}\right)\right] \\ &\le \frac{1}{2}\left[\frac{1}{n(n+1)} + \left(\frac{n^2+2n}{n^2+2n+1} - 1\right)\right] \\ &= \frac{1}{2n(n+1)^2} < 0.001 \end{aligned}$$

if $n = 8$. Thus,

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n^2} &= 1 + s_8^* = 1 + s_8 + \frac{1}{2}(A_9 + A_8) \\ &= 1 + \left[\frac{1}{2} + \frac{1}{2^2(3)} + \frac{1}{3^2(4)} + \cdots + \frac{1}{8^2(9)}\right] + \\ &\quad \frac{1}{2}\left[\left(\frac{1}{9} - \ln\frac{10}{9}\right) + \left(\frac{1}{8} - \ln\frac{9}{8}\right)\right] \\ &= 1.6450 \end{aligned}$$

with error less than 0.001.

45. $s = \sum_{n=1}^{\infty} 1/(2^n + 1)$.

(a) We have

$$\begin{aligned} 0 < s - s_n &= \sum_{i=1}^{\infty} \frac{1}{2^i + 1} = \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + \frac{1}{2^{n+3}} + \cdots \\ &= \frac{1}{2^{n+1}}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots\right) \\ &= \frac{1}{2^n} < \frac{1}{1{,}000} \quad \text{if} \quad 2^n > 1{,}000. \end{aligned}$$

Since $2^{10} = 1{,}024$, s_{10} will approximate s to within 0.001.

(b) Let $S_n = \sum_{i=1}^{n} b_i$, where $b_n = \frac{1}{2^n} - \frac{1}{2^n + 1}$. Since

$$0 < b_n = \frac{2^n + 1 - 2^n}{2^n(2^n + 1)} < \frac{1}{4^n},$$

we have

$$\begin{aligned} 0 < \sum_{i=1}^{\infty} b_i - S_n &= b_{n+1} + b_{n+2} + b_{n+3} + \cdots \\ &< \frac{1}{4^{n+1}}\left(1 + \frac{1}{4} + \frac{1}{4^2} + \cdots\right) \\ &= frac14^{n+1} \times \frac{4}{3} = \frac{1}{3 \times 4^n} < \frac{1}{1{,}000} \end{aligned}$$

provided $4^n > 1{,}000/3$. Thus $n = 5$ will do (but $n = 4$ is insufficient). S_5 approximates $\sum_{n=1}^{\infty} b_n$ to within 0.001.

(c) Since $\sum_{n=1}^{\infty} 1/2^n = 1$, we have

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{2^n + 1} &= \sum_{n=1}^{\infty} \frac{1}{2^n} - \sum_{n=1}^{\infty} b_n \\ &\approx 1 - \sum_{n=1}^{5} b_n \\ &= 1 - \left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{1}{4} - \frac{1}{5}\right) - \left(\frac{1}{8} - \frac{1}{9}\right) \\ &\quad - \left(\frac{1}{16} - \frac{1}{17}\right) - \left(\frac{1}{32} - \frac{1}{33}\right) \\ &\approx 0.765 \quad \text{with error less than } 0.001. \end{aligned}$$

Section 9.4 Absolute and Conditional Convergence (page 501)

1. $\sum \frac{(-1)^n}{\sqrt{n}}$ converges by the alternating series test (since the terms alternate in sign, decrease in size, and approach 0). However, the convergence is only conditional, since $\sum \frac{1}{\sqrt{n}}$ diverges to infinity.

2. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 + \ln n}$ converges absolutely since $\left|\frac{(-1)^n}{n^2 + \ln n}\right| \le \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

3. $\sum \frac{\cos(n\pi)}{(n+1)\ln(n+1)} = \sum \frac{(-1)^n}{(n+1)\ln(n+1)}$ converges by the alternating series test, but only conditionally since $\sum \frac{1}{(n+1)\ln(n+1)}$ diverges to infinity (by the integral test).

4. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{2n}}{2^n}=\sum_{n=1}^{\infty}\frac{1}{2^n}$ is a positive, convergent geometric series so must converge absolutely.

5. $\displaystyle\sum\frac{(-1)^n(n^2-1)}{n^2+1}$ diverges since its terms do not approach zero.

6. $\displaystyle\sum_{n=1}^{\infty}\frac{(-2)^n}{n!}$ converges absolutely by the ratio test since

$$\lim\left|\frac{(-2)^{n+1}}{(n+1)!}\cdot\frac{n!}{(-2)^n}\right|=2\lim\frac{1}{n+1}=0.$$

7. $\displaystyle\sum\frac{(-1)^n}{n\pi^n}$ converges absolutely, since, for $n\geq 1$,

$$\left|\frac{(-1)^n}{n\pi^n}\right|\leq\frac{1}{\pi^n},$$

and $\displaystyle\sum\frac{1}{\pi^n}$ is a convergent geometric series.

8. $\displaystyle\sum_{n=0}^{\infty}\frac{-n}{n^2+1}$ diverges to $-\infty$ since all terms are negative and $\displaystyle\sum_{n=0}^{\infty}\frac{n}{n^2+1}$ diverges to infinity by comparison with $\displaystyle\sum_{n=0}^{\infty}\frac{1}{n}$.

9. $\displaystyle\sum(-1)^n\frac{20n^2-n-1}{n^3+n^2+33}$ converges by the alternating series test (the terms are ultimately decreasing in size, and approach zero), but the convergence is only conditional since $\displaystyle\sum\frac{20n^2-n-1}{n^3+n^2+33}$ diverges to infinity by comparison with $\displaystyle\sum\frac{1}{n}$.

10. $\displaystyle\sum_{n=1}^{\infty}\frac{100\cos(n\pi)}{2n+3}=\sum_{n=1}^{\infty}\frac{100(-1)^n}{2n+3}$ converges by the alternating series test but only conditionally since

$$\left|\frac{100(-1)^n}{2n+3}\right|=\frac{100}{2n+3}$$

and $\displaystyle\sum_{n=1}^{\infty}\frac{100}{2n+3}$ diverges to infinity.

11. $\displaystyle\sum\frac{n!}{(-100)^n}$ diverges since $\displaystyle\lim\frac{n!}{100^n}=\infty$.

12. $\displaystyle\sum_{n=10}^{\infty}\frac{\sin(n+\frac{1}{2})\pi}{\ln\ln n}=\sum_{n=10}^{\infty}\frac{(-1)^n}{\ln\ln n}$ converges by the alternating series test but only conditionally since $\displaystyle\sum_{n=10}^{\infty}\frac{1}{\ln\ln n}$ diverges to infinity by comparison with $\displaystyle\sum_{n=10}^{\infty}\frac{1}{n}$.
($\ln\ln n<n$ for $n\geq 10$.)

13. If $\displaystyle s=\sum_{k=1}^{\infty}(-1)^{k-1}\frac{k}{k^2+1}$, and $\displaystyle s_n=\sum_{k=1}^{n}(-1)^{k-1}\frac{k}{k^2+1}$, then

$$|s-s_n|<\frac{n+1}{(n+1)^2+1}<0.001$$

if $n=999$, because the series satisfies the conditions of the alternating series test.

14. Since the terms of the series $s=\sum_{n=0}^{\infty}\dfrac{(-1)^n}{(2n)!}$ are alternating in sign and decreasing in size, the size of the error in the approximation $s\approx s_n$ does not exceed that of the first omitted term:

$$|s-s_n|\leq\frac{1}{(2n+2)!}<0.001$$

if $n=3$. Hence $s\approx 1-\dfrac{1}{2!}+\dfrac{1}{4!}-\dfrac{1}{6!}$; four terms will approximate s with error less than 0.001 in absolute value.

15. If $\displaystyle s=\sum_{k=1}^{\infty}(-1)^{k-1}\frac{k}{2^k}$, and $\displaystyle s_n=\sum_{k=1}^{n}(-1)^{k-1}\frac{k}{2^k}$, then

$$|s-s_n|<\frac{n+1}{2^{n+1}}<0.001$$

if $n=13$, because the series satisfies the conditions of the alternating series test from the second term on.

16. Since the terms of the series $s=\sum_{n=0}^{\infty}(-1)^n\dfrac{3^n}{n!}$ are alternating in sign and ultimately decreasing in size (they decrease after the third term), the size of the error in the approximation $s\approx s_n$ does not exceed that of the first omitted term (provided $n\geq 3$): $|s-s_n|\leq\dfrac{3^{n+1}}{(n+1)!}<0.001$ if $n=12$. Thus twelve terms will suffice to approximate s with error less than 0.001 in absolute value.

17. Applying the ratio test to $\displaystyle\sum\frac{x^n}{\sqrt{n+1}}$, we obtain

$$\rho=\lim\left|\frac{x^{n+1}}{\sqrt{n+2}}\cdot\frac{\sqrt{n+1}}{x^n}\right|=|x|\lim\sqrt{\frac{n+1}{n+2}}=|x|.$$

Hence the series converges absolutely if $|x|<1$, that is, if $-1<x<1$. The series converges conditionally for $x=-1$, but diverges for all other values of x.

18. Let $a_n = \dfrac{(x-2)^n}{n^2 2^{2n}}$. Apply the ratio test

$$\rho = \lim \left| \frac{(x-2)^{n+1}}{(n+1)^2 2^{2n+2}} \times \frac{n^2 2^{2n}}{(x-2)^n} \right| = \frac{|x-2|}{4} < 1$$

if and only if $|x-2| < 4$, that is $-2 < x < 6$. If $x = -2$, then $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$, which converges absolutely. If $x = 6$, then $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$, which also converges absolutely. Thus, the series converges absolutely if $-2 \le x \le 6$ and diverges elsewhere.

19. Apply the ratio test to $\sum (-1)^n \dfrac{(x-1)^n}{2n+3}$:

$$\rho = \lim \left| \frac{(x-1)^{n+1}}{2n+5} \cdot \frac{2n+3}{(x-1)^n} \right| = |x-1|.$$

The series converges absolutely if $|x-1| < 1$, that is, if $0 < x < 2$, and converges conditionally if $x = 2$. It diverges for all other values of x.

20. Let $a_n = \dfrac{1}{2n-1}\left(\dfrac{3x+2}{-5}\right)^n$. Apply the ratio test

$$\rho = \lim \left| \frac{1}{2n+1}\left(\frac{3x+2}{-5}\right)^{n+1} \times \frac{2n-1}{1}\left(\frac{3x+2}{-5}\right)^{-n} \right| = \left| \frac{3x+2}{5} \right| < 1$$

if and only if $\left|x + \frac{2}{3}\right| < \frac{5}{3}$, that is $-\frac{7}{3} < x < 1$. If $x = -\frac{7}{3}$, then $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{2n-1}$, which diverges. If $x = 1$, then $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1}$, which converges conditionally. Thus, the series converges absolutely if $-\frac{7}{3} < x < 1$, converges conditionally if $x = 1$ and diverges elsewhere.

21. Apply the ratio test to $\sum \dfrac{x^n}{2^n \ln n}$:

$$\rho = \lim \left| \frac{x^{n+1}}{2^{n+1}\ln(n+1)} \cdot \frac{2^n \ln n}{x^n} \right| = \frac{|x|}{2} \lim \frac{\ln n}{\ln(n+1)} = \frac{|x|}{2}.$$

(The last limit can be evaluated by l'Hôpital's Rule.) The given series converges absolutely if $|x| < 2$, that is, if $-2 < x < 2$. By the alternating series test, it converges conditionally if $x = -2$. It diverges for all other values of x.

22. Let $a_n = \dfrac{(4x+1)^n}{n^3}$. Apply the ratio test

$$\rho = \lim \left| \frac{(4x+1)^{n+1}}{(n+1)^3} \times \frac{n^3}{(4x+1)^n} \right| = |4x+1| < 1$$

if and only if $-\frac{1}{2} < x < 0$. If $x = -\frac{1}{2}$, then $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$, which converges absolutely. If $x = 0$, then $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n^3}$, which also converges absolutely. Thus, the series converges absolutely if $-\frac{1}{2} \le x \le 0$ and diverges elsewhere.

23. Apply the ratio test to $\sum \dfrac{(2x+3)^n}{n^{1/3} 4^n}$:

$$\rho = \lim \left| \frac{(2x+3)^{n+1}}{(n+1)^{1/3} 4^{n+1}} \cdot \frac{n^{1/3} 4^n}{(2x+3)^n} \right| = \frac{|2x+3|}{4} = \frac{\left|x + \frac{3}{2}\right|}{2}.$$

The series converges absolutely if $\left|x + \frac{3}{2}\right| < 2$, that is, if $-\frac{7}{2} < x < \frac{1}{2}$. By the alternating series test it converges conditionally at $x = -\frac{7}{2}$. It diverges elsewhere.

24. Let $a_n = \dfrac{1}{n}\left(1 + \dfrac{1}{x}\right)^n$. Apply the ratio test

$$\rho = \lim \left| \frac{1}{n+1}\left(1 + \frac{1}{x}\right)^{n+1} \times \frac{n}{1}\left(1 + \frac{1}{x}\right)^{-n} \right| = \left|1 + \frac{1}{x}\right| < 1$$

if and only if $|x+1| < |x|$, that is, $-2 < \frac{1}{x} < 0 \Rightarrow x < -\frac{1}{2}$. If $x = -\frac{1}{2}$, then $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, which converges conditionally. Thus, the series converges absolutely if $x < -\frac{1}{2}$, converges conditionally if $x = -\frac{1}{2}$ and diverges elsewhere. It is undefined at $x = 0$.

25. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(n\pi/2)}{n} = 1 + 0 - \frac{1}{3} + 0 + \frac{1}{5} + 0 - \frac{1}{7} + 0 + \cdots$

The alternating series test does not apply directly, but does apply to the modified series with the zero terms deleted. Since this latter series converges conditionally, the given series also converges conditionally.

26. If

$$a_n = \begin{cases} \dfrac{10}{n^2}, & \text{if } n \text{ is even;} \\ \dfrac{-1}{10n^3}, & \text{if } n \text{ is odd;} \end{cases}$$

then $|a_n| \le \dfrac{10}{n^2}$ for every $n \ge 1$. Hence, $\displaystyle\sum_{n=1}^{\infty} a_n$ converges absolutely by comparison with $\displaystyle\sum_{n=1}^{\infty} \frac{10}{n^2}$.

27. a) "$\sum a_n$ converges implies $\sum(-1)^n a_n$ converges" is FALSE. $a_n = \dfrac{(-1)^n}{n}$ is a counterexample.

b) "$\sum a_n$ converges and $\sum(-1)^n a_n$ converges implies $\sum a_n$ converges absolutely" is FALSE. The series of Exercise 25 is a counterexample.

c) "$\sum a_n$ converges absolutely implies $\sum(-1)^n a_n$ converges absolutely" is TRUE, because $|(-1)^n a_n| = |a_n|$.

28. a) We have

$$\begin{aligned}\ln(n!) &= \ln 1 + \ln 2 + \ln 3 + \cdots + \ln n\\ &= \text{sum of area of the shaded rectangles}\\ &> \int_1^n \ln t\,dt = (t\ln t - t)\Big|_1^n\\ &= n\ln n - n + 1.\end{aligned}$$

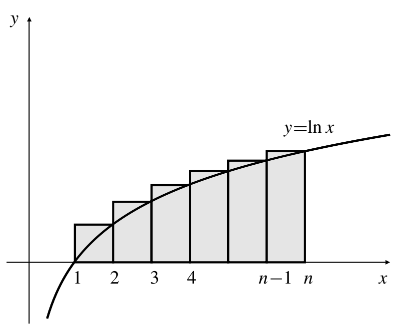

Fig. 9.4.28

b) Let $a_n = \dfrac{n!x^n}{n^n}$. Apply the ratio test

$$\begin{aligned}\rho &= \lim\left|\frac{(n+1)!x^{n+1}}{(n+1)^{n+1}} \times \frac{n^n}{n!x^n}\right|\\ &= \lim \frac{|x|}{\left(1+\frac{1}{n}\right)^n} = \frac{|x|}{e} < 1\end{aligned}$$

if and only if $-e < x < e$. If $x = \pm e$, then, by (a),

$$\begin{aligned}\ln\left|\frac{n!e^n}{n^n}\right| &= \ln(n!) + \ln e^n - \ln n^n\\ &> (n\ln n - n + 1) + n - n\ln n = 1.\\ \Rightarrow \left|\frac{n!e^n}{n^n}\right| &> e.\end{aligned}$$

Hence, $\displaystyle\sum_{n=1}^{\infty} a_n$ converges absolutely if $-e < x < e$ and diverges elsewhere.

29. Applying the ratio test to $\displaystyle\sum \frac{(2n)!x^n}{2^{2n}(n!)^2} = \sum a_n x^n$, we obtain

$$\rho = \lim |x| \frac{(2n+2)(2n+1)}{4(n+1)^2} = |x|.$$

Thus $\sum a_n x^n$ converges absolutely if $-1 < x < 1$, and diverges if $x > 1$ or $x < -1$. In Exercise 36 of Section 9.3 it was shown that $a_n \ge \dfrac{1}{2n}$, so the given series definitely diverges at $x = 1$ and may at most converge conditionally at $x = -1$. To see whether it does converge at -1, we write, as in Exercise 36 of Section 9.3,

$$\begin{aligned}a_n &= \frac{(2n)!}{2^{2n}(n!)^2} = \frac{1\times 2\times 3\times 4\times\cdots\times 2n}{(2\times 4\times 6\times 8\times\cdots\times 2n)^2}\\ &= \frac{1\times 3\times 5\times\cdots\times(2n-1)}{2\times 4\times 6\times\cdots\times(2n-2)\times 2n}\\ &= \frac{1}{2}\times\frac{3}{4}\times\cdots\times\frac{2n-3}{2n-2}\times\frac{2n-1}{2n}\\ &= \left(1-\frac{1}{2}\right)\left(1-\frac{1}{4}\right)\cdots\left(1-\frac{1}{2n-2}\right)\left(1-\frac{1}{2n}\right).\end{aligned}$$

It is evident that a_n decreases as n increases. To see whether $\lim a_n = 0$, take logarithms and use the inequality $\ln(1+x) \le x$:

$$\begin{aligned}\ln a_n &= \ln\left(1-\frac{1}{2}\right) + \ln\left(1-\frac{1}{4}\right) + \cdots + \ln\left(1-\frac{1}{2n}\right)\\ &\le -\frac{1}{2} - \frac{1}{4} - \cdots - \frac{1}{2n}\\ &= -\frac{1}{2}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right) \to -\infty \text{ as } n\to\infty.\end{aligned}$$

Thus $\lim a_n = 0$, and the given series converges conditionally at $x = -1$ by the alternating series test.

