THE EVOLUTION OF . . .

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The Birth of Literal Algebra

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Translated from the Russian by Abe Shenitzer

1. MATHEMATICS IN THE FIRST CENTURIES AD. DIOPHANTUS. The Babylonians developed a kind of numerical algebra. Then came Greek geometric algebra.

The third—very important—stage of the development of algebra began in the first centuries AD and came to an end at the turn of the 17th century. Its beginning was marked by the introduction of *literal symbolism* by Diophantus of Alexandria and its end, by the creation of *literal calculus* in the works of Viète and Descartes. It was then that algebra acquired its own distinctive language, which we use today.

The first century BC was a period of Roman conquests and of Roman civil wars. Both took place in the territories of the Hellenistic states and the Roman provinces and were accompanied by physical and economic devastation. One after another, these states lost their independence. The last to fall was Egypt (30 BC). The horrors of war and the loss of faith in a secure tomorrow promoted the spread of religious and mystical teachings and undermined interest in the exact sciences, and in abstract problems in mathematics and astronomy. In Cicero's dialogue *On the state* one of the participants proposes a discussion of why two Suns were seen in the sky. But the topic is rejected, for "even if we acquired profound insight into this matter, we would not become better or happier."

In the second half of the first century BC mathematical investigations came to a virtual halt and there was an interruption in the transmission of the scientific tradition.

At the beginning of the new era, economic conditions in the Hellenistic countries, now turned Roman provinces, gradually improved, and there was a revival of literature, art, and science. In fact, the 2nd century came to be known as the Greek Renaissance. It was the age of writers such as Plutarch and Lucian and of scholars such as Claudius Ptolemy.

Alexandria continued its role as the cultural and scientific center of antiquity and, in this respect, Rome was never its rival. Nor did it ever develop an interest in the depths of Hellenistic science. As noted by Cicero in his *Tusculanae disputationes*, the Romans, unlike the Greeks, did not appreciate geometry; just as in the case of arithmetic, they stopped at narrow, practical knowledge of this subject.

Translator's note. This article is the third chapter of an essay by I. G. Bashmakova and G. S. Smirnova devoted to the rise and evolution of algebra. The whole essay is being translated by Abe Shenitzer and is being reviewed for publication by the Mathematical Association of America.

The first two sentences of this article were added by the translator.

They had little regard for all of mathematics. Even accounting, surveying, and astronomical observations were left to the Greeks, the Syrians, and other conquered nations. According to Vergil, the destiny of Romans was wise government of the world.

The revival of the Alexandrian school was accompanied by a fundamental change of orientation of its mathematical research. During the Hellenistic period geometry was the foundation of Greek mathematics; algebra had not, as yet, become an independent science but developed within the framework of geometry, and even the arithmetic of whole numbers was constructed geometrically. Now number became the foundation. This resulted in the arithmetization of all mathematics, the elimination of geometric justifications, and the emergence and independent evolution of algebra.

We encounter the return to numerical algebra already in the works of the outstanding mathematician, mechanician, and engineer Heron of Alexandria (1st century AD). In his books *Metrica*, *Geometrica*, and others, books that resemble in many respects our handbooks for engineers, one finds rules for the computation of areas and volumes, solutions of numerical quadratic equations, and interesting problems that reduce to indeterminate equations. In particular, they contain the famous "Heron formula" for the computation of the area of a triangle given its sides a, b, c:

$$S = \sqrt{p(p-a)(p-b)(p-c)},$$

where $p = (a+b+c)/2$. Here the expression under the square root sign is a product of four segments, and thus an expression totally inadmissible in geometric algebra. It is clear that Heron thought of segments as numbers, whose products are likewise numbers.

In his famous book, known under its Arabized name *Almagest*, Claudius Ptolemy, when computing tables of chords, identified ratios of magnitudes with numbers, and the operation of "composition" of ratios—defined in Euclid's *Elements*—with ordinary multiplication.

The new tendencies found their clearest expression in the works of Diophantus of Alexandria, who founded two disciplines: algebra and Diophantine analysis.

