A Surrounded Set: 10608

Victor Zalgaller; John Arkinstall

The American Mathematical Monthly, Vol. 106, No. 1. (Jan., 1999), pp. 75-76.

Stable URL:
http://links.jstor.org/sici?sici=0002-9890%28199901%29106%3A1%3C75%3AASS1%3E2.0.CO%3B2-B

The American Mathematical Monthly is currently published by Mathematical Association of America.

for some vector $\mathbf{e}$ with integer coordinates. But we already know that $(\mathbf{b}-\mathbf{a})-(\mathbf{a}-\mathbf{c}) = 3\mathbf{d}$. Hence both $\mathbf{b}-\mathbf{a}$ and $\mathbf{a}-\mathbf{c}$ are integer vectors multiplied by 3. Thus s and t are divisible by 3, contradicting $r+s+t \equiv 2 \pmod 3$. It follows that one of r, s, or t is congruent to 1 modulo 3, and the others are congruent to 2. Also D must be divisible by $3rs$, $3st$, and $3rt$.

We may assume that $r \geq s \geq t$. Now $3rs \leq D = 4 + r + s + t$, and so

$$(3t-1)^2 \leq (3r-1)(3s-1) = 9rs - 3r - 3s + 1 \leq 13 + 3t.$$

This inequality is false for $t \geq 2$, so $t = 1$. Therefore $r \equiv s \equiv 2 \pmod 3$ and also $(3r-1)(3s-1) \leq 16$. Together, these imply that $r = s = 2$, and so $D = 9$. Now D is not divisible by $3rs$, so we have a contradiction.

Editorial comment. The proposer discovered the following result: Let a triangle have vertices at integer lattice points $(0,0)$, (b_1, b_2), and (c_2, c_2). Let $\alpha = \gcd(b_1 - c_2, b_2 - c_2)$, $\beta = \gcd(b_2, b_2)$, and $\gamma = \gcd(c_1, c_2)$. The center of mass is a lattice point if and only if either **(i)** 3 is a divisor of all three numbers α, β, and γ; or **(ii)** 3 is a divisor of none of the three numbers α, β, and γ, but 3 is a divisor of the double area $D = b_1c_2 - b_2c_1$.

Only partial solutions were received for part **(b)**. Searches by John H. Lindsey II and by the GCHQ Problems Group found the values $i < 1000$ satisfying the condition. The two lists are the same, except that 906 appears in one list and not the other. The remaining values found were: 3, 6, 15, 18, 30, 36, 48, 51, 63, 78, 90, 108, 120, 138, 150, 156, 168, 210, 228, 270, 300, 303, 336, 360, 378, 408, 426, 438, 480, 510, 528, 531, 630, 660, 723, 738, 750, 780, 888, 930, 990, 996.

Part **(a)** also solved by J. H. Lindsey II, GCHQ Problems Group (U. K.), and the proposer.

A Surrounded Set

10608 [1997, 664]. *Proposed by Victor Zalgaller, Steklov Mathematical Institute, St. Petersberg, Russia.* Let S be a compact convex set in the plane. If l is any line of support for S, let $f(l)$ be the length of the shortest curve that begins and ends on l and that together with l surrounds S. Prove that if $f(l)$ is independent of l, then S is a circle.

Solution by John Arkinstall, Monash University, Australia. Let l' be the support line parallel to l on the opposite side of S. Since $f(l) + f(l')$ is independent of l but also equals the perimeter of S plus twice the width of S perpendicular to l, S is a set of constant width w. A line l'' parallel to l and l' whose intersection with S is of maximum length is called a *diameter* of S in the direction of l. Because S has constant width w, we may conclude that the length of such a diameter is w, that it joins two points where support lines perpendicular to l touch S, that these two support lines, together with l and l', form a square of side w, and that each support line touches S in a unique point. A theorem of Khassa (Relation between maximal chords and symmetry for convex sets, *J. London Math. Soc.* **15** (1977) 541–546) states that a convex curve of constant width in which the diameter in each direction is midway between the two support lines in that direction must be a circle. Thus it suffices to prove this "midway" property.

When S is a set in the plane of constant width w, $2A(l) - wP(l)$ is independent of l, where $A(l)$ is the area of the portion of S between the opposite support line l' and the diameter l'', and $P(l)$ is the length of that part of the perimeter of S on the same side of l'' as l' (L. Beretta & A. Maxia, "Insiemi convessi e orbiformi," *Univ. Roma e Ist. Naz. Alta Mat. Rend. Mat.* (5) **1** (1940) 1–64). Let $r(l)$ denote the distance from l to the diameter in the direction of l. Since $r(l)$ is the length of the supporting line segment from l to the boundary of S on the shortest curve from l surrounding S, we have $f(l) = 2r(l) + P(l)$. Since this is independent of l by hypothesis, so is $A(l) + wr(l)$. This is the area of the convex hull of S and the two supporting line segments from l on the shortest curve from l. The complement R of this convex hull in the supporting square to S with side length w and edge along l therefore also has area independent of the direction of l.

