On Rational Function Approximations to Square Roots

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Interest in methods for calculating square roots exists partly because the speed with which a computer can evaluate them contributes a measure of its overall performance [**1**]. Newton's well known method for improving estimates of a square root uses a simple rational function approximation and converges in second order; an iterative method converges in second order if it generates a sequence $\{s_n\}$ that tends to limit s and, in which the error $(s_{n+1} - s)$ tends to $K(s_n - s)^2$ for some K, independent of n, as n tends to infinity. This note presents a rational function approximation method with faster convergence. It uses a result of Frank [**2**] on periodic continued fractions; a formula given in the 15th century by Al-Kalsadi [**3**, p. 111] is also based on continued fractions and is a special case.

Frank studied properties of the convergents (or approximants) to the pure periodic continued fraction representing the quadratic surd

$$L + \sqrt{C} = \left[b_0, \overline{b_1, \ldots, b_p}\right] = b_0 + \frac{1}{b_1} + \frac{1}{b_2} + \frac{1}{b_3} + \cdots, \tag{1}$$

where L and C are rational numbers with C positive, the b's are positive integers, p is the period and the overbar indicates the periodic part of the continued fraction. There are restrictions on the values of L and C in order that the representing continued fraction be pure periodic [**4**, p. 101], but they are satisfied if L is zero and C is an integer (when $b_p = 2b_0$, although this is not important here). A special case of a property given by Frank ([**2**, (2.2)] with $s = 1$, $C = N$, $L = 0$) is

$$x_{i+j} = (x_i x_j + N)/(x_i + x_j) \quad \text{for } i, j > 0, \tag{2}$$

where x_i is the $(pi - 1)$th convergent to the continued fraction. It can be shown by induction, from Pascal's triangle rule for binomial coefficients $\binom{i}{j}$, that the convergents satisfy

$$x_k = F_k(x_1) \quad \text{for } k > 0, \tag{3}$$

where, if k is odd $(= 2m + 1)$,

$$F_k(x) = \sum_{i=0}^{i=m} \binom{k}{k-2i} x^{k-2i} N^i \div \sum_{i=0}^{i=m} \binom{k}{k-2i-1} x^{k-2i-1} N^i$$

and, if k is even $(= 2m)$,

$$F_k(x) = \sum_{i=0}^{i=m} \binom{k}{k-2i} x^{k-2i} N^i \div \sum_{i=0}^{i=m-1} \binom{k}{k-2i-1} x^{k-2i-1} N^i. \tag{4}$$

The function $F_k(x)$ has a fixed point at $\sqrt{N}$. Equation (3) gives x_k in terms of x_1. By considering the period of the continued fraction to be the multiple $k^n p$ of p instead of p itself we obtain the same formula giving $x_{k^{n+1}}$ in terms of x_{k^n}. Thus function (4) generates a series of approximations to $\sqrt{N}$ which form a subsequence of the convergents to the continued fraction if one starts with $x = x_1$.

We can use function (4) in an iterative scheme for finding the square root of an integer. To find the square root of a rational number q/r, say, we calculate $\sqrt{qr}$ and divide by r.

The theory of continued fractions guarantees convergence (unless $k = 1$) but we must know the value of the $(p-1)$th convergent to start the iterative sequence. This is inconvenient. However, by the following theorem, convergence is also guaranteed if we start with an arbitrary positive value x.

Theorem. *For* $x > 0$, $y_n := F_{K^n}(x) \to \sqrt{N}$.

We generalize (2) and (3) and replace (2) by

$$F_{i+j}(x) = \left[F_i(x)F_j(x) + N\right] \div \left[F_i(x) + F_j(x)\right] \quad \text{for } i, j > 0. \tag{5}$$

From (3), (4), and (5) we find

$$F_k(x) - \sqrt{N} = (x - \sqrt{N})\left[F_{k-1}(x) - \sqrt{N}\right] \div \left[x + F_{k-1}(x)\right] \quad \text{for } k > 1. \tag{6}$$