We know next to nothing about Diophantus himself. On the basis of certain indirect remarks, Paul Tannery, the eminent French historian of mathematics, concluded that Diophantus lived in the middle of the 3rd century AD. On the other hand, Renaissance scholars who discovered Diophantus' works, supposed that he lived at the time of Antoninus Pius, i.e., approximately in the middle of the 2nd century. The epigram in *Anthologia Palatina* provides the following information: "Here you see the tomb containing the remains of Diophantus, it is remarkable: artfully it tells the measures of his life. God granted him to be a boy for the sixth part of his life, and adding a twelfth part to this, He clothed his cheeks with down; He lit him the light of wedlock after a seventh part, and five years after his marriage He granted him a son. Alas! late-born wretched child; after attaining the measure of half his father's life, chill Fate took him. After consoling his grief by this science of numbers for four years he ended his life. By this device of numbers tell us the extent of his life." A simple computation shows that Diophantus died at the age of 84 years. This is all we know about him.

2. DIOPHANTUS' ***Arithmetica*****. ITS DOMAIN OF NUMBERS AND SYMBOLISM.** Only two (incomplete) works of Diophantus have come down to us. One is his *Arithmetica* (six books out of thirteen; four more books in Arabic, attributed to

Diophantus, were found in 1973. They will be discussed in the sequel). The other is a collection of excerpts from his treatise *On polygonal numbers*. We discuss only the first of these works.

Arithmetica is not a theoretical work resembling Euclid's *Elements* or Apollonius' *Conic sections* but a collection of (189) problems, each of which is provided with one or more solutions and with relevant explanations. At the beginning of the first book there is a short algebraic introduction, which is basically the first account of the foundations of algebra. Here the author constructs the field of rational numbers, introduces literal symbolism, and gives rules for operating with polynomials and equations.

Already Heron regarded positive rational numbers as legitimate numbers (in classical ancient mathematics "number" denoted a collection of units, i.e., a natural number). While Diophantus defined a number as a collection of units, throughout *Arithmetica* he called every positive rational solution of one of his problems "number" (ἀριθμός), i.e., he extended the notion of number to all of $\mathbf{Q}^+$. But this was not good enough for the purposes of algebra, and so Diophantus took the next decisive step of introducing negative numbers. It was only then that he obtained a system closed under the four operations of algebra, i.e., a field.

How did Diophantus introduce these new objects? Today we would say that he used the axiomatic method: he introduced a new object called "deficiency" (λεῖψις, from λεῖπω—to lack) and stated rules for operating with it. He writes: "deficiency multiplied by deficiency yields availability (i.e., a positive number (*the authors*)); deficiency multiplied by availability yields deficiency; and the symbol for deficiency is ⋔, an inverted and shortened (letter) ψ" (Diophantus. *Arithmetica*. Definition IX). In other words, he formulated the rule of signs, which we can write as follows:

$$(-)\times(-)=(+),$$
$$(-)\times(+)=(-).$$

Diophantus did not formulate rules for addition and subtraction of the new numbers but he used them extensively in his books. Thus, while solving problem III_8 (i.e., Problem 8 in Book III), he needs to subtract $2x + 7$ from $x^2 + 4x + 1$. The result is $x^2 + 2x - 6$, i.e., here he carries out the operation $1 - 7 = -6$. In problem VI_{14}, $90 - 15x^2$ is subtracted from 54 and the result is $15x^2 - 36$. Thus here $15x^2$ is subtracted from zero; in other words, Diophantus is using the rule $-(-a) = a$.

We note that Diophantus used negative numbers only in intermediate computations and sought solutions only in the domain of positive rational numbers. A similar situation developed later in connection with the introduction of complex numbers. Initially they were regarded as just convenient symbols for obtaining results involving "genuine," i.e., real, numbers.

