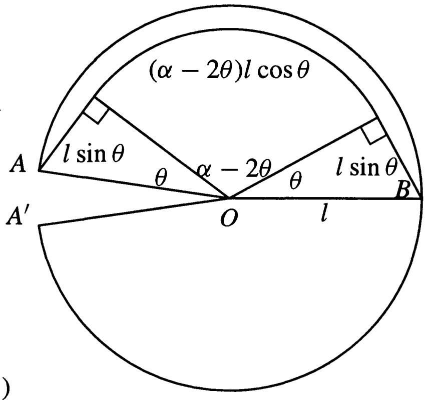
Mathematics for Hikers

10557 [1996, 902]. *Proposed by Nick MacKinnon, Winchester College, Winchester, U. K.* Naismith's rule allows walkers to compute the time for their journeys. The time is given by allowing a walking speed of 4 km/hr, but adding an extra minute for each 10m of ascent. A conical mountain has base radius 1650m and vertical height 520m. Points A and B are diametrically opposite at the base of the mountain. How should a path be constructed between between A and B on the surface of the mountain that minimizes the time taken to walk from A to B?

Solution by the proposer. The surface of the mountain can be unrolled isometrically onto a plane, forming a sector (of angle $2\alpha = 165/173 \cdot 2\pi$ radians) of a circle (of radius $l = 1730$m, the slant height of the mountain), as shown in the figure at right.



Call the time-minimising path the *Naismith geodesic* for the cone. This geodesic must reach some maximum height h. The figure shows a potential Naismith geodesic with maximum height $h = 520(1 - \cos\theta)$ meters. It is composed of a circular arc following the contour with height h, together with tangents joining the arc to A and B. No alternative path taking less time attains the maximum height h of the given path since such a path must at least touch the circular arc, must not cross the circular arc, and must leave the circular arc without subsequent reascent. The Naismith geodesic must therefore be a path of the given shape. The length of such a path is $2l \sin\theta + (\alpha - 2\theta)l \cos\theta$, and Naismith's rule gives a time of

$$t(\theta) = 0.015(2l \sin\theta + (\alpha - 2\theta)l \cos\theta) + 520(1 - \cos\theta)/10$$

minutes for the journey. The lone critical value of $t(\theta)$ for $0 < \theta < \alpha/2$ occurs when $t'(\theta) = (52 - 0.015(\alpha - 2\theta)l) \sin\theta = 0$ at $\theta^* = (\alpha - 52/(0.015l))/2 \doteq .49623$ radians. This gives the optimal time $t(\theta^*) = 76.71037$ minutes. A path around the bottom of the cone takes $t(0) = 77.75442$ minutes, while the path of shortest distance takes $t(\alpha/2) = 99.98929$ minutes.

Solved also by J. Anglesio (France), J. E. Dawson (Australia), P. G. Kirmser, J. H. Lindsey II, R. Reynolds & M. Martinez, and P. Straffin.

A Matter of Adjustment

10558 [1996, 902]. *Proposed by Zhang Chengyu, Hubei University, Wuhan, China.* Let p be a prime, and let k be a positive integer. Let $a_1, a_2, \ldots, a_{p^k}$ be any p^k integers. We define the *adjustment* of these integers to be the p^k integers $b_1, b_2, \ldots, b_{p^k}$, where $b_j = a_{j+1} + a_{j+2} + \cdots + a_{j+p}$, interpreting subscripts modulo p^k. For example, if $p = 2$ and $k = 2$, one adjustment of 1, 1, 3, 4 gives 4, 7, 5, 2. Prove that after p^k adjustments of $a_1, a_2, \ldots, a_{p^k}$, the list consists entirely of integers divisible by p.

