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LECTURES[1] ON ALAIN CONNES' NON COMMUTATIVE GEOMETRY AND APPLICATIONS TO FUNDAMENTAL INTERACTIONS

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Abstract

We introduce the reader to Alain Connes non commutative differential geometry, and sketch the applications made to date to (the lagrangian level of) fundamental physical interactions.

Key-Words : non-commutative geometry, fundamental interactions, standard model.

Number of figures : 1

July 1994
CPT-94/P.3055

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[1]To appear in the proceedings of the "First Canibbeau Spring School of Mathematics and Theoretical Physics" - Cambridge University Press, Eds. R. Coquereaux, M. Dubois-Violette.

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Lectures on Alain Connes' non commutative geometry and applications to fundamental interactions

Contents

These notes are an amplification of the four lectures delivered at the Guadeloupe school. They attempt to provide an introduction to Alain Connes' non-commutative geometry-analysis,[2] and applications to fundamental physical interactions.[3] As of today, these applications consist in designing methods based on non-commutative geometry for constructing lagrangian actions for the microworld and the cosmos within a conceptual mathematical perspective.

For the micro-world (standard model of elementary particles) the mathematical emphasis is on the "non-commutative De Rham complex", set up of the "quantum Yang-Mills algorithm" on which Alain Connes grounds his reinterpretation of the standard model (technically resting on the notion of d^+-summable K-cycle - see section B). We discuss this astounding interpretation of the full standard model in some detail (including a display of formulae and a sketch of the lengthy calculations in their up-to-date version[4]). As the reader will see, the set-up for the full standard model including chromodynamics interprets the Higgs bosons as a fifth gauge boson[5]; and suggests an enlargement of the notion of K-cycle enacting "non-commutative Poincaré duality" based on a "Poincaré dual pair of algebras" - a basic concept of non-commutative geometry[6] discovered by Alain Connes *whilst looking at the standard model*, where "Poincaré duality" correspond to the electroweak-chromodynamics duality: a beautiful instance of a crucial stride pertaining jointly to fundamental mathematical and physical features.

In addition to introducing to Alain Connes' version of the standard model, these notes attempt to sketch the developments of the last year which inaugurated

— on the one hand the differential geometry of fractals (our sketchy section E hopes to give the reader a "flavour"),

— on the other hand the beginning study of gravitation by techniques of non-commutative geometry: namely (i) the provocative fact that the Dirac operators codes the data of the Einstein-Hilbert action (§1 of section H); (ii) a quantum version of the Polyakov action for strings (§2 of section H), and a transcription of the latter to four-dimensional conformal manifolds yielding a quadratic action possibly related to gravitation (§3 of section H).

The two last of the latter subjects use a different kind of non-commutative differential geometry based on the notion of p-summable Fredholm module (see section F). The

[2]In a modern sight geometry and analysis have merged into one discipline: the study of manifolds and the operators on their vector bundles. This coalescence is in fact enhanced by Alain Connes' "quantization" of geometry-analysis.

[3]We do not treat in these lectures applications to solid-state physics (i.e. as of today to the Hall effect).

[4]This last version was preceded by (i) a first essay on the electroweak theory using the algebra $C^\infty(M)\oplus C^\infty(M)$ together with a vector bundle taylored so as to obtain the desired symmetry breaking and (ii) a first version of the present theory using formal quantum forms and leading to adynamical fields, see [1], [2], [5] I and II.

[5]of a "discrete" type not accessible to usual differential geometry

[6]Likely to become fundamental in mathematics as a piece of recognition of the basic nature of "non-commutative manifolds" - in particular opening the way to "non-commutative characteristic classes".

differential geometry of fractals also uses Fredholm modules as sketched in section G. Paragraph §1 of section H merely appeals to the notion of Wodzicki residue (the unique trace on PDOs, related to the Dixmier trace). Paragraph §5 of section E, devoted to non-commutative Poincaré duality, is placed there because of its application to the standard model - conceptually, though, it is a continuation of section B.

Before starting, I should stress that my subject is - of course - Alain Connes' intellectual property. My justification for commenting it is that I have performed in parallel to Connes and Lott the lenghty computations of the quantum-geometry Yang-Mills action for the full standard model (coupled electroweak and chromodynamics sectors) - both in the original manner where one has to deal with adynamical fields, and by means of the superior new technique using the genuine quantum forms (*quantum De Rham complex*).

The key device in Alain Connes' non-commutative geometry consists in replacing classical differentials[7]

$$\mathbf{d}a \ , \qquad a \in C^\infty(M) \ , \tag{$\star$}$$

attached to a Riemannian compact manifold M with a spin structure, by Hilbert space operators

$$[D\,,\,\underline{a}] \ , \qquad a \in A \ , \tag{$\star\star$}$$

resp.

$$[F,\underline{a}] \ , \qquad a \in A \ , \tag{$\star\star\star$}$$

where A is a now generally a non-commutative algebra (the points of the "space" M have disappeared), and where ($\star\star$), resp. ($\star\star\star$), are commutators between Hilbert-space representative $\underline{a}$ of the $a \in C^\infty(M)$ with an (appropriately defined) generalized Dirac operator D, resp. a "phase" F (the two possibilities correspond to two different kinds of non-commutative geometries, in essence metric, resp. conformal). Of course this scheme has to be implemented, given the algebra A (a dense subalgebra of a C*-algebra) by a specification of the relevant A-modules (H,D) (resp. (H,F)), H a ($\boldsymbol{Z}/2$-graded) Hilbert space carrying a bounded representation of A, and D (resp. F), a generalized Dirac operator (resp. a phase) - the reader will find the precise specification of D below in section B under the name "d^+-summable K-cycle" (respectively that of F as "p-summable Fredholm module" in section F). Defining, in this manner, the differential geometry of an algebra A produces an extension of the usual riemannian (respectively conformal) geometry with the following powerful features:

- applied to the classical case of $A = C^\infty(M)$ theories of the type ($\star\star$) reproduce the usual geometry of riemannian manifolds, comprising (modulo a slight extension of the general theory) a generalized Poincaré duality - whilst theories of the type ($\star\star\star$) reproduce the usual geometry of conformal manifolds

- this general scheme covers the case of non commutative algebras (small enough so their cohomological dimension is finite);

[7] We denote in this lecture the exterior differential by a bold-faced straight $\mathbf{d}$

- it also covers the case of spaces with non-integer (Hausdorff) dimension: fractals, whose commutative algebras of functions have a non-integer cohomological dimension.

This guided tour is organized as follows. We first describe (section A) the quantum version of classical spin^c riemannian geometry, showing how one reconstructs from it the classical De Rham complex and volume form. Extracting the quintessence thereof, we then formulate (section B) non-commutative geometry for a cohomologically finite-dimensional (non-commutative) algebra, a theory based on the notion of "d^+-summable k-cycle". We then present several examples: in section C the example of the two-point algebra already exhibiting embryonal Higgs bosons; in section D the quantum version of Yang-Mills for electrodynamics; in section E a sketch of the differential geometry of fractals (= spaces of non-integer Hausdorff dimension). This line culminates, in section E, with an exposition (with partial proofs) of Alain Connes' vision of the standard model of elementary particles. The second line of the lectures begins in section F with a description of "p-summable Fredholm modules" and their use for the description of the quantal perspective on classical conformal manifolds. We then give examples based on this notion: in section G a sketch of the differential geometry of fractals and in section H §§2 and 3 a discussion of the quantum Polyakov action. §1 of section H uses merely, as already mentioned the notion of Wodzicki residue.

A. The Dirac operator as the carrier of classical differential geometry

In what follows M is a spin Riemannian compact manifold of dimension $d = 2m$. H is the Hilbert space $H = L^2(S_M)$ of square integrable spinor fields (S_M the spinor bundle of M) acted upon by the Atiyah-Singer-Lichnerowicz Dirac operator

$$D = \gamma^\nu(\partial_\nu + \sigma_\nu) \ , \tag{A.1}$$

and by the elements of $A = C^\infty(M)$ acting multiplicatively on the spinor fields:

$$(\underline{a}\psi)(x) = \underline{a}(x)\psi(x) \ , \qquad a \in A \ , \quad \psi \in H. \tag{A.2}$$

We note the following structural features, later to be extracted as axioms:

(i) H is $Z/2$-graded: $H = H^0 \oplus H^1$; [8]

(ii) $A \ni a \to \underline{a} \in B(H)$ is a representation of the algebra A by bounded even operators on H (in fact, a $*$-representation);

(iii) D is an (unbounded) odd self-adjoint operator;

(iv) all commutators $[D, \underline{a}]$, $a \in A$, are bounded;

(v) D^{-1} is a compact operator, in fact belonging to the ideal $L^{d+}(H)$ (essentially, one has $D^{-d} \in L^{1+}$): to simplify, we assume D without zero-modes.

[8] The grading operator is chirality $\chi = \gamma^{d+1}$ (half-spinors).

Interposed comment about $L^{1+}(H)$ and $L^1(H)$: the familiar ideal $L^1(H)$ of trace-class operators consists of the compact operators T such that $|T|$ has eigenvalues λ_n with $\sum_{n=!}^{\infty}\lambda_n < \infty \ (= Tr|T|)$. $L^{1+}(H)$ is "just about above" $L^1(H)$: it consists of compact operators T with $|T|$ s.t. $\sum_{n=1}^{N}\lambda_n = O\ (\log N)$. Applying a dilation-invariant mean w to the bounded function $\sum_{n=1}^{\infty}\lambda_n/\log(N+1)$ on $\mathbf{N}$ then yields the *Dixmier trace* Tr_w, whose definition domain is L^{1+} (analogously to the familiar fact that $L^1(H)$ is the definition domain of the usual trace)[9].

The fact that the operator D encodes the information of the smooth structure is a priori shown by the provocative (easy to prove) result that the geodesic distance $d(p,q)$ between two points p,q of M is given by

$$d(p,q) = \sup\{|a(p) - a(q)|\,;\ a \in A,\ \|[D,\underline{a}]\| \leq 1\}\ . \tag{A.0}$$

We now proceed to the explicit construction of the De Rham complex $\Omega(M)$ by means of the Dirac operator. (We recall that $\Omega(M) = \sum_{n=0}^{d}\Omega(M)^d$ equipped with the exterior differential $\mathbf{d}$ is an N-graded differential algebra).

For this we need a formal construction pertaining to general unital algebras A: *the differential enveloppe of* A, namely the N-graded differential algebra $(\Omega A, \delta)$ of "formal differential forms" obtained as follows: consider linear combinations of formal symbols:

$$a_0\delta a_1\cdots\delta a_n\ , \qquad a_0,\, a_1, \cdots, a_n \in A\ , \tag{A.3}$$

with $\delta\mathbf{1} = 0$ and products $a_0\delta a_1\ldots\delta a_n \cdot b_0\delta b_1\cdots\delta b_n$ obtained by shuffling b_0 to the left by use of the Leibniz rule for the symbol δ: requiring this provides us (easy verification) with an N-graded algebra (grading given by the number of differentials). Moreover the rule

$$\delta(a_0\delta a_1\cdots\delta a_n) = 1\delta a_0\delta a_1\cdots\delta a_n \tag{A.4}$$

establishes $(\Omega A, \delta)$ as an N-graded differential algebra[10]. Mind that ΩA is a formal object carrying no real information: for instance its cohomology vanishes: $\delta\omega = 0, \omega \in \Omega A$, implies that $\omega = \delta\psi$ for some ψ.

We now consider ΩA for $A = C^\infty(M)$ and will use it for coding information via the Dirac operator D. For this we set

$$\pi_D(\omega) = (-i)^n\underline{a}_0[D,\underline{a}_1]\cdots[D,\underline{a}_n]\ , \qquad \omega = a_0\delta a_1\cdots\delta a_n \in \Omega A \tag{A.5}$$

[9]Strictly speaking there are as many Dixmier traces T_w as dilation invariant means w; the latter however coincide in all practical applications - we thus refer to them as the "the Dixmier trace".

[10]The requirement that δ be a differential (= derivation of vanishing square) compels to definition (A.4) and interprets (A.3) as the product of a_0 and the $\delta a_1, \ldots, \delta a_n$.

and note that we thus obtain a bounded *-representation π_D of ΩA *as a* $\star$*algebra*[11] - not as a differential algebra! - indeed (A.4) consists in replacing the derivation δ by the derivation $-i[D,\cdot]$.[12] We are now in a situation where we have a differential algebra ΩA represented on Hilbert space as an algebra, a situation leading naturally to a differential algebra by the (easily established fact) that, with K the kernel of π_D - an ideal of the algebra ΩA - the set $K+\delta K$ is again an ideal, but now a *differential ideal*. In fact we need a small step more, because we want to retain the $\boldsymbol{N}$-grading (in addition to the differential): so replace K by $K^* = \oplus_n K^n,\ K^n = K \cap \Omega A^n$, which is a graded ideal such that $K^* + \delta K^*$ is stable under δ and graded. We thus define

$$\Omega_D A = \Omega A / \left(K^* + \delta K^*\right) \ . \tag{A.6}$$

This now yields, together with δ which passes to the quotient, a differential algebra, amazingly *isomorphic as such to the De Rham complex*:

$$(\Omega_D A, \delta) \simeq (\Omega(M), \mathbf{d}) \ . \tag{A.7}$$

We constructed the De Rham complex by means of the Dirac operator!

For the technical computation of $\Omega_D A$ it is useful to notice that one has in grade n:

$$\Omega_D A^n = \pi_D(\Omega A^n)/\pi_D(\delta K^{n-1}) \tag{A.7a}$$

We now describe the volume-form (the device for integration). Since $D^{-d} \in L^{1+}$, since $\pi_D(\omega)$ is bounded, and since L^{1+} is an ideal, we may consider

$$\tau_D(\omega) = Tr_w(D^{-d}\pi_D(\omega)) \ , \qquad \omega \in \Omega A \ , \tag{A.8}$$

thus obtaining a trace on ΩA (by inserting in addition under Tr_w the grading involution χ, we would get the Euler characteristics).[13]

In our practical calculations we shall make use of the fundamental fact that for P a pseudodifferential operator of order $-d$, the Dixmier trace coincides with the *Wodzicki residue*, namely

$$Tr_w(P) \cong cst \int tr(\sigma_P(x,\xi))dx\delta(|\xi|^2-1)d\xi \tag{A.9}$$

where tr is the trace on the fiber and σ_P the principal symbol of P.

B. Non-commutative differential geometry via K-cycles

Let now A be a (possibly non-commutative) real or complex *-algebra. What we have seen above now justifies the claim that one will endow A with a metric differential structure by specifying a K-cycle defined as follows:

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[11] The *-operation of ΩA is defined as the extension of the *-operation of ΩA such that $(\delta\omega)^* = (-1)^n \delta(\omega^*),\ \omega \in \Omega A^n$.

[12] $\pi_F(\omega)$ with $F^2 = 1$ would yield a representation of ΩA as a differential algebra. However D is not of square one!

[13] The fact that π_D is a trace stems from the fact that $[D^{-1}, \pi_D(\omega)] \in L^1(H)$ (commutators are smoothing!), and that Tr_ω vanishes on $L^1(H)$.

Definition A d^+-*summable K-cycle* (H, D) of A consists of the following data:

(i) a $\boldsymbol{Z}/2$ graded Hilbert space $H = H^0 \oplus H^1$ (with grading involution χ, $\mathbf{1}$ on H^0, $-\mathbf{1}$ on H^1);

(ii) a *-representation $a \to \underline{a}$ of A by even bounded operators on H $\quad(\underline{a}\chi = \chi\underline{a}$ for each $a \in A)$;

(iii) a self-adjoint odd (unbounded) operator D $\quad(D\chi = -\chi D)$ such that

(iv) $[D, \underline{a}]$ is a bounded operator for all $\underline{a} \in A$

(v) D^{-1} is a compact operator[14];

(vi) $D^{-d} \in L^{1+}(H)$

Given such a d-summable K-cycle (H, D), the representation π_D of ΩA is defined as in (A.4), leading as above to the $\boldsymbol{N}$-graded differentiable algebra $\Omega_D A$ (cf. (A.5), (A.5a)) which is naturally called the *non commutative De Rham complex of A*. "Integration" is then defined by means of the trace τ_D of ΩA as given in (A.8).

At this point we dispose of all we need to define a "non commutative Yang-Mills action". For simplicity and because this is the case at hand for electrodynamics and for the standard model, we first describe this notion in the case where the bundle (or rather its module of sections) is given by the algebra itself. In that case (*metric*) *connections* are definable as odd derivations ∇ of the quantum De Rham complex $\Omega_D A$, thus of the form $\delta + \rho$, with *$i\rho$ the connection one-form (quantum potential)* a (self-adjoint) element of $\Omega_D A^1$. The corresponding *curvature* is then the endomorphism[15] ∇^2, where, as one immediately checks, $\nabla^2\omega = (\delta\rho + \rho^2)\omega$, $\omega \in \Omega A$: $\theta = \delta\rho + \rho^2 \in \Omega_D A^2$ is the *curvature two-form*. The *Yang-Mills action* is then naturally defined as

$$\text{YM}\ (\rho) = \tau_D(\theta^2) = Tr_w\,(D^{-d}\pi_D(\theta^2)) \tag{B.1}$$

The case of an arbitrary bundle, yielding in the non commutative case an arbitrary finite projective (right) A-module[16] E, is hardly more complicated. One considers the set of E-valued quantum forms:

$$E_\Omega = E \otimes_A \Omega_D A\ , \tag{B.2}$$

a $\boldsymbol{N}$-graded right module over the algebra $\Omega_D A$, and defines the connections ∇ as the grade-one graded δ-derivations of this module, i.e. the $\boldsymbol{C}$-linear $\nabla : E_\Omega \to E_\Omega$

[14] We assume for simplicity that D has no zero modes. Otherwise one should consider $(D^2 + m^2)^{\frac{1}{2}}$, instead $m > 0$. To be quite exact (vi) should be replaced by the somewhat more stringent requirement $D^{-1} \in L^{d+}(H)$. The notation (H, D) implies that H is a $\boldsymbol{Z}/2$-graded left A-module (via $a \to \underline{a}$).

[15] of the right $\Omega_D A$-module $\Omega_D A$.

[16] We remind that the modules of sections of vector bundles over a manifold M are precisely the finite-projective modules over $C^\infty(M)$. For a non-commutative A the role of (sections of) vector bundles is thus played by the finite projective modules over A.

fulfilling[17]

$$\nabla(X\omega) = (\nabla X)\omega + (-1)^{\partial X}\delta\omega \ , \qquad X \in E_\Omega, \ \ \omega \in \Omega_D A \ . \tag{B.3}$$

The curvature is then $\nabla^2 \in \mathrm{End}_{\Omega A} E_\Omega$, thus implemented by a quantum 2-form θ with values in $\mathrm{End}_{\Omega A} E_\Omega$. The Yang-Mills action is then given by (B.1) when one replaces the trace τ_D of $\Omega_D A$, by the trace of $\mathrm{End}_{\Omega(A)}\, E_\Omega$ given by

$$\tilde{\tau}_D(X\Phi) = \tau_D(\Phi(X)) \ , \qquad X \in E_\Omega \ , \ \ \Phi \in E^*_\Omega \ , \tag{B.4}$$

where Φ is the one-rank endomorphism given by $(X\Phi)U = \Phi(U)X, \ U \in E_\Omega$ (observe that E being projective-finite, the same holds for E_Ω whose endomorphisms are thus linearly generated by one-rank operators).

We end this section by a word of caution. What we have said will hold only for algebras *possessing d^+-summable K-cycles*, i.e. (by definition) having a *finite cohomological dimension*, that is "small algebras" (algebras of loop spaces, or of field theoretic operators, are "big", i.e. cohomologically infinite-dimensional).