Diophantus also introduced literal signs for an unknown and its powers. He called an unknown a "number" (ἀριθμός) and denoted it by the special symbol ς. It is possible that this symbol was introduced before him. We find it in the Michigan papyrus (2nd century AD) as well as in a table appended to Heron's *Geometrica*. But Diophantus boldly breaks with geometric algebra by introducing special symbols *for the first six positive powers of the unknown, the first six negative powers, and for its zeroth power*. While the square and cube of the unknown could be interpreted geometrically, its 4th, 5th, and 6th powers could not be so represented. Nor could the negative powers of the unknown.

Diophantus denoted the positive powers of the unknown as follows:

$$x—ς;\quad x^2—\Delta^{\upsilon};\quad x^3—K^{\upsilon};\quad x^4—\Delta^{\upsilon}\Delta;\quad x^5—\Delta K^{\upsilon};\quad x^6—K^{\upsilon}K.$$

He defined negative powers as inverses of the corresponding positive powers and denoted them by adding to the exponents of the positive powers the symbol χ. For example, he denoted $x^{-2} = 1/x^2$ by $\Delta^{v\chi}$.

He denoted the zeroth power of the unknown by the symbol $\overset{\circ}{M}$, that is by the first two letters in Μόνάς, or unity.

Then he set down a "multiplication table" for powers of the unknown that can be briefly written as follows:

$$x^m x^n = x^{m+n}, \quad -6 \le m + n \le 6.$$

He singled out two rules that correspond to the two basic axioms that we use for defining a group:

$$x^m \cdot 1 = x^m \quad \text{(definition VII)}; \tag{1}$$

$$x^m x^{-m} = 1 \quad \text{(definition VI)}. \tag{2}$$

In addition, Diophantus used the symbol $\overset{\prime}{\iota}\sigma$ for equality, and the symbol □ for an indeterminate square. All this enabled him to write equations in literal form. Since he did not use a symbol for addition, he first set down all positive terms, then the minus sign (i.e., ⋔), then the negative terms. For example, the equation

$$x^3 - 2x^2 + 10x - 1 = 5$$

was written as

$$\mathrm{K}^{v}\overline{\alpha}\varsigma\overline{\iota} \pitchfork \Delta^{v}\overline{\beta}\overset{\circ}{\mathrm{M}}\overline{\alpha}\overset{\prime}{\iota}\sigma\overset{\circ}{\mathrm{M}}\overline{\epsilon}.$$

Here $\overline{\alpha} = 1$, $\overline{\iota} = 10$, $\overline{\beta} = 2$, $\overline{\epsilon} = 5$ (we recall that the Greeks used the letters of the alphabet to denote numbers).

In the "Introduction" Diophantus formulated two basic rules of transformation of equations: 1) the rule for transfer of a term from one side of an equation to the other with changed sign and 2) reduction of like terms. Later, these two rules became well known under their Arabized names of *al − jabr* and *al − muqābala*.

Diophantus also used the rule of substitution in a masterly way but never formulated it.

We can say that in the introduction Diophantus defined the field **Q** of rational numbers, introduced symbols for an unknown and its powers, as well as symbols for equality and for negative numbers.

Before discussing the contents of *Arithmetica* we consider the possibilities and limitations of Diophantus' symbolism. Getting ahead of the story, we can say that, basically, Diophantus considered in his work indeterminate equations, i.e., equations with two or more unknowns. But he introduced symbols for just one unknown and its powers. How did he proceed when solving problems?

First he stated each problem in general form. For example: "To decompose a square into a sum of squares" (problem II_8). Now we would write this problem as

$$x^2 + y^2 = a^2.$$

How could Diophantus write this equation with just one symbol for an unknown and without symbols for parameters (in this case a)? He proceeded as follows: after the general formulation he assigned concrete values to the parameters—in the present case he put $a^2 = 16$. Then he denoted one unknown by his special symbol (we will use the letter t instead) and expressed the remaining unknowns as linear, quadratic, or more complex rational functions of that unknown and of the parameters. In case of the present example, one unknown is denoted by t and the other by $kt - a$ or, as Diophantus puts it, "a certain number of t's minus as many units as are contained in the side of 16," i.e., instead of a he takes 4 and instead of the parameter k—the number 2. But by saying "a certain number of t's" he indicates that the number 2 plays the role of an arbitrary parameter. Thus Diophantus' version of our equation is

$$t^2 + (2t - 4)^2 = 16,$$

so that

$$x = t = 16/5; \quad y = 2t - 4 = 12/5.$$

One might think that Diophantus was satisfied with finding a single solution. But this is not so. In the process of solving problem III_{19} he finds it necessary to decompose a square into two squares. In this connection he writes: "We know that a square can be decomposed into a sum of squares in infinitely many ways."