30. Let $p_n = \dfrac{1}{2n-1}$ and $q_n = -\dfrac{1}{2n}$. Then $\sum p_n$ diverges to ∞ and $\sum q_n$ diverges to $-\infty$. Also, the alternating harmonic series is the sum of all the p_ns and q_ns in a specific order:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \sum_{n=1}^{\infty}(p_n + q_n).$$

a) Rearrange the terms as follows: first add terms of $\sum p_n$ until the sum exceeds 2. Then add q_1. Then add more terms of $\sum p_n$ until the sum exceeds 3. Then add q_2. Continue in this way; at the nth stage, add new terms from $\sum p_n$ until the sum exceeds $n+1$, and then add q_n. All partial sums after the nth stage exceed n, so the rearranged series diverges to infinity.

b) Rearrange the terms of the original alternating harmonic series as follows: first add terms of $\sum q_n$ until the sum is less than -2. Then add p_1. The sum will now be greater than -2. (Why?) Then resume adding new terms from $\sum q_n$ until the sum is less than -2 again, and add p_2, which will raise the sum above -2 again. Continue in this way. After the nth stage, all succeeding partial sums will differ from -2 by less than $1/n$, so the rearranged series will converge to -2.

Section 9.5 Power Series (page 511)

1. For $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{\sqrt{n+1}}$ we have $R = \lim \left|\dfrac{\sqrt{n+2}}{\sqrt{n+1}}\right| = 1$. The radius of convergence is 1; the centre of convergence is 0; the interval of convergence is $(-1, 1)$. (The series does not converge at $x = -1$ or $x = 1$.)

2. We have $\displaystyle\sum_{n=0}^{\infty} 3n(x+1)^n$. The centre of convergence is $x = -1$. The radius of convergence is

$$R = \lim \frac{3n}{3(n+1)} = 1.$$

The series converges absolutely on $(-2, 0)$ and diverges on $(-\infty, -2)$ and $(0, \infty)$. At $x = -2$, the series is $\displaystyle\sum_{n=0}^{\infty} 3n(-1)^n$, which diverges. At $x = 0$, the series is $\displaystyle\sum_{n=0}^{\infty} 3n$, which diverges to infinity. Hence, the interval of convergence is $(-2, 0)$.

3. For $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{x+2}{2}\right)^n$ we have $R = \lim \dfrac{2^{n+1}(n+1)}{2^n n} = 2$. The radius of convergence is 2; the centre of convergence is -2. For $x = -4$ the series is an alternating harmonic series, so converges. For $x = 0$, the series is a divergent harmonic series. Therefore the interval of convergence is $[-4, 0)$.

4. We have $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4 2^{2n}} x^n$. The centre of convergence is $x = 0$. The radius of convergence is

$$\begin{aligned} R &= \lim \left|\frac{(-1)^n}{n^4 2^{2n}} \cdot \frac{(n+1)^4\, 2^{2n+2}}{(-1)^{n+1}}\right| \\ &= \lim \left|\left(\frac{n+1}{n}\right)^4 \cdot 4\right| = 4. \end{aligned}$$

At $x = 4$, the series is $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$, which converges.

At $x = -4$, the series is $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$, which also converges.

Hence, the interval of convergence is $[-4, 4]$.

5. $\displaystyle\sum_{n=0}^{\infty} n^3(2x-3)^n = \sum_{n=0}^{\infty} 2^n n^3 \left(x - \tfrac{3}{2}\right)^n$. Here $R = \lim \dfrac{2^n n^3}{2^{n+1}(n+1)^3} = \dfrac{1}{2}$. The radius of convergence is $1/2$; the centre of convergence is $3/2$; the interval of convergence is $(1, 2)$.

6. We have $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{n^3}(4-x)^n$. The centre of convergence is $x = 4$. The radius of convergence is

$$R = \lim \frac{e^n}{n^3} \cdot \frac{(n+1)^3}{e^{n+1}} = \frac{1}{e}.$$

At $x = 4 + \dfrac{1}{e}$, the series is $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$, which converges.

At $x = 4 - \dfrac{1}{e}$, the series is $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$, which also converges.

Hence, the interval of convergence is $\left[4 - \dfrac{1}{e}, 4 + \dfrac{1}{e}\right]$.

7. For $\sum_{n=0}^{\infty} \dfrac{1+5^n}{n!} x^n$ we have $R = \lim \dfrac{1+5^n}{n!} \cdot \dfrac{(n+1)!}{1+5^{n+1}} = \infty$. The radius of convergence is infinite; the centre of convergence is 0; the interval of convergence is the whole real line $(-\infty, \infty)$.

8. We have $\displaystyle\sum_{n=1}^{\infty} \frac{(4x-1)^n}{n^n} = \sum_{n=1}^{\infty} \left(\frac{4}{n}\right)^n \left(x - \frac{1}{4}\right)^n$. The centre of convergence is $x = \frac{1}{4}$. The radius of convergence is

$$\begin{aligned} R &= \lim \frac{4^n}{n^n} \cdot \frac{(n+1)^{n+1}}{4^{n+1}} \\ &= \frac{1}{4} \lim \left(\frac{n+1}{n}\right)^n (n+1) = \infty. \end{aligned}$$

Hence, the interval of convergence is $(-\infty, \infty)$.

9. By Example 5(a),

$$\begin{array}{rcccccccccccl} & 1 & + & 2x & + & 3x^2 & + & 4x^3 & + & \cdots & = & \dfrac{1}{(1-x)^2} \\ \times & 1 & + & x & + & x^2 & + & x^3 & + & \cdots & = & \dfrac{1}{1-x} \\ \hline & 1 & + & 2x & + & 3x^2 & + & 4x^3 & + & \cdots & & \\ & & & x & + & 2x^2 & + & 3x^3 & + & \cdots & & \\ & & & & & x^2 & + & 2x^3 & + & \cdots & & \\ & & & & & & & x^3 & + & \cdots & & \\ & & & & & & & & & \cdots & & \\ \hline & 1 & + & 3x & + & 6x^2 & + & 10x^3 & + & \cdots & = & \dfrac{1}{(1-x)^3} \end{array}$$

Thus

$$\frac{1}{(1-x)^3} = \sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2} x^n,$$

for $-1 < x < 1$.

10. We have

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

and

$$1 - x + x^2 - x^3 + \cdots = \frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

holds for $-1 < x < 1$. Since $a_n = 1$ and $b_n = (-1)^n$ for $n = 0, 1, 2, \ldots$, we have

$$C_n = \sum_{j=0}^{n} (-1)^{n-j} = \begin{cases} 0, & \text{if } n \text{ is odd;} \\ 1, & \text{if } n \text{ is even.} \end{cases}$$

Then the Cauchy product is

$$1 + x^2 + x^4 + \cdots = \sum_{n=0}^{\infty} x^{2n} = \frac{1}{1-x} \cdot \frac{1}{1+x} = \frac{1}{1-x^2}$$

for $-1 < x < 1$.

11. By long division:

$$\begin{array}{r|l} & 1 + 2x + 3x^2 + 4x^3 + \cdots \\ \hline 1 - 2x + x^2 & 1 \\ & \underline{1 - 2x + x^2} \\ & \quad 2x - x^2 \\ & \quad \underline{2x - 4x^2 + 2x^3 + \cdots} \\ & \qquad 3x^2 - 2x^3 + \cdots \\ & \qquad \underline{3x^2 - 6x^3 + \cdots} \\ & \qquad\qquad 4x^3 + \cdots \end{array}$$

Thus $\dfrac{1}{(1-x)^2} = \displaystyle\sum_{n=0}^{\infty} (n+1)x^n$, for $-1 < x < 1$.

12.
$$\frac{1}{2-x} = \frac{1}{2} \frac{1}{\left(1 - \dfrac{x}{2}\right)} = \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n$$
$$= \frac{1}{2} + \frac{x}{2^2} + \frac{x^2}{2^3} + \frac{x^3}{2^4} + \cdots \qquad (-2 < x < 2).$$

13.
$$\frac{1}{2-x} = \frac{1}{2} \cdot \frac{1}{1 - \dfrac{x}{2}}$$
$$= \frac{1}{2} + \frac{x}{2^2} + \frac{x^2}{2^3} + \frac{x^3}{2^4} \cdots$$

for $-2 < x < 2$. Now differentiate to get

$$\frac{1}{(2-x)^2} = \frac{1}{2^2} + \frac{2x}{2^3} + \frac{3x^2}{2^4} + \cdots$$
$$= \sum_{n=0}^{\infty} \frac{(n+1)x^n}{2^{n+2}}, \qquad (-2 < x < 2).$$

14.
$$\frac{1}{1+2x} = \sum_{n=0}^{\infty} (-2x)^n$$
$$= 1 - 2x + 2^2x^2 - 2^3x^3 + \cdots \qquad (-\tfrac{1}{2} < x < \tfrac{1}{2}).$$

15.
$$\int_0^x \frac{dt}{2-t} = \int_0^x \sum_{n=0}^{\infty} \frac{t^n}{2^{n+1}}\, dt$$
$$-\ln(2-t)\Big|_0^x = \sum_{n=0}^{\infty} \frac{t^{n+1}}{2^{n+1}(n+1)}\Bigg|_0^x$$
$$-\ln(2-x) + \ln 2 = \sum_{n=0}^{\infty} \frac{x^{n+1}}{2^{n+1}(n+1)}$$
$$\ln(2-x) = \ln 2 - \sum_{n=1}^{\infty} \frac{x^n}{2^n n}. \quad (-2 \le x < 2).$$

16. Let $y = x - 1$. Then $x = 1 + y$ and

$$\frac{1}{x} = \frac{1}{1+y} = \sum_{n=0}^{\infty} (-y)^n \qquad (-1 < y < 1)$$
$$= \sum_{n=0}^{\infty} \Big[-(x-1)\Big]^n$$
$$= 1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4 - \cdots$$

(for $0 < x < 2$).

17. Let $x + 2 = t$, so $x = t - 2$. Then

$$\frac{1}{x^2} = \frac{1}{(2-t)^2} = \sum_{n=0}^{\infty} \frac{(n+1)t^n}{2^{n+2}}$$
$$= \sum_{n=0}^{\infty} \frac{(n+1)(x+2)^n}{2^{n+2}}, \quad (-4 < x < 0).$$

18. $$\begin{aligned}\frac{1-x}{1+x} &= \frac{2}{1+x} - 1\\ &= 2(1 - x + x^2 - x^3 + \cdots) - 1\\ &= 1 + 2\sum_{n=1}^{\infty}(-x)^n \qquad (-1 < x < 1).\end{aligned}$$

19. We have

$$\begin{aligned}\frac{x^3}{1-2x^2} &= x^3\left(\sum_{n=0}^{\infty}(2x^2)^n\right)\\ &= \sum_{n=0}^{\infty}2^n x^{2n+3}, \qquad \left(-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}\right).\end{aligned}$$

20. Let $y = x - 4$. Then $x = 4 + y$ and

$$\begin{aligned}\frac{1}{x} &= \frac{1}{4+y} = \frac{1}{4}\frac{1}{\left(1+\frac{y}{4}\right)} = \frac{1}{4}\sum_{n=0}^{\infty}\left(-\frac{y}{4}\right)^n\\ &= \frac{1}{4}\sum_{n=0}^{\infty}\left[-\frac{(x-4)}{4}\right]^n\\ &= \frac{1}{4} - \frac{(x-4)}{4^2} + \frac{(x-4)^2}{4^3} - \frac{(x-4)^3}{4^4} + \cdots\end{aligned}$$

for $0 < x < 8$. Therefore,

$$\begin{aligned}\ln x &= \int_1^x \frac{dt}{t} = \int_1^4 \frac{dt}{t} + \int_4^x \frac{dt}{t}\\ &= \ln 4 + \int_4^x \left[\frac{1}{4} - \frac{(t-4)}{4^2} + \frac{(t-4)^2}{4^3} - \frac{(t-4)^3}{4^4} + \cdots\right] dt\\ &= \ln 4 + \frac{x-4}{4} - \frac{(x-4)^2}{2\cdot 4^2} + \frac{(x-4)^3}{3\cdot 4^3} - \frac{(x-4)^4}{4\cdot 4^4} + \cdots\\ &\quad \text{(for } 0 < x \le 8\text{)}.\end{aligned}$$

21. $$\begin{aligned}&1 = 4x + 16x^2 - 64x^3 + \cdots\\ =&1 + (-4x) + (-4x)^2 - (-4x)^3 + \cdots\\ =&\frac{1}{1-(-4x)} = \frac{1}{1+4x}, \quad \left(-\tfrac{1}{4} < x < \tfrac{1}{4}\right).\end{aligned}$$

22. We differentiate the series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

and multiply by x to get

$$\sum_{n=0}^{\infty} nx^n = x + 2x^2 + 3x^3 + \cdots = \frac{x}{(1-x)^2}$$

for $-1 < x < 1$. Therefore,

$$\begin{aligned}\sum_{n=0}^{\infty}(n+3)x^n &= \sum_{n=0}^{\infty}nx^n + 3\sum_{n=0}^{\infty}x^n\\ &= \frac{x}{(1-x)^2} + \frac{3}{1-x}\\ &= \frac{3-2x}{(1-x)^2} \qquad (-1 < x < 1).\end{aligned}$$

23. $$\begin{aligned}&\frac{1}{3} + \frac{x}{4} + \frac{x^2}{5} + \frac{x^3}{6} + \cdots\\ =&\frac{1}{x^3}\left(\frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \cdots\right)\\ =&\frac{1}{x^3}\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + + \cdots - x - \frac{x^2}{2}\right)\\ =&\frac{1}{x^3}\left[-\ln(1-x) - x - \frac{x^2}{2}\right]\\ =&-\frac{1}{x^3}\ln(1-x) - \frac{1}{x^2} - \frac{1}{2x}. \quad (-1 \le x < 1,\ x \ne 0).\end{aligned}$$

24. We start with

$$1 - x + x^2 - x^3 + x^4 - \cdots = \frac{1}{1+x}$$

and differentiate to get

$$-1 + 2x - 3x^3 + 4x^3 - \cdots = -\frac{1}{(1+x)^2}.$$

Now we multiply by $-x^3$:

$$x^3 - 2x^4 + 3x^5 - 4x^6 + \cdots = \frac{x^3}{(1+x)^2}.$$

Differentiating again we get

$$3x^2 - 2\times 4x^3 + 3\times 5x^4 - 4\times 6x^5 + \cdots = \frac{x^3 + 3x^2}{(1+x)^3}.$$

Finally, we remove the factor x^2:

$$3 - 2\times 4x + 3\times 5x^2 - 4\times 6x^3 + \cdots = \frac{x+3}{(1+x)^3}.$$

All steps are valid for $-1 < x < 1$.

25. Since $1 + x^2 + x^4 + x^6 + \cdots = \dfrac{1}{1-x^2}$, for $-1 < x < 1$, we obtain by differentiation

$$2x + 4x^3 + 6x^5 + 8x^7 + \cdots = \frac{2x}{(1-x^2)^2},$$

or, on division by x,

$$2 + 4x^2 + 6x^4 + 8x^6 + \cdots = \frac{2}{(1-x^2)^2},$$

for $-1 < x < 1$.

26. Since $x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots=\ln(1+x)$ for $-1<x\le 1$, therefore

$$x^2-\frac{x^4}{2}+\frac{x^6}{3}-\frac{x^8}{4}+\cdots=\ln(1+x^2)$$

for $-1\le x\le 1$, and, dividing by x^2,

$$1-\frac{x^2}{2}+\frac{x^4}{3}-\frac{x^6}{4}+\cdots=\begin{cases}\dfrac{\ln(1+x^2)}{x^2} & \text{if } -1\le x\le 1,\ x\ne 0\\ 1 & \text{if } x=0.\end{cases}$$

27. From Example 5(a),

$$\sum_{n=1}^{\infty} nx^{n-1}=\frac{1}{(1-x)^2},\quad (-1<x<1).$$

Putting $x=1/3$, we get

$$\sum_{n=1}^{\infty}\frac{n}{3^{n-1}}=\frac{1}{\left(1-\frac{1}{3}\right)^2}=\frac{9}{4}.$$

Thus $\sum_{n=1}^{\infty}\frac{n}{3^n}=\frac{1}{3}\cdot\frac{9}{4}=\frac{3}{4}.$

28. From Example 5(a) with $x=1/2$,

$$\sum_{n=0}^{\infty}\frac{n+1}{2^n}=\sum_{k=1}^{\infty}k\left(\frac{1}{2}\right)^{k-1}=\frac{1}{\left(1-\frac{1}{2}\right)^2}=4.$$

29. From Example 7, $\sum_{n=1}^{\infty}n^2x^{n-1}=\frac{1+x}{(1-x)^3}$ for $-1<x<1$. Putting $x=1/\pi$, we get

$$\sum_{n=0}^{\infty}\frac{(n+1)^2}{\pi^n}=\sum_{k=1}^{\infty}\frac{k^2}{\pi^{k-1}}=\frac{1+\frac{1}{\pi}}{(1-\frac{1}{\pi})^3}=\frac{\pi^2(\pi+1)}{(\pi-1)^3}.$$

30. From Example 5(a),

$$\sum_{n=1}^{\infty} nx^{n-1}=\frac{1}{(1-x)^2},\quad (-1<x<1).$$

Differentiate with respect to x and then replace n by $n+1$:

$$\sum_{n=2}^{\infty}n(n-1)x^{n-2}=\frac{2}{(1-x)^3},\quad (-1<x<1)$$
$$\sum_{n=1}^{\infty}(n+1)nx^{n-1}=\frac{2}{(1-x)^3},\quad (-1<x<1).$$

Now let $x=-1/2$:

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n(n+1)}{2^{n-1}}=\frac{16}{27}.$$

Finally, multiply by $-1/2$:

$$\sum_{n=1}^{\infty}(-1)^{n}\frac{n(n+1)}{2^{n}}=-\frac{8}{27}.$$

31. Since $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}=\ln(1+x)$ for $-1<x\le 1$, therefore

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n2^n}=\ln\left(1+\frac{1}{2}\right)=\ln\frac{3}{2}.$$

32. In the series for $\ln(1+x)$ in Example 5(c), put $x=-1/2$ to get

$$\sum_{n=1}^{\infty}(-1)\frac{1}{n2^n}=\sum_{k=0}^{\infty}\frac{(-1)^k}{k+1}\left(-\frac{1}{2}\right)^{k+1}=\ln\left(1-\frac{1}{2}\right)=-\ln 2.$$

Therefore

$$\sum_{n=1}^{\infty}\frac{1}{n2^n}=\ln 2$$
$$\sum_{n=3}^{\infty}\frac{1}{n2^n}=\ln 2-\frac{1}{2}-\frac{1}{8}=\ln 2-\frac{5}{8}.$$

Section 9.6 Taylor and Maclaurin Series (page 520)

1.
$$e^{3x+1}=e\cdot e^{3x}=e\left(\sum_{n=0}^{\infty}\frac{(3x)^n}{n!}\right)$$
$$=\sum_{n=0}^{\infty}\frac{e3^nx^n}{n!}\qquad\text{(for all } x).$$