C. The two-point algebra: Higgs bosons in a nutshell

We now look at the simplest possible example $A = \boldsymbol{C} \oplus \boldsymbol{C}$ (besides the trivial $\boldsymbol{C}$ itself!). An element of A is of the form $a = (f, g), \ f, g \in \boldsymbol{C}$. On the discrete object A we define the differential geometry by the following K-cycle $(H, D) : H = \boldsymbol{C}^N \oplus \boldsymbol{C}^N$ where the integer N prefigures the number of fermion families. Linear endomorphisms of H are thus described by 2×2 matrices with entries in $M_N(\boldsymbol{C})$. In this sense we have the grading involution

$$\chi = \begin{pmatrix} \mathbf{1}_N & 0 \\ 0 & -\mathbf{1}_N \end{pmatrix} , \tag{C.1}$$

and we define[18]

$$\underline{a} = \begin{pmatrix} f\mathbf{1}_N & 0 \\ 0 & g\mathbf{1}_N \end{pmatrix} , \qquad \underline{a} = (f, g) \in A \tag{C.2}$$

and

$$D = \begin{pmatrix} 0 & M^* \\ M & 0 \end{pmatrix} \tag{C.3}$$

[17] We recall that a classical connection looked at in the guise of the corresponding *exterior covariant derivative* ∇ is a grade-one graded derivation of the $\Omega(M)$-module $E_\Omega = E \otimes_A \Omega(M)$, restricting on E as $(\nabla\eta)(\xi) = \nabla_\xi \eta$, $\eta \in E$, $\nabla\xi$.

[18] In these expressions we left implicit the identity $\mathbf{1}_2$ of $\boldsymbol{C}^2$.

where the $N \times N$ matrix M prefigures the fermionic mass matrix. For $\omega = \sum_i a_0^i \delta a_1^i$, $a_0^i = (f_0^i, g_0^i)$, $a_1^i = (f_1^i, g_1^i)$ one then easily finds that

$$\pi_D(\omega) = \begin{pmatrix} 0 & -\sum_i f_0^i(f_1^i - g_1^i)M^* \\ \sum_i g_0^i(f_1^i - g_1^i)M & 0 \end{pmatrix} \tag{C.4}$$

and

$$\pi_D(\delta\omega) = \sum_i (f_0^i - g_0^i)(f_1^i - g_1^i) \begin{pmatrix} M^*M & 0 \\ 0 & MM^* \end{pmatrix} \tag{C.5}$$

This implies that K^1 consists of the ω for which $\sum_i f_0^i(f_1^i - g_1^i) = \sum_i g_0^i(f_1^i - g_1^i) = 0$, these are such that $\pi_D(\delta\omega)$, hence $\pi_D(\delta K^1)$ vanishes. Since K^0 vanishes, one has thus $\Omega_D A^1 = \pi_D(\Omega A^1)$ and $\Omega_D A^2 = \pi_D(\Omega A^2)$ (cf. (A.5a)). The metric connection one-forms are the antihermitic elements of $\pi_D(\Omega A^1)$, thus of the form

$$\rho = \begin{pmatrix} 0 & -\bar{h}M^* \\ hM & 0 \end{pmatrix} , \qquad h \in \boldsymbol{C} , \tag{C.6}$$

with, by (C.5):

$$\delta\rho = -(h + \bar{h}) \begin{pmatrix} M^*M & 0 \\ 0 & MM^* \end{pmatrix} , \tag{C.7}$$

and

$$\rho^2 = -\bar{h}h \begin{pmatrix} M^*M & 0 \\ 0 & MM^* \end{pmatrix} , \tag{C.8}$$

thus with curvature

$$\theta = -(\bar{h}h + \bar{h} + h) \begin{pmatrix} M^*M & 0 \\ 0 & MM^* \end{pmatrix} . \tag{C.9}$$

Setting $h + 1 = \phi$ one has $\bar{h}h + h + \bar{h} = |\phi|^2 - 1$. In this finite case the Dixmier trace reduces to the ordinary trace Tr. The Yang-Mills action is thus

$$Tr(\theta^2) = 2[|\phi|^2 - 1]^2 \; Tr\,(M^*M)^2 \tag{C.10}$$

displaying the characteristic form of a Higgs potential.

D. Equivalence of the quantum and the classical Yang-Mills algorithms

We formulated the "quantum Yang-Mills algorithm" for a general (cohomologically finite) A in (B.1). We now apply this to electrodynamics, where $A = C^\infty(M)$, and D is the usual Dirac operator, and show that this merges into the usual Yang-Mills. The computation is effected via the analytic form (Wodzicki residue) (A.9) of Tr_w. Let $\omega = \sum_{i=1}^{m} a_0 \delta a_1^i \in \Omega A^1$, a_0^i, $a_1^i \in C^\infty(M)$, yielding the quantum potential

$$\rho = \pi_D(\omega) = \sum_{i=1}^{m} \underline{a}_0^i \underline{\gamma}^\mu \partial_\mu a_1^i = \gamma(A) \tag{D.1}$$

where A is the classical potential[19]

$$A = \sum_i a_0^i \mathbf{d} a_1^i \tag{D.2}$$

With $\gamma(\mathbf{d}a) = \gamma^\mu \partial_\mu a$, we then have that[20]

$$\begin{aligned} \pi_D(\delta\omega) &= \sum_{i=1}^{m} \gamma(\mathbf{d}a_0^i)\gamma(\mathbf{d}a_1^i) = \gamma(\sum_i \mathbf{d}a_0^i \otimes \mathbf{d}a_1^i) \\ &= \gamma\left(\frac{1}{2}\mathbf{d}a_0^i \wedge \mathbf{d}a_1^i + \frac{1}{2}\sum_{i=1}^{m} \mathbf{d}a_0^i \vee \mathbf{d}a_1^i\right) \\ &= \gamma\left(\frac{1}{2}\mathbf{d}A\right) + \sum_i (\mathbf{d}a_0^i, \mathbf{d}a_1^i) \end{aligned} \tag{D.3}$$

and

$$\pi_D(\omega^2) = \gamma(A \otimes A) = (A, A) = A_\mu A^\mu \tag{D.4}$$

We have to evaluate the curvature $\rho^2 + \delta\rho$ as an element of $\Omega_D A^2 \simeq \Omega(M)^2$. For this we use the fact here stated without proof that the quotient modulo $\pi_D(\delta K^1)$ of a Clifford element $\gamma(T)$, T a tensor, is obtained by expressing $\gamma(T)$ in terms of antisymmetric tensors and retaining only the part of highest order. This rule leads to the fact that $\delta\rho = \gamma\left(\frac{1}{2}F\right)$, where $F = \mathbf{d}A$ and $\rho^2 = 0$.

Inserting this in (A.9), where $\theta = \gamma\left(\frac{1}{2}F\right)$ then yields the classical action. (The principal symbol $|\xi|^{-4}$ of D^{-4}, 1 on the unit sphere, drops out of the calculation).

E. The standard model of elementary particles

This section describes the method and the results, giving in detail some of the computations, but omitting the lengthiest, for which we refer to the paper [5]III which we are following here.

[19] Since the representation $a \to \underline{a}$ is faithful, we have $K^0 = 0$, hence $\Omega_D A^1 = \pi_D(\Omega A^1)$. ω should be taken self-adjoint, resulting in A being real (we do not discuss reality conditions).

[20] γ denotes the canonical map: $\mathcal{T}(T^M) \to \mathrm{Cliff}(T^M)$.

§1 Strategy We have shown in section E that for electrodynamics the quantum Yang-Mills action is another version of the classical Yang-Mills action[21]. On the other hand the (quantum) Yang-Mills action of the two-point algebra in section C had produced the Higgs phenomenon as stemming from a "discrete connection". Those two facts can now be combined thanks to an important property of the quantum Yang-Mills formalism: given two algebras A' and A'' equipped with respective K-cycles (H', D'), (H'', D''), there is indeed a canonical way for obtaining a tensor product K-cycle (H, D) of the tensor product algebra $A = A' \otimes A''$: namely[22]

$$\left\{ \begin{array}{l} H = H' \otimes H'' \\ \\ \chi = \chi' \otimes \chi'' \\ \\ \underline{a' \otimes a''} = \underline{a}' \otimes \underline{a}'' \\ \\ D \quad = D' \hat{\otimes} \mathbf{1}'' + \mathbf{1}' \hat{\otimes} D'' \\ \\ \qquad = D' \otimes \mathbf{1}'' + \chi' \otimes D'' \end{array} \right. \tag{E.1}$$

The strategy is now clear, at least for the electroweak sector of the standard model. The passage from electrodynamics to the electroweak synthesis occurs through the passage $U(1) \to U(1) \times SU(2)$. This calls for tensoring as in (E.1) a finite $U(1) \times SU(2)$ Yang Mills theory (based on the algebra $\boldsymbol{C} \oplus \boldsymbol{H}$ analogous to the former $\boldsymbol{C} \oplus \boldsymbol{C}$ having gauge group $U(1) \times U(1)$) with the electrodynamics Yang-Mills based on $C^\infty(M)$ and the Dirac operator.

Of course one has to specify the choice of differential geometry for $\boldsymbol{C} \oplus \boldsymbol{H}$ - i.e. the relevant "inner K-cycle" (appropriate analogue of the K-cycle of $\boldsymbol{C} \oplus \boldsymbol{C}$ described in section C). The choice of $\boldsymbol{C} + \boldsymbol{H}$ was dictated by its gauge group $U(1) \times SU(2)$ emerging from the phenomenology of weak interactions. As we now describe, the inner K-cycle is likewise dictated by phenomenology.

§2 The electroweak inner K-cycle The algebra $\boldsymbol{C} \oplus \boldsymbol{H}$ consists of pairs (p, q), $(p \in \boldsymbol{C})$, $q \in \boldsymbol{H}$. For reasons which appear later it is convenient to identify the complex number p with a diagonal quaternion[23]

$$\boldsymbol{C} \ni p \leftrightarrow \begin{pmatrix} \overline{p} & 0 \\ 0 & p \end{pmatrix} \in \boldsymbol{H}_{diag} \tag{E.2}$$

The electroweak inner K-cycle (H_f, D_f) (f for fermion) is now the direct sum $(H_l \oplus H_q, D_l \oplus D_q)$ of a leptonic K-cycle (H_l, D_l) and a quarkonic K-cycle (H_q, D_q). We begin

[21] A piece of the general fact that non commutative geometry applied to the classical case yields the usual smooth structure.

[22] This is the "exterior product" of Kasparov's KK-theory, $\hat{\otimes}$ denoting a skew tensor-product.

[23] We recall that quaternions are 2×2 complex matrices of the form $\begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}$, α, $\beta \in \boldsymbol{C}$.

with the description of the latter, from which the former is obtained by "splitting a corner" (*leptonic reduction*). The (finite) dimensional Hilbert space H_q is the tensor product

$$H_q = (\underset{u_R d_R}{\boldsymbol{C}^2_R} \oplus \underset{u_L d_L}{\boldsymbol{C}^2_L}) \otimes \boldsymbol{C}^N , \tag{E.3}$$

where the factor $\boldsymbol{C}^N$ stands for N fermion families, R and L stand for right, resp. left, and we indicated the particle - content, u and d standing for upper, resp. lower quarks. We now specify the grading, the representation of $\boldsymbol{C} \oplus \boldsymbol{H}$, and the generalized Dirac operator: they are:[24]

$$\chi_q = \begin{pmatrix} \mathbf{1}_2 \otimes \boldsymbol{N} & 0 \\ 0 & -\mathbf{1}_2 \otimes \mathbf{1}_N \end{pmatrix} \tag{E.4}$$

$$\pi_q(p,q) = \begin{pmatrix} p \otimes \mathbf{1}_N & 0 \\ 0 & q \otimes \mathbf{1}_N \end{pmatrix} \tag{E.5}$$

$$D = \begin{pmatrix} 0 & M_q^* \\ M_q & 0 \end{pmatrix} \quad \text{with} \quad M_q = \begin{pmatrix} M_u & 0 \\ 0 & M_d \end{pmatrix} \tag{E.6}$$

where M_u, resp. M_d, are the mass matrices for upper, resp. lower quarks. Notice that these definitions are natural: (p,q) is even and D odd as they should. For definition (E.5) we have the following guidance: since the bundle is the algebra itself, the gauge group is the group of unitaries of the latter. Now (p,q) is unitary for $p \in S^1$ whilst q is a unitary quaternion (i.e. an element of $SU(2)$).

The leptonic Hilbert space H_l is

$$H_l = (\underset{e_R}{\boldsymbol{C}^1_R} \oplus \underset{\nu_L e_L}{\boldsymbol{C}^2_L}) \otimes \boldsymbol{C}^N , \tag{E.7}$$

coding the fact that there is no right-handed neutrino: e_R, ν_L and e_L respectively stand for right-hand electron (left handed) neutrino, left-handed electron - where electron stands, generically, for electron - muon - tau. Notice that (E.7) is analogous to (E.3) with the first entry chopped-off. Correspondingly, χ_l, π_l and D_l are obtained from χ_q, π_q and D_q written as 4×4 matrices with entries in $M_N(C^2)$ by effecting the changes $u_L \to 0$, $d_R \to e_R$, $u_L \to \nu_L$, $d_L \to 0$, $M_u \to 0$, $M_d \to M_e$; and then chopping off the first row and column of the matrix. This procedure (*leptonic reduction*) allows to evolve the computation for the leptonic from those for the quarkonic K-cycle. We will thus only treat the latter.

We now describe the quantum one- and two-forms. From (E.5) and (E.6) we have, successively, for p, $\pi \in \boldsymbol{H}_{diag}$, $q, \chi \in \boldsymbol{H}$, writing $M_q = M$:

$$i\pi_q(\delta(p,q)) = [D,\, \pi_q(p,q)] = \begin{pmatrix} 0 & \boldsymbol{M}^*[(q-p) \otimes \mathbf{1}_N\,] \\ [(p-q) \otimes \mathbf{1}_N]\ \boldsymbol{M} & 0 \end{pmatrix} \tag{E.8}$$

[24] We describe elements of $\text{End}_{\boldsymbol{C}} H_q$ as 2×2 matrices with entries in $M_2(\boldsymbol{C}) \oplus M_N(\boldsymbol{C})$; and use (E.2).

(we used the fact that $p \otimes \mathbf{1}_N$ commutes with $\boldsymbol{M}$ and $\boldsymbol{M}^*$)

$$i\pi_q(p,q)\delta(\pi,\chi) = \begin{pmatrix} 0 & \boldsymbol{M}^*[p(\chi-\pi)\otimes \mathbf{1}_N] \\ [q(\pi-\chi)\otimes \mathbf{1}_N]\,\boldsymbol{M} & 0 \end{pmatrix} \tag{E.9}$$

$$\pi_q(\delta(p,q)\delta(\pi,\chi)) = \begin{pmatrix} \boldsymbol{M}^*[(p-q)(\pi-\chi)\otimes \mathbf{1}_N]\boldsymbol{M} & 0 \\ 0 & [(p-q)\otimes \mathbf{1}_N]\boldsymbol{M}\boldsymbol{M}^*[(\pi-\chi)\otimes \mathbf{1}_N] \end{pmatrix} \tag{E.10}$$

$$\pi_q((s,r)\delta(p,q)\delta(\pi,\chi) = \begin{pmatrix} \boldsymbol{M}^*[s(p-q)(\pi-\chi)\otimes \mathbf{1}_N]\boldsymbol{M} & 0 \\ 0 & [r(p-q)\otimes \mathbf{1}_N]\boldsymbol{M}\boldsymbol{M}^*[(\pi-\chi)\otimes \mathbf{1}_N] \end{pmatrix} \tag{E.11}$$

Note here that one has from (E.6a), with K the third basic quaternion

$$\boldsymbol{M}\boldsymbol{M}^* = \mathbf{1}\otimes\Sigma - iK\otimes\Delta\ , \quad \begin{cases} \Sigma = \dfrac{1}{2}(M_u M_u^* + M_d M_d^*) \\ \Delta = \dfrac{1}{2}(M_u M_u^* - M_d M_d^*) \end{cases} \tag{E.12}$$

We then conclude that one has for $\omega = \sum_i (p_i, q_i)\delta(\pi_i, \chi_i) \in \Omega A^1_{ew}$:

$$i\pi_q(\omega) = \begin{pmatrix} 0 & \boldsymbol{M}^*(Q'\otimes \mathbf{1}_N) \\ (Q\otimes \mathbf{1}_N)\boldsymbol{M} & 0 \end{pmatrix} \tag{E.13}$$

where the couple

$$\begin{cases} Q = \sum_i q_i(\pi_i - \chi_i) \\ Q' = \sum_i p_i(\chi_i - \pi_i) \end{cases} \tag{E.13a}$$

ranges through $\boldsymbol{H}\times\boldsymbol{H}$; and, for $\theta = \sum_i (s_i, r_i)\delta(p_i, q_i)\delta(\pi_i, \chi_i)$:

$$\pi_q(\theta) = \begin{pmatrix} \boldsymbol{M}^*(Q\otimes \mathbf{1}_N)\boldsymbol{M} & 0 \\ 0 & Q'\otimes\Sigma - iQ''\otimes\Delta \end{pmatrix} \tag{E.14}$$

where the triple

$$\begin{cases} Q = \sum_i s_i(p_i - q_i)(\pi_i - \chi_i) \\ Q' = \sum_i r_i(p_i - q_i)(\pi_i - \chi_i) \\ Q'' = \sum_i r_i(p_i - q_i)K(\pi_i - \chi_i) \end{cases} \tag{E.14a}$$

ranges through $\boldsymbol{H} \times \boldsymbol{H} \times \boldsymbol{H}$: this is obvious for Q and Q' and will follow for Q'' from the forthcoming argument. Note that whilst the general quantum one-forms are given in (E.13) (since $\Omega_D A^1_{ew} = \pi_D(\Omega A^1_{ew})$ owing to $K^0 = 0$), the quantum two-forms will be the classes of elements (E.14) modulo the set $\pi_D(\delta K^1)$, which we now investigate. The kernel K^1 in grade one consists of the ω in (E.13) fulfilling

$$\sum_i q_i(\pi_i - \chi_i) = \sum_i p_i(\pi_i - \chi_i) = 0 \ , \tag{E.15}$$

$\pi_D(\delta\omega)$ being given by (E.14) where on makes $s_i = r_i$ for all i. Looking at (E.14a) we see that the conditions (E.15) then imply the following shape of the matrix (E.14)

$$\begin{pmatrix} 0 & 0 \\ 0 & -iQ'' \otimes \Delta \end{pmatrix} \quad \text{with } Q'' = \sum_i (p_i - q_i)K(\pi_i - \chi_i) \ , \tag{E.16}$$

with the range of such Q'' including the set of $\sum_i q_i K \chi_i$, q_i, χ_i arbitrary quaternions: since this set is obviously an ideal of the quaternions, thus the whole quaternions, we see that one has

$$\pi_D(\delta K') = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & iQ'' \otimes \Delta \end{pmatrix} \ , \quad Q'' \in \boldsymbol{H} \right\} \ , \tag{E.17}$$

and thus the quantum two-forms are of the type

$$\begin{pmatrix} \boldsymbol{M}^*(Q \otimes \mathbf{1}_N)\boldsymbol{M} & 0 \\ 0 & Q' \otimes \Sigma \end{pmatrix} \ , \quad Q, \, Q' \in \boldsymbol{H} \ . \tag{E.18}$$

It is interesting that quotienting through $\pi_D(\delta K^1)$ precisely leaves us with the quaternionic part in (E.14).