The use of a concrete number to denote an arbitrary parameter has the virtue of simplicity. Sometimes it turned out that the parameter could not be selected arbitrarily, that it had to satisfy additional conditions. In such cases Diophantus determined these conditions. Thus problem VI_8 reduces to the system

$$x_1^3 + x_2 = y^3, \quad x_1 + x_2 = y.$$

Diophantus puts $x_2 = t$, $x_1 = \beta t$, where $\beta = 2$. Then from the second equation we obtain $y = (\beta + 1)t$, and from the first

$$t^2 = \frac{1}{(\beta + 1)^3 - \beta^3}.$$

Since $\beta = 2$, $t^2 = 1/19$, i.e., t is not rational. In order to obtain a rational solution Diophantus looks at the way t^2 is expressed in terms of the parameter β. The expression in question is a fraction whose numerator, 1, is a square. But then the denominator must also be a square:

$$(\beta + 1)^3 - \beta^3 = \square.$$

Diophantus took as the new unknown $\beta = \tau$ (he denoted it by the same symbol as the original unknown x_2) and obtained

$$3\tau^2 + 3\tau + 1 = \square.$$

Solving this equation by his method (which we will describe in detail in the next section) Diophantus obtained

$$\tau = \frac{3 + 2\lambda}{\lambda^2 - 3},$$

i.e., the parameter could only be chosen from the class of numbers $\{(3 + 2\lambda)/(\lambda^2 - 3)\}$. Diophantus takes $\lambda = 2$ and obtains $\beta = 7$. Then he goes back to solving the original problem.

Diophantus often deliberately chooses for parameters numbers that do not lead to solutions. He does this in order to show how to analyze problems.

Thus concrete numbers play two roles in *Arithmetica*. One role is that of ordinary numbers and the other is that of symbols for arbitrary parameters. Numbers were destined to play the latter role almost to the end of the 16th century.

Time to sum up. Diophantus was first to reduce determinate and indeterminate problems to equations. We may say that for a large class of problems of arithmetic and algebra he did the same thing that Descartes was later to do for problems of geometry, namely he reduced them to setting up and solving algebraic equations.

Indeed, in order to solve problems—arithmetical in the case of Diophantus and geometric in the case of Descartes—both set up algebraic equations that they subsequently transformed and solved in accordance with the rules of algebra. Also, the transformations involved— such as elimination of unknowns, substitutions, and reduction of similar terms—had no direct arithmetic or geometric significance and were not subject to extensive interpretations. In both cases such interpretations were reserved for the final results. We are used to associating this important step with Descartes' creation of analytic geometry, but long before Descartes this step was taken by Diophantus in his *Arithmetica*. Before exposure to *Arithmetica*, none of the scholars of the period between the 13th and 16th centuries entertained the idea of applying algebra to the solution of number-theoretic problems.

3. THE CONTENTS OF *Arithmetica*. DIOPHANTUS' METHODS. We have already mentioned that *Arithmetica* is a collection of problems with solutions. This may create the impression that it is not a theoretical work. But a more careful reading makes it clear that the purpose of the painstaking choice and deliberate placement of problems was to illustrate the application of specific general methods. It is a characteristic of ancient mathematics that methods were not formulated apart from problems but were disclosed in the process of their solution. We recall that the famous "method of exhaustion"—the first variant of the theory of limits—was not set down in pure form either by its author Eudoxus of Cnidus or by Archimedes. It was mathematicians of the 16th and 17th centuries who isolated it by analyzing Euclid's *Elements* and Archimedes' quadratures and formulated it in general terms. The same applies to Diophantus' *Arithmetica*. As we show in the sequel, his methods were isolated in the 16th and 17th centuries by Italian and French mathematicians. Following them, we will try to isolate some of these methods and state them in general form.