2.
$$\cos(2x^3)=1-\frac{(2x^3)^2}{2!}+\frac{(2x^3)^4}{4!}-\frac{(2x^3)^6}{6!}+\cdots$$
$$=1-\frac{2^2x^6}{2!}+\frac{2^4x^{12}}{4!}-\frac{2^6x^{18}}{6!}+\cdots$$
$$=\sum_{n=0}^{\infty}\frac{(-1)^n4^n}{(2n)!}x^{6n}\qquad\text{(for all } x).$$

3. $$\sin\left(x-\frac{\pi}{4}\right) = \sin x\cos\frac{\pi}{4} - \cos x\sin\frac{\pi}{4}$$
$$= \frac{1}{\sqrt{2}}\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} - \frac{1}{\sqrt{2}}\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$$
$$= \frac{1}{\sqrt{2}}\sum_{n=0}^{\infty}(-1)^n\left[-\frac{x^{2n}}{(2n)!} + \frac{x^{2n+1}}{(2n+1)!}\right] \qquad \text{(for all } x).$$

4. $$\cos(2x-\pi) = -\cos(2x)$$
$$= -1 + \frac{2^2x^2}{2!} - \frac{2^4x^4}{4!} + \frac{2^6x^6}{6!} - \cdots$$
$$= -\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(2x)^{2n}$$
$$= \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(2n)!}4^n(x)^{2n} \qquad \text{(for all } x).$$

5. $$x^2\sin\frac{x}{3} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+3}}{3^{2n+1}(2n+1)!} \quad \text{(for all } x).$$

6. $$\cos^2\left(\frac{x}{2}\right) = \frac{1}{2}(1+\cos x)$$
$$= \frac{1}{2}\left(1 + 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)$$
$$= 1 + \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n} \qquad \text{(for all } x).$$

7. $$\sin x\cos x = \frac{1}{2}\sin(2x)$$
$$= \sum_{n=0}^{\infty}(-1)^n\frac{2^{2n}x^{2n+1}}{(2n+1)!} \qquad \text{(for all } x).$$

8. $$\tan^{-1}(5x^2) = (5x^2) - \frac{(5x^2)^3}{3} + \frac{(5x^2)^5}{5} - \frac{(5x^2)^7}{7} + \cdots$$
$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)}(5x^2)^{2n+1}$$
$$= \sum_{n=0}^{\infty}\frac{(-1)^n 5^{2n+1}}{(2n+1)}x^{4n+2}$$
$$\left(\text{for } -\frac{1}{\sqrt{5}} \le x \le \frac{1}{\sqrt{5}}\right).$$

9. $$\frac{1+x^3}{1+x^2} = (1+x^3)\left(1 - x^2 + x^4 - x^6 + \cdots\right)$$
$$= 1 - x^2 + x^3 + x^4 - x^5 - x^6 + x^7 + x^8 - \cdots$$
$$= 1 - x^2 + \sum_{n=2}^{\infty}(-1)^n\left(x^{2n-1} + x^{2n}\right) \qquad (|x| < 1).$$

10. $$\ln(2+x^2) = \ln 2\left(1 + \frac{x^2}{2}\right)$$
$$= \ln 2 + \ln\left(1 + \frac{x^2}{2}\right)$$
$$= \ln 2 + \left[\frac{x^2}{2} - \frac{1}{2}\left(\frac{x^2}{2}\right)^2 + \frac{1}{3}\left(\frac{x^2}{2}\right)^3 - \cdots\right]$$
$$= \ln 2 + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\cdot\frac{x^{2n}}{2^n}$$
$$\text{(for } -\sqrt{2} \le x \le \sqrt{2}).$$

11. $$\ln\frac{1+x}{1-x} = \ln(1+x) - \ln(1-x)$$
$$= \sum_{n=1}^{\infty}\frac{x^n}{n} - \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$$
$$= 2\sum_{n=1}^{\infty}\frac{x^{2n-1}}{2n-1} \qquad (-1 < x < 1).$$

12. $$\frac{e^{2x^2}-1}{x^2} = \frac{1}{x^2}\left(e^{2x^2} - 1\right)$$
$$= \frac{1}{x^2}\left(1 + 2x^2 + \frac{(2x^2)^2}{2!} + \frac{(2x^2)^3}{3!} + \cdots - 1\right)$$
$$= 2 + \frac{2^2x^2}{2!} + \frac{2^3x^4}{3!} + \frac{2^4x^6}{4!} + \cdots$$
$$= \sum_{n=0}^{\infty}\frac{2^{n+1}}{(n+1)!}x^{2n} \qquad \text{(for all } x \ne 0).$$

13. $$\cosh x - \cos x = \sum_{n=0}^{\infty}\left[1 - (-1)^n\right]\frac{x^{2n}}{(2n)!}$$
$$= 2\left(\frac{x^2}{2!} + \frac{x^6}{6!} + \frac{x^{10}}{10!} + \cdots\right)$$
$$= 2\sum_{n=0}^{\infty}\frac{x^{4n+2}}{(4n+2)!} \qquad \text{(for all } x).$$

14. $$\sinh x - \sin x = \sum_{n=0}^{\infty}\left[1 - (-1)^n\right]\frac{x^{2n+1}}{(2n+1)!}$$
$$= 2\left(\frac{x^2}{2!} + \frac{x^6}{6!} + \frac{x^{10}}{10!} + \cdots\right)$$
$$= 2\sum_{n=0}^{\infty}\frac{x^{4n+3}}{(4n+3)!} \qquad \text{(for all } x).$$

15. Let $t = x + 1$, so $x = t - 1$. We have
$$f(x) = e^{-2x} = e^{-2(t-1)}$$
$$= e^2\sum_{n=0}^{\infty}\frac{(-2)^n t^n}{n!}$$
$$= e^2\sum_{n=0}^{\infty}\frac{(-1)^n 2^n (x+1)^n}{n!} \qquad \text{(for all } x).$$

16. Let $y = x - \frac{\pi}{2}$; then $x = y + \frac{\pi}{2}$. Hence,

$$\begin{aligned}\sin x &= \sin\left(y + \frac{\pi}{2}\right) = \cos y \\ &= 1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \cdots \qquad \text{(for all } y) \\ &= 1 - \frac{1}{2!}\left(x - \frac{\pi}{2}\right)^2 + \frac{1}{4!}\left(x - \frac{\pi}{2}\right)^4 - \cdots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(x - \frac{\pi}{2}\right)^{2n} \qquad \text{(for all } x).\end{aligned}$$

17. Let $t = x - \pi$, so $x = t + \pi$. Then

$$\begin{aligned}f(x) &= \cos x = \cos(t + \pi) = -\cos t = -\sum_{n=0}^{\infty}(-1)^n \frac{t^{2n}}{(2n)!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n)!}(x - \pi)^{2n} \qquad \text{(for all } x).\end{aligned}$$

18. Let $y = x - 3$; then $x = y + 3$. Hence,

$$\begin{aligned}\ln x &= \ln(y + 3) = \ln 3 + \ln\left(1 + \frac{y}{3}\right) \\ &= \ln 3 + \frac{y}{3} - \frac{1}{2}\left(\frac{y}{3}\right)^2 + \frac{1}{3}\left(\frac{y}{3}\right)^3 - \frac{1}{4}\left(\frac{y}{3}\right)^4 + \cdots \\ &= \ln 3 + \frac{(x-3)}{3} - \frac{(x-3)^2}{2 \cdot 3^2} + \frac{(x-3)^3}{3 \cdot 3^3} - \frac{(x-3)^4}{4 \cdot 3^4} + \cdots \\ &= \ln 3 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n \cdot 3^n}(x - 3)^n \qquad (0 < x \le 6).\end{aligned}$$

19.

$$\begin{aligned}\ln(2 + x) &= \ln[4 + (x - 2)] = \ln\left[4\left(1 + \frac{x-2}{4}\right)\right] \\ &= \ln 4 + \ln\left(1 + \frac{x-2}{4}\right) \\ &= \ln 4 + \sum_{n=1}^{\infty}(-1)^{n-1}\frac{(x-2)^n}{n4^n} \qquad (-2 < x \le 6).\end{aligned}$$

20. Let $t = x + 1$. Then $x = t - 1$, and

$$\begin{aligned}e^{2x+3} &= e^{2t+1} = e\,e^{2t} \\ &= e\sum_{n=0}^{\infty} \frac{2^n t^n}{n!} \qquad \text{(for all } t) \\ &= \sum_{n=0}^{\infty} \frac{e2^n(x+1)^n}{n!} \qquad \text{(for all } x).\end{aligned}$$

21. Let $t = x - (\pi/4)$, so $x = t + (\pi/4)$. Then

$$\begin{aligned}f(x) &= \sin x - \cos x \\ &= \sin\left(t + \frac{\pi}{4}\right) - \cos\left(t + \frac{\pi}{4}\right) \\ &= \frac{1}{\sqrt{2}}\Big[(\sin t + \cos t) - (\cos t - \sin t)\Big] \\ &= \sqrt{2}\sin t = \sqrt{2}\sum_{n=0}^{\infty}(-1)^n \frac{t^{2n+1}}{(2n+1)!} \\ &= \sqrt{2}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(x - \frac{\pi}{4}\right)^{2n+1} \qquad \text{(for all } x).\end{aligned}$$

22. Let $y = x - \frac{\pi}{8}$; then $x = y + \frac{\pi}{8}$. Thus,

$$\begin{aligned}\cos^2 x &= \cos^2\left(y + \frac{\pi}{8}\right) \\ &= \frac{1}{2}\left[1 + \cos\left(2y + \frac{\pi}{4}\right)\right] \\ &= \frac{1}{2}\left[1 + \frac{1}{\sqrt{2}}\cos(2y) - \frac{1}{\sqrt{2}}\sin(2y)\right] \\ &= \frac{1}{2} + \frac{1}{2\sqrt{2}}\left[1 - \frac{(2y)^2}{2!} + \frac{(2y)^4}{4!} - \cdots\right] \\ &\quad - \frac{1}{2\sqrt{2}}\left[2y - \frac{(2y)^3}{3!} + \frac{(2y)^5}{5!} - \cdots\right] \\ &= \frac{1}{2} + \frac{1}{2\sqrt{2}}\left[1 - 2y - \frac{(2y)^2}{2!} + \frac{(2y)^3}{3!} \right. \\ &\quad \left. + \frac{(2y)^4}{4!} - \frac{(2y)^5}{5!} - \cdots\right] \\ &= \frac{1}{2} + \frac{1}{2\sqrt{2}}\left[1 - 2\left(x - \frac{\pi}{8}\right) - \frac{2^2}{2!}\left(x - \frac{\pi}{8}\right)^2 \right. \\ &\quad \left. + \frac{2^3}{3!}\left(x - \frac{\pi}{8}\right)^3 + \frac{2^4}{4!}\left(x - \frac{\pi}{8}\right)^4 - \frac{2^5}{5!}\left(x - \frac{\pi}{8}\right)^5 - \cdots\right] \\ &= \frac{1}{2} + \frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}\sum_{n=1}^{\infty}(-1)^n\left[\frac{2^{2n-1}}{(2n-1)!}\left(x - \frac{\pi}{8}\right)^{2n-1} \right. \\ &\quad \left. + \frac{2^{2n}}{(2n)!}\left(x - \frac{\pi}{8}\right)^{2n}\right] \qquad \text{(for all } x).\end{aligned}$$

23. Let $t = x + 2$, so $x = t - 2$. We have

$$\begin{aligned}f(x) &= \frac{1}{x^2} = \frac{1}{(t-2)^2} = \frac{1}{4\left(1 - \frac{t}{2}\right)^2} \\ &= \frac{1}{4}\sum_{n=1}^{\infty} n\frac{t^{n-1}}{2^{n-1}} \qquad (-2 \le t < 2) \\ &= \frac{1}{4}\sum_{n=1}^{\infty} \frac{n(x+2)^{n-1}}{2^{n-1}} \\ &= \frac{1}{4}\sum_{n=0}^{\infty} \frac{(n+1)(x+2)^n}{2^n} \qquad (-4 < x < 0).\end{aligned}$$

24. Let $y = x - 1$; then $x = y + 1$. Thus,

$$\begin{aligned}
\frac{x}{1+x} &= \frac{1+y}{2+y} = 1 - \frac{1}{2\left(1+\frac{y}{2}\right)} \\
&= 1 - \frac{1}{2}\left[1 - \frac{y}{2} + \left(\frac{y}{2}\right)^2 - \left(\frac{y}{2}\right)^3 + \cdots\right] \\
&= \frac{1}{2}\left[1 + \frac{y}{2} - \frac{y^2}{2^2} + \frac{y^3}{2^3} - \frac{y^4}{2^4} + \cdots\right] \qquad (-1 < y < 1) \\
&= \frac{1}{2} + \frac{1}{2^2}(x-1) - \frac{1}{2^3}(x-1)^2 + \frac{1}{2^4}(x-1)^3 - \cdots \\
&= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^{n+1}}(x-1)^n \qquad (\text{for } 0 < x < 2).
\end{aligned}$$

25. Let $u = x - 1$. Then $x = 1 + u$, and

$$\begin{aligned}
x \ln x &= (1+u)\ln(1+u) \\
&= (1+u)\sum_{n=1}^{\infty}(-1)^{n-1}\frac{u^n}{n} \qquad (-1 < u \le 1) \\
&= \sum_{n=1}^{\infty}(-1)^{n-1}\frac{u^n}{n} + \sum_{n=1}^{\infty}(-1)^{n-1}\frac{u^{n+1}}{n}.
\end{aligned}$$

Replace n by $n-1$ in the last sum.

$$\begin{aligned}
x \ln x &= \sum_{n=1}^{\infty}(-1)^{n-1}\frac{u^n}{n} + \sum_{n=2}^{\infty}(-1)^{n-2}\frac{u^n}{n-1} \\
&= u + \sum_{n=2}^{\infty}(-1)^{n-1}\left(\frac{1}{n} - \frac{1}{n-1}\right)u^n \\
&= (x-1) + \sum_{n=2}^{\infty}\frac{(-1)^n}{n(n-1)}(x-1)^n \qquad (0 \le x \le 2).
\end{aligned}$$

26. Let $u = x + 2$. Then $x = u - 2$, and

$$\begin{aligned}
xe^x &= (u-2)e^{u-2} \\
&= (u-2)e^{-2}\sum_{n=0}^{\infty}\frac{u^n}{n!} \qquad (\text{for all } u) \\
&= \sum_{n=0}^{\infty}\frac{e^{-2}u^{n+1}}{n!} - \sum_{n=0}^{\infty}\frac{2e^{-2}u^n}{n!}.
\end{aligned}$$

In the first sum replace n by $n-1$.

$$\begin{aligned}
xe^x &= \sum_{n=1}^{\infty}\frac{e^{-2}u^n}{(n-1)!} - \sum_{n=0}^{\infty}\frac{2e^{-2}u^n}{n!} \\
&= -\frac{2}{e^2} + \sum_{n=1}^{\infty}\frac{1}{e^2}\left(\frac{1}{(n-1)!} - \frac{2}{n!}\right)u^n \\
&= -\frac{2}{e^2} + \sum_{n=1}^{\infty}\frac{1}{e^2}\left(\frac{1}{(n-1)!} - \frac{2}{n!}\right)(x+2)^n \qquad (\text{for all } x).
\end{aligned}$$

27. $\cos x = 1 - \dfrac{x^2}{2} + \dfrac{x^4}{24} - \cdots.$

$$\begin{array}{r|l}
 & 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \cdots \\ \hline
1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots & 1 \\
 & 1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots \\ \cline{2-2}
 & \qquad \frac{x^2}{2} - \frac{x^4}{24} + \cdots \\
 & \qquad \frac{x^2}{2} - \frac{x^4}{4} + \cdots \\ \cline{2-2}
 & \qquad\qquad \frac{5x^4}{24} - \cdots
\end{array}$$

Thus $\sec x = 1 + \dfrac{x^2}{2} + \dfrac{5x^4}{24} + \cdots.$

28. If we divide the first four terms of the series

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots$$

into 1 we obtain

$$\sec x = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \cdots.$$

Now we can differentiate and obtain

$$\sec x \tan x = x + \frac{5x^3}{6} + \frac{61x^5}{120} + \cdots.$$

(Note: the same result can be obtained by multiplying the first three nonzero terms of the series for $\sec x$ (from Exercise 25) and $\tan x$ (from Example 6(b)).)

29.

$$\begin{aligned}
e^x - 1 &= x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots \\
\tan^{-1}(e^x - 1) &= (e^x - 1) - \frac{(e^x-1)^3}{3} \\
&\quad + \frac{(e^x-1)^5}{5} - \cdots \\
&= x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots \\
&\quad - \frac{1}{3}\left(x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots\right)^3 \\
&\quad + \frac{1}{5}\left(x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots\right)^5 + \cdots \\
&= x + \frac{x^2}{2} + \frac{x^3}{6} - \frac{1}{3}\left(x^3 + \cdots\right) + \cdots \\
&= x + \frac{x^2}{2} - \frac{x^3}{6} + \cdots
\end{aligned}$$

30. We have

$$\begin{aligned}e^{\tan^{-1}x}-1 &= \exp\left[x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots\right]-1\\ &= 1+\left(x-\frac{x^3}{3}+\frac{x^5}{5}-\cdots\right)+\frac{1}{2!}\left(x-\frac{x^3}{3}+\cdots\right)^2\\ &\quad+\frac{1}{3!}(x-\cdots)^3+\cdots-1\\ &= x-\frac{x^3}{3}+\frac{x^2}{2}+\frac{x^3}{6}+\text{higher degree terms}\\ &= x+\frac{x^2}{2}-\frac{x^3}{6}+\cdots.\end{aligned}$$

31. Let $\sqrt{1+x}=1+ax+bx^2+\cdots$.
Then $1+x=1+2ax+(a^2+2b)x^2+\cdots$, so $2a=1$, and $a^2+2b=0$. Thus $a=1/2$ and $b=-1/8$.
Therefore $\sqrt{1+x}=1+(x/2)-(x^2/8)+\cdots$.

32. $\csc x$ does not have a Maclaurin series because $\lim_{x\to 0}\csc x$ does not exist.
Let $y=x-\dfrac{\pi}{2}$. Then $x=y+\dfrac{\pi}{2}$ and $\sin x=\cos y$. Therefore, using the result of Exercise 25,

$$\begin{aligned}\csc x=\sec y &= 1+\frac{y^2}{2}+\frac{5y^4}{24}+\cdots\\ &= 1+\frac{1}{2}\left(x-\frac{\pi}{2}\right)^2+\frac{5}{24}\left(x-\frac{\pi}{2}\right)^4+\cdots.\end{aligned}$$

33. $1+x^2+\dfrac{x^4}{2!}+\dfrac{x^6}{3!}+\cdots=e^{x^2}$ (for all x).