§3. The electroweak sector We now effect the tensor product as described in (E.1), taking for (H', D') the classical K-cycle ($H = L^2(\delta_M)$, Dirac operator) of $A = C^\infty(M)$, and taking for (H'', D'') the quarkonic K-cycle (H_q, D_q) of A_{ew} as described in the last section. We thus obtain a 4^+-summable K-cycle[25] $(\mathcal{D}_q, \mathcal{H}_q = H \otimes H_q)$ of the compound algebra

$$\begin{aligned} \mathcal{A} = \quad & A \otimes A_{ew} = C^\infty(M, \boldsymbol{H}_{diag}) \oplus C^\infty(M, \boldsymbol{H}) \\ & = \{(f,q);\ f \in C^\infty(M, \boldsymbol{H}_{diag}),\ q \in C^\infty(M, \boldsymbol{H})\} \end{aligned} \tag{E.19}$$

Note that, considering now endomorphisms of $\mathcal{H}_q$ as 2×2 matrices with entries endomorphisms of $H \otimes C^2 \otimes C^N$, we have that

$$\pi_{\mathcal{D}_q}(f \otimes (p,q)) = \begin{pmatrix} f \otimes p \otimes \mathbf{1}_N & 0 \\ 0 & f \otimes q \otimes \mathbf{1}_N \end{pmatrix} \tag{E.20}$$

$$\mathcal{D}_q = \begin{pmatrix} D \otimes \mathbf{1}_2 \otimes \mathbf{1}_N & \gamma^5 \otimes \boldsymbol{M}^* \\ \gamma^5 \otimes \boldsymbol{M} & D \otimes \mathbf{1}_2 \otimes \mathbf{1}_N \end{pmatrix} \tag{E.21}$$

$$\chi_{\mathcal{D}_q} = \begin{pmatrix} \gamma^5 \otimes \mathbf{1}_2 \otimes \mathbf{1}_N & 0 \\ 0 & -\gamma^5 \otimes \mathbf{1}_2 \otimes \mathbf{1}_N \end{pmatrix} \tag{E.22}$$

For describing the quantum one- and two-forms we are now guided by a tensor-product structure discussed in detail in [6] for the systems compound of space-time and a finite dimensional inner space. Stated in our present case the structure is as follows. The differential envelope $\Omega\mathcal{A}$ projects homomorphically onto the skew tensor product $\Omega\mathcal{A}\hat{\otimes}\Omega A_{ew}$ of differential envelopes. The skew product tensor is perfect at the level of the representation $\pi_{\mathcal{D}_q}$: one has for one-forms (formal or quantum)[26]

$$\pi_{\mathcal{D}_q}(\Omega\mathcal{A}^1) = \pi_D(\Omega A^1)\hat{\otimes}\pi_q(A_{ew}) + \pi_D(A)\hat{\otimes}\pi_{\mathcal{D}_q}(\Omega A^1_{ew}) \ , \tag{E.23}$$

i.e.

$$\begin{aligned} \Omega_{D_q}\mathcal{A}^1 \quad & = \Omega_D A^1 \otimes A_{ew} \oplus A \otimes \Omega_{D_q}(A_{ew}) \ , \\ (\quad & = \text{“}\Omega_{D_q}\mathcal{A}_{[1,0]} \oplus \Omega_{D_q}\mathcal{A}_{[0,1]}\text{”}) \end{aligned} \tag{E.23a}$$

and for representatives of formal two-forms:

$$\begin{aligned} \pi_{\mathcal{D}_q}(\Omega\mathcal{A}^2) \quad & = \pi_D(\Omega A^2)\hat{\otimes}\pi_q(A_{ew}) + \pi_D(\Omega A^1)\hat{\otimes}\pi_q(\Omega A^1_{ew}) + \pi_D(A)\hat{\otimes}\pi_q(\Omega A^2_{ew}) \\ (\quad & = \text{“}\ \pi_{\mathcal{D}_q}(\Omega\mathcal{A}_{[2,0]}) + \pi_{\mathcal{D}_q}(\Omega\mathcal{A}^2_{[1,1]}) + \pi_{\mathcal{D}_q}(\Omega\mathcal{A}^2_{[0,2]})\text{ ” }) \end{aligned} \tag{E.24}$$

[25] We leave aside, momentarily, the leptonic K-cycle (H_l, D_l), relying on leptonic reduction.
[26] Note that $\pi_{\mathcal{D}_q}$ is faithful. One will thus have $\Omega_{\mathcal{D}_q}\mathcal{A}^1 = \pi_{\mathcal{D}_q}(\Omega\mathcal{A}^1)$.

Furthermore the set by which one has to devide in order to get the quantum one-forms also tensorially decompose

$$\pi_{\mathcal{D}_q}(\delta K^1_{\mathcal{A}}) = \pi_D(\delta K^1_A) \otimes \pi_q(A_{ew}) + \pi_D(A) \otimes \pi_q(\delta K^1_{A_{ew}}) \tag{E.25}$$

This structure then implies the following: the quantum one-forms $\rho \in \pi_{\mathcal{D}_q}(\mathcal{A}^1) = \Omega_{D_q}\mathcal{A}^1$ are given as follows in terms of quadruples $i\rho = (\mathbf{a},\, \mathbf{b},\, H,\, H')$, $\mathbf{a} \in \Omega(\boldsymbol{M}, \boldsymbol{C})^1$, $\mathbf{b}^{\cdot}. \in (\Omega(\boldsymbol{M}, i\boldsymbol{H})^1$, $H, H' \in C^\infty(\boldsymbol{M}, \boldsymbol{H})$: one has for the bihomogeneous components the following 2×2 matrices with entries acting on the fibers of $L^2(\boldsymbol{S}_{\boldsymbol{M}}) \otimes \boldsymbol{C}^N$:

$$i\rho = (\boldsymbol{a}, \boldsymbol{b}, H, H') : \begin{cases} i\rho_{[1,0]} = \begin{pmatrix} \underline{\gamma}(\boldsymbol{a}^{\cdot}.) \otimes \mathbf{1}_N & 0 \\ 0 & \underline{\gamma}(\boldsymbol{b}^{\cdot}.) \otimes \mathbf{1}_N \end{pmatrix} \\ i\rho_{[0,1]} = \begin{pmatrix} 0 & \boldsymbol{M}^*(\gamma^5 \underline{H'} \otimes \mathbf{1}_N) \\ (\gamma^5 \underline{H} \otimes \mathbf{1}_N)\boldsymbol{M} & 0 \end{pmatrix} \end{cases} \tag{E.26}$$

where we used the shorthands

$$\begin{cases} \gamma(\boldsymbol{a}^{\cdot}.) = \begin{pmatrix} -\underline{\gamma}(\overline{\boldsymbol{a}}) & 0 \\ 0 & \underline{\gamma}(\boldsymbol{a}) \end{pmatrix} \;, \quad \boldsymbol{a} \in \Omega(\boldsymbol{M}, \boldsymbol{C})^1 \; (\boldsymbol{a}^{\cdot}. = \begin{pmatrix} -\overline{\boldsymbol{a}} & 0 \\ 0 & \boldsymbol{a} \end{pmatrix} \in \Omega(\boldsymbol{M}, i\boldsymbol{H}_{diag})^1) \;, \\ \underline{\gamma}(\boldsymbol{b}^{\cdot}.) = \begin{pmatrix} \underline{\gamma}(\boldsymbol{b}^1{}_1) & \underline{\gamma}(\boldsymbol{b}^1{}_2) \\ \underline{\gamma}(\boldsymbol{b}^2{}_1) & \underline{\gamma}(\boldsymbol{b}^2{}_2) \end{pmatrix} \;, \quad \boldsymbol{b}^{\cdot}. = \begin{pmatrix} \boldsymbol{b}^1{}_1 & \boldsymbol{b}^1{}_2 = \boldsymbol{b}^2{}_1 \\ \boldsymbol{b}^2{}_1 & \boldsymbol{b}^2{}_2 = \boldsymbol{b}^1{}_1 \end{pmatrix} \in \Omega(\boldsymbol{M},\, i\boldsymbol{H})^1 \;. \end{cases} \tag{E.27}$$

The elements of $\pi_{\mathcal{D}_q}(\delta K^1)$ have the form (statement synonimous with (E.25)):

$$\begin{cases} \begin{pmatrix} (\underline{\boldsymbol{S}}^i{}_k) \otimes \mathbf{1}_N & 0 \\ 0 & (\underline{\mathbf{T}}^i{}_k \otimes \mathbf{1}_N \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & i\underline{\boldsymbol{R}} \otimes \Delta \end{pmatrix} \\ \text{where} \quad \begin{cases} (\mathbf{S}^i{}_k) \in C^\infty(\boldsymbol{M}, \boldsymbol{H}_{diag}) \\ (\mathbf{T}^i{}_k) \in C^\infty(\boldsymbol{M}, \boldsymbol{H}) \\ \boldsymbol{R} \in C^\infty(\boldsymbol{M}, \boldsymbol{H}) \end{cases} \end{cases} \tag{E.28}$$

The vector potentials (antihermitean elements of $\Omega_{\mathcal{D}_q}\mathcal{A}^1$, singled out as the antihermitean $V = \rho$ above are accordingly specified by triples $(\boldsymbol{a},\, \boldsymbol{b},\, H)$ of a hermitean $U(1)$-connection-one-form $\boldsymbol{a}$, a hermitean $SU(2)$-connection-one-form $\boldsymbol{b}^{\cdot}.$, and a doublet field $H \in C^\infty(\boldsymbol{M}, \boldsymbol{H})$: one has[27]

[27] Note that a quaternion is antihermitean iff it is traceless.

$$iV = (\boldsymbol{a},\, \boldsymbol{b}, H,\, H^*) : \tag{E.29}$$

$$\begin{cases} iV_{[1,0]} = \begin{pmatrix} \underline{\gamma}(\boldsymbol{a}^{\cdot}.) \otimes \mathbf{1}_N & 0 \\ 0 & \underline{\gamma}(\boldsymbol{b}^{\cdot}.) \otimes \mathbf{1}_N \end{pmatrix} \\ \qquad\qquad \left(\begin{cases} \boldsymbol{a} = \overline{\boldsymbol{a}} \\ \boldsymbol{b} = \boldsymbol{b}^* \ \text{ i.e. } \boldsymbol{b}^i{}_k = \boldsymbol{b}^k{}_i, \quad i,k = 1,2, \end{cases} \right) \\ iV_{[0,1]} = \begin{pmatrix} 0 & \boldsymbol{M}^*(\gamma^5 \underline{H}^* \otimes \mathbf{1}_N) \\ \gamma^5 \underline{H} \otimes \mathbf{1}_N)\boldsymbol{M} & 0 \end{pmatrix} \end{cases}$$

The field $H \in C^\infty(\boldsymbol{M}, \boldsymbol{H})$ is identified with a doublet field according to

$$H = \begin{pmatrix} \overline{H}^2 & H^1 \\ -\overline{H}^1 & H^2 \end{pmatrix} \Leftrightarrow H^{\cdot} = (H^1, H^2) \Leftrightarrow H_{.} = (\overline{H}^1, \overline{H}^2).$$

Under the gauge transformation $\boldsymbol{u} = (u, v = (v^i{}_k))$ the $\mathcal{D}_q$-quantum-one-form $-i(\boldsymbol{a}, \boldsymbol{b}, H, H')$ becomes $-i({}^{\boldsymbol{u}}\boldsymbol{a},\, {}^{\boldsymbol{u}}\boldsymbol{b},\, {}^{\boldsymbol{u}}H,\, {}^{\boldsymbol{u}}H')$ with:

$$\begin{cases} {}^{\boldsymbol{u}}\boldsymbol{a} = \boldsymbol{a} + iu\mathbf{d}u^* \\ {}^{\boldsymbol{u}}\boldsymbol{b} = v\boldsymbol{b}v^* + iv\mathbf{d}v^* \\ {}^{\boldsymbol{u}}(H + \mathbf{1}) = u^* v(H + \mathbf{1}) \\ {}^{\boldsymbol{u}}(H' + \mathbf{1}) = u^* v(H' + \mathbf{1}) \end{cases} \tag{E.30}$$

In the particular case of vector potentials, $\boldsymbol{a}$, and $(\boldsymbol{a}^i{}_k)$ thus behave as the respective one-forms of a $U(1)$- and a $SU(2)$-connection, whilst $\Phi_{.} = H_{.} + (0,1)$ *and* $\Phi^{\cdot} = H^{\cdot} + (0,1)$ behave as respective covariant and contravariant $SU(2)$-doublets.

The elements of $\pi_{\mathcal{D}_q}(\Omega\mathcal{A}^2)$ are of the type[28]:

$$\left\{\begin{array}{l} \eta_{[2,0]} = \begin{pmatrix} [(\underline{\gamma}(\lambda^i{}_k)) - (\underline{X}^i{}_k)] \otimes \mathbf{1}_N & 0 \\ 0 & [(\underline{\gamma}(\mu^i{}_k)) - (\underline{Y}^i{}_k)] \otimes 1_N \end{pmatrix} \\ \eta_{[1,1]} = \begin{pmatrix} 0 & \boldsymbol{M}^* \left[i\underline{\gamma}(\boldsymbol{q}')\gamma^5 \otimes \mathbf{1}_N\right] \\ \left[i\underline{\gamma}(\boldsymbol{q})\gamma^5 \otimes \mathbf{1}_N\right] \boldsymbol{M} & 0 \end{pmatrix} \\ \eta_{[0,2]} = \begin{pmatrix} \boldsymbol{M}^*(\boldsymbol{Q} \otimes \mathbf{1}_N)\boldsymbol{M} & 0 \\ 0 & \boldsymbol{Q}' \otimes \Sigma + i\boldsymbol{Q}'' \otimes \Delta \end{pmatrix} \end{array}\right. \tag{E.31}$$

where $\left\{\begin{array}{l} (\lambda^i{}_k) \in \Omega(\boldsymbol{M}, \boldsymbol{H}_{diag})^2 \\ (X^i{}_k) \in C^\infty(\boldsymbol{M}, \boldsymbol{H}_{diag}) \\ (\mu^i{}_k) \in \Omega(\boldsymbol{M}, \boldsymbol{H})^2 \\ (Y^i_k) \in C^\infty(\boldsymbol{M}, \boldsymbol{H}) \\ \boldsymbol{q}, \boldsymbol{q}' \in \Omega(\boldsymbol{M}, \boldsymbol{H})^1 \\ \boldsymbol{Q}, \boldsymbol{Q}', \boldsymbol{Q}'' \in C^\infty(\boldsymbol{M}, \boldsymbol{H}) \end{array}\right.$

with the following action of P_2:

$$\left\{\begin{array}{l} (P_2\eta)_{[2,0]} = \begin{pmatrix} [(\underline{\gamma}(\lambda^i{}_k)) - (2N)^{-1} Tr(\mu_u + \mu_d) Q_{diag}] \otimes \mathbf{1}_N & 0 \\ 0 & [(\underline{\gamma}(\mu^i{}_k)) - (2N)^{-1} Tr(\mu_u + \mu_d) \cdot Q'] \otimes \mathbf{1}_N \end{pmatrix} \\ (P_2\eta)_{[1,1]} = \begin{pmatrix} 0 & \boldsymbol{M}^*[i\underline{\gamma}(\boldsymbol{q}')\gamma^5 \otimes \mathbf{1}_N] \\ \left[i\underline{\gamma}(\boldsymbol{q})\gamma^5 \otimes \mathbf{1}_N\right] \boldsymbol{M} & 0 \end{pmatrix} = \eta_{[1,1]} \\ (P_2\eta)_{[0,2]} = \begin{pmatrix} \boldsymbol{M}^*(\boldsymbol{Q} \otimes 1_N)\boldsymbol{M} & 0 \\ 0 & \boldsymbol{Q}' \otimes \Sigma \end{pmatrix} \end{array}\right. \tag{E.32}$$

[28] Whilst $\rho_{[1,0]}$ and $\rho_{[0,1]}$ are linearly independent, this does not hold for $\eta_{[20]}$, $\eta_{[11]}$ and $\eta_{[02]}$ (in the subscript$_{[i,j]}$, i refers to the γ^5- and j to the χ_q-parity).

The next step (which we do not describe, referring to [5] III for details) consists in adapting the results above to a situation incorporating the leptons, replacing the inner K-cycle (H_q, D_q) above by a convex combination $\alpha_l(H_l, D_l) + \alpha_q(H_q, D_q)$, $\alpha_l = (1+x)/2$, $\alpha_q = (1-x)/2$, $-1 \leq x \leq 1$. Using leptonic reduction, one gets the following structure: the projection P_2 is modified as follows, where we set $\underline{L} = Tr[\alpha_l\mu_l + \alpha_q(\mu_u + \mu_d)]$, $\mu_l = M_l M_l^*$, $\mu_u = M_u M_u^*$, $\mu_d = M_d M_d^*$:

$$\left\{\begin{array}{l} (P_2\eta)_{[2,0]} = \begin{pmatrix} [(\underline{\gamma}(\lambda^i{}_k)) - (\alpha_l + 2\alpha_q)^{-1}\underline{L}Q_{diag}] \otimes 1_N & 0 \\ 0 & [(\underline{\gamma}(\mu^i{}_k)) - (\alpha_l + 3\alpha_q)^{-1}(2\boldsymbol{N})^{-1}\underline{L}\cdot Q'] \otimes \mathbf{1}_N \end{pmatrix} \\ (P_2\eta)_{[1,1]} = \begin{pmatrix} 0 & \boldsymbol{M}^*\left[i\underline{\gamma}(\boldsymbol{q})\gamma^5 \otimes \mathbf{1}_N\right] \\ \left[i\underline{\gamma}(\boldsymbol{q})\gamma^5 \otimes \mathbf{1}_N\right]\boldsymbol{M} & 0 \end{pmatrix} = \eta_{[1,1]} \\ (P_2\eta)_{[0,2]} = \begin{pmatrix} \boldsymbol{M}^*(\boldsymbol{Q} \otimes \mathbf{1}_N)\boldsymbol{M} & 0 \\ 0 & \boldsymbol{Q'} \otimes \Sigma \end{pmatrix} \end{array}\right. \tag{E.33}$$

the action of P_2 on the leptonic reduction of η yielding the leptonic reduction of $P_2\eta$.