In Book I Diophantus solved particular determinate linear and quadratic equations. The remaining books deal with the solution of indeterminate equations, i.e., equations of the form

$$F(x_1,\dots,x_n) = 0, \quad n \geq 2,$$

$F(x_1,\dots,x_n)$ a polynomial, or of systems of such equations:

$$\begin{cases} F_1(x_1,\dots,x_n) = 0; \\ \dots\dots\dots\dots\dots\dots \\ F_k(x_1,\dots,x_n) = 0, \quad k < n. \end{cases}$$

Diophantus looks for positive rational solutions $x_1^0, x_2^0, \dots, x_n^0$, $x_i^0 \in \mathbf{Q}^+$, of such equations or of such systems.

It is clear that to solve his determinate equations Diophantus needed only symbols for x and x^2 and not for x^n, $-6 \leq n \leq 6$. In other words, he extended his domain of numbers and introduced most of his symbols to investigate and solve indeterminate equations, where he really needed higher powers of the unknown as well as its negative powers.

Thus the birth of literal algebra was connected not with determinate but with indeterminate equations.

Here we present just one of Diophantus' methods, namely his method for finding the rational solutions of a quadratic equation in two unknowns:

$$F_2(x,y) = 0, \tag{3}$$

where $F_2(x,y)$ is a quadratic polynomial with rational coefficients.

Basically, Diophantus proves the following theorem: if equation (3) has a rational solution (x_0, y_0) then it has infinitely many such solutions (x, y), and x and y are both rational functions (with rational coefficients) of a single parameter:

$$x = \varphi(k), \quad y = \psi(k). \tag{4}$$

When presenting his methods we use modern algebraic symbolism. This is by now a standard procedure in historical-mathematical literature.

Diophantus began by considering quadratic equations of the form

$$y^2 = ax^2 + bx + c, \quad a, b, c \in \mathbf{Q}, \tag{5}$$

and put $c = m^2$ (in other words, he assumed that the equation had two rational solutions $(0, m)$ and $(0, -m)$). To find solutions he made the substitution

$$y = kx \pm m \tag{6}$$

and obtained

$$x = \frac{b \mp 2km}{k^2 - a}, \qquad y = \frac{b \mp 2km}{k^2 - a} \pm m.$$

By assigning to k all possible rational values (Diophantus took only values that yielded positive x and y) we obtain infinitely many solutions of equation (5).

We note that the substitutions (6) are the famous Euler substitutions that are applied to integrals of the form

$$\int \frac{dx}{\sqrt{ax^2 + bx + c}}.$$

We mentioned earlier problem II_8, which reduces to the equation

$$x^2 + y^2 = a^2, \tag{7}$$

and recall that Diophantus solved it by making the substitution

$$x = t; \quad y = kt - a, \tag{8}$$

and obtained (we are replacing his numerical values by appropriate letters)

$$x = t = a\frac{2k}{1 + k^2}; \qquad y = a\frac{k^2 - 1}{k^2 + 1}.$$

To see the sense of this solution and to appreciate its generality we must look at its geometric interpretation. Equation (7) determines a circle of radius a centered at the origin, and the substitution (8) is the equation of a straight line with slope k passing through the point $A(0, -a)$ on that circle (Figure 1). It is clear that the straight line (8) intersects the circle (7) in another point B with rational coordinates. Conversely, if there is a point B_1 with rational coordinates (x_1, y_1) on the circle (7) then AB_1 is a straight line of the pencil (8) with rational slope k. Thus to every rational k there corresponds a rational point on the circle (7) and to every rational point on the circle (7) there corresponds a rational value of k. Hence Diophantus' method yields all rational solutions of equation (7).