34.
$$\begin{aligned}&x^3-\frac{x^9}{3!\times 4}+\frac{x^{15}}{5!\times 16}-\frac{x^{21}}{7!\times 64}+\frac{x^{27}}{9!\times 256}-\cdots\\ &=2\left[\frac{x^3}{2}-\frac{1}{3!}\left(\frac{x^3}{2}\right)^3+\frac{1}{5!}\left(\frac{x^3}{2}\right)^5-\cdots\right]\\ &=2\sin\left(\frac{x^3}{2}\right)\qquad\text{(for all } x).\end{aligned}$$

35.
$$\begin{aligned}&1+\frac{x^2}{3!}+\frac{x^4}{5!}+\frac{x^6}{7!}+\cdots\\ &\qquad=\frac{1}{x}\sinh x=\frac{e^x-e^{-x}}{2x}\end{aligned}$$
if $x\neq 0$. The sum is 1 if $x=0$.

36.
$$\begin{aligned}&1+\frac{1}{2\times 2!}+\frac{1}{4\times 3!}+\frac{1}{8\times 4!}+\cdots\\ &=2\left[\frac{1}{2}+\frac{1}{2!}\left(\frac{1}{2}\right)^2+\frac{1}{3!}\left(\frac{1}{2}\right)^3+\cdots\right]\\ &=2\left(e^{1/2}-1\right).\end{aligned}$$

37. $P(x)=1+x+x^2$.

a) The Maclaurin series for $P(x)$ is $1+x+x^2$ (for all x).

b) Let $t=x-1$, so $x=t+1$. Then
$P(x)=P(t+1)=1+t+1+(t+1)^2=3+3t+t^2$.
The Taylor series for $P(x)$ about 1 is $3+3(x-1)+(x-1)^2$.

38. If $a\neq 0$ and $|x-a|<|a|$, then

$$\begin{aligned}\frac{1}{x}&=\frac{1}{a+(x-a)}=\frac{1}{a}\,\frac{1}{1+\dfrac{x-a}{a}}\\ &=\frac{1}{a}\left[1-\frac{x-a}{a}+\frac{(x-a)^2}{a^2}-\frac{(x-a)^3}{a^3}+\cdots\right].\end{aligned}$$

The radius of convergence of this series is $|a|$, and the series converges to $1/x$ throughout its interval of convergence. Hence, $1/x$ is analytic at a.

39. If $a>0$ and $t=x-a$, then $x=t+a$ and

$$\begin{aligned}\ln x&=\ln(a+t)=\ln a+\ln\left(1+\frac{t}{a}\right)\\ &=\ln a+\sum_{n=1}^{\infty}(-1)^{n-1}\frac{t^n}{a^n}\qquad(-a<t\le a)\\ &=\ln a+\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(x-a)^n}{a^n}\qquad(0<x<2a).\end{aligned}$$

Since the series converges to $\ln x$ on an interval of positive radius (a), centred at a, ln is analytic at a.

40. If
$$f(x)=\begin{cases}e^{-1/x^2}, & \text{if } x\neq 0;\\ 0, & \text{if } x=0;\end{cases}$$
then the Maclaurin series for $f(x)$ is the identically zero series $0+0x+0x^2+\cdots$ since $f^{(k)}(0)=0$ for every k. The series converges for every x, but converges to $f(x)$ only at $x=0$, since $f(x)\neq 0$ if $x\neq 0$. Hence, f cannot be analytic at 0.

41.
$$\begin{aligned}e^xe^y&=\left(\sum_{n=0}^{\infty}\frac{x^n}{n!}\right)\left(\sum_{m=0}^{\infty}\frac{y^m}{m!}\right)\\ e^{x+y}&=\sum_{k=0}^{\infty}\frac{(x+y)^k}{k!}=\sum_{k=0}^{\infty}\frac{1}{k!}\sum_{j=0}^{k}\frac{k!}{j!(k-j)!}x^jy^{k-j}\\ &=\sum_{j=0}^{\infty}\frac{x^j}{j!}\sum_{k=j}^{\infty}\frac{y^{k-j}}{(k-j)!}\qquad(\text{let } k-j=m)\\ &=\sum_{j=0}^{\infty}\frac{x^j}{j!}\sum_{m=0}^{\infty}\frac{y^m}{m!}=e^xe^y.\end{aligned}$$

42. We want to prove that $f(x)=P_n(x)+E_n(x)$, where P_n is the nth-order Taylor polynomial for f about c and

$$E_n(x)=\frac{1}{n!}\int_c^x(x-t)^nf^{(n+1)}(t)\,dt.$$

(a) The Fundamental Theorem of Calculus written in the form

$$f(x) = f(c) + \int_c^x f'(t)\,dt = P_0(x) + E_0(x)$$

is the case $n = 0$ of the above formula. We now apply integration by parts to the integral, setting

$$\begin{aligned} U &= f'(t), & dV &= dt, \\ dU &= f''(t)\,dt, & V &= -(x-t). \end{aligned}$$

(We have broken our usual rule about not including a constant of integration with V. In this case we have included the constant $-x$ in V in order to have V vanish when $t = x$.) We have

$$\begin{aligned} f(x) &= f(c) - f'(t)(x-t)\Big|_{t=c}^{t=x} + \int_c^x (x-t) f''(t)\,dt \\ &= f(c) + f'(c)(x-c) + \int_c^x (x-t) f''(t)\,dt \\ &= P_1(x) + E_1(x). \end{aligned}$$

We have now proved the case $n = 1$ of the formula.

(b) We complete the proof for general n by mathematical induction. Suppose the formula holds for some $n = k$:

$$\begin{aligned} f(x) &= P_k(x) + E_k(x) \\ &= P_k(x) + \frac{1}{k!}\int_c^x (x-t)^k f^{(k+1)}(t)\,dt. \end{aligned}$$

Again we integrate by parts. Let

$$\begin{aligned} U &= f^{(k+1)}(t), & dV &= (x-t)^k\,dt, \\ dU &= f^{(k+2)}(t)\,dt, & V &= \frac{-1}{k+1}(x-t)^{k+1}. \end{aligned}$$

We have

$$\begin{aligned} f(x) &= P_k(x) + \frac{1}{k!}\left(-\frac{f^{(k+1)}(t)(x-t)^{k+1}}{k+1}\Big|_{t=c}^{t=x} \right. \\ &\qquad \left. + \int_c^x \frac{(x-t)^{k+1} f^{(k+2)}(t)}{k+1}\,dt\right) \\ &= P_k(x) + \frac{f^{(k+1)}(c)}{(k+1)!}(x-c)^{k+1} \\ &\qquad + \frac{1}{(k+1)!}\int_c^x (x-t)^{k+1} f^{(k+2)}(t)\,dt \\ &= P_{k+1}(x) + E_{k+1}(x). \end{aligned}$$

Thus the formula is valid for $n = k+1$ if it is valid for $n = k$. Having been shown to be valid for $n = 0$ (and $n = 1$), it must therefore be valid for every positive integer n for which $E_n(x)$ exists.

43. If $f(x) = \ln(1+x)$, then

$$f'(x) = \frac{1}{1+x}, \quad f''(x) = \frac{-1}{(1+x)^2}, \quad f'''(x) = \frac{2}{(1+x)^3},$$
$$f^{(4)}(x) = \frac{-3!}{(1+x)^4}, \quad \ldots, \quad f^{(n)} = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}$$

and

$$\begin{gathered} f(0) = 0, \quad f'(0) = 1, \quad f''(0) = -1, \quad f'''(0) = 2, \\ f^{(4)}(0) = -3!, \quad \ldots, \quad f^{(n)}(0) = (-1)^{n-1}(n-1)!. \end{gathered}$$

Therefore, the Taylor Formula is

$$\begin{aligned} f(x) = x &+ \frac{-1}{2!}x^2 + \frac{2}{3!}x^3 + \frac{-3!}{4!}x^4 + \cdots + \\ &\frac{(-1)^{n-1}(n-1)!}{n!}x^n + E_n(x) \end{aligned}$$

where

$$\begin{aligned} E_n(x) &= \frac{1}{n!}\int_0^x (x-t)^n f^{(n+1)}(t)\,dt \\ &= \frac{1}{n!}\int_0^x (x-t)^n \frac{(-1)^n n!}{(1+t)^{n+1}}\,dt \\ &= (-1)^n \int_0^x \frac{(x-t)^n}{(1+t)^{n+1}}\,dt. \end{aligned}$$

If $0 \le t \le x \le 1$, then $1 + t \ge 1$ and

$$|E_n(x)| \le \int_0^x (x-t)^n\,dt = \frac{x^{n+1}}{n+1} \le \frac{1}{n+1} \to 0$$

as $n \to \infty$.
If $-1 < x \le t \le 0$, then

$$\left|\frac{x-t}{1+t}\right| = \frac{t-x}{1+t} \le |x|,$$

because $\dfrac{t-x}{1+t}$ increases from 0 to $-x = |x|$ as t increases from x to 0. Thus,

$$|E_n(x)| < \frac{1}{1+x}\int_0^{|x|} |x|^n\,dt = \frac{|x|^{n+1}}{1+x} \to 0$$

as $n \to \infty$ since $|x| < 1$. Therefore,

$$f(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n},$$

for $-1 < x \le 1$.

44. We follow the steps outlined in the problem:

(a) Note that $\ln(j-1) < \int_{j-1}^{j} \ln x\,dx < \ln j$, $j = 1, 2, \ldots$. For $j = 0$ the integral is improper but convergent. We have

$$\begin{aligned} n\ln n - n &= \int_0^n \ln x\,dx < \ln(n!) < \int_1^{n+1} \ln x\,dx \\ &= (n+1)\ln(n+1) - n - 1 < (n+1)\ln(n+1) - n. \end{aligned}$$

(b) If $c_n = \ln(n!) - \left(n+\frac{1}{2}\right)\ln n + n$, then

$$\begin{aligned} c_n - c_{n+1} &= \ln\frac{n!}{(n+1)!} - \left(n+\tfrac{1}{2}\right)\ln n \\ &\quad + \left(n+\tfrac{3}{2}\right)\ln(n+1) - 1 \\ &= \ln\frac{1}{n+1} - \left(n+\tfrac{1}{2}\right)\ln n \\ &\quad + \left(n+\tfrac{1}{2}\right)\ln(n+1) + \ln(n+1) - 1 \\ &= \left(n+\tfrac{1}{2}\right)\ln\frac{n+1}{n} - 1 \\ &= \left(n+\tfrac{1}{2}\right)\ln\frac{1+\frac{1}{2n+1}}{1-\frac{1}{2n+1}} - 1. \end{aligned}$$

(c) $\ln\dfrac{1+t}{1-t} = 2\left(t + \dfrac{t^3}{3} + \dfrac{t^5}{5} + \cdots\right)$ for $-1 < t < 1$. Thus

$$\begin{aligned} 0 < c_n - c_{n+1} &= (2n+1)\left(\frac{1}{2n+1} + \frac{1}{3(2n+1)^3}\right. \\ &\qquad \left. + \frac{1}{5(2n+1)^5} + \cdots\right) - 1 \\ &< \frac{1}{3}\left(\frac{1}{(2n+1)^2} + \frac{1}{(2n+1)^4} + \cdots\right) \\ &\qquad \text{(geometric)} \\ &= \frac{1}{3(2n+1)^2}\,\frac{1}{1-\dfrac{1}{(2n+1)^2}} \\ &= \frac{1}{12(n^2+n)} \\ &= \frac{1}{12}\left(\frac{1}{n} - \frac{1}{n+1}\right). \end{aligned}$$

These inequalities imply that $\{c_n\}$ is decreasing and $\left\{c_n - \frac{1}{12n}\right\}$ is increasing. Thus $\{c_n\}$ is bounded below by $c_1 - \frac{1}{12} = \frac{11}{12}$ and so $\lim_{n\to\infty} c_n = c$ exists. Since $e^{c_n} = n!n^{-(n+1/2)}e^n$, we have

$$\lim_{n\to\infty} \frac{n!}{n^{n+1/2}e^{-n}} = \lim_{n\to\infty} e^{c_n} = e^c$$

exists. It remains to show that $e^c = \sqrt{2\pi}$.

(d) The Wallis Product,

$$\lim_{n\to\infty} \frac{2}{1}\frac{2}{3}\frac{4}{3}\frac{4}{5}\frac{6}{5}\cdots\frac{2n}{2n-1}\frac{2n}{2n+1} = \frac{\pi}{2}$$

can be rewritten in the form

$$\lim_{n\to\infty} \frac{2^n n!}{1\cdot 3\cdot 5\cdots(2n-1)\sqrt{2n+1}} = \sqrt{\frac{\pi}{2}},$$

or, equivalently,

$$\lim_{n\to\infty} \frac{2^{2n}(n!)^2}{(2n)!\sqrt{2n+1}} = \sqrt{\frac{\pi}{2}}.$$

Substituting $n! = n^{n+1/2}e^{-}ne^{c_n}$ and a similar expression for $(2n)!$, we obtain

$$\lim_{n\to\infty} \frac{2^{2n}n^{2n+1}e^{-2n}e^{2c_n}}{2^{2n+1/2}n^{2n+1/2}e^{-2n}e^{c_{2n}}\sqrt{2n}} = \frac{e^{2c}}{2e^c} = \frac{e^c}{2}.$$

Thus $e^c/2 = \sqrt{\pi}2$, and $e^c = \sqrt{2\pi}$, which completes the proof of Stirling's Formula.

Section 9.7 Applications of Taylor and Maclaurin Series (page 524)

1. If $f(x) = \sin x$, then $P_5(x) = x - \dfrac{x^3}{6} + \dfrac{x^5}{120}$.

METHOD I. (using an alternating series bound)

$$|f(0.2) - P_5(0.2)| \le \frac{(0.2)^7}{7!} < 2.6\times 10^{-9}.$$

METHOD II. (using Taylor's Theorem) Since $P_5(x) = P_6(x)$ (Maclaurin polynomials for sin have only odd degree terms) we are better off using the remainder E_6.

$$|f(0.2) - P_5(0.2)| = |E_6(0.2)| = \frac{|f^{(7)}(s)|}{7!}(0.2)^7,$$

for some s between 0 and 0.2. Now $f^{(7)}(x) = -\cos x$, so

$$|f(0.2) - P_5(0.2)| < \frac{1}{7!}\times(0.2)^7 < 2.6\times 10^{-9}.$$

2. If $f(x) = \ln x$, then $f'(x) = 1/x$, $f''(x) = -1/x^2$, $f'''(x) = 2/x^3$, $f^{(4)}(x) = -6/x^4$, and $f^{(5)}(x) = 24/x^5$. If $P_4(x)$ is the Taylor polynomial for f about $x = 2$, then for some s between 1.95 and 2 we have (using Taylor's Theorem)

$$\begin{aligned} |f(1.95) - P_4(1.95)| &= \frac{24}{s^5}\cdot\frac{(0.05)^5}{5!} \\ &\le \frac{24(0.05)^5}{(1.95)^5 120} < 2.22\times 10^{-9}. \end{aligned}$$

3. $e^{0.2} \approx 1 + 0.2 + \frac{(0.2)^2}{2!} + \cdots + \frac{(0.2)^n}{n!} = s_n$

Error estimate:

$$\begin{aligned} 0 < e^{0.2} - s_n &= \frac{(0.2)^{n+1}}{(n+1)!} + \frac{(0.2)^{n+2}}{(n+2)!} + \cdots \\ &\le \frac{(0.2)^{n+1}}{(n+1)!}\left[1 + \frac{0.2}{n+2} + \frac{(0.2)^2}{(n+2)^2} + \cdots\right] \\ &= \frac{(0.2)^{n+1}}{(n+1)!} \cdot \frac{10n+20}{10n+18} < 5 \times 10^{-5} \text{ if } n = 4. \end{aligned}$$

$$\begin{aligned} e^{0.2} &\approx 1 + 0.2 + \frac{(0.2)^2}{2!} + \frac{(0.2)^3}{3!} + \frac{(0.2)^4}{4!} \\ &\approx 1.221400 \end{aligned}$$

4. We have

$$\frac{1}{e} = e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots$$

which satisfies the conditions for the alternating series test, and the error incurred in using a partial sum to approximate e^{-1} is less than the first omitted term in absolute value. Now $\frac{1}{(n+1)!} < 5 \times 10^{-5}$ if $n = 7$, so

$$\frac{1}{e} \approx \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} - \frac{1}{5040} \approx 0.36786$$

with error less than 5×10^{-5} in absolute value.

5. $e^{1.2} = ee^{0.2}$. From Exercise 1: $e^{0.2} \approx 1.221400$, with error less than $\frac{(0.2)^5}{5!} \cdot \frac{60}{58} \approx 0.000003$. Since $e = 2.718281828\cdots$, it follows that $e^{1.2} \approx 3.3201094\cdots$, with error less than $3 \times 0.000003 = 0.000009 < \frac{1}{20,000}$. Thus $e^{1.2} \approx 3.32011$ with error less than 1/20,000.

6. We have

$$\sin(0.1) = 0.1 - \frac{(0.1)^3}{3!} + \frac{(0.1)^5}{5!} - \frac{(0.1)^7}{7!} + \cdots.$$

Since $\frac{(0.1)^5}{5!} = 8.33 \times 10^{-8} < 5 \times 10^{-5}$, therefore

$$\sin(0.1) = 0.1 - \frac{(0.1)^3}{3!} \approx 0.09983$$

with error less than 5×10^{-5} in absolute value.

7.

$$\begin{aligned} \cos 5^\circ &= \cos\frac{5\pi}{180} = \cos\frac{\pi}{36} \\ &\approx 1 - \frac{1}{2!}\left(\frac{\pi}{36}\right)^2 + \frac{1}{4!}\left(\frac{\pi}{36}\right)^4 - \cdots + \frac{(-1)^n}{(2n)!}\left(\frac{\pi}{36}\right)^{2n} \\ |\text{Error}| &< \frac{1}{(2n+2)!}\left(\frac{\pi}{36}\right)^{2n+2} \\ &< \frac{1}{(2n+2)!9^{2n+2}} < 0.00005 \quad \text{if } n = 1. \end{aligned}$$

$$\cos 5^\circ \approx 1 - \frac{1}{2!}\left(\frac{\pi}{36}\right)^2 \approx 0.996192$$

with error less than 0.00005.

8. We have

$$\begin{aligned} \ln\left(\frac{6}{5}\right) &= \ln\left(1 + \frac{1}{5}\right) \\ &= \frac{1}{5} - \frac{1}{2}\left(\frac{1}{5}\right)^2 + \frac{1}{3}\left(\frac{1}{5}\right)^3 - \frac{1}{4}\left(\frac{1}{5}\right)^4 + \cdots. \end{aligned}$$

Since $\frac{1}{n}\left(\frac{1}{5}\right)^n < 5 \times 10^{-5}$ if $n = 6$, therefore

$$\begin{aligned} \ln\left(\frac{6}{5}\right) &\approx \frac{1}{5} - \frac{1}{2}\left(\frac{1}{5}\right)^2 + \frac{1}{3}\left(\frac{1}{5}\right)^3 - \frac{1}{4}\left(\frac{1}{5}\right)^4 + \frac{1}{5}\left(\frac{1}{5}\right)^5 \\ &\approx 0.18233 \end{aligned}$$

with error less than 5×10^{-5} in absolute value.