We now describe the curvature corresponding to the vector-potential specified in (E.29): it is specified as follows by its quark-component θ_q:

$$\left\{\begin{array}{l} -\theta_{q[2,0]} = \begin{pmatrix} \left[\frac{i}{2}\underline{\gamma}(f\cdot) - (\alpha_l + 2\alpha_q)^{-1}N^{-1}\underline{L}v_\Phi\mathbf{1}\right] \otimes \mathbf{1}_N & 0 \\ 0 & \left[\frac{i}{2}\underline{\gamma}(\boldsymbol{h}\cdot) - (2N)^{-1}\underline{L}v_\Phi\mathbf{1}\right] \otimes \mathbf{1}_N \end{pmatrix} \\ -\theta_{q[1,1]} = \begin{pmatrix} 0 & \boldsymbol{M}^*\left[\underline{\gamma}(i\boldsymbol{D}\Phi^*)\gamma^5 \otimes \mathbf{1}_N\right] \\ \left[\underline{\gamma}(i\boldsymbol{D}\Phi)\gamma^5 \otimes \mathbf{1}_N\right]\boldsymbol{M} & 0 \end{pmatrix} \\ -\theta_{q[0,2]} = \begin{pmatrix} v_\Phi \boldsymbol{M}^*\boldsymbol{M} & 0 \\ 0 & v_\Phi\mathbf{1} \otimes \Sigma \end{pmatrix} \end{array}\right. \tag{E.34}$$

the leptonic component θ_l resulting by leptonic reduction. Here we use the shorthands[29]:

$$\begin{cases} \underline{\gamma}(\mathbf{f}^{\cdot}.) = \begin{pmatrix} -\underline{\gamma}(\mathbf{f}) & 0 \\ 0 & \underline{\gamma}(\mathbf{f}) \end{pmatrix} , & \mathbf{f} = \mathbf{d}\boldsymbol{a} \in C^{\infty}(\boldsymbol{M}, \boldsymbol{C}) \ , \\ \underline{\gamma}(\boldsymbol{h}^{\cdot}.) = \begin{pmatrix} \underline{\gamma}(\boldsymbol{h}^1{}_1) & \underline{\gamma}(\boldsymbol{h}^1{}_2) \\ \underline{\gamma}(\boldsymbol{h}^2{}_1) & \underline{\gamma}(\boldsymbol{h}^2{}_2) \end{pmatrix} , & \boldsymbol{h}^j{}_k = \mathbf{d}\boldsymbol{b}^j{}_k - i\boldsymbol{b}^j{}_m \wedge \boldsymbol{b}^m_k \ , \qquad {}^{30} \\ \Phi = \boldsymbol{H} + \mathbf{1} \, , \quad v_\Phi = \Phi_i \Phi^i - 1 \ , & \boldsymbol{H} \in C^{\infty}(\boldsymbol{M}, \boldsymbol{H}) \ , \\ \boldsymbol{D}\Phi = \mathbf{d}\Phi + i(\Phi \boldsymbol{a}^{\cdot}. - \boldsymbol{b}^{\cdot}.\Phi) \ , & \boldsymbol{D}\Phi^* = (\boldsymbol{D}\Phi)^* = \mathbf{d}\Phi^* - i(\boldsymbol{a}^{\cdot}.\Phi^* - \Phi^* \boldsymbol{b}^{\cdot}.) \ . \end{cases} \tag{E.35}$$

Here $\mathbf{f}, . \, \boldsymbol{h}^{\cdot}. = (\boldsymbol{h}^i{}_j)$, $i, j = 1, 2$ are the respective curvatures of the $U(1)$-connection $\boldsymbol{a}$ and the $SU(2)$-connection $\boldsymbol{b}^{\cdot}. = (\boldsymbol{b}^i{}_j) = \boldsymbol{b}^a \frac{\tau_a}{2}$, $\alpha = 1, 2, 3$:

$$\begin{cases} \mathbf{f} = \mathbf{d}\boldsymbol{a} = \frac{1}{2} \mathbf{f}_{\mu\nu} dx^\mu \wedge dx^\nu \\ \text{with } \mathbf{f}_{\mu\nu} = \partial_\mu \boldsymbol{a}_\nu - \partial_\nu \boldsymbol{a}_\mu \end{cases} , \tag{E.36}$$

$$\begin{cases} \boldsymbol{h}^{\cdot}. = \nabla \boldsymbol{b}^{\cdot}. = \frac{1}{2} \boldsymbol{h}_{\mu\nu} \mathbf{d}\boldsymbol{x}^\mu \wedge \mathbf{d}x^\nu (\boldsymbol{h}^i{}_k) = \boldsymbol{h}^a \frac{\tau_a}{2} \\ \text{with } \boldsymbol{h}^i{}_k = \mathbf{d}\boldsymbol{b}^j{}_k - i\boldsymbol{b}^j{}_m \wedge \boldsymbol{b}^m_k \ , \quad \text{i.e. } \boldsymbol{h}^j{}_{k\mu\nu} = \partial_\mu \boldsymbol{b}^j{}_{k\nu} - \partial_\mu \boldsymbol{b}^j{}_{km} - i[\boldsymbol{b}_\mu, \boldsymbol{b}_\nu]^j{}_k \ , \\ \text{or } \ \boldsymbol{h}_{\mu\nu} = \partial_\mu \boldsymbol{b}_\nu - \partial_\nu \boldsymbol{b}_\mu - i[\boldsymbol{b}_\mu, \boldsymbol{b}_\nu] \, , \quad \text{i.e. } \boldsymbol{g}^a{}_{\mu\nu} = \partial_\mu \boldsymbol{b}^a{}_\nu - \partial_\nu \boldsymbol{b}^a{}_\mu + \varepsilon_{abc} \boldsymbol{b}^b \ \mu \boldsymbol{b}^c{}_\nu \end{cases} \tag{E.37}$$

The usual combined $U(1) \times SU(2)$-covariant derivatives

$$\begin{cases} \boldsymbol{D}\Phi^{\cdot} : \boldsymbol{D}\Phi^j = \mathbf{d}\Phi^j + i(\boldsymbol{a}\Phi^j - \boldsymbol{b}^j{}_k \Phi^k) \ , \quad \text{or} \ \ \boldsymbol{D}\Phi^{\cdot} = \mathbf{d}\Phi + i\left(\boldsymbol{a} - \boldsymbol{b}^a \frac{\tau_a}{2}\right)\Phi^{\cdot} \\ \boldsymbol{D}\Phi. : \boldsymbol{D}\Phi_j = \mathbf{d}\Phi_j - i(\boldsymbol{a}\Phi_j - \boldsymbol{b}^k{}_j \Phi_k) = \boldsymbol{D}\overline{\Phi}^j \ , \quad \text{or } \boldsymbol{D}\Phi. = \mathbf{d}\Phi. - i\Phi. \left(\boldsymbol{a} - \boldsymbol{b}^a \frac{\tau_a}{2}\right) \end{cases} \tag{E.38}$$

has the following relationship with the above quaternion covariant derivatives: for

[29] We denoted covariant derivatives by bold-faced letters to avoid confusion with the Dirac operator. We recall that $\Sigma = \frac{1}{2}(\mu_u + \mu_d)$ with $\mu_u = M_u M_{u^*}$ and $\mu_d = M_d M_{d^*}$.

[30] i.e. $\boldsymbol{h}_{\mu\nu} = \partial_\mu \boldsymbol{b}_\nu - \partial_\nu \boldsymbol{b}_\mu - i \, [\boldsymbol{b}_\mu, \boldsymbol{b}_\nu]$.

$\Phi = \begin{pmatrix} \overline{\Phi}^2 & \Phi^1 \\ -\overline{\Phi}^1 & \Phi^2 \end{pmatrix} \leftrightarrow \Phi^{\cdot} = (\Phi^1, \Phi^2) \leftrightarrow \Phi_{\cdot} = (\overline{\Phi}^1, \overline{\Phi}^2)$ one has:

$$\boldsymbol{D}\Phi = \mathbf{d}\Phi + i(\Phi(\boldsymbol{a}^{\cdot}_{\cdot}) - (\boldsymbol{b}^{\cdot}_{\cdot})\Phi = \begin{pmatrix} \boldsymbol{D}\Phi_2 & \boldsymbol{D}\Phi^1 \\ -\boldsymbol{D}\Phi_1 & \boldsymbol{D}\Phi^2 \end{pmatrix} . \tag{E.39}$$

Having at hand the Yang-Mills action of the full electroweak sector one could think of exploiting it physically (electroweak physics). But the corresponding theory *leads to wrong hypercharges*, this being in fact a success of Connes' approach, which thus *entails the necessity of putting together the electroweak and chromodynamics sectors*[31].

§4. Appending the chromodynamics sector On the chromodynamics side the group of inner degrees of freedom is $SU(3)$ and the interaction mediated by the gluons is of the Yang-Mills type with unbroken symmetry. The relevant "inner space" algebra is

$$B_{chrom} = \boldsymbol{C} \oplus M(\boldsymbol{C}^3) \ , \tag{E.40}$$

the best we can do to accomodate $SU(3)$[32]. The overall chromodynamics algebra, counterpart of $\mathcal{A}$ in (E.19), is then, with $A = C^\infty(M)$ as above:

$$\begin{aligned} \mathcal{B} &= A \otimes B_{chrom} = \boldsymbol{C}^\infty(M, \boldsymbol{C}) \oplus C^\infty(M, M_3)(\boldsymbol{C})) \\ &= \{(g', m);\ g' \in C^\infty(M, \boldsymbol{C}),\ m \in C^\infty(M, M_3(\boldsymbol{C})\} \ . \end{aligned} \tag{E.41}$$

The relevant K-cycle is now, in terms of the previous H_q, H_l:

$$H \otimes (H_q \otimes \boldsymbol{C}^3_{colour} \oplus H_l \otimes \boldsymbol{C}) \ , \qquad H = L^2(\boldsymbol{S}_M) \ , \tag{E.42}$$

in conformity with the fact that the quarks acquire a threefold colour degree of freedom, the leptons being colourless. On this new Hilbert space enriched by colour

— the electroweak algebra acts trivially on the colour tensorial factors $\boldsymbol{C}^3_{colour}$ and $\boldsymbol{C}$, acting as above on the factors $H_q \otimes H$ and $H_l \otimes H$;

— the generalized Dirac operator also proceeds in this way from the former $\mathcal{D}_q \oplus \mathcal{D}_l$ (indifference to colour);

— the algebra $\mathcal{B}$ acts by its space-time factor on H in the usual way, and by its B_{chrom}-factor, as naturally expected: the $\boldsymbol{C}$-part acting on the $\boldsymbol{C}$ factor of $H_l \otimes C$, and the $M(\boldsymbol{C}^3)$-part on the $\boldsymbol{C}^3_{colour}$-factor of $H_q \otimes \boldsymbol{C}^3_{colour}$.

[31] This aspect is in fact enhanced by interpreting the duality electroweak-strong as a Poincaré duality.

[32] We are not here in the fortunate situation of the electroweak $U(1) \otimes SU(2)$, unitary group of $\boldsymbol{C} \oplus \boldsymbol{H}$. We are compelled to work with $U(1)$ and the $U(3)$ of $M(\boldsymbol{C}^3)$, and later collapse three $U(1)$ group into one.

It is important to note that one has *commutation of the actions of* $\mathcal{A}$ *and* $\mathcal{B}$ (in other terms an action of $\mathcal{A}\otimes\mathcal{B}$) and *commutation of the action of* $\mathcal{B}$ *with the discrete part of the generalized Dirac operator*, this producing the setting of "non-commutative Poincaré duality" [3a] for which we refer to §5.

The quantum one-forms of the chromodynamics sector and their curvatures have in fact been already computed - modulo tensorization and non-commutativity of the gauge group - in section D describing electrodynamics. The respective lepton and quark contribution are

$$iV_l' = \begin{array}{c} \\ e_R \\ \nu_L \\ e_L \end{array} \begin{array}{c} \begin{array}{ccc} e_R & \nu_L & e_L \end{array} \\ \begin{pmatrix} \gamma(\boldsymbol{a}')\otimes\mathbf{1}_N & 0 & 0 \\ 0 & - & 0 \\ 0 & 0 & - \end{pmatrix} \end{array}, \tag{E.43}$$

$$-\theta_l' = \begin{array}{c} \\ e_R \\ \nu_L \\ e_L \end{array} \begin{array}{c} \begin{array}{ccc} e_R & \nu_L & e_L \end{array} \\ \begin{pmatrix} \frac{i}{2}\gamma(f')\otimes\mathbf{1}_N & 0 & 0 \\ 0 & - & 0 \\ 0 & 0 & - \end{pmatrix} \end{array}, \tag{E.44}$$

and[33]

$$\begin{aligned} iV_q' &= \begin{array}{c} \\ u_R \\ d_r \\ u_L \\ d_L \end{array} \begin{array}{c} \begin{array}{cccc} u_R & d_R & u_L & d_L \end{array} \\ \begin{pmatrix} \gamma({c^j}_k)\otimes\mathbf{1}_N & 0 & 0 & 0 \\ 0 & - & 0 & 0 \\ 0 & 0 & - & 0 \\ 0 & 0 & 0 & - \end{pmatrix} \end{array} \otimes {e^k}_j \\ &= \begin{pmatrix} \gamma(c^0)\otimes\mathbf{1}_N & 0 & 0 & 0 \\ 0 & - & 0 & 0 \\ 0 & 0 & - & 0 \\ 0 & 0 & 0 & - \end{pmatrix} \otimes \mathbf{1}_3 + \begin{pmatrix} \gamma(c^a)\otimes\mathbf{1}_N & 0 & 0 & 0 \\ 0 & - & 0 & 0 \\ 0 & 0 & - & 0 \\ 0 & 0 & 0 & - \end{pmatrix} \otimes \frac{\lambda_a}{2} \end{aligned} \tag{E.45}$$

$$-\theta_q' = \begin{array}{c} \\ u_R \\ d_R \\ u_L \\ d_L \end{array} \begin{array}{c} \begin{array}{cccc} u_R & d_R & u_L & d_L \end{array} \\ \begin{pmatrix} \frac{i}{2}\gamma({g^i}_k)\otimes\mathbf{1}_N & 0 & 0 & 0 \\ 0 & - & 0 & 0 \\ 0 & 0 & - & 0 \\ 0 & 0 & 0 & - \end{pmatrix} \end{array} \otimes {e^k}_j\ , \tag{E.46}$$

[33]Here a' and c^0 are $U(1)$-potentials with curvatures f', resp. g^0, c^a being a $SU(3)$-potential with curvature g^a, $a = 1,\ldots,8$.

$$= \begin{pmatrix} \frac{i}{2}\gamma(g^0)\otimes \mathbf{1}_N & 0 & 0 & 0 \\ 0 & - & 0 & 0 \\ 0 & 0 & - & 0 \\ 0 & 0 & 0 & - \end{pmatrix} \otimes \mathbf{1}_3 + \begin{pmatrix} \frac{i}{2}\gamma(g^a)\otimes \mathbf{1}_N & 0 & 0 & 0 \\ 0 & - & 0 & 0 \\ 0 & 0 & - & 0 \\ 0 & 0 & 0 & - \end{pmatrix} \otimes \frac{\lambda_a}{2}$$

where the second lines correspond to decompositions of 3×3 matrices as sums

$$m = (m^i_k) = m^0\mathbf{1} + m^a\frac{\lambda_a}{2} \ , \tag{E.47}$$

with the λ_a, $a = 1, \ldots, 8$ the $SU(3)$ Gell-Mann matrices. This alternative corresponds to the isolation of the $U(1)$-part of the group $U(3)$, necessary in order to remove two related unwanted features: a plethoral gauge group $U(1) \times SU(2) \times U(1) \times U(3)$ and an excess by two of the number of gauge fields. These two drawbacks can be removed by imposing a "*modularity condition*" (see [3], [4] and §6 for details) leading to the following coalescence of the gauge fields a, a' and c^0:[34]

$$\begin{cases} a' = a \\ c^0 = -\frac{1}{3}a \end{cases} \tag{E.48}$$

with, correspondingly:

$$\begin{cases} f' = f \\ g^0 = -\frac{1}{3}f \ . \end{cases} \tag{E.49}$$

Once this is done, calculation of the Yang-Mills action corresponding to the leptonic, resp. quark percentages $\alpha_l = (1-x)/2$ and $\alpha_q = (1-x)/2$ yields the following standard-model bosonic Lagrangian density

$$\begin{aligned} \mathcal{L}_B \ = \ & -\frac{1}{2}(1-x)\boldsymbol{N} g_{a\mu\nu} g^{a\mu\nu} - \frac{1}{3}(10-x)\boldsymbol{N} f_{\mu\nu} f^{\mu\nu} - \frac{1}{4}(2-x)\boldsymbol{N}\boldsymbol{h}^s{}_{\mu\nu}\boldsymbol{h}^{\mu\nu}_s \\ & +2L(\boldsymbol{D}\Phi_j)(\boldsymbol{D}\Phi^j) + K(\Phi_i\Phi^i - 1)^2 \ , \end{aligned} \tag{E.50}$$

where

$$\begin{cases} K = \frac{3}{2}F - 6\alpha_q Tr(\mu_u\mu_d) - \boldsymbol{N}^{-1}[2^{-1}(\alpha_1 + 3\alpha_q)^{-1} + (\alpha_l + 6\alpha_q)^{-1}]L^2 \\ F = Tr[\alpha_l\mu_e^2 + 3\alpha_q(\mu_d + \mu_u)^2] \\ L = Tr[\alpha_l\mu_e + 3\alpha_q(\mu_u + \mu_d)] \end{cases} \tag{E.51}$$

[34] Modular connections are couples (V, V') of an electroweak and a chromodynamic connection for which one makes the identifications (E.48), (E.49). The corresponding curvature is the sum "$\theta + \theta'$" with those identifications. We must in addition take account of the colour - tripling the quark multiplicity by making the change $\alpha_q \to 3\alpha_q$ in θ_q of (E.34) as well as in its leptonic reduction θ_l.

$$
\begin{cases} D\Phi^{\cdot} = \mathbf{d}\Phi + i\left(a - b^a \dfrac{\tau_a}{2}\right)\Phi^{\cdot} \\ \\ D\Phi. = \mathbf{d}\Phi. - i\Phi.\left(a - b^a \dfrac{\tau_a}{2}\right) \quad \text{i.e. } D\Phi_j = \overline{D\Phi^j}\,, \quad i = 1,2 \end{cases} \tag{E.52}
$$

now correspond to correct hypercharge assignements. These satisfactory covariant derivatives also appear in the fermionic part of the Lagrangian density (see E §7).

We conclude with two remarks:

— Caveat: this astonishing reinterpretation of the standard model in non-commutative geometry (embodying the "inner degrees of freedom" as features of a (mildly) non commutative space inseparable from the elementary particle structure) is for the moment *confined to the classical (Lagrangian) level.* Field quantization and renormalization studies still lie ahead.

— A preliminary calculation at tree level (without deep significance) exhibits for the most symmetric Ansatz $x = 0$ results with a grand unification flavour, and for lepton-dominance ($x = 1$) results resembling the realistic situation [7] (details in [5]). We reproduce this results borrowed from [7]:

x	-1	0	$\frac{1}{2}$	0.99	1
$(g_3/g_2)^2$	$\frac{3}{4}$	1	$\frac{3}{2}$	50.5	∞
$\sin^2\theta_W$	$\frac{9}{20}$	$\frac{3}{8}$	$\frac{9}{28}$	0.252	$\frac{1}{4}$
m_t/m_W	$\sqrt{3}$	2	$\sqrt{6}$	14.2	∞
m_H/m_W	2.65	3.14	3.96	24.5	∞

We note that the ratio m_H/m_t shows little variationfrom 1.53 to $\sqrt{3}$. The table suggests the following remarks:

(i) all tabulated functions are monotonic in x.

(ii) the value $x = 0$ seems to correspond to a situation of the "unification" type.

(iii) for the limit value $x = 1$, i.e. $\alpha_q = 0$, the Weinberg angle is near its experimental value, whilst strong interactions prevail. Indication of lepton dominance at experimental energies? Connected with confinement?

In fact the theory presented here is not the least constraining: the most general choice of coupling constant exhausting the degeneracy of the K-cycle $H_l \oplus H_q$, plainly restitutes the 17 constants of the usual standard model [4].

§5 Dual pair of quantum spaces In section B we defined the "non-commutative differential geometry" of a (cohomologically finite-dimensional) algebra by specifying a d^+-summable K-cycle, this procedure generalizing to the non-commutative frame the fact that usual differential geometry is coded in the Dirac operator. However, the mathematical object consisting of a couple of an algebra and a d^+-summable K-cycle does not yet fully deserve the name of "non-commutative riemannian manifold" (we propose below to call it a "riemannian quantum space"): indeed it still lacks the important feature of "Poincaré duality". Broadly speaking Poincaré duality for classical manifolds consists in the isomorphism of homology and cohomology. And it technically arises, in the appropriate KK-theoretic langage, via the presence of a module over the algebra of smooth functions. Now a module over an abelian algebra is a bimodule (in a non-agressive way!): Alain Connes realized[35] that the appropriate non-commutative generalization consists in considering a couple $(\mathbf{A}', \mathbf{A}'')$ of algebras and a $\mathbf{A}' - \mathbf{A}''$-bimodule, in other terms a $\mathbf{A}' \otimes \mathbf{A}''$-module, with conditions ensuring the equality of the homology of $\mathbf{A}'$ and the cohomology of $\mathbf{A}''$. It is a fascinating fact that the physical "duality" of the electroweak and chromodynamics sectors of elementary particle physics can be viewed as an example (indeed the historically first example!) of the non-commutative generalization of Poincaré duality.