Solution I by Thomas Jager, Calvin College, Grand Rapids, MI. We prove a stronger statement. Given an integer vector $v = (v_1, \ldots, v_{p^k})$, define the v-adjustment of $(a_1, \ldots, a_{p^k})^T$ to be $(b_1, \ldots, b_{p^k})^T$, where $b_j = v_1 a_{j+1} + v_2 a_{j+2} + \cdots + v_{p^k} a_{j+p^k}$, again treating subscripts modulo p^k. As a transformation, the v-adjustment is represented by the matrix $A = v_1 S + v_2 S^2 + \cdots + v_{p^k} S^{p^k}$, where S is the permutation matrix for a cyclic shift by one position. Hence

$$A^{p^k} = \left(v_1 S + \cdots + v_{p^k} S^{p^k}\right)^{p^k} \equiv v_1^{p^k} I + \cdots + v_{p^k}^{p^k} I \equiv (v_1 + \cdots + v_{p^k})^{p^k} I \pmod{p}.$$

Thus if $v_1 + \cdots + v_{p^k} \equiv 0$ modulo p, then p^k applications of the v-adjustment matrix produces a vector of integers divisible by p. In the problem statement, the vector v consists of p ones and $p^k - p$ zeros.

Solution II by J. H. van Lint, Eindhoven University of Technology, Eindhoven, the Netherlands. Starting with a_0, we construct an infinite sequence with $a_i = a_{i+p^k}$. Over the field $\mathbb{F}_p$, we consider the formal power series $f(x) = \sum_{i=0}^{\infty} a_i x^i = A(x)/(1 - x^{p^k})$, where $A(x) = \sum_{i=0}^{p^k-1} a_i x^i$ is a polynomial of degree less than p^k.

After one adjustment, the terms $b_0, b_1, \ldots$ are the coefficients of $x^{p+1}, x^{p+2}, \ldots$ in the formal power series for

$$(x + x^2 + \cdots + x^p) f(x) = \frac{x(1 - x^p)}{1 - x} f(x) = x(1 - x)^{p-1} f(x).$$

The result of n adjustments is the list of coefficients of $x^{n(p+1)}, x^{n(p+1)+1}, \ldots$ in the formal power series for

$$x^n (1 - x)^{n(p-1)} f(x) = \frac{x^n (1 - x)^{n(p-1)}}{1 - x^{p^k}} A(x),$$

which is a polynomial of degree less than np if $n(p-1) \geq p^k$. Thus the list consists entirely of integers divisible by p after n adjustments if $n \geq p^k/(p-1)$. Noting that

$$\frac{p^k - 1}{p - 1} + 1 = \frac{p^k}{p - 1} + \frac{p - 2}{p - 1}$$

is the least integer greater than or equal to $p^k/(p-1)$, we see that the list consists entirely of integers divisible by p after n adjustments if $n \geq (p^k - 1)/(p - 1) + 1$. As this is at most p^k, the desired result follows.

Editorial comment. David Callan proved that for positive m the list consists entirely of integers divisible by p^{m-1} after mp^{k-1} adjustments. In particular, after p^k adjustments the list consists entirely of integers divisible by p^{p-1}. Another consequence is that the list consists entirely of integers divisible by p after $2p^{k-1}$ adjustments, but this is not as strong as the result proved by van Lint.

Solved also by D. Beckwith, A. E. Caicedo Núñez (Colombia), D. Callan, R. J. Chapman (U. K.), J. E. Dawson (Australia), W. Janous (Austria), K. S. Kedlaya, J. H. Lindsey II, R. Martin (Germany), A. Nijenhuis, J. C. Smith, H.-T. Wee (Singapore), GCHQ Problems Group (U. K.), and the proposer.

Sets with Fixed Nim-Sum

10564 [1997, 68]. *Proposed by Proposed by Aviezri Fraenkel, Weizmann Institute of Science, Rehovot, Israel.* The *Nim-sum* of two positive integers with binary expansions $\sum_{i \geq 0} a_i 2^i$ and $\sum_{i \geq 0} b_i 2^i$ is the number with binary expansion $\sum_{i \geq 0} c_i 2^i$, where a_i, b_i, c_i are in $\{0, 1\}$ and $c_i \equiv a_i + b_i \bmod 2$. Let n be a positive integer, and let j be a nonnegative integer. How many of the 2^n subsets of $\{1, 2, \ldots, n\}$ have the property that their elements have Nim-sum equal to j?

Solution by Reiner Martin, Deutsche Bank, London, U. K. Let $[n] = \{1, 2, \ldots, n\}$, and let Δ denote the symmetric difference operation. Let $k = \lceil \log_2(n + 1) \rceil$. There exists a subset of $[n]$ whose elements have Nim-sum j only if $0 \leq j < 2^k$. We claim that the number of such subsets does not depend upon j and thus that this number is 2^{n-k} for each such j.