



Lattice Points inside a Triangle: 10600

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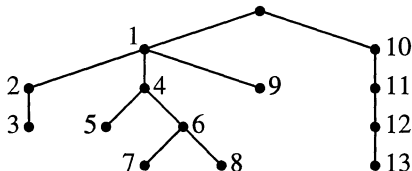
For the inverse map, index the blocks in partition π in increasing order of their least elements: A_1 , possibly A_2 , possibly A_3 . Build the corresponding tree traversal (starting with 1 as the leftmost vertex at level 1) as follows. Suppose that the label just processed was label k , belonging to A_i . Let A_j be the block containing label $k + 1$.

Case 1: $j \leq i$. Make $k + 1$ a new rightmost leaf at level j .

Case 2: $j = i + 1$. Make $k + 1$ the first child of the vertex with label k .

Case 3: $j = 3$ and $i = 1$. Because $\min A_2 < \min A_3$, there exists a label less than k in A_2 . Let l be the largest such label less than k . Let m be the least label such that all labels from m to k lie in A_1 ; note that $m > l$. Remove m, \dots, k from level 1. Make m the rightmost child of l (at level 3). Make $m + 1, \dots, k + 1$ children of m (at level 4).

Applying the original map to the resulting tree puts each label back into its block in π . As an example, the tree corresponding to $A_1 = \{1, 6, 7, 10, 12\}$, $A_2 = \{2, 4, 9, 11\}$, and $A_3 = \{3, 5, 8, 13\}$ is shown.



Solved also by D. Callan, R. Holzsager, the Anchorage Math Solutions Group, and the proposer.

Lattice Points Inside a Triangle

10600 [1997, 566]. *Proposed by Franz Rothe, University of North Carolina, Charlotte, NC.*

(a) Suppose a triangle has its vertices at integer lattice points in the plane and contains exactly 3 integer lattice points in its interior. Show that the center of mass of the triangle is not an integer lattice point.

(b)* Find all values i such that, if a triangle has its vertices at integer lattice points in the plane and contains exactly i integer lattice points in its interior, then the center of mass of the triangle cannot be an integer lattice point.

Solution of part (a) by Robin J. Chapman, University of Exeter, Exeter, U. K. Let the vertices of the triangle be A , B , and C , with position vectors \mathbf{a} , \mathbf{b} , and \mathbf{c} , respectively. Suppose that the centroid of the triangle has integer coordinates. This centroid is $(1/3)(\mathbf{a} + \mathbf{b} + \mathbf{c})$. Let D denote twice the area of the triangle. Then

$$D = |(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})| = |(\mathbf{c} - \mathbf{b}) \times (\mathbf{a} - \mathbf{b})| = |(\mathbf{a} - \mathbf{c}) \times (\mathbf{b} - \mathbf{c})| = |\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}|$$

where \times denotes the vector product. By assumption $(1/3)(\mathbf{a} + \mathbf{b} + \mathbf{c})$ has integer coordinates; therefore, so has $(1/3)(\mathbf{a} + \mathbf{b} + \mathbf{c}) - \mathbf{a} = (1/3)((\mathbf{b} - \mathbf{a}) + (\mathbf{c} - \mathbf{a}))$. Hence $(\mathbf{c} - \mathbf{a}) = -(\mathbf{b} - \mathbf{a}) + 3\mathbf{d}$, where \mathbf{d} has integer coordinates, and so $D = |(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})| = 3|(\mathbf{b} - \mathbf{a}) \times \mathbf{d}|$ is a multiple of 3. Let r , s , and t be the largest integers such that $(1/r)(\mathbf{b} - \mathbf{c})$, $(1/s)(\mathbf{c} - \mathbf{a})$, and $(1/t)(\mathbf{a} - \mathbf{b})$ have integer coordinates. Then the interiors of the sides BC , CA , and AB contain respectively $r - 1$, $s - 1$, and $t - 1$ lattice points. By Pick's Theorem $D = 2N_1 + N_2 - 2$, where N_1 is the number of lattice points in the interior of the triangle and N_2 is the number of lattice points on its boundary (including the vertices). Consequently $D = 4 + r + s + t$. Also

$$D = st \left| \frac{\mathbf{b} - \mathbf{a}}{t} \times \frac{\mathbf{c} - \mathbf{a}}{s} \right|,$$

and so D is divisible by st . Similarly, D is divisible by rs and rt .