We begin with definitions fixing terminology (whereby (i) just renames the lanscape of section B not yet deserving the name of quantum manifolds).

(i) A *riemannian quantum space* is a couple (A, H) of a unital *-algebra A and a K-cyle $H = (H, D, \chi)$ over A. The quantum space (A, H) is d-dimensional whenever the K-cyle H is d^+-summable.

(ii) A *dual riemannian quantum space* is a riemannian quantum space $(A \otimes A', H)$, where $A' \otimes A''$ is the algebraic tensor product of the unital*-algebras A', A'' (with respective units $\mathbb{1}'$ and $\mathbb{1}''$); and where $H = (H, D, \chi)$ is a d^+-summable K-cyle of $A' \otimes A''$ fullfilling the algebraic condition:

$$[[D, \underline{a' \otimes \mathbb{1}}],\ \underline{\mathbb{1} \otimes a''}] = 0 \quad ,\ a' \in A',\ a'' \in A'', \tag{E.53}$$

(iii) A d-dimensional *riemannian quantum manifold* is a dual riemannian quantum space $(A' \otimes A'', H)$ whose d^+-summable K-cyle $H = (H, D, \chi)$ has its "Hochschild obstruction" vanishing under the operator B, i.e.

$$Tr_\omega \left\{ \chi \underline{a}_0 [D, \underline{a}_1] \ldots [D, \underline{a}_k] |D|^{-k} \right\} = 0 \quad ,\ a_0, a_1, \ldots, a_k \in A. \tag{E.54}$$

(In order to alleviate notation, we shall consider $\mathbf{A}'$ and $\mathbf{A}''$ as acting on H by commuting representations $a' \to \underline{a'} = \underline{a' \otimes \mathbb{1}}$ and $a'' \to \underline{a''} = \underline{\mathbb{1} \otimes a''}$, then reading

$$[[D, \underline{a'}],\ \underline{a''}] = 0 \quad ,\ a' \in A',\ a'' \in A'')\,. \tag{E.55}$$

(iv) The quantum space (A, H), $H = (H, D, \chi)$, is *self-dual* whenever the operator algebra $\{\underline{a}, a \in A\}$ is commutative, and one has:

$$[[D, \underline{a}],\ \underline{b}] = 0 \quad ,\ a, b \in A. \tag{E.56}$$

[35] in fact whilst looking at the standard model!

The *gauge group of the dual riemannian quantum space* $A' \otimes A''$, H) is by definition the product $\mathcal{G}' \times \mathcal{G}''$ of the respective gauge group $\mathcal{G}'$, $\mathcal{G}''$ (= groups of unitaries of A', resp. A'').

Remarks:[36]

(a) With $(A' \otimes A'', H)$ a dual riemannian quantum space we have that:

$$[\pi_D(\Omega A'),\ \pi_D(A'')] = [\pi_D(\Omega A''),\ \pi_D(A')] = 0, \tag{E.57}$$

(b) There is a bijection between self-dual quantum spaces (A, H) and quantum manifolds $(A \otimes A, H)$.

(c) The couple of the algebra of smooth functions over a compact spin^c riemannian manifold, and its Dirac K-cycle is (the archetype of) a self-dual quantum space.

The main mathematical point of the definition (iii) of a d-dimensional riemannian quantum manifold is that it allows to construct maps between homology and cohomology. We do not discuss this point in these lectures devoted to physics and refer to [3a] for it . For our physical applications we shall in fact only use the structure (ii) of what we call a dual riemannian quantum space. We shall be concerned with two items: tensor products on the one hand, (bi)connections on the other.

Tensor products of dual riemannian quantum spaces. We prove the following result showing that the (algebraic condition of) Poincaré duality met in Section E § 6 boils down to Poincaré duality for the inner space:

The (tensor) product of two dual riemannian quantum spaces is a dual riemannian quantum space. With $(A_1 = A_1' \otimes A_1'', \mathbf{H}_1)$, $\mathbf{H}_1 = (H_1, D_1, \chi_1)$, $(A_2 = A_2' \otimes A_2'', \mathbf{H}_2)\mathbf{H}_2 = (H_2, D_2, \chi_2)$, dual riemannian quantum spaces, we consider the riemannian quantum space $(A_1 \otimes A_2, H_1 \otimes H_2)$ tensor product of $(A_1,\ H_1)$ and $(A_2,\ H_2)$ as we defined it in E § 1 (cf.(E.1)). With $a_1' \in A_1'$, $a_1'' \in A_1''$, $a_2' \in A_2'$, $a_2'' \in A_2''$, we want to check that one has:

$$\left[\left[D, \underline{a_1'} \otimes \underline{a_2'}\right],\ \underline{a_1''} \otimes \underline{a_2''}\right] = 0 \tag{E.58}$$

We have, since $\underline{a_1'}$ is even:

$$(D_1 \otimes id + \chi_1 \otimes D_2)\left(\underline{a_1'} \otimes \underline{a_2'}\right) = D_1\underline{a_1'} \otimes \underline{a_2'} + \chi_1\underline{a_1'} \otimes D_2\underline{a_2'}, \tag{E.59}$$

$$\left(\underline{a_1'} \otimes \underline{a_2'}\right)(D_1 \otimes id + \chi_1 \otimes D_2) = \underline{a_1'}D_1 \otimes \underline{a_2'} + \chi_1\underline{a_1'} \otimes \underline{a_2'}D_2, \tag{E.60}$$

hence

$$\left[D, \underline{a_1'} \otimes \underline{a_2'}\right] = \left[D_1, \underline{a_1'}\right] \otimes \underline{a_2'} + \chi_1\underline{a_1'} \otimes \left[D_2, \underline{a_2'}\right], \tag{E.61}$$

and further

$$\left[D, \underline{a_1'} \otimes \underline{a_2'}\right]\left(\underline{a_1''} \otimes \underline{a_2''}\right) = \left[D_1, \underline{a_1'}\right]\underline{a_1''} \otimes \underline{a_2'}\underline{a_2''} + \chi_1\underline{a_1'}\underline{a_1''} \otimes \left[D_2, \underline{a_2'}\right]\underline{a_2''}, \tag{E.62}$$

[36] Note that condition (E.55) is symmetric in $'$ and $''$.

$$\left(\underline{a_1''} \otimes \underline{a_2''}\right)\left[D, \underline{a_1'} \otimes \underline{a_2'}\right] = \underline{a_1''}\left[D_1, \underline{a_1'}\right] \otimes \underline{a_2''}\,\underline{a_2'} + \chi_1 \underline{a_1''}\,\underline{a_1'} \otimes \underline{a_2''}\left[D_2, \underline{a_2'}\right], \tag{E.63}$$

hence, since $\underline{a_1'}$ and $\underline{a_1''}$ commute:

$$\left[\left[D, \underline{a_1'} \otimes \underline{a_2'}\right],\ \underline{a_1''} \otimes \underline{a_2''}\right] = \left[\left[D_1, \underline{a_1'}\right],\ \underline{a_1''}\right] \otimes \underline{a_2'}\,\underline{a_2''} + \chi_1 \underline{a_1'}\,\underline{a_1''} \otimes \left[\left[D_2, \underline{a_2'}\right],\ \underline{a_2''}\right], \tag{E.64}$$

vanishing by the duality property of the systems 1 and 2.

Biconnections. Our problem is the definition of connections of dual riemannian quantum spaces. Indeed the passage from riemannian quantum spaces involving one algebra to dual quantum spaces involving a commuting[37] couple of algebras (A', A'') (i.e. their tensor product $A = A' \otimes A''$ together with its tensorial splitting) raises questions as to the maintenance in this enlarged[38] frame of the previous philosophy pertaining to riemannian quantum spaces (and of the latter's implications on connections, the quantum Yang-Mills algorithm, etc.). Taking the tensor product $A = A' \otimes A''$ as the new "basic" algebra, one is at first tempted to consider the differential envelope $(\Omega A, \delta)$ as the "basic" differential algebra. However the latter does not contain the information of the commutativity of A' and A'' and of the vanishing of commutators (E.58) in the representation π_D. This information is incorporated in each of the following items:

(1) Definition. Let $(A, \mathbf{H})$, $A = A' \otimes A''$, $\mathbf{H} = (H, D, \chi)$, be a dual riemannian quantum space, with $(\Omega A, \delta)$ the unital differential envelope of A; and consider the following subsets of ΩA^1:[39]

$$\mathbf{S}' = \left\{[\delta(a' \otimes \mathbb{1}''),\ \mathbb{1}' \otimes a''] \in \Omega A^1; a' \in A', a'' \in A''\right\} = -\mathbf{S}'' \tag{E.65}$$

resp.

$$\mathbf{S}'' = \left\{[\delta(\mathbb{1}' \otimes a''),\ a' \otimes \mathbb{1}''] \in \Omega A^1; a' \in A', a'' \in A''\right\}. \tag{E.66}$$

We denote by $\mathbf{j}_P$ the ideal of ΩA generated by $\mathbf{S}'$ (or, for that matter, by $\mathbf{S}''$), and define the *Poincaré ideal* as:

$$\mathbf{k}_P = \mathbf{j}_P + \delta \mathbf{j}_P. \tag{E.67}$$

(2) Definition. Let A' and A'' be unital (real or complex) *-algebra, with respective units $\mathbb{1}'$ and $\mathbb{1}''$, and with respective unital differential envelopes $(\Omega A', \delta')$ and $(\Omega A'', \delta'')$. Let $A = A' \otimes A''$, with unit $\mathbb{1}$ and unital differential envelope $(\Omega A, \delta)$, be the algebraic tensor-product of A' and A''. Set $\Omega A_\otimes = \Omega A' \widehat{\otimes} \Omega A''$, with differential $\delta_\otimes$, for the skew tensor product of $\Omega A'$ and $\Omega A''$ defined as follows: we have, for $\omega', \psi' \in A', \omega'', \psi'' \in A''$:

[37] in the representation π_D.

[38] enlarged in view of Remark (c).

[39] Note that $\mathbf{S}' = -\mathbf{S}''$ because $\delta\,[\mathbb{1}' \otimes a'', a' \otimes \mathbb{1}''] = 0 = [\delta(\mathbb{1}' \otimes a''),\ a' \otimes \mathbb{1}''] + [\mathbb{1}' \otimes a'', \delta(a' \otimes \mathbb{1}'')]$.

$$\begin{cases} \Omega A^n_{\otimes} = \sum_{p+q=n} \Omega A'^p \otimes \Omega A''^q, \\ (\omega' \otimes \omega'')(\psi' \otimes \psi'') = \omega'\psi' \otimes \omega''\psi'' \\ \delta_{\otimes}(\omega' \otimes \omega'') = \delta'\omega' \otimes \omega'' + \omega' \otimes \delta\omega'' \end{cases} . \tag{E.68}$$

We now have the following consequences of Definition (1):

(i): $\mathbf{j}_P$ is a graded ideal of ΩA.

(ii): $\mathbf{k}_P$ is a graded differential ideal of ΩA, with $\mathbf{k}^n_P = \mathbf{j}^n_P + \delta \mathbf{j}^{n-1}_P$, hence $\Omega A/\mathbf{k}_P$ is a $\mathbf{N}$-graded differential algebra, with differential d obtained from that of ΩA by passage to the quotient through the ideal $\mathbf{k}_P$.

(iii): The quantum DeRham complex ΩA_D is a quotient of $\Omega A/\mathbf{k}_P$ as a $\mathbf{N}$-graded differential algebra.

(iv): One has $[\delta(a' \otimes 1\!\!1''), \delta(1\!\!1' \otimes a'')] \in \mathbf{k}^2_P,\ a' \in A', a'' \in A''$.

(v): The ideal $\mathbf{k}_P$ is generated by the elements $[\delta(a' \otimes 1\!\!1''), 1\!\!1' \otimes a'']$ and $[\delta(a' \otimes 1\!\!1''), \delta(1\!\!1' \otimes a'')]\,,\ a' \in A', a'' \in A''$.

and the following consequence of Definition (2):
The map $\bar{i} : \Omega A \to \Omega A_{\otimes}$ specified as:

$$\begin{aligned} &\bar{i}\left[(a'_0\widehat{\otimes}a''_0)\delta(a'_1\widehat{\otimes}a''_1)\ldots\delta(a'_n\widehat{\otimes}a''_n)\right] = (a'_0\widehat{\otimes}a''_0)\delta_{\otimes}(a'_1\widehat{\otimes}a''_1)\ldots\delta_{\otimes}(a'_n\widehat{\otimes}a''_n) \\ &= (a'_0\widehat{\otimes}a''_0)\left[\delta'a'_1\widehat{\otimes}a''_1 + a'_1\widehat{\otimes}\delta''a''_1\right]\ldots\left[\delta'a'_n\widehat{\otimes}a''_n + a'_n\widehat{\otimes}\delta''a''_n\right], \end{aligned} \tag{E.69}$$

is a homomorphism of $\mathbf{N}$-graded differential algebras.

Furthermore we have the important fact that *the kernel of $\bar{i}$ coincides with the Poincaré ideal $\mathbf{k}_P$. Consequently we have the following isomorphism of $\mathbf{N}$-graded differential algebras:*

$$(\Omega\mathbf{A}/\mathbf{k},\ \delta) \cong (\Omega\mathbf{A}_{\otimes},\ \delta_{\otimes}). \tag{E.70}$$

and the canonical map $\psi : \Omega\mathbf{A} \to \Omega\mathbf{A}_D$ factors through the canonical map $\phi : \Omega\mathbf{A} \to \Omega\mathbf{A}_{\otimes}$ as $\psi = \xi \circ \phi$, where $\xi : \Omega\mathbf{A}_{\otimes} \to \Omega\mathbf{A}_D$ is a homomorphism of $\mathbf{N}$-graded differential algebras.

We are now in a position to discuss the issue of "connections" of the dual riemannian quantum space $(A = A' \otimes \mathbf{A}'', \mathbf{H}), \mathbf{H} = (H, D, \chi)$. The quantum forms of $\mathbf{A}$, if we forget about its tensorial decomposition $\mathbf{A} = \mathbf{A}' \otimes \mathbf{A}''$, are the elements of the quantum DeRham complex $(\Omega\mathbf{A}_D, \mathbf{d})$, its connections are thus the grade-one $\mathbf{d}$-derivations ∇ of $\Omega\mathbf{A}_D$. Now, utilizing the homomorphism $\xi : \Omega\mathbf{A}_{\otimes} \to \Omega\mathbf{A}_D$ (cf.[6]), the latter can be considered as ξ-images of δ-derivations of $\Omega\mathbf{A}_{\otimes}$ ("preconnexions"), this according to the scheme:
Let $\phi : (\Omega, \delta) \to (\widetilde{\Omega}, \widetilde{\delta})$ be an epimorphism of $\mathbf{N}$-graded differential algebras, and denote by $Der_{\delta}\Omega$ the set of grade-one δ-derivations of Ω considered as a right Ω-module (resp.

by $Der_{\widetilde{\delta}}\widetilde{\Omega}$ the set of $\widetilde{\delta}$-derivations of $\widetilde{\Omega}$ considered as a right $\widetilde{\Omega}$-module). Then, given $\widetilde{\nabla} \in Der_{\widetilde{\delta}}\widetilde{\Omega}$, there is $\nabla \in Der_{\delta}\Omega$ such that ϕ intertwines ∇ and $\widetilde{\nabla} : \phi \circ \nabla = \widetilde{\nabla} \circ \phi$. Specifically, for $\widetilde{\nabla} = \widetilde{\delta} + \widetilde{\rho}, \rho' \in \widetilde{\Omega}^1$, one has $\nabla = \delta + \rho$, picking $\rho \in \Omega^1$ such that $\phi\rho = \widetilde{\rho}$.[40] The respective curvatures θ and $\widetilde{\theta}$ of ∇ and $\widetilde{\nabla}$ are related by $\widetilde{\theta} = \phi\theta$.

Our problem was to select, amongst all connexions of A, the ones matching the tensorial decomposition $A = A' \otimes A''$. Looking at the preconnections acting on $\Omega A_{\otimes} = \Omega A' \widehat{\otimes} \Omega A''$, one is now naturally led to the following notion of "biconnection".

Definitions. Let $(A = A' \otimes A'', \mathbf{H})$, with $\mathbf{H} = (H, D, \chi)$, be a dual riemannian quantum space, and consider the above $\mathbf{N}$-graded differential algebras $(\Omega\mathbf{A}_{\otimes}, \delta)$ and $(\Omega\mathbf{A}_D, \mathbf{d})$.

(i): a *preconnection of* $(\mathbf{A}, \mathbf{H})$ is a grade-one graded δ-derivation ∇ of the right $\Omega\mathbf{A}_{\otimes}$-module $\Omega\mathbf{A}_{\otimes}$, with *curvature* is $\theta = \nabla^2$. The *preconnection-one form* $\rho \in \Omega\mathbf{A}^1_{\otimes}$ arises by asking $\nabla = \delta + \rho$ (one has thus $\theta = \delta\rho + \rho^2$). The action $\nabla \to \nabla^{\mathbf{u}}$ of the element $\mathbf{u}$ of the gauge group $\mathcal{G}$ of $\mathbf{A}$ on the preconnection is by definition:[41]

$$\nabla^{\mathbf{u}}\omega = \mathbf{u}\nabla(\mathbf{u}^{\star}\omega) \tag{E.71}$$

(one has then $\nabla^{\mathbf{u}} = \delta + \rho^{\mathbf{u}}$ with

$$\rho^{\mathbf{u}} = \mathbf{u}\rho\mathbf{u}^{\star} + \mathbf{u}\delta\mathbf{u}^{\star}). \tag{E.72}$$

(ii): a *biconnection* is a preconnection of the form $\nabla = \nabla' \widehat{\otimes} id + id \widehat{\otimes} \nabla''$, where ∇' and ∇'' are respective connections of $\mathbf{A}'$ and $\mathbf{A}''$.[42] (one has thus $\rho = \rho' \otimes \mathbb{1}'' + \mathbb{1}' \otimes \rho''$ for the preconnection-one form ρ of ∇)

(iii): a *connection of the dual riemannian quantum space* $(\mathbf{A} = \mathbf{A}' \otimes \mathbf{A}'', \mathbf{H})$ is a grade-one graded δ-derivation ∇ of the right $\Omega\mathbf{A}_D$-module $\Omega\mathbf{A}_D$ which is the image of a biconnection in the homomorphism $\xi : \Omega\mathbf{A}_{\otimes} \to \Omega\mathbf{A}_D$.

We check that $\nabla = \nabla' \widehat{\otimes} id + id \widehat{\otimes} \nabla''$ is a preconnection with preconnection-one form $\rho = \rho' \otimes \mathbb{1}'' + \mathbb{1}' \otimes \rho''$: for homogeneous $\omega', \psi' \in \Omega\mathbf{A}', \omega'', \psi'' \in \Omega\mathbf{A}''$, we have:

$$\begin{aligned} \nabla(\omega' \widehat{\otimes} \omega'') &= \nabla'\omega' \widehat{\otimes} \omega'' + (-1)^{\partial\omega'} \omega' \widehat{\otimes} \nabla''\omega'' \\ &= (\delta' + \rho')\omega' \widehat{\otimes} \omega'' + (-1)^{\partial\omega'} \omega' \widehat{\otimes} (\delta'' + \rho'')\omega'' \\ &= \delta(\omega' \widehat{\otimes} \omega'') + (\rho' \otimes \mathbb{1}'' + \mathbb{1}' \otimes \rho'')(\omega' \widehat{\otimes} \omega''). \end{aligned} \tag{E.73}$$

We then have the following satisfying set of facts:

Let $\nabla = \nabla' \widehat{\otimes} id + id \widehat{\otimes} \nabla'' = \delta + \rho$, $\rho = \rho' \widehat{\otimes} \mathbb{1}'' + \mathbb{1}' \widehat{\otimes} \rho''$, *be a biconnection of* $(A = A' \otimes A'', \mathbf{H})$. *Then*

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[40] In the expression $\nabla = \delta + \rho$ the symbol ρ has to be interpreted as usual as denoting multiplication from the left by ρ.