9.

$$\begin{aligned} \ln(0.9) &= \ln(1 - 0.1) \\ &\approx -0.1 - \frac{(0.1)^2}{2} - \frac{(0.1)^3}{3} - \cdots - \frac{(0.1)^n}{n} \\ |\text{Error}| &< \frac{(0.1)^{n+1}}{n+1} + \frac{(0.1)^{n+2}}{n+2} + \cdots \\ &< \frac{(0.1)^{n+1}}{n+1}\left[1 + 0.1 + (0.1)^2 + \cdots\right] \\ &= \frac{(0.1)^{n+1}}{n+1} \cdot \frac{10}{9} < 0.00005 \quad \text{if } n = 3. \end{aligned}$$

$$\ln(0.9) \approx -0.1 - \frac{(0.1)^2}{2} - \frac{(0.1)^3}{3} \approx -0.10533$$

with error less than 0.00005.

10. We have

$$\begin{aligned} \sin 80^\circ &= \cos 10^\circ = \cos\left(\frac{\pi}{18}\right) \\ &= 1 - \frac{1}{2!}\left(\frac{\pi}{18}\right)^2 + \frac{1}{4!}\left(\frac{\pi}{18}\right)^4 - \cdots. \end{aligned}$$

Since $\frac{1}{4!}\left(\frac{\pi}{18}\right)^4 < 5 \times 10^{-5}$, therefore

$$\sin 80^\circ \approx 1 - \frac{1}{2!}\left(\frac{\pi}{18}\right)^2 \approx 0.98477$$

with error less than 5×10^{-5} in absolute value.

11.

$$\begin{aligned} \cos 65^\circ &= \cos\left(\frac{\pi}{3} + \frac{5\pi}{180}\right) \\ &= \frac{1}{2}\cos\frac{5\pi}{180} - \frac{\sqrt{3}}{2}\sin\frac{5\pi}{180} \end{aligned}$$

From Exercise 5, $\cos(5\pi/180) \approx 0.996192$ with error less than 0.000003. Also

$$\sin\frac{5\pi}{180} = \frac{5\pi}{180} - \frac{1}{3!}\left(\frac{5\pi}{180}\right)^3 \approx 0.0871557$$

with error less than $\dfrac{5^5\pi^5}{5!180^5} < 0.00000005$. Thus

$$\cos 65^\circ \approx \frac{0.996192}{2} - \frac{\sqrt{3}(0.0871557)}{2} \approx 0.42262$$

with error less than 0.00005.

12. We have

$$\tan^{-1}(0.2) = 0.2 - \frac{(0.2)^3}{3} + \frac{(0.2)^5}{5} - \frac{(0.2)^7}{7} + \cdots.$$

Since $\dfrac{(0.2)^7}{7} < 5 \times 10^{-5}$, therefore

$$\tan^{-1}(0.2) \approx 0.2 - \frac{(0.2)^3}{3} + \frac{(0.2)^5}{5} \approx 0.19740$$

with error less than 5×10^{-5} in absolute value.

13. $\cosh 1 \approx 1 + \dfrac{1}{2!} + \dfrac{1}{4!} + \cdots + \dfrac{1}{(2n)!}$ with error less than

$$\begin{aligned}&\frac{1}{(2n+2)!}\left[1 + \frac{1}{(2n+3)^2} + \frac{1}{(2n+3)^4} + \cdots\right]\\ &\quad = \frac{1}{(2n+2)!} \cdot \frac{1}{1 - \dfrac{1}{(2n+3)^2}} < 0.00005 \quad \text{if } n = 3.\end{aligned}$$

Thus $\cosh 1 \approx 1 + \dfrac{1}{2} + \dfrac{1}{24} + \dfrac{1}{720} \approx 1.54306$ with error less than 0.00005.

14. We have

$$\begin{aligned}\ln\left(\frac{3}{2}\right) &= \ln\left(1 + \frac{1}{2}\right)\\ &= \frac{1}{2} - \frac{1}{2}\left(\frac{1}{2}\right)^2 + \frac{1}{3}\left(\frac{1}{2}\right)^3 - \frac{1}{4}\left(\frac{1}{2}\right)^4 + \cdots.\end{aligned}$$

Since $\dfrac{1}{n}\left(\dfrac{1}{2}\right)^n < \dfrac{1}{20000}$ if $n = 11$, therefore

$$\begin{aligned}\ln\left(\frac{3}{2}\right) &\approx \frac{1}{2} - \frac{1}{2}\left(\frac{1}{2}\right)^2 + \frac{1}{3}\left(\frac{1}{2}\right)^3 - \cdots - \frac{1}{10}\left(\frac{1}{2}\right)^{10}\\ &\approx 0.40543\end{aligned}$$

with error less than 5×10^{-5} in absolute value.

15.
$$\begin{aligned}I(x) &= \int_0^x \frac{\sin t}{t}\,dt\\ &= \int_0^x \left[1 - \frac{t^2}{3!} + \frac{t^4}{5!} - \frac{t^6}{7!} + \cdots\right] dt\\ &= x - \frac{x^3}{3 \times 3!} + \frac{x^5}{5 \times 5!} - \cdots\\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)(2n+1)!} \qquad \text{for all } x.\end{aligned}$$

16.
$$\begin{aligned}J(x) &= \int_0^x \frac{e^t - 1}{t}\,dt\\ &= \int_0^x \left(1 + \frac{t}{2!} + \frac{t^2}{3!} + \frac{t^3}{4!} + \cdots\right) dt\\ &= x + \frac{x^2}{2! \cdot 2} + \frac{x^3}{3! \cdot 3} + \frac{x^4}{4! \cdot 4} + \cdots\\ &= \sum_{n=1}^{\infty} \frac{x^n}{n! \cdot n}.\end{aligned}$$

17.
$$\begin{aligned}K(x) &= \int_1^{1+x} \frac{\ln t}{t - 1}\,dt \qquad \text{let } u = t - 1\\ &= \int_0^x \frac{\ln(1+u)}{u}\,du\\ &= \int_0^x \left[1 - \frac{u}{2} + \frac{u^2}{3} - \frac{u^3}{4} + \cdots\right] du\\ &= x - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \frac{x^4}{4^2} + \cdots\\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{(n+1)^2} \qquad (-1 \le x \le 1)\end{aligned}$$

18.
$$\begin{aligned}L(x) &= \int_0^x \cos(t^2)\,dt\\ &= \int_0^x \left(1 - \frac{t^4}{2!} + \frac{t^8}{4!} - \frac{t^{12}}{6!} + \cdots\right) dt\\ &= x - \frac{x^5}{2! \cdot 5} + \frac{x^9}{4! \cdot 9} - \frac{x^{13}}{6! \cdot 13} + \cdots\\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+1}}{(2n)! \cdot (4n+1)}.\end{aligned}$$

19.
$$\begin{aligned}M(x) &= \int_0^x \frac{\tan^{-1}(t^2)}{t^2}\,dt\\ &= \int_0^x \left[1 - \frac{t^4}{3} + \frac{t^8}{5} - \frac{t^{12}}{7} + \cdots\right] dt\\ &= x - \frac{x^5}{3 \times 5} + \frac{x^9}{5 \times 9} - \frac{x^{13}}{7 \times 13} + \cdots\\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+1}}{(2n+1)(4n+1)} \qquad (-1 \le x \le 1)\end{aligned}$$

20. We have

$$L(0.5) = 0.5 - \frac{(0.5)^5}{2! \cdot 5} + \frac{(0.5)^9}{4! \cdot 9} - \frac{(0.5)^{13}}{6! \cdot 13} + \cdots.$$

Since $\dfrac{(0.5)^{4n+1}}{(2n)! \cdot (4n+1)} < 5 \times 10^{-4}$ if $n = 2$, therefore

$$L(0.5) \approx 0.5 - \frac{(0.5)^5}{2! \cdot 5} \approx 0.497$$

rounded to 3 decimal places.

21. From Exercise 13:

$$I(x) = x - \frac{x^3}{3!3} + \frac{x^5}{5!5} - \cdots$$
$$I(1) \approx 1 - \frac{1}{3!3} + \frac{1}{5!5} - \cdots + (-1)^n \frac{1}{(2n+1)!(2n+1)}$$
$$|\text{Error}| \le \frac{1}{(2n+3)!(2n+3)} < 0.0005 \quad \text{if } n = 2.$$

Thus $I(1) \approx 1 - \frac{1}{3!3} + \frac{1}{5!5} \approx 0.946$ correct to three decimal places.

22.
$$\lim_{x\to 0} \frac{\sin(x^2)}{\sinh x} = \lim_{x\to 0} \frac{x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \cdots}{x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots} = \lim_{x\to 0} \frac{x - \frac{x^5}{3!} + \frac{x^9}{5!} - \cdots}{1 + \frac{x^2}{3!} + \frac{x^4}{5!} + \cdots} = 0.$$

23.
$$\lim_{x\to 0} \frac{1-\cos(x^2)}{(1-\cos x)^2} = \lim_{x\to 0} \frac{1 - 1 + \frac{x^4}{2!} - \frac{x^8}{4!} + \cdots}{\left(1 - 1 + \frac{x^2}{2!} - \frac{x^4}{4!} + \cdots\right)^2} = \lim_{x\to 0} \frac{\frac{1}{2!} + O(x^2)}{\frac{1}{4} + O(x^2)} = 2.$$

24. We have

$$\lim_{x\to 0} \frac{(e^x - 1 - x)^2}{x^2 - \ln(1+x^2)} = \lim_{x\to 0} \frac{\left(\frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right)^2}{\frac{x^4}{2} - \frac{x^6}{3} + \frac{x^8}{4} - \cdots}$$
$$= \lim_{x\to 0} \frac{\frac{x^4}{4}\left(1 + \frac{x}{3} + \frac{x^2}{12} + \cdots\right)^2}{\frac{x^4}{2} - \frac{x^6}{3} + \frac{x^8}{4} - \cdots} = \frac{\left(\frac{1}{4}\right)}{\left(\frac{1}{2}\right)} = \frac{1}{2}.$$

25.
$$\lim_{x\to 0} \frac{2\sin 3x - 3\sin 2x}{5x - \tan^{-1} 5x}$$
$$= \lim_{x\to 0} \frac{2\left(3x - \frac{3^3x^3}{3!} + \cdots\right) - 3\left(2x - \frac{2^3x^3}{3!} + \cdots\right)}{5x - \left(5x - \frac{5^3x^3}{3} + \cdots\right)}$$
$$= \lim_{x\to 0} \frac{-9 + 4 + O(x^2)}{\frac{125}{3} + O(x^2)} = -\frac{5\times 3}{125} = -\frac{3}{25}.$$

26. We have

$$\lim_{x\to 0} \frac{\sin(\sin x) - x}{x[\cos(\sin x) - 1]}$$
$$= \lim_{x\to 0} \frac{\left(\sin x - \frac{1}{3!}\sin^3 x + \frac{1}{5!}\sin^5 x - \cdots\right) - x}{x\left[1 - \frac{1}{2!}\sin^2 x + \frac{1}{4!}\sin^4 x - \cdots - 1\right]}$$
$$= \lim_{x\to 0} \frac{\left(x - \frac{x^3}{3!} + \cdots\right) - \frac{1}{3!}\left(x - \frac{x^3}{3!} + \cdots\right)^3 + \frac{1}{5!}\left(x - \cdots\right)^5 - \cdots - x}{x\left[-\frac{1}{2!}\left(x - \frac{x^3}{3!} + \cdots\right)^2 + \frac{1}{4!}\left(x - \cdots\right)^4 - \cdots\right]}$$
$$= \lim_{x\to 0} \frac{-\frac{2}{3!}x^3 + \text{higher degree terms}}{-\frac{1}{2!}x^3 + \text{higher degree terms}} = \frac{\frac{2}{3!}}{\frac{1}{2!}} = \frac{2}{3}.$$

27.
$$\lim_{x\to 0} \frac{\sinh x - \sin x}{\cosh x - \cos x}$$
$$= \lim_{x\to 0} \frac{\left(x + \frac{x^3}{3!} + \cdots\right) - \left(x - \frac{x^3}{3!} + \cdots\right)}{\left(1 + \frac{x^2}{2!} + \cdots\right) - \left(1 - \frac{x^2}{2!} + \cdots\right)}$$
$$= \lim_{x\to 0} \frac{\frac{x^3}{3} + O(x^5)}{x^2 + O(x^4)} = 0.$$

Section 9.8 The Binomial Theorem and Binomial Series (page 528)

1.
$$\sqrt{1+x} = (1+x)^{1/2}$$
$$= 1 + \frac{x}{2} + \frac{1}{2}\left(-\frac{1}{2}\right)\frac{x^2}{2!} + \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\frac{x^3}{3!} + \cdots$$
$$= 1 + \frac{x}{2} + \sum_{n=2}^{\infty} (-1)^{n-1} \frac{1\cdot 3\cdot 5\cdots(2n-3)}{2^n n!} x^n$$
$$= 1 + \frac{x}{2} + \sum_{n=2}^{\infty} (-1)^{n-1} \frac{(2n-2)!}{2^{2n-1}(n-1)!n!} x^n \quad (-1 < x < 1).$$

2.
$$x\sqrt{1-x} = x(1-x)^{1/2}$$
$$= x - \frac{x^2}{2} + \frac{1}{2}\left(-\frac{1}{2}\right)\frac{(-1)^2x^3}{2!}$$
$$+ \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\frac{(-1)^3x^4}{3!} + \cdots$$
$$= x - \frac{x^2}{2} - \sum_{n=2}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-3)}{2^n n!} x^{n+1}$$
$$= x - \frac{x^2}{2} - \sum_{n=2}^{\infty} (-1)^{n-1} \frac{(2n-2)!}{2^{2n-1}(n-1)!n!} x^{n+1} \quad (-1 < x < 1).$$

3. $$\sqrt{4+x} = 2\sqrt{1+\frac{x}{4}}$$
$$= 2\left[1 + \frac{1}{2}\cdot\frac{x}{4} + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}\left(\frac{x}{4}\right)^2 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}\left(\frac{x}{4}\right)^3 + \cdots\right]$$
$$= 2 + \frac{x}{4} + 2\sum_{n=2}^{\infty}(-1)^{n-1}\frac{1\cdot 3\cdot 5\cdots(2n-3)}{2^{3n}n!}x^n$$
$$= 2 + \frac{x}{4} + 2\sum_{n=2}^{\infty}(-1)^{n-1}\frac{(2n-1)!}{2^{4n-1}n!(n-1)!}x^n$$
$$(-4 < x < 4).$$

4. $$\frac{1}{\sqrt{4+x^2}} = \frac{1}{2\sqrt{1+\left(\frac{x}{2}\right)^2}} = \frac{1}{2}\left[1+\left(\frac{x}{2}\right)^2\right]^{-1/2}$$
$$= \frac{1}{2}\left[1 + \left(-\frac{1}{2}\right)\left(\frac{x}{2}\right)^2 + \frac{1}{2!}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(\frac{x}{2}\right)^4 + \frac{1}{3!}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(\frac{x}{2}\right)^6 + \cdots\right]$$
$$= \frac{1}{2} - \frac{1}{2^4}x^2 + \frac{3}{2^7 2!}x^4 - \frac{3\times 5}{2^{10}3!}x^6 + \cdots$$
$$= \frac{1}{2} + \sum_{n=1}^{\infty}(-1)^n\frac{1\times 2\times 3\times\cdots\times(2n-1)}{2^{3n+1}n!}x^{2n}$$
$$(-2 \le x \le 2).$$

5. $$(1-x)^{-2}$$
$$= 1 - 2(-x) + \frac{(-2)(-3)}{2!}(-x)^2 + \frac{(-2)(-3)(-4)}{3!}(-x)^3 + \cdots$$
$$= 1 + 2x + 3x^2 + 4x^3 + \cdots = \sum_{n=1}^{\infty}nx^{n-1} \qquad (-1 < x < 1).$$

6. $$(1+x)^{-3} = 1 - 3x + \frac{(-3)(-4)}{2!}x^2 + \frac{(-3)(-4)(-5)}{3!}x^3 + \cdots$$
$$= 1 - 3x + \frac{(3)(4)}{2}x^2 - \frac{(4)(5)}{2}x^3 + \cdots$$
$$= \sum_{n=0}^{\infty}(-1)^n\frac{(n+2)(n+1)}{2}x^n \qquad (-1 < x < 1).$$

7. i) $\binom{n}{0} = \frac{n!}{0!n!} = 1, \quad \binom{n}{n} = \frac{n!}{n!0!} = 1.$

ii) If $0 \le k \le n$, then
$$\binom{n}{k-1} + \binom{n}{k} = \frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{k!(n-k)!}$$
$$= \frac{n!}{k!(n-k+1)!}\big(k + (n-k+1)\big)$$
$$= \frac{(n+1)!}{k!(n+1-k)!} = \binom{n+1}{k}.$$

8. The formula $(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k$

holds for $n=1$; it says $a+b = a+b$ in this case. Suppose the formula holds for $n=m$, where m is some positive integer. Then

$$(a+b)^{m+1} = (a+b)\sum_{k=0}^{m}\binom{m}{k}a^{m-k}b^k$$
$$= \sum_{k=0}^{m}\binom{m}{k}a^{m+1-k}b^k + \sum_{k=0}^{m}\binom{m}{k}a^{m-k}b^{k+1}$$
(replace k by $k-1$ in the latter sum)
$$= \sum_{k=0}^{m}\binom{m}{k}a^{m+1-k}b^k + \sum_{k=1}^{m+1}\binom{m}{k-1}a^{m+1-k}b^k$$
$$= a^{m+1} + \sum_{k=1}^{m}\left[\binom{m}{k} + \binom{m}{k-1}\right]a^{m+1-k}b^k + b^{m+1}$$
(by #13(i))
$$= a^{m+1} + \sum_{k=1}^{m}\binom{m+1}{k}a^{m+1-k}b^k + b^{m+1} \qquad \text{(by #13(ii))}$$
$$= \sum_{k=0}^{m+1}\binom{m+1}{k}a^{m+1-k}b^k \qquad \text{(by #13(i) again).}$$

Thus the formula holds for $n = m+1$. By induction it holds for all positive integers n.