From the definition of $F_k(x)$, if x is positive so is $F_k(x)$ for any k. By induction from (3), (4), and (6) we find

$$F_k(x) \geq \sqrt{N} \quad \text{if} \quad x \geq \sqrt{N}, \tag{7a}$$

$$F_k(x) < x \quad \text{if} \quad x > \sqrt{N}. \tag{7b}$$

These inequalities imply that repeated application of the function $F_k(x)$ generates a strictly monotonic decreasing sequence whose greatest lower bound is $\sqrt{N}$ if the starting value exceeds $\sqrt{N}$. Hence convergence is guaranteed for any starting value exceeding $\sqrt{N}$. An argument similar to that leading to inequality (7a) shows that

$$F_k(x) \geq \sqrt{N} \quad \text{if} \quad x \leq \sqrt{N} \text{ for } k \text{ even}, \tag{8a}$$

$$F_k(x) \leq \sqrt{N} \quad \text{if} \quad x \leq \sqrt{N} \text{ for } k \text{ odd}. \tag{8b}$$

For even k and starting value smaller that $\sqrt{N}$ the first generated value exceeds $\sqrt{N}$ and convergence is guaranteed by the argument of the preceding paragraph. If k is odd it can be shown from (6) and (8) that the function $F_k(x)$ is strictly monotonic increasing with least upper bound equal to $\sqrt{N}$ for $x \leq \sqrt{N}$. Thus convergence is guaranteed for any positive starting value. Equation (6) shows that the convergence is of order k.

The first two estimates of $\sqrt{2}$ (1.414213562) obtained with

$$F_4(x) = (x^4 + 6x^2N + N^2)/(4x^3 + 4xN) \tag{9}$$

and starting value 1 are $17/12 = 1.416666667$ and $665857/470832 = 1.414213562$, rounded to nine decimal places; convergence is rapid.

The Newton method is

$$F_2(x) = (x^2 + N)/2x. \tag{10}$$

The approximation of Al-Kalsadi is

$$\sqrt{(a^2 + b)} \approx (4a^3 + 3ab)/(4a^2 + b), \tag{11}$$

which is $F_3(a)$ with N replaced by $a^2 + b$; with starting value 1 ($a = 1$) the first estimate of $\sqrt{2}$ is $7/5 = 1.4$.

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A Note on Jacobi Symbols and Continued Fractions

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1. INTRODUCTION. It is well known that the continued fraction expansion of a real quadratic irrational is periodic. Here we relate the expansion for $\sqrt{rs}$, under the assumption that $rX^2 - sY^2 = \pm 1$ has a solution in integers X and Y, to that of $\sqrt{r/s}$ and to the Jacobi symbols $\left(\frac{r}{s}\right)$, which appear in the theory of quadratic residues.

We have endeavoured to make our remarks self-contained to the extent of providing a brief reminder of the background theory together with a cursory sketch of the proofs of the critical assertions. For extensive detail the reader can refer to [**5**], the bible of the subject. The introductory remarks following in Sections 2–3 below are *inter alia* detailed in [**1**].

Let p and q denote distinct odd primes. In [**3**], Friesen proved connections between the value of the Legendre symbol $\left(\frac{p}{q}\right)$ and the length of the period of the continued fraction expansion of $\sqrt{pq}$. These results, together with those of Schinzel in [**6**], provided a solution to a conjecture of Chowla and Chowla in [**2**].

We report a generalization of those results to the evaluation of Jacobi symbols $\left(\frac{r}{s}\right)$, and, in the context of there being a solution in integers X, Y to the equation $rX^2 - sY^2 = \pm 1$, remark on the continued fraction expansion of $\sqrt{r/s}$ *vis à vis* that of $\sqrt{rs}$.

Theorem 1. *Let r and s be squarefree positive integers with $r > s > 1$, such that the equation $rX^2 - sY^2 = \pm 1$ has a solution in positive integers X, Y. Suppose the continued fraction expansion of $\sqrt{rs}$ is $[a_0, \overline{a_1, a_2, \dots, a_l}]$. Then both the length of the period $l = 2h$, and the 'central' partial quotient a_h, are even, and the continued fraction expansion of $\sqrt{r/s}$ is*

$$\left[\tfrac{1}{2}a_h, \overline{a_{h+1}, \dots, a_l, a_1, \dots, a_h}\right] = \left[\tfrac{1}{2}a_h, \overline{a_{h-1}, \dots, a_1, a_l, a_1, \dots, a_{h-1}, a_h}\right].$$