Since D is divisible by 3, we have $r + s + t \equiv 2 \pmod{3}$. We now show that none of r , s , t is divisible by 3. Suppose that r is divisible by 3. Then $\mathbf{b} - \mathbf{c} = (\mathbf{b} - \mathbf{a}) + (\mathbf{a} - \mathbf{c}) = 3\mathbf{e}$

for some vector \mathbf{e} with integer coordinates. But we already know that $(\mathbf{b}-\mathbf{a})-(\mathbf{a}-\mathbf{c})=3\mathbf{d}$. Hence both $\mathbf{b}-\mathbf{a}$ and $\mathbf{a}-\mathbf{c}$ are integer vectors multiplied by 3. Thus s and t are divisible by 3, contradicting $r+s+t\equiv 2\pmod{3}$. It follows that one of r, s , or t is congruent to 1 modulo 3, and the others are congruent to 2. Also D must be divisible by $3rs, 3st$, and $3rt$.

We may assume that $r\geq s\geq t$. Now $3rs\leq D=4+r+s+t$, and so

$$(3t-1)^2\leq(3r-1)(3s-1)=9rs-3r-3s+1\leq 13+3t.$$

This inequality is false for $t\geq 2$, so $t=1$. Therefore $r\equiv s\equiv 2\pmod{3}$ and also $(3r-1)(3s-1)\leq 16$. Together, these imply that $r=s=2$, and so $D=9$. Now D is not divisible by $3rs$, so we have a contradiction.

Editorial comment. The proposer discovered the following result: Let a triangle have vertices at integer lattice points $(0,0)$, (b_1,b_2) , and (c_2,c_2) . Let $\alpha=\gcd(b_1-c_2,b_2-c_2)$, $\beta=\gcd(b_2,b_2)$, and $\gamma=\gcd(c_1,c_2)$. The center of mass is a lattice point if and only if either (i) 3 is a divisor of all three numbers α, β , and γ ; or (ii) 3 is a divisor of none of the three numbers α, β , and γ , but 3 is a divisor of the double area $D=b_1c_2-b_2c_1$.

Only partial solutions were received for part (b). Searches by John H. Lindsey II and by the GCHQ Problems Group found the values $i < 1000$ satisfying the condition. The two lists are the same, except that 906 appears in one list and not the other. The remaining values found were: 3, 6, 15, 18, 30, 36, 48, 51, 63, 78, 90, 108, 120, 138, 150, 156, 168, 210, 228, 270, 300, 303, 336, 360, 378, 408, 426, 438, 480, 510, 528, 531, 630, 660, 723, 738, 750, 780, 888, 930, 990, 996.

Part (a) also solved by J. H. Lindsey II, GCHQ Problems Group (U. K.), and the proposer.

A Surrounded Set

10608 [1997, 664]. *Proposed by Victor Zalgaller, Steklov Mathematical Institute, St. Petersburg, Russia.* Let S be a compact convex set in the plane. If l is any line of support for S , let $f(l)$ be the length of the shortest curve that begins and ends on l and that together with l surrounds S . Prove that if $f(l)$ is independent of l , then S is a circle.

Solution by John Arkinstall, Monash University, Australia. Let l' be the support line parallel to l on the opposite side of S . Since $f(l)+f(l')$ is independent of l but also equals the perimeter of S plus twice the width of S perpendicular to l , S is a set of constant width w . A line l'' parallel to l and l' whose intersection with S is of maximum length is called a *diameter* of S in the direction of l . Because S has constant width w , we may conclude that the length of such a diameter is w , that it joins two points where support lines perpendicular to l touch S , that these two support lines, together with l and l' , form a square of side w , and that each support line touches S in a unique point. A theorem of Khassa (Relation between maximal chords and symmetry for convex sets, *J. London Math. Soc.* **15** (1977) 541–546) states that a convex curve of constant width in which the diameter in each direction is midway between the two support lines in that direction must be a circle. Thus it suffices to prove this “midway” property.

When S is a set in the plane of constant width w , $2A(l)-wP(l)$ is independent of l , where $A(l)$ is the area of the portion of S between the opposite support line l' and the diameter l'' , and $P(l)$ is the length of that part of the perimeter of S on the same side of l'' as l' (L. Beretta & A. Maxia, “Insiemi convessi e orbiformi,” *Univ. Roma e Ist. Naz. Alta Mat. Rend. Mat.* (5) **1** (1940) 1–64). Let $r(l)$ denote the distance from l to the diameter in the direction of l . Since $r(l)$ is the length of the supporting line segment from l to the boundary of S on the shortest curve from l surrounding S , we have $f(l)=2r(l)+P(l)$. Since this is independent of l by hypothesis, so is $A(l)+wr(l)$. This is the area of the convex hull of S and the two supporting line segments from l on the shortest curve from l . The complement R of this convex hull in the supporting square to S with side length w and edge along l therefore also has area independent of the direction of l .