This argument shows that a conic with a rational point is birationally equivalent to a rational straight line.

Next Diophantus considered the more general case when equation (5) has a rational point but the coefficient c is not a square. He first considered this case in problem II_9, which reduces to the equation

$$x^2 + y^2 = a^2 + b^2 \tag{9}$$

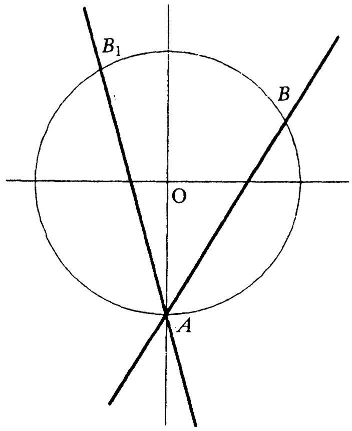

Figure 1

(Diophantus put $a = 2$, $b = 3$). It is clear that equation (9) has the following four solutions: (a, b), $(-a, b)$, $(a, -b)$, and $(-a, -b)$. Diophantus makes the substitution

$$x = t + a, \qquad y = kt - b \tag{10}$$

and obtains $t = 2(bk - a)/(1 + k^2)$. Applying a geometric interpretation analogous to the one just used we see that, essentially, he is leading a straight line with slope k through $(a, -b)$ on the circle (9).

Diophantus considered a more general case in lemma 2, proposition VI_{12} and in the lemma for proposition VI_{15}: assuming that equation (5) has a rational solution (x_0, y_0) he made the substitution $x = t + x_0$ and obtained

$$y^2 = at^2 + (2ax_0 + b)t + y_0^2,$$

i.e., he reduced the problem to the case $c = m^2$.

Finally, he considered equation (5) in the case when $a = \alpha^2$. He made the substitution (easily recognized as Euler's "second substitution" (*the authors*))

$$y = \alpha t \pm k \tag{11}$$

and obtained

$$t = \frac{c - k^2}{2\alpha k - b}.$$

This case calls for a separate discussion. To understand why the straight line (11) intersects the conic section (5) in just one point we introduce projective coordinates (U, V, W) by putting $x = U/W$, $y = V/W$, i.e., we consider our conic in the projective plane P^2. Then equation (5) takes the form

$$\alpha^2U^2 + bUW + cW^2. \tag{12}$$

The curve L so defined intersects the line at infinity $W = 0$ in two rational points: $(1, \alpha, 0)$ and $(1, -\alpha, 0)$. The straight line (11), whose equation in projective coordinates is

$$V = \alpha U + kW,$$

passes through the first of these points.

In summary, we can say that Diophantus carried out a complete investigation of a quadratic indeterminate equation in two unknowns. Later, his analysis served as a model for the investigation of the question of rational points on curves of genus 0.

Diophantus used more complex and more sophisticated methods to solve equations of the form

$$y^2 = ax^3 + bx^2 + cx + d,$$
$$y^3 = ax^3 + bx^2 + cx + d,$$
$$y^2 = ax^4 + bx^3 + cx^2 + dx + f,$$

and systems of the form

$$\begin{cases} ax^2 + bx + c = y^2, \\ a_1x^2 + b_1x + c_1 = z^2, \end{cases}$$

which he called "double equalities." Readers interested in getting a deeper understanding of Diophantus' methods should consult the book by Bashmakova and Slavutin: *A history of Diophantine analysis from Diophantus to Fermat* (Russian) which contains further references to the literature. The history of Diophantus' methods extends all the way to the papers of Poincaré that appeared at the beginning of the 20th century. It was on the basis of these methods that Poincaré constructed an arithmetic of algebraic curves—an area of mathematics that is being intensively developed at the present time.