[41] $\mathcal{G}$ is the group of unitaries of $\mathbf{A} = \Omega\mathbf{A}^0_{\otimes}$.

[42] Thus ∇' is a grade-one δ'-derivation of $\Omega\mathbf{A}'$, with $\nabla' = \delta' + \rho', \rho' \in \Omega\mathbf{A}'^1_{\otimes}$, and ∇'' is a grade-one δ''-derivation of $\Omega\mathbf{A}''$ with $\nabla'' = \delta'' + \rho'', \rho'' \in \Omega\mathbf{A}''^1_{\otimes}$.

(i): *The curvature θ of ∇ is the sum of the curvatures θ' and θ'' of ρ' and ρ'' in the following sense: one has:*

$$\nabla^2 = \nabla'^2 \widehat{\otimes} id + id \widehat{\otimes} \nabla''^2, \tag{E.74}$$

$$\theta = \theta' \widehat{\otimes} \mathbb{1}'' + \mathbb{1}' \widehat{\otimes} \theta'', \tag{E.75}$$

$$\rho^2 = \rho'^2 \widehat{\otimes} \mathbb{1}'' + \mathbb{1}' \widehat{\otimes} \rho''^2. \tag{E.76}$$

(ii): *The set of biconnections is stable under the gauge group $\mathcal{G}' \times \mathcal{G}''$ of the dual riemannian quantum space ($A = A' \otimes A'', \mathbf{H}$) one has, for $\mathbf{u}' \in \mathcal{G}', \mathbf{u}'' \in \mathcal{G}'', \mathbf{u} = \mathbf{u}' \widehat{\otimes} \mathbf{u}''$:*

$$\nabla^{\mathbf{u}} = \nabla'^{\mathbf{u}'} \widehat{\otimes} id + id \widehat{\otimes} \nabla''^{\mathbf{u}''}, \tag{E.77}$$

$$\rho^{\mathbf{u}} = \rho'^{\mathbf{u}'} \widehat{\otimes} \mathbb{1}'' + \mathbb{1}' \widehat{\otimes} \rho''^{\mathbf{u}''}. \tag{E.78}$$

§6. The modularity condition We recall that, putting together in Section E §4 the electroweak and chromodynamics sectors, we had the following commuting representations of the electroweak algebra $\mathcal{A}$:

$$\pi_l((f,q)) = \begin{array}{c} \quad e_R \qquad \nu_L e_L \\ \begin{pmatrix} \underline{f} \otimes \mathbf{1}_N & 0 \\ 0 & \underline{q} \otimes \mathbf{1}_N \end{pmatrix} \end{array} \begin{array}{l} e_R \\ \nu_L e_L \end{array} \quad , \ (f,q) \in \mathcal{A}, \tag{E.79}$$

$$\pi_q((f,q)) = \begin{array}{c} \quad u_R \qquad d_R \qquad u_L d_L \\ \begin{pmatrix} \underline{\overline{f}} \otimes \mathbf{1}_N & 0 & 0 \\ 0 & \underline{f} \otimes \mathbf{1}_N & 0 \\ 0 & 0 & \underline{q} \otimes \mathbf{1} \end{pmatrix} \end{array} \begin{array}{l} u_R \\ d_R \\ u_L \\ d_L \end{array} \quad , \ (f,q) \in \mathcal{A}, \tag{E.80}$$

resp. the chromodynamics algebra $\mathcal{B}$:

$$\pi_l((f',m)) = \pi_l((f',0)) = \begin{array}{c} \quad e_R \qquad \nu_L e_L \\ \begin{pmatrix} \underline{f'} \otimes \mathbf{1}_N & 0 \\ 0 & \underline{f'} \otimes \mathbf{1}_N \end{pmatrix} \end{array} \begin{array}{l} e_R \\ \nu_L \\ e_L \end{array} \quad , \ (f',m) \in \mathcal{B}, \tag{E.81}$$

$$\pi_q((f',m)) = \pi_q((0,m)) = \begin{array}{c} \begin{array}{ccc} \quad u_R & \quad d_R & \quad u_L d_L \end{array} \\ \begin{pmatrix} m^j_k \otimes \mathbf{1}_N & 0 & 0 \\ 0 & m^j_k \otimes \mathbf{1}_N & 0 \\ 0 & 0 & m^j_k \otimes \mathbf{1}_N \end{pmatrix} \end{array} \otimes e^k_j \begin{array}{c} u_R \\ d_R \\ u_L \\ d_L \end{array} \quad , \; (f',m) \in \mathcal{B}, \tag{E.82}$$

with the corresponding respective representations of their gauge groups $\mathcal{G}' = U(1) \times SU(2) = \{(u,v); u \in C^\infty(\mathbf{M}, U(1)),\ v \in C^\infty(\mathbf{M}, SU(2))\}$:

$$\pi_l((u,v)) = \begin{array}{c} \begin{array}{cc} \quad e_R & \quad \nu_L e_L \end{array} \\ \begin{pmatrix} \underline{u} \otimes \mathbf{1}_N & 0 \\ 0 & \underline{q} \otimes \mathbf{1}_N \end{pmatrix} \end{array} \begin{array}{c} e_R \\ \nu_L \\ e_L \end{array} \quad , \; (u,v) \in \mathcal{G}', \tag{E.83}$$

$$\pi_q((u,v)) = \begin{array}{c} \begin{array}{ccc} \quad u_R & \quad d_R & \quad u_L d_L \end{array} \\ \begin{pmatrix} \overline{\underline{u}} \otimes \mathbf{1}_N & 0 & 0 \\ 0 & \underline{u} \otimes \mathbf{1}_N & 0 \\ 0 & 0 & \underline{q} \otimes \mathbf{1}_N \end{pmatrix} \end{array} \otimes \mathbf{1}_3 \begin{array}{c} u_R \\ d_R \\ u_L \\ d_L \end{array} \quad , \; (u,v) \in \mathcal{G}', \tag{E.84}$$

respectively of $\mathcal{G}'' = U(1) \times U(1) \times SU(3) = \{(u',u'',v'); u',u'' \in C^\infty(\mathbf{M}, U(1)), v' \in C^\infty(\mathbf{M}, SU(3))\}$:

$$\pi_l((u',u'',v')) = \pi_l((u',0)) = \begin{array}{c} \begin{array}{cc} \quad e_R & \quad \nu_L e_L \end{array} \\ \begin{pmatrix} \underline{u'} \otimes \mathbf{1}_N & 0 \\ 0 & \underline{u'} \otimes \mathbf{1}_N \end{pmatrix} \end{array} \begin{array}{c} e_R \\ \nu_L \\ e_L \end{array} \quad , \; (u',u'',v') \in \mathcal{G}'', \tag{E.85}$$

$$\pi_q((u',u'',v')) = \pi_q((0,u'',v')) = \begin{array}{c} \begin{array}{ccc} \quad u_R & \quad d_R & \quad u_L d_L \end{array} \\ \begin{pmatrix} \underline{u''} \otimes \mathbf{1}_N & 0 & 0 \\ 0 & \underline{u''} \otimes \mathbf{1}_N & 0 \\ 0 & 0 & \underline{u'} \otimes \mathbf{1}_N \end{pmatrix} \end{array} \otimes \underline{v'} \begin{array}{c} u_R \\ d_R \\ u_L \\ d_L \end{array} \quad , \; (u',u'',v') \in \mathcal{G}'', \tag{E.86}$$

We have at this point the plethoral inner symmetry group $U(1) \times SU(2) \times U(1) \times U(1) \times SU(3)$, which phenomenology commands us to reduce to the usual $U(1) \times SU(2) \times SU(3)$: we must thus coalesce the threefold $U(1) \times U(1) \times U(1)$ to a single $U(1)$, a task which will be achieved by imposing the "modularity condition" below. Before describing the latter, let us however derive the way in which the coalescence should arise: we can read this off from the table:

	Leptons			**Quarks**			
	e_R	ν_L	e_L	u_R	d_R	u_L	d_L
Y	-2	-1	-1	$4/3$	$-2/3$	$1/3$	$1/3$
U	-1	0	0	1	-1	0	0
U'	-1	-1	-1	0	0	0	0
U''	0	0	0	-1	-1	-1	-1

(E.87)

where we plotted the hypercharge Y and the infinitesimal generators U, U', U'' defined by $u = e^{iUt}$, $u' = e^{iU't}$, $u'' = e^{iU''t}$: we see that we need:

$$\begin{cases} U = U' = -Y \\ U'' = -\frac{1}{3}Y \end{cases} \quad \text{i.e.} \quad \begin{cases} u = u' = e^{-iYt} \\ u'' = e^{-\frac{i}{3}Yt} \end{cases} . \tag{E.88}$$

We now discuss the modularity condition. For the latter we shall use the following definition of the **phase $\phi(u)$ of the determinant** of a unitary u belonging to a unital $C^\star$-algebra $\mathcal{C}$ endowed with a hermitean normalized trace τ:[43]

$$\phi(u) = \frac{1}{2\pi i}\int_0^1 \tau\left[u(t)'u(t)^{-1}\right] dt \tag{E.89}$$
$(t \to u(t)$ a continuous path of unitaries from 1 to $u)$.

This formula defines $\phi(u)$ for u in the connected component $U_0(\mathcal{C})$ of the group of unitaries $U(\mathcal{C})$ of $\mathcal{C}$ (in fact up to the image $\tau(K_0(\mathcal{C}))$ of the K_0-group of $\mathcal{C}$ under the trace τ - a countable subgroup of $\mathbf{R}$). By the unitarity invariance of the trace τ, one has, for all $s \in U(\mathcal{C}), \phi(\text{sus}^{-1}) = \phi(u)$, hence $\{u \in U_0(\mathcal{A}); \phi(u) = 1\}$ is a normal subgroup of $U_0(\mathcal{C})$. We set:

$$U_\tau(\mathcal{C}) = \text{ component of the unit of} \{u \in U_0(\mathcal{C}); \phi(u) = 1\}. \tag{E.90}$$

The name we gave to $\phi(u)$ is heuristically justified as follows from the familiar formula giving the determinant of the exponential of a self-adjoint h acting on the Hilbert space $\mathbf{C}^n$ in terms of $\tau = \frac{1}{n}Tr$:

$$Det\ e^{2\pi i h} = e^{2\pi i \tau(h)}. \tag{E.91}$$

[43] i.e. $\tau(a^\star) = \overline{\tau(a')}$ and $\tau(\mathbf{1}) = 1$.

With $u = e^{2\pi i h}$, we have indeed $h = \frac{1}{2\pi i} lnu$, hence $\phi(u) = \tau(h) = \frac{1}{2\pi i}\tau(lnu)$, leading to (E.89).[44]

We now apply the notion (E.89) to the situation of the full standard model, with $\mathcal{C} = \Omega_D(\mathcal{A} \otimes \mathcal{B})$, and the following family of traces of $\mathcal{C}$.

$$\tau_\rho(a) = Tr_\omega\{D^{-4}\rho x\} \qquad\qquad ,x \in \mathcal{C}, \tag{E.92}$$

where ρ ranges through the self-adjoint elements of the center of $\mathcal{A}$. Setting:

$$U_{\mathcal{A}}(\mathcal{C}) = \bigcap_\rho U_{\tau_\rho}(\mathcal{C}), \tag{E.93}$$

we get a subgroup of the unitarity group of $\Omega_D(\mathcal{A} \otimes \mathcal{B})$, i.e. a subgroup of the gauge group $\mathcal{G}' \times \mathcal{G}''$. Now *this subgroup turns out to be the group* $U(1) \times SU(2) \times SU(3)$ *obtained through the mechanism* (E.88) *necessary to match the hypercharge.*

§7. The fermionic action By definition the fermionic action equals:[45]

$$\frac{1}{8\pi^2}\mathbf{L}_F dv \tag{E.94}$$

with the fermionic Lagrangian:

$$\mathbf{L}_F = \mathbf{L}_{Fl} + \mathbf{L}_{Fq} = (\Psi, \mathbf{D}_{\Delta l}\Psi) + (\mathbf{Q}, \mathbf{D}_{\Delta q}\mathbf{Q}), \tag{E.95}$$

specified as follows:

— the leptonic field $\Psi \in \mathbf{H}_l = H_l$, is

$$\Psi = \Psi = \left(\Psi^R, \Psi^L\right) \text{ with } \begin{cases} \Psi^R = \left(\Psi^R_f\right)_{f=1,\dots,N} \in H^R_l \otimes \mathbf{C}^N \\ \\ \Psi^L = \left(\Psi^{L1}_f, \Psi^{L2}_f\right)_{f=1,\dots,N} \in H^L_l \otimes \mathbf{C}^2_{iso} \otimes \mathbf{C}^N \end{cases} ; \tag{E.96}$$

whilst the operator $\mathbf{D}_{\Delta l}$ of $\mathbf{H}_l = H_l$ is given by:

$$\mathbf{D}_{\Delta l} = \mathbf{D} + i\left(V_l \overset{\approx}{+} V'_l\right), \tag{E.97}$$

— the quark field $\mathbf{Q} \in \mathbf{H}_q = H_q \otimes \mathbf{C}^3_{color}$, is:

$$\mathbf{Q} = \left(\mathbf{Q}^R,\ \mathbf{Q}^L\right) = \left(\mathbf{Q}^{Ru}_f,\ \mathbf{Q}^{Rd}_f,\ \mathbf{Q}^{L1}_f,\ \mathbf{Q}^{L2}_f\right)_{f=1,\dots,N}, \tag{E.98}$$

[44] For a rigourous exposition of (E.89) we refer to the paper by P. De la Harpe and G. Skandalis, Ann. Inst. Fourier (1986).
[45] F in $\mathbf{L}_F$ stands for fermionic.

with
$$\begin{cases} \mathbf{Q}_f^{Ru} = \mathbf{Q}_f^{Rum} \otimes e_m \\ \\ \mathbf{Q}_f^{Rd} = \mathbf{Q}_f^{Rdm} \otimes e_m \\ \\ \mathbf{Q}_f^{L1} = \mathbf{Q}_f^{L1m} \otimes e_m \\ \\ \mathbf{Q}_f^{L2} = \mathbf{Q}_f^{L2m} \otimes e_m \end{cases} \text{(summation on } m \text{ from 1 to 3)}, \tag{E.99}$$
with $\{e_m\}$ the canonical basis of $\mathbf{C}^3_{color}$; whilst the operator $\mathbf{D}_{\Delta q}$ of $\mathbf{H}_q = H_q \otimes \mathbf{C}^3_{color}$ is given by:
$$\mathbf{D}_{\Delta q} = \mathbf{D} + i\left(V_q \overset{\approx}{+} V'_q\right) = D_{\Delta q} \otimes \mathbb{1}_3 + iV'_q. \tag{E.100}$$

The letter f denotes the fermion-family index, R and L stand respectively for right and left, and the spinor fields correspond to the following particle-types:
$$\begin{cases} \text{Leptons:} \quad \Psi^R : e_R \qquad \qquad \Psi_f^{L1} : \nu_L \quad \Psi_f^{L2} : e_R \\ \\ \text{Quarks:} \quad \mathbf{Q}_f^{Ru} : u_R \quad \mathbf{Q}_f^{Rd} : d_L \quad \mathbf{Q}_f^{L1} : u_L \quad \mathbf{Q}_f^{L2} : d_L \end{cases}. \tag{E.101}$$

One then has the fermionic Lagrangian $\mathbf{L}_F = \mathbf{L}_{Fl} + \mathbf{L}_{Fq}$, *with the leptonic term:*
$$\begin{aligned} \mathbf{L}_{Fl} = \quad & \Sigma_{f=1,\dots,N}\left\{\overline{\Psi}_f^R i\gamma^\mu D_\mu^{Rl}\Psi_f^R + \overline{\Psi}_f^L i\gamma^\mu D_\mu^{Rl}\Psi_f^L\right\} \\ & + \Sigma_{f_1 f_2=1,\dots,N}\left\{M^\star_{ef_1f_2}\overline{\Psi}_{f_1}^R\overline{\Phi}\gamma^5\Psi_{f_2}^L + M_{ef_1f_2}\overline{\Psi}_{f_1}^L\Phi\gamma^5\Psi^R\right\} \end{aligned} \tag{E.102}$$
with the covariant derivatives
$$\begin{cases} D_\mu^{Rl} = \nabla_\mu - 2i\mathbf{a}_\mu \\ \\ D_\mu^{Ll} = \nabla_\mu - i\mathbf{a}_\mu - i\mathbf{b}_\mu = \nabla_\mu - ia_\mu - i\mathbf{b}_\mu^a\frac{\tau_a}{2} \end{cases}, \tag{E.103}$$
and the quark term:
$$\begin{aligned} \mathbf{L}_{Fq} = \quad & \Sigma_{f=1,\dots,N}\left\{\overline{\mathbf{Q}}_{fm}^{Ru} i\gamma^\mu D_\mu^{Ru}\mathbf{Q}_f^{Rum} + \overline{\mathbf{Q}}_{fm}^{Rd} i\gamma^\mu D_\mu^{Rd}\mathbf{Q}_f^{Rdm}\right. \\ & \qquad \qquad \left. + \overline{\mathbf{Q}}_{fm}^{L} i\gamma^\mu D_\mu^{Lq}\mathbf{Q}_f^{Lm}\right\} \\ & + \Sigma_{f_1 f_2=1,\dots,N}\left\{M^\star_{df_1f_2}\overline{\mathbf{Q}}_{f_1m}^{Rd}\overline{\Phi}\gamma^5\mathbf{Q}_{f_2}^{Lm} + M_{ef_1f_2}\mathbf{Q}_{f_1m}^{L}\Phi\gamma^5\mathbf{Q}_{f_2}^{Rdm}\right\} \\ & + \Sigma_{f_1 f_2=1,\dots,N}\left\{M^\star_{uf_1f_2}\overline{\mathbf{Q}}_{f_1m}^{Ru}\overline{\widetilde{\Phi}}\gamma^5\mathbf{Q}_{f_2}^{Lm} + M_{uf_1f_2}\overline{\mathbf{Q}}_{f_1m}^{L}\widetilde{\Phi}\gamma^5\mathbf{Q}_{f_2}^{Rum}\right\} \end{aligned} \tag{E.104}$$
with the covariant derivatives
$$\begin{cases} D_\mu^{Ru} = \nabla_\mu + \frac{4}{3}i\mathbf{a}_\mu - i\mathbf{c}_\mu = \nabla_\mu + \frac{4}{3}i\mathbf{a}_\mu - i\mathbf{c}_\mu^a\frac{\lambda_a}{2} \\ \\ D_\mu^{Rd} = \nabla_\mu - \frac{2}{3}i\mathbf{a}_\mu - i\mathbf{c}_\mu = \nabla_\mu - \frac{2}{3}i\mathbf{a}_\mu - i\mathbf{c}_\mu^a\frac{\lambda_a}{2} \\ \\ D^{Lq} = \nabla_\mu - i\mathbf{b}_\mu + \frac{1}{3}i\mathbf{a}_\mu - i\mathbf{c}_\mu = \nabla_\mu - i\mathbf{b}_\mu^i\frac{\tau_i}{2} + \frac{1}{3}i\mathbf{a}_\mu - i\mathbf{c}_\mu^a\frac{\lambda_a}{2} \end{cases}. \tag{E.105}$$

Here both sides of these formulae refer to space-time fields dependent on a (non specified) space-time point $x \in M$. With $\mathbf{C}^4_{spin}$, $\mathbf{C}^2_{iso}$, and $\mathbf{C}^3_{color}$ the respective spaces of $O(4)$-spinors, $SU(2)$-vectors, and $SU(3)$-vectors, we have $\Psi^R \in \mathbf{C}^4_{spin}$, $\Psi^L \in \mathbf{C}^4_{spin} \otimes \mathbf{C}^2_{iso}$, $\mathbf{Q}^{Rd}$, $\mathbf{Q}^{Ru} \in \mathbf{C}^4_{spin} \otimes \mathbf{C}^3_{color}$, $\mathbf{Q}^L \in \mathbf{C}^4_{spin} \otimes \mathbf{C}^2_{iso} \otimes \mathbf{C}^3_{color}$, $\Phi, \widetilde{\Phi} \in \mathbf{C}^2_{iso}$. Correspondingly, symbols like $\overline{\Psi}^R \Psi^R$ (resp. $\overline{\Psi}^L \Psi^R$, $\overline{\mathbf{Q}}^R_m \mathbf{Q}^{Rm}$, $\overline{\mathbf{Q}}^L_m \mathbf{Q}^{Lm}$) denote a scalar product in $\mathbf{C}^4_{spin}$ (resp. $\mathbf{C}^4_{spin} \otimes \mathbf{C}^2_{iso}$, $\mathbf{C}^4_{spin} \otimes \mathbf{C}^3_{color}$, $\mathbf{C}^4_{spin} \otimes \mathbf{C}^2_{iso} \otimes \mathbf{C}^3_{color}$). Symbols like $\overline{\mathbf{Q}}^R \overline{\Phi} \mathbf{Q}^L$ (resp. $\overline{Q}^L \Phi Q^R$) denote a scalar product of elements $\overline{\mathbf{Q}}^R \otimes \overline{\Phi}$ and $\mathbf{Q}^L$ (resp. $\overline{\mathbf{Q}}^L$ and $\Phi \otimes \mathbf{Q}^R$) in $\mathbf{C}^4_{spin} \otimes \mathbf{C}^2_{iso}$ (given by $\mathbf{Q}^R_\alpha \Phi_k \mathbf{Q}^{L\alpha k}$ (resp. $\mathbf{Q}^L_{\alpha k} \Phi^k \mathbf{Q}^{R a}$) in terms of coordinates). The operators included in the above tensor products (e.g. γ_μ, γ^5, $\mathbf{a}_\mu$) are understood to be tensorized by the appropriate unit operators.