Now consider how the area of R changes when the direction of l is changed by a small angle ϕ. If the point of support on l' divides its side of the supporting square in the ratio $u : w - u$, then the area of R changes by four small approximately triangular regions: It decreases by $(1/2)r(l')\phi + o(\phi)$, increases by $(1/2)u\phi + o(\phi)$, decreases by $(1/2)(w - u)\phi + o(\phi)$, and increases by $(1/2)r(l')\phi + o(\phi)$. Thus, the area of R changes by the sum $(1/2)(2u - w)\phi + o(\phi)$. Since this is 0, we have $2u - w = 0$, and thus the support point on l' is midway between the two support lines perpendicular to l'.

Solved also by S. S. Kim (Korea), J. G. Merickel, GCHQ Problems Group (U. K.), and the proposer.

Tight Bounds for the Normal Distribution

10611 [1997, 665]. *Proposed by Zoltán Sasvári, Technical University of Dresden, Dresden, Germany.* Find the largest value of a and the smallest value of b for which the inequalities

$$\frac{1+\sqrt{1-e^{-ax^2}}}{2} < \Phi(x) < \frac{1+\sqrt{1-e^{-bx^2}}}{2}$$

hold for all $x > 0$, where $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\, dy$.

Solution by Hongwei Chen, Christopher Newport University, Newport News, VA. We show that $a = 1/2$ and $b = 2/\pi$ are the best possible constants for which the stated inequalities hold. Since $\int_{-\infty}^{0} e^{-y^2/2}\, dy = \int_0^\infty e^{-y^2/2}\, dy = \sqrt{\pi/2}$, the stated inequalities are equivalent to

$$\frac{\sqrt{1-e^{-ax^2}}}{2} < f(x) < \frac{\sqrt{1-e^{-bx^2}}}{2}, \tag{1}$$

where $f(x) = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-y^2/2}\, dy$. If the second inequality of (1) holds for all $x > 0$, then

$$0 < \frac{\sqrt{1-e^{-bx^2}}}{2} - f(x) = \left(\frac{\sqrt{b}}{2} - \frac{1}{\sqrt{2\pi}}\right) x + O(x^3)$$

as $x \to 0$, which implies $b \geq 2/\pi$. Similarly, if the first inequality of (1) holds for all $x > 0$, then

$$0 < f(x) - \frac{\sqrt{1-e^{-ax^2}}}{2} = \frac{1}{4} e^{-ax^2} + O\left(e^{-2ax^2}\right) - \frac{e^{-x^2/2}}{\sqrt{2\pi}x} + O\left(\frac{e^{-x^2/2}}{x^2}\right)$$

as $x \to \infty$. Dividing each side by $e^{-x^2/2}$ yields $a \leq 1/2$.

To show that inequalities (1) hold for all $x > 0$ when $a = 1/2$ and $b = 2/\pi$, we write

$$(f(x))^2 = \frac{1}{2\pi} \int_0^x \int_0^x e^{-(y^2+z^2)/2}\, dy dz.$$

Let $D = [0, x]^2$, $D_1 = \{ (y, z) : 0 \leq y,\ 0 \leq z,\ y^2 + z^2 \leq x^2 \}$, and $D_2 = \{ (y, z) : 0 \leq y,\ 0 \leq z,\ y^2 + z^2 \leq (4/\pi)x^2 \}$. We have the inequalities

$$\frac{1}{2\pi} \iint_{D_1} e^{-(y^2+z^2)/2}\, dydz < \frac{1}{2\pi} \iint_{D} e^{-(y^2+z^2)/2}\, dydz < \frac{1}{2\pi} \iint_{D_2} e^{-(y^2+z^2)/2}\, dydz, \tag{2}$$

the first because $D_1 \subset D$, and the second because D and D_2 have the same area and $e^{-(y^2+z^2)/2} \leq e^{-(2/\pi)x^2}$ for $(y, z) \in D - D_2$ while $e^{-(y^2+z^2)/2} \geq e^{-(2/\pi)x^2}$ for $(y, z) \in D_2 - D$. Evaluating the outer integrals in (2) in polar coordinates, we obtain

$$\frac{1-e^{-x^2/2}}{4} < f(x)^2 < \frac{1-e^{-2x^2/\pi}}{4},$$

which is equivalent to (1).

Solved also by P. Alsholm (Denmark), J. Anglesio (France), P. Bracken (Canada), B. Burdick, G. G. Chappell, P. Devaraj (India), G. Keselman, K.-W. Lau (Hong Kong), J. H. Lindsey II, A. Stadler, GCHQ Problems Group (U. K.), NCCU Problems Group, NSA Problems Group, WMC Problems Group, and the proposer.