We conclude our survey by considering Diophantus' problem III_{19}. This problem reduces to a system of 8 equations in 12 unknowns:

$$\begin{cases} (x_1 + x_2 + x_3 + x_4)^2 + x_i = y_i^2, \\ (x_1 + x_2 + x_3 + x_4)^2 - x_i = z_i^2; \quad i = 1,2,3,4. \end{cases}$$

Diophantus notes that "in every right triangle the square of the hypotenuse remains a square if we add to it, or subtract from it, twice the product of its legs." This means that he must find four right triangles with the same hypotenuse. Indeed, let the sides of the four triangles be a_i, b_i, c, $i = 1,2,3,4$. Then it suffices to put $x_1 + x_2 + x_3 + x_4 = ct$, $x_i = 2a_ib_it^2$, $i = 1,\dots,4$. Thus the problem reduces to finding a number c that can be written as a sum of two squares in four different ways. Diophantus solves this essentially number-theoretic problem as follows: he takes two right triangles with sides 3, 4, 5 and 5, 12, 13 respectively, and multiplies the sides of each of them by the hypotenuse of the other. As a result he obtains two right triangles with the same hypotenuse: 39, 42, 65 and 25, 60, 65. Now $5 = 1^2 + 2^2$ and $13 = 2^2 + 3^2$. Using the rule for composition of forms $u^2 + v^2$ known already to the Babylonians, namely

$$(u^2 + v^2)(\alpha^2 + \beta^2) = (\alpha u - \beta v)^2 + (\alpha v + \beta u)^2$$
$$= (\alpha u + \beta v)^2 + (\alpha v - \beta u)^2,$$

he obtains

$$65 = 5 \cdot 13 = (1^2 + 2^2)(2^2 + 3^2) = 4^2 + 7^2 = 8^2 + 1^2.$$

Using Euclid's formulas for the general solution of $x^2 + y^2 = z^2$ (i.e., $z = p^2 + q^2$; $x = p^2 - q^2$; $y = 2pq$) we obtain two more right triangles with hypotenuse 65: 33, 56, 65 and 63, 16, 65. This completes the solution of the problem.

In connection with this problem Fermat stated that a prime of the form $4n + 1$ could be written as a sum of squares in just one way. Then he gave a formula for the determination of the number of ways in which a given number can be written as a sum of squares. Thus problems involving indeterminate equations led to number-theoretic insights.

Did Diophantus know the theorems formulated by Fermat? It is possible that he did. Jacobi offered a reconstruction of Diophantus' conjectured proofs, but the answer to this question remains hypothetical.

One can hardly overestimate the significance of Diophantus' *Arithmetica* for the subsequent history of algebra. It is no exaggeration to say that its role was comparable to the role of Archimedes' treatises in the history of the differential and integral calculus. We will see that it was the starting point for all mathematicians up to Bombelli and Viète, and that its importance for number theory and for indeterminate equations can be traced up to the present.

4. ALGEBRA AFTER DIOPHANTUS. The period from the 4th to the 6th centuries AD was marked by the precipitous decline of ancient society and learning. But eminent commentators, such as Theon of Alexandria (second half of the 4th century) and his daughter Hypatia (murdered in 418 by a fanatical Christian mob), were still active. In the 5th century there was an exodus of scholars from Alexandria to Athens. Finally, in the 6th century, Eutocius and Simplicius, the last of the great commentators, were expelled from Athens and settled in Persia.

We can turn to the question of the Arabic translations of four books attributed to Diophantus. An analysis of these books, translated at the end of the 9th century from Greek to Arabic by Costa ibn Luca (i.e., the Greek Constantin, son of Luca) shows that they are a reworked version of Diophantus' *Arithmetica*. They contain problems, possibly due to Diophantus, as well as extensive additions and commentaries to them. According to Suidas' Byzantine dictionary, Hypatia wrote commentaries on *Arithmetica*. It is therefore very likely that the four books translated into Arabic are books edited and provided with commentaries by Hypatia. These books contain no new methods, but the material is presented in a complete and systematic manner. Their author went beyond Diophantus by introducing the 8th and 9th powers of the unknown.

The subsequent development of mathematics, including that of algebra, was connected with the Arabic East. Scholars from Syria, Egypt, Persia, and other regions conquered by the Arabs wrote scientific treatises in Arabic.

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