9. Consider the Leibniz Rule:
$$(fg)^{(n)} = \sum_{k=0}^{n}\binom{n}{k}f^{(n-k)}g^{(k)}.$$

This holds for $n = 1$; it says $(fg)' = f'g + fg'$ in this case. Suppose the formula holds for $n = m$, where m is some positive integer. Then

$$\begin{aligned}(fg)^{(m+1)} &= \frac{d}{dx}(fg)^{(m)}\\ &= \frac{d}{dx}\sum_{k=0}^{m}\binom{m}{k} f^{(m-k)}g^{(k)}\\ &= \sum_{k=0}^{m}\binom{m}{k} f^{(m+1-k)}g^{(k)} + \sum_{k=0}^{m}\binom{m}{k} f^{(m-k)}g^{(k+1)}\Big]\\ &\qquad \text{(replace } k \text{ by } k-1 \text{ in the latter sum)}\\ &= \sum_{k=0}^{m}\binom{m}{k} f^{(m+1-k)}g^{(k)} + \sum_{k=1}^{m+1}\binom{m}{k-1} f^{(m+1-k)}g^{(k)}\\ &= f^{(m+1)}g^{(0)} + \sum_{k=1}^{m}\left[\binom{m}{k}+\binom{m}{k-1}\right]\\ &\qquad \times f^{(m+1-k)}g^{(k)} + f^{(0)}g^{(m+1)}\\ &\qquad \text{(by Exercise 7(i))}\end{aligned}$$

$$\begin{aligned}&= f^{(m+1)}g^{(0)} + \sum_{k=1}^{m}\binom{m+1}{k} f^{(m+1-k)}g^{(k)} + f^{(0)}g^{(m+1)}\\ &\qquad \text{(by Exercise 7(ii))}\\ &= \sum_{k=0}^{m+1}\binom{m+1}{k} f^{(m+1-k)}g^{(k)} \qquad \text{(by 7(i) again).}\end{aligned}$$

Thus the Rule holds for $n = m+1$. By induction, it holds for all positive integers n.

Section 9.9 Fourier Series (page 534)

1. $f(t) = \sin(3t)$ has fundamental period $2\pi/3$ since $\sin t$ has fundamental period 2π:

$$\begin{aligned}f\left(t+\tfrac{2\pi}{3}\right) &= \sin\left(3\left(t+\tfrac{2\pi}{3}\right)\right) = \sin(3t+2\pi)\\ &= \sin(3t) = f(t).\end{aligned}$$

2. $g(t) = \cos(3+\pi t)$ has fundamental period 2 since $\cos t$ has fundamental period 2π:

$$\begin{aligned}g(t+2) &= \cos\Big(3+\pi(t+2)\Big) = \cos(3+\pi t+2\pi)\\ &= \cos(3+\pi t) = g(t).\end{aligned}$$

3. $h(t) = \cos^2 t = \frac{1}{2}(1+\cos 2t)$ has fundamental period π:

$$h(t+\pi) = \frac{1+\cos(2t+2\pi)}{2} = \frac{1+\cos 2t}{2} = h(t).$$

4. Since $\sin 2t$ has periods $\pi, 2\pi, 3\pi, \ldots$, and $\cos 3t$ has periods $\frac{2\pi}{3}, \frac{4\pi}{3}, \frac{6\pi}{3} = 2\pi, \frac{8\pi}{3}, \ldots$, the sum $k(t) = \sin(2t) + \cos(3t)$ has periods $2\pi, 4\pi, \ldots$. Its fundamental period is 2π.

5. Since $f(t) = t$ is odd on $(-\pi, \pi)$ and has period 2π, its cosine coefficients are 0 and its sine coefficients are given by

$$b_n = \frac{2}{2\pi}\int_{-\pi}^{\pi} t\,\sin(nt)\,dt = \frac{2}{\pi}\int_0^{\pi} t\,\sin(nt)\,dt.$$

This integral can be evaluated by a single integration by parts. Instead we used Maple to do the integral:

$$b_n = -\frac{2}{n}\cos(n\pi) = (-1)^{n+1}\frac{2}{n}.$$

The Fourier series of f is $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{n}\sin(nt)$.

6. $f(t) = \begin{cases} 0 & \text{if } 0 \le t < 1\\ 1 & \text{if } 1 \le t < 2\end{cases}$, f has period 2.

The Fourier coefficients of f are as follows:

$$\begin{aligned}\frac{a_0}{2} &= \frac{1}{2}\int_0^2 f(t)\,dt = \frac{1}{2}\int_1^2 dt = \frac{1}{2}\\ a_n &= \int_0^2 f(t)\cos(n\pi t)\,dt = \int_1^2 \cos(n\pi t)\,dt\\ &= \frac{1}{n\pi}\sin(n\pi t)\Big|_1^2 = 0, \qquad (n \ge 1)\\ b_n &= \int_1^2 \sin(n\pi t)\,dt = -\frac{1}{n\pi}\cos(n\pi t)\Big|_1^2\\ &= -\frac{1-(-1)^n}{n\pi} = \begin{cases} -\dfrac{2}{n\pi} & \text{if } n \text{ is odd}\\ 0 & \text{if } n \text{ is even}\end{cases}\end{aligned}$$

The Fourier series of f is

$$\frac{1}{2} - \sum_{n=1}^{\infty}\frac{2}{(2n-1)\pi}\sin\Big((2n-1)\pi t\Big).$$

7. $f(t) = \begin{cases} 0 & \text{if } -1 \le t < 0\\ t & \text{if } 0 \le t < 1\end{cases}$, f has period 2.

The Fourier coefficients of f are as follows:

$$\begin{aligned}\frac{a_0}{2} &= \frac{-1}{1}\int_{-1}^{1} f(t)\,dt = \frac{1}{2}\int_0^1 t\,dt = \frac{1}{4}\\ a_n &= \int_{-1}^{1} f(t)\cos(n\pi t)\,dt = \int_0^1 t\cos(n\pi t)\,dt\\ &= \frac{(-1)^n - 1}{n^2\pi^2} = \begin{cases} -2/(n\pi)^2 & \text{if } n \text{ is odd}\\ 0 & \text{if } n \text{ is even}\end{cases}\\ b_n &= \int_0^1 t\,\sin(n\pi t)\,dt\\ &= -\frac{(-1)^n}{n\pi}.\end{aligned}$$

The Fourier series of f is

$$\frac{1}{4}-\frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\Big((2n-1)\pi t\Big)-\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin(n\pi t).$$

8. $f(t)=\begin{cases} t & \text{if } 0\le t<1 \\ 1 & \text{if } 1\le t<2, \\ 3-t & \text{if } 2\le t<3 \end{cases}$ f has period 3.

f is even, so its Fourier sine coefficients are all zero. Its cosine coefficients are

$$\begin{aligned}
\frac{a_0}{2} &= \frac{1}{2}\cdot\frac{2}{3}\int_0^3 f(t)\,dt = \frac{2}{3}(2) = \frac{2}{3} \\
a_n &= \frac{2}{3}\int_0^3 f(t)\,\cos\frac{2n\pi t}{3}\,dt \\
&= \frac{2}{3}\bigg[\int_0^1 t\cos\frac{2n\pi t}{3}\,dt + \int_1^2 \cos\frac{2n\pi t}{3}\,dt \\
&\qquad + \int_2^3 (3-t)\cos\frac{2n\pi t}{3}\,dt\bigg] \\
&= \frac{3}{2n^2\pi^2}\left[\cos\frac{2n\pi}{3} - 1 - \cos(2n\pi) + \cos\frac{4n\pi}{3}\right].
\end{aligned}$$

The latter expression was obtained using Maple to evaluate the integrals. If $n=3k$, where k is an integer, then $a_n=0$. For other integers n we have $a_n=-9/(2\pi^2 n^2)$. Thus the Fourier series of f is

$$\frac{2}{3}-\frac{9}{2\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\frac{2n\pi t}{3}+\frac{1}{2\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos(2n\pi t).$$

9. The even extension of $h(t)=1$ on $[0,1]$ to $[-1,1]$ has the value 1 everywhere. Therefore all the coefficients a_n and b_n are zero except a_0, which is 2. The Fourier series is $a_0/2=1$.

10. The Fourier sine series of $g(t)=\pi-t$ on $[0,\pi]$ has coefficients

$$b_n=\frac{2}{\pi}\int_0^{\pi}(\pi-t)\sin nt\,dt=\frac{2}{n}.$$

The required Fourier sine series is

$$\sum_{n=1}^{\infty}\frac{2}{n}\sin nt.$$

11. The Fourier sine series of $f(t)=t$ on $[0,1]$ has coefficients

$$b_n=2\int_0^1 t\,\sin(n\pi t)\,dt=-2\frac{(-1)^n}{n\pi}.$$

The required Fourier sine series is

$$\sum_{n=1}^{\infty}\frac{2(-1)^n}{n\pi}\sin(n\pi t).$$

12. The Fourier cosine series of $f(t)=t$ on $[0,1]$ has coefficients

$$\begin{aligned}
\frac{a_0}{2} &= \int_0^1 t\,dt = \frac{1}{2} \\
a_n &= 2\int_0^1 t\,\cos(n\pi t)\,dt \\
&= \frac{2(-1)^n-2}{n^2\pi^2} = \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{-4}{n^2\pi^2} & \text{if } n \text{ is odd.}\end{cases}
\end{aligned}$$

The required Fourier cosine series is

$$\frac{1}{2}-\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos\Big((2n-1)\pi t\Big)}{(2n-1)^2}.$$

13. From Example 3,

$$\frac{\pi}{2}+\sum_{n=1}^{\infty}\frac{4}{\pi(2n-1)^2}\cos\Big((2n-1)\pi t\Big)=\pi-|t|$$

for $-\pi\le t\le\pi$. Putting $t=\pi$, we obtain

$$\frac{\pi}{2}+\sum_{n=1}^{\infty}\frac{4}{\pi(2n-1)^2}(-1)=0.$$

Thus $\displaystyle\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=\frac{\pi}{2}\cdot\frac{\pi}{4}=\frac{\pi}{8}$.

14. If f is even and has period T, then

$$\begin{aligned}
b_n &= \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin\frac{2n\pi t}{T}\,dt \\
&= \frac{2}{T}\left[\int_{-T/2}^{0} f(t)\sin\frac{2n\pi t}{T}\,dt + \int_0^{T/2} f(t)\sin\frac{2n\pi t}{T}\,dt\right].
\end{aligned}$$

In the first integral in the line above replace t with $-t$. Since $f(-t)=f(t)$ and sine is odd, we get

$$\begin{aligned}
b_n &= \frac{2}{T}\bigg[\int_{T/2}^{0} f(t)\left(-\sin\frac{2n\pi t}{T}\right)(-dt) \\
&\qquad + \int_0^{T/2} f(t)\sin\frac{2n\pi t}{T}\,dt\bigg] \\
&= \frac{2}{T}\left[-\int_0^{T/2} f(t)\sin\frac{2n\pi t}{T}\,dt + \int_0^{T/2} f(t)\sin\frac{2n\pi t}{T}\,dt\right] \\
&= 0.
\end{aligned}$$

Similarly,

$$\begin{aligned} a_n &= \frac{2}{T}\left[\int_{-T/2}^{0} f(t)\cos\frac{2n\pi t}{T}\,dt + \int_0^{T/2} f(t)\cos\frac{2n\pi t}{T}\,dt\right] \\ &= \frac{2}{T}\left[\int_{T/2}^{0} f(t)\cos\frac{2n\pi t}{T}\,(-dt) + \int_0^{T/2} f(t)\cos\frac{2n\pi t}{T}\,dt\right] \\ &= \frac{4}{T}\int_0^{T/2} f(t)\cos\frac{2n\pi t}{T}\,dt. \end{aligned}$$

The corresponding result for an odd function f states that $a_n = 0$ and

$$b_n = \frac{4}{T}\int_0^{T/2} f(t)\sin\frac{2n\pi t}{T}\,dt,$$

and is proved similarly.

Review Exercises 9 (page 534)

1. $\lim_{n\to\infty}\frac{(-1)^n e^n}{n!} = 0$. The sequence converges.

2. $\lim_{n\to\infty}\frac{n^{100}+2^n\pi}{2^n} = \lim_{n\to\infty}\left(\pi + \frac{n^{100}}{2^n}\right) = \pi$.
The sequence converges.

3. $\lim_{n\to\infty}\frac{\ln n}{\tan^{-1} n} \ge \lim_{n\to\infty}\frac{\ln n}{\pi/2} = \infty$.
The sequence diverges to infinity.

4. $\lim_{n\to\infty}\frac{(-1)^n n^2}{\pi n(n-\pi)} = \lim_{n\to\infty}\frac{(-1)^n}{1-(\pi/n)}$ does not exist.
The sequence diverges (oscillates).

5. Let $a_1 > \sqrt{2}$ and $a_{n+1} = \frac{a_n}{2} + \frac{1}{a_n}$.

If $f(x) = \frac{x}{2} + \frac{1}{x}$, then $f'(x) = \frac{1}{2} - \frac{1}{x^2} > 0$ if $x > \sqrt{2}$.
Since $f(\sqrt{2}) = \sqrt{2}$, we have $f(x) > \sqrt{2}$ if $x > \sqrt{2}$.
Therefore, if $a_n > \sqrt{2}$, then $a_{n+1} = f(a_n) > \sqrt{2}$.
Thus $a_n > \sqrt{2}$ for all $n \ge 1$, by induction.

$$\begin{aligned} a_n > \sqrt{2} &\Rightarrow 2 < a_n^2 \Rightarrow a_n^2 + 2 < 2a_n^2 \\ &\Rightarrow \frac{a_n^2+2}{2a_n} < a_n \Rightarrow a_{n+1} < a_n. \end{aligned}$$

Thus $\{a_n\}$ is decreasing and $a_n > \sqrt{2}$ for all n.

Being decreasing and bounded below by $\sqrt{2}$, $\{a_n\}$ must converge by the completeness axiom. Let $\lim_{n\to\infty} a_n = a$. Then $a \ge \sqrt{2}$, and

$$\begin{aligned} \lim_{n\to\infty} a_{n+1} &= \lim_{n\to\infty}\left(\frac{a_n}{2} + \frac{1}{a_n}\right) \\ a &= \frac{a}{2} + \frac{1}{a}. \end{aligned}$$

Thus $a/2 = 1/a$, so $a^2 = 2$, and $\lim_{n\to\infty} a_n = a = \sqrt{2}$.

6. By l'Hôpital's Rule,

$$\lim_{x\to\infty}\frac{\ln(x+1)}{\ln x} = \lim_{x\to\infty}\frac{1/(x+1)}{1/x} = \lim_{x\to\infty}\frac{x}{x+1} = 1.$$

Thus

$$\lim_{n\to\infty}\Big(\ln\ln(n+1) - \ln\ln n\Big) = \lim_{n\to\infty}\ln\frac{\ln(n+1)}{\ln n} = \ln 1 = 0.$$

7.
$$\begin{aligned} \sum_{n=1}^{\infty} 2^{-(n-5)/2} &= 2^2\left(1 + \frac{1}{\sqrt{2}} + \frac{1}{2} + \cdots\right) \\ &= \frac{4}{1-(1/\sqrt{2})} = \frac{4\sqrt{2}}{\sqrt{2}-1}. \end{aligned}$$

8.
$$\begin{aligned} \sum_{n=0}^{\infty}\frac{4^{n-1}}{(\pi-1)^{2n}} &= \frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{4}{(\pi-1)^2}\right)^n \\ &= \frac{1}{4}\cdot\frac{1}{1-\dfrac{4}{(\pi-1)^2}} = \frac{(\pi-1)^2}{4(\pi-1)^2-16}, \end{aligned}$$
since $(\pi-1)^2 > 4$.

9.
$$\begin{aligned} \sum_{n=1}^{\infty}\frac{1}{n^2-\frac{1}{4}} &= \sum_{n=1}^{\infty}\left(\frac{1}{n-\frac{1}{2}} - \frac{1}{n+\frac{1}{2}}\right) \quad \text{(telescoping)} \\ &= 2 - \lim_{N\to\infty}\frac{1}{N+\frac{1}{2}} = 2. \end{aligned}$$

10.
$$\begin{aligned} \sum_{n=1}^{\infty}\frac{1}{n^2-\frac{9}{4}} &= \sum_{n=1}^{\infty}\frac{1}{3}\left(\frac{1}{n-\frac{3}{2}} - \frac{1}{n+\frac{3}{2}}\right) \quad \text{(telescoping)} \\ &= \frac{1}{3}\bigg[\frac{1}{-1/2} - \frac{1}{5/2} + \frac{1}{1/2} - \frac{1}{7/2} \\ &\qquad + \frac{1}{3/2} - \frac{1}{9/2} + \frac{1}{5/2} - \frac{1}{11/2} + \cdots\bigg] \\ &= \frac{1}{3}\left[-2 + 2 + \frac{2}{3}\right] = \frac{2}{9}. \end{aligned}$$

11. Since $0 \le \frac{n-1}{n^3} \le \frac{1}{n^2}$ for $n \ge 1$ and $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges, $\sum_{n=1}^{\infty}\frac{n-1}{n^3}$ must also converge.

12. $\sum_{n=1}^{\infty}\frac{n+2^n}{1+3^n}$ converges by comparison with the convergent geometric series $\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n$ because

$$\lim_{n\to\infty}\frac{\dfrac{n+2^n}{1+3^n}}{(2/3)^n} = \lim_{n\to\infty}\frac{(n/2^n)+1}{(1/3^n)+1} = 1.$$

13. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(1+n)(1+n\sqrt{n})}$ converges by comparison with the convergent p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ because

$$\lim_{n\to\infty} \frac{\dfrac{n}{(1+n)(1+n\sqrt{n})}}{\dfrac{1}{n^{3/2}}} = \lim_{n\to\infty} \frac{1}{\left(\dfrac{1}{n}+1\right)\left(\dfrac{1}{n^{3/2}+1}\right)} = 1.$$

14. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{(1+2^n)(1+n\sqrt{n})}$ converges by comparison with the convergent series $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{2^n}$ (which converges by the ratio test) because

$$\lim_{n\to\infty} \frac{\dfrac{n^2}{(1+2^n)(1+n\sqrt{n})}}{\dfrac{\sqrt{n}}{2^n}} = \lim_{n\to\infty} \frac{1}{\left(\dfrac{1}{2^n}+1\right)\left(\dfrac{1}{n^{3/2}+1}\right)} = 1.$$

15. $\displaystyle\sum_{n=1}^{\infty} \frac{3^{2n+1}}{n!}$ converges by the ratio test, because

$$\lim_{n\to\infty} \frac{3^{2(n+1)+1}}{(n+1)!} \cdot \frac{n!}{3^{2n+1}} = \lim_{n\to\infty} \frac{9}{n+1} = 0 < 1.$$

16. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{(n+2)!+1}$ converges by comparison with the convergent p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$, because

$$0 \le \frac{n!}{(n+2)!+1} < \frac{n!}{(n+2)!} = \frac{1}{(n+2)(n+1)} < \frac{1}{n^2}.$$

17. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{1+n^3}$ converges absolutely by comparison with the convergent p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$, because

$$0 \le \left|\frac{(-1)^{n-1}}{1+n^3}\right| \le \frac{1}{n^3}.$$

18. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n-n}$ converges absolutely by comparison with the convergent geometric series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$, because

$$\lim_{n\to\infty} \frac{\left|\dfrac{(-1)^n}{2^n-n}\right|}{\dfrac{1}{2^n}} = \lim_{n\to\infty} \frac{1}{1-\dfrac{n}{2^n}} = 1.$$

19. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\ln\ln n}$ converges by the alternating series test, but the convergence is only conditional since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\ln\ln n}$ diverges to infinity by comparison with the divergent harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$. (Note that $\ln\ln n < n$ for all $n \ge 1$.)

20. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2\cos(n\pi)}{1+n^3}$ converges by the alternating series test (note that $\cos(n\pi) = (-1)^n$), but the convergence is only conditional because

$$\left|\frac{n^2\cos(n\pi)}{1+n^3}\right| = \frac{n^2}{1+n^3} \ge \frac{1}{2n}$$

for $n \ge 1$, and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n}$ is a divergent harmonic series.

21. $$\lim_{n\to\infty} \left|\frac{\dfrac{(x-2)^{n+1}}{3^{n+1}\sqrt{n+1}}}{\dfrac{(x-2)^n}{3^n\sqrt{n}}}\right| = \lim_{n\to\infty} \frac{|x-2|}{3}\sqrt{\frac{n}{n+1}} = \frac{|x-2|}{3}.$$

$\displaystyle\sum_{n=1}^{\infty} \frac{(x-2)^n}{3^n\sqrt{n}}$ converges absolutely if $\dfrac{|x-2|}{3} < 1$, that is, if $-1 < x < 5$, and diverges if $x < -1$ or $x > 5$.

If $x = -1$ the series is $\displaystyle\sum \frac{(-1)^n}{\sqrt{n}}$, which converges conditionally.

If $x = 5$ the series is $\displaystyle\sum \frac{1}{\sqrt{n}}$, which diverges (to ∞).

22. $$\lim_{n\to\infty} \left|\frac{\dfrac{(5-2x)^{n+1}}{n+1}}{\dfrac{(5-2x)^n}{n}}\right| = \lim_{n\to\infty} |5-2x|\frac{n}{n+1} = |5-2x|.$$

$\displaystyle\sum_{n=1}^{\infty} \frac{(5-2x)^n}{n}$ converges absolutely if $|5-2x| < 1$, that is, if $2 < x < 3$, and diverges if $x < 2$ or $x > 3$.

If $x = 2$ the series is $\sum \frac{1}{n}$, which diverges.

If $x = 3$ the series is $\sum \frac{(-1)^n}{n}$, which converges conditionally.

23. Let $s = \sum_{k=1}^{\infty} \frac{1}{k^3}$ and $s_n = \sum_{k=1}^{n} \frac{1}{k^3}$. Then

$$\int_{n+1}^{\infty} \frac{dt}{t^3} < s - s_n < \int_{n}^{\infty} \frac{dt}{t^3}$$
$$s_n + \frac{1}{2(n+1)^2} < s < s_n + \frac{1}{2n^2}.$$

Let

$$s_n^* = \frac{1}{2}\left[s_n + \frac{1}{2(n+1)^2} + s_n + \frac{1}{2n^2}\right] = s_n + \frac{n^2 + (n+1)^2}{4n^2(n+1)^2}.$$

Then $s \approx s_n^*$ with error satisfying

$$|s - s_n^*| < \frac{1}{2}\left[\frac{1}{2n^2} - \frac{1}{2(n+1)^2}\right] = \frac{2n+1}{4n^2(n+1)^2}.$$

This error is less than 0.001 if $n \ge 8$. Hence

$$s \approx \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \frac{1}{6^3} + \frac{1}{7^3} + \frac{1}{8^3} + \frac{64+81}{4(64)(81)} \approx 1.202$$

with error less than 0.001.

24. Let $s = \sum_{k=1}^{\infty} \frac{1}{4+k^2}$ and $s_n = \sum_{k=1}^{n} \frac{1}{4+k^2}$. Then

$$\int_{n+1}^{\infty} \frac{dt}{4+t^2} < s - s_n < \int_{n}^{\infty} \frac{dt}{4+t^2}$$
$$s_n + \frac{\pi}{4} - \frac{1}{2}\tan^{-1}\frac{n+1}{2} < s < s_n + \frac{\pi}{4} - \frac{1}{2}\tan^{-1}\frac{n}{2}.$$

Let

$$s_n^* = s_n + \frac{\pi}{4} - \frac{1}{4}\left[\tan^{-1}\frac{n+1}{2} + \tan^{-1}\frac{n}{2}\right].$$

Then $s \approx s_n^*$ with error satisfying

$$|s - s_n^*| < \frac{1}{4}\left[\tan^{-1}\frac{n+1}{2} - \tan^{-1}\frac{n}{2}\right].$$

This error is less than 0.001 if $n \ge 22$. Hence

$$s \approx \sum_{k=1}^{22} \frac{1}{4+k^2} + \frac{\pi}{4} - \frac{1}{4}\left[\tan^{-1}\frac{23}{2} + \tan^{-1}(11)\right] \approx 0.6605$$

with error less than 0.001.

25.
$$\frac{1}{3-x} = \frac{1}{3\left(1 - \frac{x}{3}\right)} = \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n = \sum_{n=0}^{\infty} \frac{x^n}{3^{n+1}} \quad (-3 < x < 3).$$

26. Replace x with x^2 in Exercise 25 and multiply by x to get

$$\frac{x}{3-x^2} = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{3^{n+1}} \quad (-\sqrt{3} < x < \sqrt{3}).$$

27.
$$\ln(e + x^2) = \ln e + \ln\left(1 + \frac{x^2}{e}\right) = \ln e + \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n}}{ne^n} \quad (-\sqrt{e} < x \le \sqrt{e}).$$

28.
$$\frac{1 - e^{-2x}}{x} = \frac{1}{x}\left(1 - 1 - \sum_{n=1}^{\infty}\frac{(-2x)^n}{n!}\right) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{2^n x^{n-1}}{n!} \quad \text{(for all } x \ne 0\text{)}.$$

29.
$$x\cos^2 x = \frac{x}{2}(1 + \cos(2x)) = \frac{x}{2}\left(1 + \sum_{n=0}^{\infty}(-1)^n\frac{(2x)^{2n}}{(2n)!}\right) = x + \sum_{n=1}^{\infty}(-1)^n\frac{2^{2n-1}x^{2n+1}}{(2n)!} \quad \text{(for all } x\text{)}.$$

30.
$$\sin\left(x + \frac{\pi}{3}\right) = \sin x\cos\frac{\pi}{3} + \cos x\sin\frac{\pi}{3} = \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} + \frac{\sqrt{3}}{2}\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} = \sum_{n=0}^{\infty}\frac{(-1)^n}{2}\left(\frac{\sqrt{3}x^{2n}}{(2n)!} + \frac{x^{2n+1}}{(2n+1)!}\right) \quad \text{(for all } x\text{)}.$$

31.
$$(8+x)^{-1/3} = \frac{1}{2}\left(1 + \frac{x}{8}\right)^{-1/3} = \frac{1}{2}\left[1 - \frac{1}{3}\left(\frac{x}{8}\right) + \frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)}{2!}\left(\frac{x}{8}\right)^2 + \frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)\left(-\frac{7}{3}\right)}{3!}\left(\frac{x}{8}\right)^3 + \cdots\right]$$
$$= \frac{1}{2} + \sum_{n=1}^{\infty}(-1)^n\frac{1\cdot 4\cdot 7\cdots(3n-2)}{2\cdot 3^n\cdot 8^n\cdot n!}x^n \quad (-8 < x < 8).$$

(Remark: Examining the ln of the absolute value of the nth term at $x = 8$ shows that this term $\to 0$ as $n \to \infty$. Therefore the series also converges at $x = 8$.)

32. $$(1+x)^{1/3} = 1 + \frac{1}{3}x + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2!}x^2 + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{3!}x^3 + \cdots$$
$$= 1 + \frac{x}{3} + \sum_{n=2}^{\infty}(-1)^{n-1}\frac{2\cdot 5\cdot 8\cdots(3n-4)}{3^n n!}x^n \quad (-1 < x < 1).$$
(Remark: the series also converges at $x = 1$.)

33. $$\frac{1}{x} = \frac{1}{\pi + (x-\pi)} = \frac{1}{\pi}\cdot\frac{1}{1+\frac{x-\pi}{\pi}}$$
$$= \frac{1}{\pi}\sum_{n=0}^{\infty}(-1)^n\left(\frac{x-\pi}{\pi}\right)^n$$
$$= \sum_{n=0}^{\infty}(-1)^n\frac{(x-\pi)^n}{\pi^{n+1}} \quad (0 < x < 2\pi).$$

34. Let $u = x - (\pi/4)$, so $x = u + (\pi/4)$. Then
$$\sin x + \cos x = \sin\left(u + \frac{\pi}{4}\right) + \cos\left(u + \frac{\pi}{4}\right)$$
$$= \frac{1}{\sqrt{2}}\Big((\sin u + \cos u) + (\cos u - \sin u)\Big)$$
$$= \sqrt{2}\cos u = \sqrt{2}\sum_{n=0}^{\infty}(-1)^n\frac{u^{2n}}{(2n)!}$$
$$= \sqrt{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(x-\frac{\pi}{4}\right)^{2n} \quad \text{(for all } x).$$

35. $$e^{x^2+2x} = e^{x^2}e^{2x}$$
$$= (1 + x^2 + \cdots)\left(1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!} + \cdots\right)$$
$$= 1 + 2x + 2x^2 + \frac{4}{3}x^3 + x^2 + 2x^3 + \cdots$$
$$P_3(x) = 1 + 2x + 3x^2 + \frac{10}{3}x^3.$$

36. $$\sin(1+x) = \sin(1)\cos x + \cos(1)\sin x$$
$$= \sin(1)\left(1 - \frac{x^2}{2!} + \cdots\right) + \cos(1)\left(x - \frac{x^3}{3!} + \cdots\right)$$
$$P_3(x) = \sin(1) + \cos(1)x - \frac{\sin(1)}{2}x^2 - \frac{\cos(1)}{6}x^3.$$

37. $$\cos(\sin x) = 1 - \frac{\left(x - \frac{x^3}{3!} + \cdots\right)^2}{2!} + \frac{(x - \cdots)^4}{4!} - \cdots$$
$$= 1 - \frac{1}{2}\left(x^2 - \frac{x^4}{3} + \cdots\right) + \frac{x^4}{24} + \cdots$$
$$P_4(x) = 1 - \frac{1}{2}x^2 + \frac{5}{24}x^4.$$

38. $$\sqrt{1+\sin x} = 1 + \frac{1}{2}\sin x + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}(\sin x)^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}(\sin x)^3 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{4!}(\sin x)^4 + \cdots$$
$$= 1 + \frac{1}{2}\left(x - \frac{x^3}{6} + \cdots\right) - \frac{1}{8}\left(x - \frac{x^3}{6} + \cdots\right)^2 + \frac{1}{16}(x - \cdots)^3 - \frac{5}{128}(x - \cdots)^4 + \cdots$$
$$= 1 + \frac{x}{2} - \frac{x^3}{12} - \frac{x^2}{8} + \frac{x^4}{24} + \frac{x^3}{16} - \frac{5x^4}{128} + \cdots$$
$$P_4(x) = 1 + \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{48} + \frac{x^4}{384}.$$

39. The series $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{(2n)!}$ is the Maclaurin series for $\cos x$ with x^2 replaced by x. For $x > 0$ the series therefore represents $\cos\sqrt{x}$. For $x < 0$, the series is $\displaystyle\sum_{n=0}^{\infty}\frac{|x|^n}{(2n)!}$, which is the Maclaurin series for $\cosh\sqrt{|x|}$. Thus the given series is the Maclaurin series for
$$f(x) = \begin{cases}\cos\sqrt{x} & \text{if } x \ge 0\\ \cosh\sqrt{|x|} & \text{if } x < 0.\end{cases}$$

40. Since
$$1 + \sum_{n=1}^{\infty}\frac{x^{2n}}{n^2} = \sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}x^k$$
for x near 0, we have, for $n = 1, 2, 3, \ldots$
$$f^{(2n)}(0) = \frac{(2n)!}{n^2}, \quad f^{(2n-1)}(0) = 0.$$

41. $$\sum_{n=0}^{\infty}x^n = \frac{1}{1-x}$$
$$\sum_{n=0}^{\infty}nx^{n-1} = \sum_{n=1}^{\infty}nx^{n-1} = \frac{1}{(1-x)^2}$$
$$\sum_{n=0}^{\infty}nx^n = \frac{x}{(1-x)^2}$$
$$\sum_{n=0}^{\infty}\frac{n+1}{\pi^n} = \frac{\frac{1}{\pi}}{\left(1-\frac{1}{\pi}\right)^2} + \frac{1}{1-\frac{1}{\pi}}$$
$$= \frac{\pi}{(\pi-1)^2} + \frac{\pi}{\pi-1} = \left(\frac{\pi}{\pi-1}\right)^2.$$

42. $$\sum_{n=0}^{\infty} nx^n = \frac{x}{(1-x)^2} \quad \text{as in Exercise 23}$$
$$\sum_{n=0}^{\infty} n^2x^{n-1} = \frac{d}{dx}\frac{x}{(1-x)^2} = \frac{1+x}{(1-x)^3}$$
$$\sum_{n=0}^{\infty} n^2x^n = \frac{x(1+x)}{(1-x)^3}$$
$$\sum_{n=0}^{\infty} \frac{n^2}{\pi^n} = \frac{\frac{1}{\pi}\left(1+\frac{1}{\pi}\right)}{\left(1-\frac{1}{\pi}\right)^3} = \frac{\pi(\pi+1)}{(\pi-1)^3}.$$

43. $$\sum_{n=1}^{\infty} \frac{x^n}{n} = -\ln(1-x)$$
$$\sum_{n=1}^{\infty} \frac{1}{ne^n} = -\ln\left(1-\frac{1}{e}\right) = 1-\ln(e-1).$$

44. $$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^{2n-1}}{(2n-1)!} = \sin x$$
$$\sum_{n=1}^{\infty} \frac{(-1)^n\pi^{2n-1}}{(2n-1)!} = -\sin\pi = 0$$
$$\sum_{n=2}^{\infty} \frac{(-1)^n\pi^{2n-4}}{(2n-1)!} = \frac{1}{\pi^3}\left(0-\frac{(-1)\pi}{1!}\right) = \frac{1}{\pi^2}.$$

45. $$S(x) = \int_0^x \sin(t^2)\,dt = \int_0^x \left(t^2 - \frac{t^6}{3!} + \cdots\right)dt = \frac{x^3}{3} - \frac{x^7}{7\cdot 3!} + \cdots$$
$$\lim_{x\to 0}\frac{x^3-3S(x)}{x^7} = \lim_{x\to 0}\frac{x^3-x^3+\frac{x^7}{14}-\cdots}{x^7} = \frac{1}{14}.$$

46. $$\lim_{x\to 0}\frac{(x-\tan^{-1}x)(e^{2x}-1)}{2x^2-1+cos(2x)}$$
$$= \lim_{x\to 0}\frac{\left(x-x+\frac{x^3}{3}-\frac{x^5}{5}+\cdots\right)\left(2x+\frac{4x^2}{2!}+\cdots\right)}{2x^2-1+1-\frac{4x^2}{2!}+\frac{16x^4}{4!}-\cdots}$$
$$= \lim_{x\to 0}\frac{x^4\left(\frac{2}{3}+\cdots\right)}{x^4\left(\frac{2}{3}+\cdots\right)} = 1.$$

47. $$\int_0^{1/2} e^{-x^4}\,dx = \int_0^{1/2}\sum_{n=0}^{\infty}\frac{(-x^4)^n}{n!}\,dx = \sum_{n=0}^{\infty}\frac{(-1)^n x^{4n+1}}{(4n+1)n!}\bigg|_0^{1/2} = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{4n+1}(4n+1)n!}.$$

The series satisfies the conditions of the alternating series test, so if we truncate after the term for $n = k-1$, then the error will satisfy

$$|\text{error}| \le \frac{1}{2^{4k+1}(4k+1)k!}.$$

This is less than 0.000005 if $2^{4k+1}(4k+1)k! > 200,000$, which happens if $k \ge 3$. Thus, rounded to five decimal places,

$$\int_0^{1/2} e^{-x^4}\,dx \approx \frac{1}{2\cdot 1\cdot 1} - \frac{1}{32\cdot 5\cdot 1} + \frac{1}{512\cdot 9\cdot 2} \approx 0.49386.$$

48. If $f(x) = \ln(\sin x)$, then calculation of successive derivatives leads to

$$f^{(5)}(x) = 24\csc^4 x\cot x - 8\csc^2\cot x.$$

Observe that $1.5 < \pi/2 \approx 1.5708$, that $\csc x \ge 1$ and $\cot x \ge 0$, and that both functions are decreasing on that interval. Thus

$$|f^{(5)}(x)| \le 24\csc^4(1.5)\cot(1.5) \le 2$$

for $1.5 \le x \le \pi/2$. Therefore, the error in the approximation

$$\ln(\sin 1.5) \approx P_4(x),$$

where P_4 is the 4th degree Taylor polynomial for $f(x)$ about $x = \pi/2$, satisfies

$$|\text{error}| \le \frac{2}{5!}\left|1.5-\frac{\pi}{2}\right|^5 \le 3\times 10^{-8}.$$

49. The Fourier sine series of $f(t) = \pi - t$ on $[0,\pi]$ has coefficients

$$b_n = \frac{2}{\pi}\int_0^{\pi}(\pi-t)\sin(nt)\,dt = \frac{2}{n}.$$

The series is $\displaystyle\sum_{n=1}^{\infty}\frac{2}{n}\sin(nt)$.