F. The quantum version of classical conformal manifolds

In section A we showed how the differential geometry of a smooth spin-manifold can be described via the Dirac operator, thus opening the way to a non-commutative generalization. An analogous situation prevails for conformal manifolds: there the conformal structure can again be described in a Hilbert space setting generalizable to the non-commutative frame, but now simply in terms of a bounded operator F of square one, archetype of the following general notion:[46]
Definition: Let A be a *-algebra over $\mathbf{C}$.[47] A *Fredholm module over A* is a pair (H, F) of

— a Hilbert space H which is a left A-module via a bounded *-representation π of A:

$$A \ni a \to \pi(a) = \underline{a} \in B(H), \tag{F.1}$$

— and a bounded linear operator $F \in B(H)$,

with the following properties:

$$(F^2 - \mathbf{1})\underline{a} \in K(H) \quad , \; a \in A, \tag{F.2}$$

$$[F, \underline{a}] \in K(H) \quad , \; a \in A. \tag{F.3}$$

The Fredholm module (H, F) is:

— *involutive* whenever $F^2 = \mathbf{1}$,

— *self-adjoint* whenever $F = F^\star$,

— *even* whenever H is $Z/2$-graded, F is odd, and π is of grade zero: $H = H^0 + H^1$ with grading involution ε, $\underline{a}H^i \in H^i$, and $FH^i \in H^{i+1}$, $i \in Z/2$.

[46] simpler
[47] possibly Z/2-graded.

— *odd* otherwise,

— *p-summable*, $p \in [1;\infty)$, whenever $[F,a] \in L^p(H), a \in A$.

(here $L^p(H)$, $p \in [1;\infty)$, is the Schatten ideal of bounded operators A s.t. $|A|^p$ is trace-class, i.e. belongs to $L^1(H)$).
The *n-character* χ_n^F of the $(p+1)$-summable Fredholm module (H,F), is defined for $N \geq p+1$ as:

$$\chi_n^F(a_0, a_1, \ldots, a_n) = cste \cdot Tr\left\{\underline{a_0}[F, \underline{a_1}] \ldots [F, a_n]\right\} \tag{F.4}$$

We shall envisage conformal manifolds as conformal classes of riemannian manifolds. Our setting and notation will be the following: $\mathbf{M}$ is an orientable smooth pseudo-riemannian manifold of even dimension $d = 2m$ with metric g of signature[48] s. We set $C^\infty(\mathbf{M}) = A$, and denote the DeRham complex of $\mathbf{M}$ by $(\Omega(\mathbf{M}) = \oplus_{p=1,\ldots,d}\Omega(\mathbf{M})^p,\ \mathbf{d})$. The components of differential forms and the scalar product of $\Omega(\mathbf{M})$ are defined as follows: with $\varepsilon^i \in \Omega(\mathbf{M})^1,\ i = 1,\ldots,d$, a orthonormal frame of $T^{\mathbf{M}\star}$, we write:

$$\lambda = \lambda_I \varepsilon^I = \frac{1}{p!}\lambda_{i_1\ldots i_p}\varepsilon^{i_1} \wedge \ldots \wedge \varepsilon^{i_p} = \lambda_{i_1\ldots i_p}\varepsilon^{i_1} \oplus \ldots \oplus \varepsilon^{i_p} \quad ,\ \lambda \in \Omega(\mathbf{M})^p, \tag{F.5}$$

and

$$(\lambda, \mu) = \Sigma_I \lambda_I \mu_I = \frac{1}{p!}\Sigma_{i_1\ldots i_p}\lambda_{i_1\ldots i_p}\mu_{i_1\ldots i_p} \quad ,\ \lambda,\ \mu \in \Omega(\mathbf{M})^p, \tag{F.6}$$

with the λ_I lexicographic and the $\lambda_{i_1\ldots i_p}$ totally antisymmetric. The volume form is $dv = \varepsilon^1 \wedge \varepsilon^2 \wedge \ldots \wedge \varepsilon^d$ for any direct orthonormal frame $\varepsilon^1,\ \varepsilon^2, \ldots, \varepsilon^d$ of $T^{\mathbf{M}\star}$: in the coordinate patch x^μ:

$$\mathbf{d}v = \left[((-1)^{d-s})\det(g_{\mu\nu})\right]^{1/2} \mathbf{d}x^1 \wedge \ldots \wedge \mathbf{d}x^d. \tag{F.7}$$

The *Hodge involution* $\gamma : \Omega(\mathbf{M}) \to \Omega(\mathbf{M})$ (such that $\gamma^2 = id$), defined by:[49]

$$\lambda \wedge^{\star} \mu = (-1)^{dp + \frac{p(p-1)}{2}} i^m (\lambda, \mu) dv \quad ,\ \lambda,\ \mu \in \Omega(\mathbf{M})^p,\ 0 \leq p \leq d, \tag{F.8}$$

is such that

$$\gamma \mathbf{d} \gamma = -\mathbf{d}^\star, \tag{F.9}$$

and undergoes the following change in the change of metric $g \to g' = e^{2\phi} g$:

$$\gamma_{p'} = e^{(d-2p)\phi}\gamma_p \quad \left(\gamma_p = \gamma|_{\Omega(\mathbf{M})^p}\right). \tag{F.10}$$

Reminder.

---

[48] number of positive eigen values

[49] The canonical identification of $Cl(\mathbf{M})$ to $\Omega(\mathbf{M})$ identifies γ with the involution γ^{d+1}.

(i) Hodge theory: With $\Delta = \mathbf{dd}^\star + \mathbf{d}^\star\mathbf{d}$ the Laplace-Beltrami operator, and denoting by Δ_i and d_i the respective restrictions of Δ and d to $\Omega(\mathbf{M})^i$, we have the following location of vector spaces[50]:

$$\begin{array}{rl} Ker\Delta_i = & Ker\mathbf{d}_i \cap Ker\mathbf{d}^\star_{i-1} \\ & \\ \uparrow & Ker\mathbf{d}^\star_{i-1} \\ Ker\mathbf{d}_i & \longrightarrow Im\mathbf{d}^\star_i \\ \swarrow & Im\Delta_i \\ Im\mathbf{d}_{i-1} & \end{array} \tag{F.11}$$

with $\dim Ker\Delta_i = \dim(Ker\mathbf{d}_i / Im\mathbf{d}_i) < \infty$.

(ii) : description of $Ker^\perp\Delta$: The most general element $\omega \in Ker^\perp\Delta$ has $\gamma-{\text{even} \atop \text{odd}}$ components of the form:

$$\begin{cases} \omega^+ = \frac{1+\gamma}{2}\mathbf{d}\alpha^+ \\ \\ \omega^- = \frac{1-\gamma}{2}\mathbf{d}\alpha^- \end{cases} \quad \text{with} \quad \alpha^+,\ \alpha^- \in \Omega(\mathbf{M}). \tag{F.12}$$

For $\alpha \in \Omega(\mathbf{M})$, each of the forms $d\alpha$, $\frac{1+\gamma}{2}d\alpha$ and $\frac{1-\gamma}{2}d\alpha$ determines the other two: one has:

$$\begin{cases} \frac{1-\gamma}{2}\mathbf{d}\alpha = F\frac{1+\gamma}{2}\mathbf{d}\alpha \\ \\ \frac{1+\gamma}{2}\mathbf{d}\alpha = F\frac{1-\gamma}{2}\mathbf{d}\alpha \end{cases}, \tag{F.13}$$

and

$$\mathbf{d}\alpha = (1+F)\frac{1+\gamma}{2}\mathbf{d}\alpha = (1+F)\frac{1-\gamma}{2}\mathbf{d}\alpha, \tag{F.14}$$

with $F : Ker^\perp\Delta \to Ker^\perp\Delta$ the involution:

$$F = \frac{\mathbf{dd}^\star - \mathbf{d}^\star\mathbf{d}}{\mathbf{dd}^\star + \mathbf{d}^\star\mathbf{d}}. \tag{F.15}$$

In what follows we shall be concerned with the differential forms of $\mathbf{M}$ of order half the dimension: indeed the latter are connected with the conformal structure of $\mathbf{M}$ (aimed to be described in terms of non-commutative concepts) due to the fact that the Hodge involution commutes with conformal changes of the metric exclusively in restriction to $\Omega(\mathbf{M})^m$, $d = 2m$. We now have the following results due to Alain Connes:
Let $H_0 = H_m \oplus H^\perp$ be the Hilbert completion of $\Omega(\mathbf{M})^m = Ker\Delta_m \oplus Ker^\perp\Delta_m$, $d = 2m$, (thus H_m is the Hilbert space of harmonic m-forms with orthogonal complement $H^\perp$ in the Hilbert space of all m-forms). If one defines as follows the triple (H, F, γ):

— H is the Hilbert space $H = H'_m \oplus H_0 = H'_m \oplus H_m \oplus H^\perp$, H'_m a *second copy of* H_m.

[50] figurative tridimensional representation.

— γ is the involution of H restricting to the Hodge involution on H_0, and to minus the Hodge involution on H'_m.

— the action of $A = C^\infty(\mathbf{M})$ on H restricts to the usual action on H_0 and to the zero action on H'_m.

— the operator F on H restricts to F as given by (F.13) and (F.15) *on* $H^\perp$, and restricts to the matrix $\begin{pmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix}$ on $H'_m \oplus H_m$.

one has that (H, F, γ) is an even unital self-adjoint Fredholm module over A, moreover p-summable for all $p > 2m$. Furthermore the Fredholm module (H, F, γ) uniquely determines the oriented conformal structure of $\mathbf{M}$.

Note that in restriction to $H^\perp = \frac{1+\gamma}{2}H^\perp \oplus \frac{1-\gamma}{2}H^\perp$ one has $F = P \oplus P^\star$, with

$$\begin{cases} P\frac{1-\gamma}{2}\mathbf{d}\alpha = \frac{1+\gamma}{2}\mathbf{d}\alpha \\ \\ P^\star\frac{1+\gamma}{2}\mathbf{d}\alpha = \frac{1-\gamma}{2}\mathbf{d}\alpha \end{cases} . \tag{F.16}$$

G. Fractals

We sketch a situation inaugurating the differential geometry of fractals. Let Σ and Σ' be two Riemann surfaces of the same genus > 1. According to Beers, they can be uniformized jointly on the Riemann sphere (by means of quasi-fuchsian groups) with a common limit set forming an irregular equator (called a quasi cercle).

By stereographic projection the quasi cercle projects on a Jordan curve Γ of the complex plane $\boldsymbol{C}$. Conferring this with the usual uniformization of Σ within the unit disk D, the Riemann mapping theorem yields a map Z from the closed disk to $\boldsymbol{C}$, analytic conformal in the interior of the disk and bicontinuous on the boundary S^1, in restriction to which Z is a bicontinuous (highly non differentiable) complex function with range Γ. Let now F be the phase of the Dirac operator $i\frac{d}{d\varphi}$ of S^1: F is the identity on the Hardy space, and minus the identity on its orthogonal complement, in other words F is the *Hilbert transform*. Alain Connes then defines the quantum differential as $dZ = [F, Z]$. Taking the power $(dZ)^p$ of the latter, with p the (non integer) Hausdorff dimension of the fractal Γ, $(dZ)^p$ happens to lie in $L^{1+}(S^1)$ and one then has the following astonishing analytic expression for the Hausdorff measure μ_H of the fractal Γ

$$\mu_H(f) = Tr_w\{f(Z)\,(dZ)^p\}\ . \tag{G.1}$$

H. Essays on gravitation

Recently Connes' non-commutative geometry turned out — besides yielding the above reinterpretation of the standard model — to be relevant to gravitation.

§1 The Dirac operator and gravitation Alain Connes first made the challenging observation, that the Wodzicki residue of the inverse square of the (Atiyah-Singer-Lichnérowicz) Dirac operator yields the Einstein-Hilbert action of general relativity, a fact which he left unpublished, but mentioned verbally in different talks. The Wodzicki residue is a (in fact the unique, thus canonical) trace on the pseudo-differential operators (concentrated on pseudo-differential operators of order - the dimension of the manifold).

In the paper [9] we computed the Wodzicki residue of D^{-2}, first computing this object for the pure Dirac operator (built with the spin connection of a riemannian spin manifold).

We thus work in the setting of a 4-dimensional oriented riemannian spin manifold $\mathbf{M}$ with riemannian metric g. The Dirac operator D is locally given as follows in terms of an orthonormal section e_i (with dual section θ^k) of the frame bundle of $\mathbf{M}$: one has

$$\begin{cases} D = i\gamma^i\widetilde{\nabla}_i = i\gamma^i(e_i + \sigma_i) \\ \text{with } \sigma_i(x) = \frac{1}{4}\gamma_{ij,k}(x)\gamma^j\gamma^k = \frac{1}{8}\gamma_{ij,k}(x)\left[\gamma^j\gamma^k - \gamma^k\gamma^j\right] \end{cases}\ , \tag{H.1}$$

where the $\gamma_{ij,k}$ represent the Levi-Civita connection ∇ with spin connection $\widetilde{\nabla}$, specifically :

$$\begin{cases} \gamma_{ij,k} = -\gamma_{ik,j} = \frac{1}{2}[c_{ij,k} + c_{ki,j} + c_{kj,i}] \qquad ,\ i,\ j,\ k = 1,\ \ldots,\ 4. \\ \text{with } c^k_{ij} = \theta^k([e_i, e_j]) \end{cases} \tag{H.2}$$

(The γ^i are constant self-adjoint Dirac matrices s.t. $\gamma^i\gamma^j + \gamma^j\gamma^i = 2\delta^{ij}$). In terms of local coordinates x^μ inducing the alternative vierbein $\partial_\mu = S^i_\mu(x)e_i$ (with dual vierbein dx^μ)

we have $\gamma^i e_i = \gamma^\mu \partial_\mu$, the γ^μ being now x-dependent Dirac matrices s.t. $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = g^{\mu\nu}$ (we use latin sub-(super-)scripts for the basis e_i and greek sub-(super-)scripts for the basis ∂_μ, the type of sub-(super-)scripts specifying the type of Dirac matrices). The specification of the Dirac operator in the greek basis is then :

$$\begin{cases} D = i\gamma^\mu \widetilde{\nabla}_\mu = i\gamma^\mu(\partial_\mu + \sigma_\mu) \\ \text{with } \sigma_\mu(x) = S^i_\mu(x)\sigma_i(x) \end{cases} . \tag{H.1a}$$

In what follows the notation D^{-1} refers to an inverse modulo smoothing operators. One then finds that the value of the Wodzicki residue on the inverse square of the Dirac operator, namely :

$$I = 4Tr_\omega\{\sigma_{-4}(x,\xi)\} = 4(2\pi)^{-4}\int_{\xi\in S^3} tr\{\sigma_{-4}(x,\xi)\}d^3\xi dv, \tag{H.3}$$

(tr the normalized Clifford trace) where :

$$\sigma_{-4}(x,\xi) = \text{part of order} - 4 \text{ of the total symbol } \sigma(x,\xi) \text{ of } D^{-2}, \tag{H.4}$$

coincides up to a constant with the Hilbert-Einstein action $\int \mathcal{L}_g dv$ of general relativity, where :

$$\mathcal{L}_g = R_{\mu\nu} \wedge \star(\mathbf{d}x^\mu \wedge \mathbf{d}x^\nu) \tag{H.5}$$

(specifically

$$\mathcal{L}_g = \frac{1}{2}R_{ikmn}(\mathbf{d}x^m \wedge \mathbf{d}x^n,\ \mathbf{d}x^i \wedge \mathbf{d}x^k) = (g^{im}g^{nk} - g^{in}g^{mk})R_{ikmn} = s, \tag{H.5a}$$

s the scalar curvature). One has $I = -\frac{1}{24\pi}\int \mathcal{L}_g dv$.

Our proof is a brute-force computation performed in arbitrary coordinate patches. We start from the Lichnérowicz formula for the square of the Dirac operator :

$$\begin{aligned} D^2 &= -g^{\mu\nu}\left(\widetilde{\nabla}_\mu\widetilde{\nabla}_\nu - \Gamma^\alpha_{\mu\nu}\widetilde{\nabla}_\alpha\right) + \tfrac{1}{4}s \\ &= -g^{\mu\nu}\left[\partial^x_\mu\partial^x_\nu + 2\sigma_\mu\cdot\partial^x_\nu - \Gamma^\alpha_{\mu\nu}\partial_\alpha + \partial^x_\mu\sigma_\nu + \sigma_\mu\sigma_\nu - \Gamma^\alpha_{\mu\nu}\sigma_\alpha\right] + \tfrac{1}{4}s. \end{aligned} \tag{H.6}$$

Our computations are based on the algorithm yielding the principal symbol of a product of pseudo-differential operators in terms of the principal symbols of the factors, namely, with the shorthands $\partial^\alpha_\xi = \partial^\alpha/\partial\xi_\alpha,\ \partial^x_\alpha = \partial_\alpha/\partial x^\alpha$:

$$\sigma^{PQ}(x,\xi) = \Sigma_\alpha\frac{(-i)^{|\alpha|}}{\alpha!}\partial^\alpha_\xi\sigma^P(x,\xi)\cdot\partial^x_\alpha\sigma^Q(x,\xi). \tag{H.7}$$

We needed to compute the total symbol $\sigma(x,\xi)$ of D^{-2} up to order -4, the computation is involved because of the need to “dive by two orders” in evaluating full symbols.