50. $f(t) = \begin{cases} 1 & \text{if } -\pi < t \le 0 \\ t & \text{if } 0 < t \le \pi \end{cases}$ has period 2π. Its Fourier coefficients are

$$\begin{aligned}
\frac{a_0}{2} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt \\
&= \frac{1}{2\pi}\left[\int_{-\pi}^{0} dt + \int_0^{\pi} t\,dt\right] = \frac{1}{2} + \frac{\pi}{4} \\
a_n &= \frac{1}{\pi}\left[\int_{-\pi}^{0} \cos(nt)\,dt + \int_0^{\pi} t\cos(nt)\,dt\right] \\
&= \frac{1}{\pi}\int_0^{\pi}(1+t)\cos(nt)\,dt \\
&= \frac{(-1)^n - 1}{\pi n^2} = \begin{cases} -2/(\pi n^2) & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases} \\
b_n &= \frac{1}{\pi}\left[\int_{-\pi}^{0} \sin(nt)\,dt + \int_0^{\pi} t\sin(nt)\,dt\right] \\
&= \frac{1}{\pi}\int_0^{\pi}(t-1)\sin(nt)\,dt \\
&= -\frac{1 + (-1)^n(\pi - 1)}{\pi n} = \begin{cases} (\pi - 2/(\pi n) & \text{if } n \text{ is odd} \\ -(1/n) & \text{if } n \text{ is even.} \end{cases}
\end{aligned}$$

The required Fourier series is, therefore,

$$\frac{2+\pi}{4} - \sum_{n=1}^{\infty}\left[\frac{2\cos\big((2n-1)t\big)}{\pi(2n-1)^2} + \frac{(2-\pi)\sin\big((2n-1)t\big)}{\pi(2n-1)} + \frac{\sin(2nt)}{2n}\right].$$

Challenging Problems 9 (page 535)

1. If $a_n > 0$ and $\dfrac{a_{n+1}}{a_n} > \dfrac{n}{n+1}$ for all n, then

$$\begin{aligned}
\frac{a_2}{a_1} &> \frac{1}{2} &\Rightarrow\quad & a_2 > \frac{a_1}{2} \\
\frac{a_3}{a_2} &> \frac{2}{3} &\Rightarrow\quad & a_3 > \frac{2a_2}{3} > \frac{a_1}{3} \\
&\vdots \\
\frac{a_n}{a_{n-1}} &> \frac{n-1}{n} &\Rightarrow\quad & a_n > \frac{a_1}{n}.
\end{aligned}$$

(This can be verified by induction.)
Therefore $\sum_{n=1}^{\infty} a_n$ diverges by comparison with the harmonic series $\sum_{n=1}^{\infty} \dfrac{1}{n}$.

2. a) If $s_n = \sum_{k=1}^{n} v_k$ for $n \ge 1$, and $s_0 = 0$, then $v_k = s_k - s_{k-1}$ for $k \ge 1$, and

$$\sum_{k=1}^{n} u_k v_k = \sum_{k=1}^{n} u_k s_k - \sum_{k=1}^{n} u_k s_{k-1}.$$

In the second sum on the right replace k with $k+1$:

$$\begin{aligned}
\sum_{k=1}^{n} u_k v_k &= \sum_{k=1}^{n} u_k s_k - \sum_{k=0}^{n-1} u_{k+1} s_k \\
&= \sum_{k=1}^{n}(u_k - u_{k+1})s_k - u_1 s_0 + u_{n+1}s_n \\
&= u_{n+1}s_n + \sum_{k=1}^{n}(u_k - u_{k+1})s_k.
\end{aligned}$$

b) If $\{u_n\}$ is positive and decreasing, and $\lim_{n\to\infty} u_n = 0$, then

$$\begin{aligned}
\sum_{k=1}^{n}(u_k - u_{k+1}) &= u_1 - u_2 + u_2 - u_3 + \cdots + u_n - u_{n+1} \\
&= u_1 - u_{n+1} \to u_1 \text{ as } n \to \infty.
\end{aligned}$$

Thus $\sum_{k=1}^{n}(u_k - u_{k+1})$ is a convergent, positive, telescoping series.

If the partial sums s_n of $\{v_n\}$ are bounded, say $|s_n| \le K$ for all n, then

$$|(u_n - u_{n+1})s_n| \le K(u_n - u_{n+1}),$$

so $\sum_{n=1}^{\infty}(u_n - u_{n+1})s_n$ is absolutely convergent (and therefore convergent) by the comparison test. Therefore, by part (a),

$$\begin{aligned}
\sum_{k=1}^{\infty} u_k v_k &= \lim_{n\to\infty}\left(u_{n+1}s_n + \sum_{k=1}^{n}(u_k - u_{k+1})s_k\right) \\
&= \sum_{k=1}^{\infty}(u_k - u_{k+1})s_k
\end{aligned}$$

converges.

3. If $x = m\pi$ for some integer m, then all the terms of the series $\sum_{n=1}^{\infty}(1/n)\sin(nx)$ are 0, so the series converges to 0.

If $x \ne m\pi$ for any integer m, then $\sin(x/2) \ne 0$. Using the addition formulas we obtain

$$\sin(nx)\sin(x/2) = \frac{1}{2}\left[\cos\Big((n - \tfrac{1}{2})x\Big) - \cos\Big((n + \tfrac{1}{2})x\Big)\right].$$

Therefore, using the telescoping property of these terms,

$$\begin{aligned}
\sum_{n=1}^{N}\sin(nx) &= \sum_{n=1}^{N}\frac{\left[\cos\Big((n-\frac{1}{2})x\Big) - \cos\Big((n+\frac{1}{2})x\Big)\right]}{2\sin(x/2)} \\
&= \frac{\cos(x/2) - \cos\Big((N+\frac{1}{2})x\Big)}{2\sin(x/2)}.
\end{aligned}$$

Therefore, the partial sums of $\sum_{n=1}^{\infty} \sin(nx)$ are bounded. Since the sequence $\{1/n\}$ is positive, decreasing, and has limit 0, part (b) of Problem 2 shows that $\sum_{n=1}^{\infty} \sin(nx)/n$ converges in this case too. Therefore the series converges for all x.

4. Let a_n be the nth integer that has no zeros in its decimal representation. The number of such integers that have m digits is 9^m. (There are nine possible choices for each of the m digits.) Also, each such m-digit number is greater than 10^{m-1} (the smallest m-digit number). Therefore the sum of all the terms $1/a_n$ for which a_n has m digits is less than $9^m/(10^{m-1})$. Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{a_n} < 9 \sum_{m=1}^{\infty} \left(\frac{9}{10}\right)^{m-1} = 90.$$

5. $\displaystyle\int_{k-1/2}^{k+1/2} f(x)\,dx - f(k) = \frac{f''(c)}{24}$, for some c in the interval $[k-\frac{1}{2}, k+\frac{1}{2}]$.

a) By the Mean-Value Theorem,

$$f'\left(k+\tfrac{3}{2}\right) - f'\left(k+\tfrac{1}{2}\right) = \left(\tfrac{3}{2}-\tfrac{1}{2}\right) f''(u) = f''(u)$$

for some u in $[k+\frac{1}{2}, k+\frac{3}{2}]$. Similarly,

$$f'\left(k-\tfrac{1}{2}\right) - f'\left(k-\tfrac{3}{2}\right) = \left(-\tfrac{1}{2}+\tfrac{3}{2}\right) f''(v) = f''(v)$$

for some v in $[k-\frac{3}{2}, k-\frac{1}{2}]$. Since f'' is decreasing and $v \le c \le u$, we have $f''(u) \le f''(c) \le f''(v)$, and so

$$f'\left(k+\tfrac{3}{2}\right) - f'\left(k+\tfrac{1}{2}\right) \le f''(c) \le f'\left(k-\tfrac{1}{2}\right) - f'\left(k-\tfrac{3}{2}\right).$$

b) If f'' is decreasing, $\int_{N+\frac{1}{2}}^{\infty} f(x)\,dx$ converges, and $f'(x) \to 0$ as $x \to \infty$, then

$$\begin{aligned}
&\sum_{n=N+1}^{\infty} f(n) - \int_{N+\frac{1}{2}}^{\infty} f(x)\,dx \\
&= \sum_{n=N+1}^{\infty} \left(f(n) - \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} f(x)\,dx \right) \\
&= -\frac{1}{24} \sum_{n=N+1}^{\infty} f''(c_n),
\end{aligned}$$

for some numbers c_n in $[n-\frac{1}{2}, n+\frac{1}{2}]$. Using the result of part (a), we see that

$$\begin{aligned}
&\sum_{n=N+1}^{\infty} \left[f'(n+\tfrac{3}{2}) - f'(n+\tfrac{1}{2})\right] \le \sum_{n=N+1}^{\infty} f''(c_n) \\
&\le \sum_{n=N+1}^{\infty} \left[f'(n-\tfrac{1}{2}) - f'(n-\tfrac{3}{2})\right] \\
&- f'(N+\tfrac{3}{2}) \le \sum_{n=N+1}^{\infty} f''(c_n) \le -f'(N-\tfrac{1}{2}) \\
&\frac{f'(N-\frac{1}{2})}{24} \le \sum_{n=N+1}^{\infty} f(n) - \int_{N+\frac{1}{2}}^{\infty} f(x)\,dx \le \frac{f'(N+\frac{3}{2})}{24}.
\end{aligned}$$

c) Let $f(x) = 1/x^2$. Then $f'(x) = -2/x^3 \to 0$ as $x \to \infty$, $f''(x) = 6/x^4$ is decreasing, and

$$\int_{N+\frac{1}{2}}^{\infty} f(x)\,dx = \int_{N+\frac{1}{2}}^{\infty} \frac{dx}{x^2} = \frac{1}{N+\frac{1}{2}}$$

converges. From part (b) we obtain

$$\left|\sum_{n=N+1}^{\infty} \frac{1}{n^2} - \frac{1}{N+\frac{1}{2}}\right| \le \frac{1}{12\left(N-\frac{1}{2}\right)^3}.$$

The right side is less than 0.001 if $N = 5$. Therefore

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{5} \frac{1}{n^2} + \frac{1}{5.5} \approx 1.6454$$

correct to within 0.001.

6. a) Since $e = \displaystyle\sum_{j=0}^{\infty} \frac{1}{j!}$, we have

$$\begin{aligned}
0 < e - \sum_{j=0}^{n} \frac{1}{j!} &= \sum_{j=n+1}^{\infty} \frac{1}{j!} \\
&= \frac{1}{(n+1)!}\left(1 + \frac{1}{n+2} + \frac{1}{(n+2)(n+3)} + \cdots\right) \\
&\le \frac{1}{(n+1)!}\left(1 + \frac{1}{n+2} + \frac{1}{(n+2)^2} + \cdots\right) \\
&= \frac{1}{(n+1)!} \cdot \frac{1}{1-\frac{1}{n+2}} = \frac{n+2}{(n+1)!(n+1)} < \frac{1}{n!n}.
\end{aligned}$$

The last inequality follows from $\dfrac{n+2}{(n+1)^2} < \dfrac{1}{n}$, that is, $n^2+2n < n^2+2n+1$.

b) Suppose e is rational, say $e = M/N$ where M and N are positive integers. Then $N!e$ is an integer and $N!\sum_{j=0}^{N}(1/j!)$ is an integer (since each $j!$ is a factor of $N!$). Therefore the number

$$Q = N!\left(e - \sum_{j=0}^{N} \frac{1}{j!}\right)$$

is a difference of two integers and so is an integer.

c) By part (a), $0 < Q < \dfrac{1}{N} \le 1$. By part (b), Q is an integer. This is not possible; there are no integers between 0 and 1. Therefore e cannot be rational.

7. Let $f(x) = \sum_{k=0}^{\infty} a_k x^{2k+1}$, where $a_k = \dfrac{2^{2k} k!}{(2k+1)!}$.

a) Since

$$\begin{aligned}\lim_{k\to\infty} \left| \frac{a_{k+1} x^{2k+3}}{a_k x^{2k+1}} \right| &= |x|^2 \lim_{k\to\infty} \frac{2^{2k+2}}{2^{2k}} \cdot \frac{(k+1)!}{k!} \cdot \frac{(2k+1)!}{(2k+3)!} \\ &= |x|^2 \lim_{k\to\infty} \frac{4k+4}{(2k+3)(2k+2)} = 0\end{aligned}$$

for all x, the series for $f(x)$ converges for all x. Its radius of convergence is infinite.

b) $f'(x) = \sum_{k=0}^{\infty} \dfrac{2^{2k} k!}{(2k+1)!} (2k+1) x^{2k} = 1 + \sum_{k=1}^{\infty} \dfrac{2^{2k} k!}{(2k)!} x^{2k}$

$$\begin{aligned}1 + 2xf(x) &= 1 + \sum_{k=0}^{\infty} \frac{2^{2k+1} k!}{(2k+1)!} x^{2k+2} \\ &\qquad \text{(replace } k \text{ with } k-1\text{)} \\ &= 1 + \sum_{k=1}^{\infty} \frac{2^{2k-1}(k-1)!}{(2k-1)!} x^{2k} \\ &= 1 + \sum_{k=1}^{\infty} \frac{2^{2k} k!}{(2k)!} x^{2k} = f'(x).\end{aligned}$$

c) $\dfrac{d}{dx}\left(e^{-x^2} f(x)\right) = e^{-x^2}\left(f'(x) - 2xf(x)\right) = e^{-x^2}$.

d) Since $f(0) = 0$, we have

$$\begin{aligned}e^{-x^2} f(x) - f(0) &= \int_0^x \frac{d}{dt}\left(e^{-t^2} f(t)\right) dt = \int_0^x e^{-t^2}\, dt \\ f(x) &= e^{x^2} \int_0^x e^{-t^2}\, dt.\end{aligned}$$

8. Let f be a polynomial and let

$$g(x) = \sum_{j=0}^{\infty} (-1)^j f^{(2j)}(x).$$

This "series" is really just a polynomial since sufficiently high derivatives of f are all identically zero.

a) By replacing j with $j-1$, observe that

$$\begin{aligned}g''(x) &= \sum_{j=0}^{\infty} (-1)^j f^{(2j+2)}(x) \\ &= \sum_{j=1}^{\infty} (-1)^{j-1} f^{(2j)}(x) = -\left(g(x) - f(x)\right).\end{aligned}$$

Also

$$\begin{aligned}&\frac{d}{dx}\left(g'(x)\sin x - g(x)\cos x\right) \\ &= g''(x)\sin x + g'(x)\cos x - g'(x)\cos x + g(x)\sin x \\ &= \left(g''(x) + g(x)\right)\sin x = f(x)\sin x.\end{aligned}$$

Thus

$$\int_0^{\pi} f(x)\sin x\, dx = \left(g'(x)\sin x - g(x)\cos x\right)\Big|_0^{\pi} = g(\pi) + g(0).$$

b) Suppose that $\pi = m/n$, where m and n are positive integers. Since $\lim_{k\to\infty} x^k/k! = 0$ for any x, there exists an integer k such that $(\pi m)^k/k! < 1/2$. Let

$$f(x) = \frac{x^k (m - nx)^k}{k!} = \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} m^{k-j} (-n)^j x^{j+k}.$$

The sum is just the binomial expansion.
For $0 < x < \pi = m/n$ we have

$$0 < f(x) < \frac{\pi^k m^k}{k!} < \frac{1}{2}.$$

Thus $0 < \int_0^{\pi} f(x)\sin x\, dx < \dfrac{1}{2}\displaystyle\int_0^{\pi} \sin x\, dx = 1$, and so $0 < g(\pi) + g(0) < 1$.

c) $$\begin{aligned}f^{(i)}(x) &= \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} m^{k-j} (-n)^j \\ &\qquad \times (j+k)(j+k-1)\cdots(j+k-i+1) x^{j+k-i} \\ &= \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} m^{k-j} (-n)^j \frac{(j+k)!}{(j+k-i)!} x^{j+k-i}.\end{aligned}$$

d) Evidently $f^{(i)}(0) = 0$ if $i < k$ or if $i > 2k$.
If $k \le i \le 2k$, the only term in the sum for $f^{(i)}(0)$ that is not zero is the term for which $j = i - k$. This term is the constant

$$\frac{1}{k!} \binom{k}{i-k} m^{k-j} (-n)^j \frac{i!}{0!}.$$

This constant is an integer because the binomial coefficient $\dbinom{k}{i-k}$ is an integer and $i!/k!$ is an integer. (The other factors are also integers.) Hence $f^{(i)}(0)$ is an integer, and so $g(0)$ is an integer.

e) Observe that $f(\pi - x) = f((m/n) - x) = f(x)$ for all x. Therefore $f^{(i)}(\pi)$ is an integer (for each i), and so $g(\pi)$ is an integer. Thus $g(\pi) + g(0)$ is an integer, which contradicts the conclusion of part (b). (There is no integer between 0 and 1.) Therefore, π cannot be rational.

9. Let $x > 0$, and let

$$I_k = \int_0^x t^k e^{-1/t}\, dt$$

$$\begin{array}{ll} U = t^{k+2} & dV = \dfrac{1}{t^2} e^{-1/t}\, dt \\ dU = (k+2)t^{k+1}\, dt & V = e^{-1/t} \end{array}$$

$$= t^{k+2} e^{-1/t}\Big|_0^x - (k+2)\int_0^x t^{k+1} e^{-1/t}\, dt$$

$$I_k = x^{k+2} e^{-1/x} - (k+2) I_{k+1}.$$

Therefore,

$$\begin{aligned}
\int_0^x e^{-1/t}\, dt &= I_0 = x^2 e^{-1/x} - 2I_1 \\
&= x^2 e^{-1/x} - 2\Big(x^3 e^{-1/x} - 3I_2\Big) \\
&= e^{-1/x}[x^2 - 2!x^3] + 3!\Big(x^4 e^{-1/x} - 4I_3\Big) \\
&= e^{-1/x}[x^2 - 2!x^3 + 3!x^4] - 4!\Big(x^5 e^{-1/x} - 5I_4\Big) \\
&\ \ \vdots \\
&= e^{-1/x} \sum_{n=2}^{N} (-1)^n (n-1)!x^n \\
&\quad + (-1)^{N+1} N! \int_0^x t^{N-1} e^{-1/t}\, dt.
\end{aligned}$$

The Maclaurin series for $e^{-1/t}$ does not exist. The function is not defined at $t = 0$.
For $x = 0.1$ and $N = 5$, the approximation

$$\begin{aligned}
I = \int_0^{0.1} e^{-1/t}\, dt &\approx e^{-10} \sum_{n=2}^{5} (-1)^n (n-1)!(0.1)^n \\
&= e^{-10}\Big((0.1)^2 - 2(0.1)^3 + 6(0.1)^4 - 24(0.1)^5\Big) \\
&\approx 0.00836e^{-10}
\end{aligned}$$

has error E given by

$$E = (-1)^6 5! \int_0^{0.1} t^4 e^{-1/t}\, dt.$$

Since $e^{-1/t} \le e^{-10}$ for $0 \le t \le 0.1$, we have

$$|E| \le 120e^{-10} \int_0^{0.1} t^4\, dt \approx 2.4 \times 10^{-4} e^{-10},$$

which is about 3% of the size of I.

For $N = 10$, the error estimate is

$$|E| \le 10!e^{-10} \int_0^{0.1} t^9\, dt \approx 3.6 \times 10^{-5} e^{-10},$$

which is about 0.4% of the size of I.
For $N = 20$, the error estimate is

$$|E| \le 20!e^{-10} \int_0^{0.1} t^{19}\, dt \approx 1.2 \times 10^{-3} e^{-10},$$

which is about 15% of the size of I.
Observe, therefore, that the sum for $N = 10$ does a better job of approximating I than those for $N = 5$ or $N = 20$.