Since the Einstein-Hilbert action and the action of the standard model are both obtained by algorithms based on the Dixmier trace, one naturally wishes to obtain these two actions within a single procedure. Along this line the first natural object to investigate is the Wodzicki residue of $\mathbf{D}^{-2}$, $\mathbf{D}$ the compound Dirac operator built with the tensor product of the spin connection σ_μ and the electrodynamics $U(1)$-connection a_μ. But computation of this object yields the same result as that stated above: the connection a_μ drops out of the calculation [9].[51]

We thus conclude that the present algorithms of non-commutative geometry yielding the respective lagrangians of the microworld and the cosmos seem (superficially) to tend to repel each other : whilst a_μ drops out of the Wodzicki residue of the inverse square of the compound Dirac operator, σ_μ drops out of the non-commutative Yang-Mills algorithm.[52]

§2 The quantum Polyakov action in two dimensions We recall the expression of the (classical) Polyakov action

$$I_d(X^\alpha) = \sum_{\alpha,\beta=1}^{m} \int \eta_{\alpha\beta} \mathbf{d}X^\alpha \wedge *\mathbf{d}X^\beta \tag{H.8}$$

where the X^α, $\alpha = 1, \ldots, m$ are smooth functions on a Riemann surface Σ (with metric g_{ik}) and $\eta_{\alpha\beta}$ is an Euclidean metric of R^m.

Alain Connes obtains the action (F.1) from the following "quantum Polyakov action"

$$I_{quant}(X^\alpha) = Tr_w\{\eta_{\alpha\beta}[F, X^\alpha][F, X^\beta]\} \tag{H.9}$$

specified as follows. Take as Hilbert space the set $\Omega(M)^1$ completed for the scalar product stemming from the Riemannian metric. And take[53]

$$F = \frac{\mathbf{d}^*\mathbf{d} - \mathbf{d}\mathbf{d}^*}{\mathbf{d}^*\mathbf{d} + \mathbf{d}\mathbf{d}^*} \tag{H.10}$$

Evaluating the Dixmier trace Tr_w as a Wodzicki residue then restitutes the classical Polyakov action (H.2) of which (H.3) is thus a quantum version. We now describe this computation.

Quantum differentials are defined as follows:

$$df = [F, f] \qquad , \ f \in C^\infty(\Sigma). \tag{H.11}$$

Note that, with $\Phi = (\mathbf{d}\mathbf{d}^\star - \mathbf{d}^\star\mathbf{d}),\ \Delta = \mathbf{d}\mathbf{d}^\star + \mathbf{d}^\star\mathbf{d},\ q = \mathbf{d} + \mathbf{d}^\star,\ r = i(\mathbf{d} - \mathbf{d}^\star)$, we have

$$\begin{cases} \Delta = \mathbf{d}\mathbf{d}^\star + \mathbf{d}^\star\mathbf{d} = q^2 = r^2, \\ \Phi = \mathbf{d}\mathbf{d}^\star - \mathbf{d}^\star\mathbf{d} = iqr = -irq \\ F = -iqr^{-1} = irq^{-1}. \end{cases} \tag{H.12}$$

[51]In fact, since our calculation is based on the Lichnérowicz formula for the square or the Dirac operator holding in the case of general Dirac operators stemming from Clifford connections on Clifford bundles, our result naturally generalizes to this frame

[52]Indeed σ_μ drops out of the commutators $[D,\ a],\ a \in C^\infty(\mathbf{M})$.

[53]We may ignore the subset of harmonic forms on which the denominator vanishes.

thus F is involutive: $F^2 = \mathbb{1}$. (we ignored the fact that inverses are not defined on $Ker\Delta = Ker\mathbf{d} \cap Ker\mathbf{d}^\star = Kerq \cap Kerr = Ker\Phi \cap Ker\Delta = Ker\mathbf{d} \cap (Im\mathbf{d})^\perp = Ker\mathbf{d}^\star \cap (Im\mathbf{d}^\star)^\perp$)

Note that definition (H.9), is well taken: indeed $F,\ X^\mu,\ [F, X^\mu]$, and $[F, X^\mu][F, X^\nu]$ are pseudodifferential operators of respective orders 0,0,-1, and $-2 = -\dim\Sigma$: $[F, X^\mu][F, X^\nu]$ thus belongs to the definition ideal L^{1+} of the Dixmier trace.

We first compute the principal symbol $\sigma^F(x, \xi)$ of F. We recall that the total symbols (adjoint of each other) of $\mathbf{d}$ and $\mathbf{d}^\star$ are given by:

$$\begin{cases} \sigma^{\mathbf{d}}(x,\xi) = \ \mathrm{i}e_\xi \\ \\ \sigma^{\mathbf{d}^\star}(x,\xi) = -ii_{g^{-1}\xi} \end{cases}, \tag{H.13}$$

where e_ξ denotes an exterior, and i_u an inner product, $\xi \in T_x^{\Sigma\star}$, and $u \in T_x^\Sigma$. We have thus the principal symbols:

$$\begin{cases} \sigma^\Phi(x,\xi) = e_\xi i_{g^{-1}\xi} - i_{g^{-1}\xi} e_\xi \\ \\ \sigma^\Delta(x,\xi) = e_\xi i_{g^{-1}\xi} + i_{g^{-1}\xi} e_\xi = ||\xi||^{-2} \end{cases}, \tag{H.14}$$

and

$$\sigma^F(x,\xi) = (e_\xi i_{g^{-1}\xi} - i_{g^{-1}\xi} e_\xi)||\xi||^2. \tag{H.15}$$

these formulae following from the rule for computing the total symbol of a product of two pseudodifferential operators in terms of the total symbols of the factors, namely:[54]

$$\sigma^{PQ} = \Sigma_\alpha \frac{1}{\alpha!} \partial_\xi^\alpha \sigma^P \cdot D_\alpha^x \sigma^Q \tag{H.16}$$

We now show that the principal symbol $\sigma^{[F,f]}(x,\xi)$ of $[F, f]$ is given by:

$$\sigma^{[F,f]}(x,\xi) = 2\left(e_{\mathbf{d}f^\perp} i_\xi - i_\xi e_{\mathbf{d}f^\perp}\right)||\xi||^{-2}, \tag{H.17}$$

where $\mathbf{d}f^\perp = \mathbf{d}f - (\xi, \mathbf{d}f)||\xi||^{-2} \cdot \xi$, the projection of $\mathbf{d}f$ on the plane orthogonal to ξ. Indeed, applying (H.7), keeping the first non vanishing order, we get, taking account

[54] We recall the notation $D_\alpha^x = (-i)^{|\alpha|}\partial_x^\alpha$.

of the fact that $O(\xi,\eta) = \left(e_\xi i_{g^{-1}\eta} - i_{g^{-1}\xi} e_\eta\right)$ is a symmetric bilinear form in ξ and η:

$$\begin{aligned}
\sigma^{[F,f]}(x,\xi) &= \sigma^F \cdot f + \Sigma_{|\alpha|=1} \partial^\alpha_\xi \sigma^F D^x_\alpha f - f \cdot \sigma^F = \Sigma_{|\alpha|=1} \partial^\alpha_\xi \sigma^F D^x_\alpha f \\
&= \Sigma_{i=1,2} \frac{\partial}{\xi_i} \sigma^F \partial_i f = \Sigma_{i=1,2} \frac{\partial}{\xi_i} \left[\xi_k \xi_l ()^{kl} ||\xi||^{-2}\right] \partial_i f \\
&= \Sigma_{i=1,2} O^{kl} ||\xi||^{-2} \left\{\delta^i_k \xi_l + \xi_k \delta^i_l - ||\xi||^{-2}_k \xi_k \xi^{st}_{lg} \left[\delta^i_s \xi_t + \xi_s \delta^i_t\right]\right\} \partial_i f \\
&= \Sigma_{i=1,2} O^{kl} ||\xi||^{-2} \left\{[\xi_l \partial_k f + \xi_k \partial_l f] - 2\xi_k \xi_l ||\xi||^{-2} (\mathbf{d}f, \xi)\right\} \\
&= \Sigma_{i=1,2} O^{kl} ||\xi||^{-2} \left\{\left[\xi_k \left[\partial_l f - \xi_l ||\xi||^{-2} (\mathbf{d}f,\xi)\right]\right]\right. \\
&\qquad\qquad \left. + \xi_l \left[\partial_k f - \xi_k ||\xi||^{-2} (\mathbf{d}f,\xi)\right]\right\} \\
&= \Sigma_{i=1,2} O^{kl} ||\xi||^{-2} \left\{\xi_l (\mathbf{d}f^\perp)_k + \xi_k (\mathbf{d}f^\perp)_l\right\} \\
&= 2\left(e_{\mathbf{d}f^\perp} i_\xi - i_\xi e_{\mathbf{d}f^\perp}\right) ||\xi||^{-2}.
\end{aligned} \tag{H.18}$$

The principal symbol σ of $f_0[F,f_1][F,f_2]$ is therefore given by the product:

$$\sigma(x,\xi) = 4\left(e_{\mathbf{df}_1^\perp} i_\xi - i_\xi e_{\mathbf{df}_1^\perp}\right)\left(e_{\mathbf{df}_2^\perp} i_\xi - i_\xi e_{\mathbf{df}_2^\perp}\right) ||\xi||^{-4}. \tag{H.19}$$

We now show that the trace of σ on the fiber is given by:

$$tr\sigma(x,\xi) = 8||\xi||^{-d} f_0 \left(\mathbf{d}f_1^\perp, \mathbf{d}f_2^\perp\right). \tag{H.20}$$

Let indeed $\eta_1 = \mathbf{d}f_1^\perp = \eta_{1k}\varepsilon^k$, $\eta_2 = \mathbf{d}f_2^\perp = \eta_{2k}\varepsilon^k$, with $\left(e_k,\ \varepsilon^k\right)$ a dual basis of $\left(T^M_x = T^{M\star\star}_x,\ T^{M\star}_x\right)$: using the fact that we have, for a one-form α, with $g^{-1}\xi = \overline{\xi}$ and $g^{-1}\eta = \overline{\eta}$:

$$\left(e_\eta i_\xi - e_\xi i_\eta\right)\alpha = \alpha(\overline{\xi})\eta - \alpha(\overline{\eta})\xi, \tag{H.21}$$

we get (summation over k):

$$\begin{aligned}
\tfrac{1}{4} tr\sigma(x,\xi) &= e_k \left(e_{\eta_1} i_\xi - e_\xi i_{\eta_1}\right)\left(e_{\eta_2} i_\xi - e_\xi i_{\eta_2}\right)\varepsilon^k = \\
&= e_k \left\{\xi^k \left[\eta_2(\overline{\xi})\eta_1 - \eta_2(\overline{\eta}_1)\xi\right] - \eta_2^k \left[\xi(\overline{\xi})\eta_1 - \xi(\overline{\eta}_1)\xi\right]\right\} \\
&= (\xi,\eta_1)(\xi,\eta_2) - (\eta_1,\eta_2)||\xi||^2 - ||\xi||^2 (\eta_1,\eta_2) + (\xi,\eta_1)(\xi,\eta_2) \\
&= 2\left[(\xi,\eta_1)(\xi,\eta_2) + ||\xi||^2 (\eta_1,\eta_2)\right] = -2||\xi||^2 \left(\mathbf{d}f_1^\perp, \mathbf{d}f_2^\perp\right)
\end{aligned} \tag{H.22}$$

Integrating this in ξ over the unit ball yields:

$$-2\int_{S^1} \left(\mathbf{d}f_1^\perp, \mathbf{d}f_2^\perp\right) \mathbf{d}v_{S^1} = \frac{8\pi}{3}\left(\mathbf{d}f_1, \mathbf{d}f_2\right). \tag{H.23}$$

One has indeed, for $||\xi|| = 1$:

$$\left(\mathbf{d}f_1^{\perp}, \mathbf{d}f_2^{\perp}\right) = (\mathbf{d}f_1 - (\xi, \mathbf{d}f_1)\xi), (\mathbf{d}f_2 - (\xi, \mathbf{d}f_2)\xi)$$

$$= (\mathbf{d}f_1, \mathbf{d}f_2) - (\xi, \mathbf{d}f_1)(\xi, \mathbf{d}f_2) = (\mathbf{d}f_1, \mathbf{d}f_2) - (\xi^i, \mathbf{d}f_{1_i})(\xi_j, \mathbf{d}f_{2^j}), \tag{H.24}$$

hence

$$\begin{aligned} \textstyle\int_{S^1} \left(\mathbf{d}f_1^{\perp}, \mathbf{d}f_2^{\perp}\right) \mathbf{d}v_{S^1} &= \textstyle\int_{S^1} \left[(\mathbf{d}f_1, \mathbf{d}f_2) - \xi^i \xi_j \mathbf{d}f_{1_i} \mathbf{d}f_{2^j}\right] \mathbf{d}v_{S^1} \\ &= 2\pi(\mathbf{d}f_1, \mathbf{d}f_2) - \delta^i_j \textstyle\int_{S^1} \xi^l \xi_l \mathbf{d}f_{1_i} \mathbf{d}f_{2^j} \\ &= 2\pi\left(l - \tfrac{1}{3}\right)(\mathbf{d}f_1, \mathbf{d}f_2) = \tfrac{4\pi}{3}(\mathbf{d}f_1, \mathbf{d}f_2) \end{aligned} \tag{H.25}$$

Plugging this result in (A.7a) then yields:

$$Tr_\omega \left\{f_0[F, f_1][F, f_2]\right\} = -(3\pi)^{-1} \int_\Sigma f_0(\mathbf{d}f_1, \mathbf{d}f_2) dv = -(3\pi)^{-1} \int_\Sigma f_0 \mathbf{d}f_1 \wedge^\star \mathbf{d}f_2 \tag{H.26}$$

our result then following from the replacement s: $f_0 \to l,\ f_1 \to X^\mu,\ f_2 \to X^\nu$.

§3 The quantum Polyakov action in four dimensions

In two dimensions the quantum Polyakov action turned out to be equal to the classical action. However the quantum version of the action has an advantage over the classical version: it is adaptable, mutatis mutandis, to a four-dimensional conformal manifolds Σ. The setting is now the following:

— The basic algebra is $C^\infty(\Sigma)$, Σ a conformal 4-dimensional manifold equipped with a riemannian metric g^2.

— The $(X^\mu)_{\mu=1,\ldots,N}, X^\mu \in C^\infty(\Sigma)$, take values in the euclidean space $\mathbf{R}^N$ with metric η.

— The even K-cycle is $(H = \Omega(\Sigma)^2,\ F = (\mathbf{d}\mathbf{d}^\star - \mathbf{d}^\star\mathbf{d})(\mathbf{d}\mathbf{d}^\star + \mathbf{d}^\star\mathbf{d})^{-1})$. As above, we take as Hilbert space the Hilbert completion of the set of differential forms of middle dimension. The scalar product of $\Omega(\Sigma)^2$ is induced by the riemannian structure of Σ. In defining the involutive operator F we ignore the subspace of harmonic two-forms.

The quantum Polyakov action is now as above where Res denotes the Wodzicki residue (now concentrated on pseudo-differential operators of order $-4 = -\dim\Sigma$, where it coincides with $4Tr_\omega$, Tr_ω the Dixmier trace).

As in §1, the computation is involved because of the need to "dive by two orders". Invariance considerations allow one not to drown in the calculation, which yields the following esoteric quadratic action:

$$\begin{aligned} \mathcal{L}(X^\mu, X^\nu) = -\eta_{\mu\nu}\Delta\left[(\mathbf{d}X^\mu, \mathbf{d}X^\nu)\right] &+ \eta_{\mu\nu}(\Delta\mathbf{d}X^\mu, \Delta\mathbf{d}X^\nu) - \tfrac{1}{2}\eta_{\mu\nu}\Delta\mathbf{d}X^\mu \cdot \Delta\mathbf{d}X^\nu \\ &+ \tfrac{1}{3}\mathbf{s} \cdot \eta_{\mu\nu}(\mathbf{d}X^\mu, \mathbf{d}X^\nu). \end{aligned} \tag{H.27}$$

I. Conclusion and outlook

Applications as of today to the fundamental physical interactions concern the classical (Lagrangian) level (field quantization and renormalization still lie ahead). They comprise an elaborate classical version of the full standard model (electroweak and chromodynamics sectors) and promising essays on gravitation.

Concerning the microworld Connes proposes an astounding interpretation of the full standard model in terms of a Poincaré-dual pair of (mildly) non-commutative spaces, compounds of usual space-time and the electroweak, resp. chromodynamical, degrees of freedom. The latter are interpreted as attributes of a non-commutative space synthetizing usual space time with the "inner symmetries" of elementary particles, a new notion of space combining inseparably space-time with the elementary particle structure. This theory presents the Higgs boson as a fith gauge-boson corresponding to a discrete connection of a two-sheeted space and incorporates the symmetry-breaking in the definition of the K-cycle. All the terms of the usual Glashow-Salam-Weinberg lagrangian proceed miraculously from a compact quantum Yang-Mills algorithm. The convincing interpretation of the electroweak-chromodynamical duality as a case of non-commutative Poincaré duality, with concomitant enrichment of the mathematical notion of non-commutative space, is a beautiful feature. The only flaw of the theory is the physically ad hoc (if mathematically coherent) "modularity condition". Since the theory allows restricted choices of coupling constants displaying various degrees of additional symmetry and implying constraints, it is important to know if these potentially predictive constraints can survive renormalization. In this respect it would be important to display the symmetry (at the moment not visible) associated to the Higgs boson as a fifth gauge boson - and more generally to look systematically for "quantum symmetries" - progress is expected on this point in a near future.

Concerning the cosmos, Connes' non-commutative geometry turned out lately to be relevant to gravitation. On the one hand there is the recognition that the Dirac operator embodies the lagrangian of general relativity. On the other hand Connes developped a quantum version of the usual Polyakov action (within the conformal variety of his non-commutative geometry). This quantum version has the virtue of beeing transcribable to the frame of 4-dimensional conformal manifolds, yielding a conformally invariant lagrangian possibly tied up with gravitation. The analysis of the anomalies of this lagrangian is underway.

Of course one wishes to synthetize the standard model and gravitation within a non-commutative geometry frame a priori adequate for such a project... for the moment future music! One could hope from such a development to come nearer to the explanation of the fermion masses - a problem in the face of which it is frustrating to see the formidable apparatus of present-day quantum field theory totally ineffective.

I want to conclude with the personal statement that, contrary to most of my colleagues, I do not adhere to the view that renormalization will remain as a standing feature of the future theory - I can not help feeling repelled by the ugliness and the incomplete nature of renormalized field theories (anyway confined to a perturbative approach unfit to gravitation). My hope is that a further, deeper, step into non-commutativity of the basic algebra will once wipe out the divergencies. In respect to gravitation, I sympathize with Ashtekar's philosophy, hoping for a future jointure of the latter with Alain Connes' project.

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The reader will find under I the references relevant to these lectures, whose subject-matter corresponds to the my preference for Connes' approach and reflect my personal bias when I wrote the paper [5]. I regret not to have had the leisure of including an account of Connes' non-commited study of the possible choice of coupling constants: for this I refer to [3], [3a], [4], where results are stated without details of calculations. The present lectures utilize refs. [3a], [5], [6], [7], [8], [9], [10]. They do not touch the other references quoted under I. For introductions we refer to [3], [8], [11].

We attempted in II to list the approaches of physical interactions by means of non-commutative geometry techniques other than Connes'. Refs. [38-46], concern the works of the Marseille-Mainz group whose theory of the electroweak sector of the standard model is in spirit a parent of Connes' approach (partly inspired by it) however more in the mood of Kaluza-Klein theories, and technically based on the use of a differential algebra. In [38-39], the basic object is a Z_2-graded differential algebra defined as the graded tensor product of the Z_2 graded matrix algebra of p-forms times a matrix algebra of even matrices (with a natural Z_2 grading). For a detailed comparison between this approach and Connes' see [47].

In [30 - 37], the role of the differential algebra is played by $\Omega_{Der}(A)$ which is defined as the homomorphic image of the universal differential envelope of A into the algebra of A-valued multilinear forms on the space of derivations of A. The oldest paper of this series is [30], whereas the most complete on this approach is certainly [37].

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