UNSOLVED PROBLEMS

Edited by **Richard Nowakowski**

In this department the MONTHLY presents easily stated unsolved problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial or related results. Typescripts should be sent to Richard Nowakowski, Department of Mathematics & Statistics, Dalhousie University, Halifax NS, Canada B3H 3J5, rjn@mscs.dal.ca

Unsolved Problems, 1969–1999

References in brackets are to year and page numbers of this MONTHLY, while dates in parentheses refer to publications listed at the end; other items are labelled (tbp) if they are likely to be published formally, or as written communications (wrc) if publication plans are not now known. Dates and pages in brackets are also appended to items in the bibliography indicating where the problem originally appeared in the MONTHLY.

Sommers (1998) gives some convex solutions to the sofa problem [1976, 188] and the Erikssons (1998) treat rectangular food-trolleys going round corners of any angle between corridors of different widths. A reference not made earlier, and not in **G5** of Croft, Falconer, and Guy (1991) is Davenport (1986).

In spite of exhortations [1983, 36], people continue to attempt to solve the $3x + 1$ problem. If we iterate the function $T(n) = n/2$ (n even), $(3n + 1)/2$ (n odd), then we can define the *stopping time*, $s(n)$, as the least number k of iterations that give $T^k(n) < n$, and the *maximum excursion*, $t(n)$, as the maximum value of $T^k(n)$ for $k > 0$. Are $s(n)$ and $t(n)$ always finite? Tomás Oliveira e Silva (1999) has verified the $3x + 1$ conjecture for $n \le 3 \cdot 2^{53} \approx 2.702 \cdot 10^{16}$. He lists all the record holders for $s(n)$ and $t(n)$ in this range, the largest being $s(1008932249296231) = 886$ and $t(10709980568908647) = 175294593968539094415936960141122$. There is evidence that $t(n) < n^2 f(n)$ where $f(n)$ is either constant or very slowly increasing. The highest value found of $t(n)/n^2$ is 7.527 for $n = 3716509988199$. For only 7 of the 76 record-holders is the value greater than one.

There has been a good deal written about polynomials, such as $x^3 - 33x^2 + 216x$, all of whose derivatives have integer roots [1989, 129]; Buchholz and MacDougall (tbp) give 34 references. For quartics and quintics the situation is fully understood, except that it has not been proved that there are no such polynomials with four or more distinct roots, nor quintics with three distinct roots, one of them triple.

Coxeter (1989) solved his 'challenging definite integral' [1988, 330] geometrically, while Peter Wagner (1996) includes an analytical solution in proving the more general result,

$$\frac{5}{2}\int_{\frac{1}{2}\cot^2\alpha}^{1} \frac{\arccos x}{(2x+1)\sqrt{x+1}}\left[\frac{\sqrt{2}\cos\alpha}{\sqrt{2x\sin^2\alpha-\cos 2\alpha}}+\frac{2}{\sqrt{x}}\right]dx$$

$$=\int_{\alpha_0}^{\alpha}\arccos\left(\frac{\cos 2\alpha(\sin^2 t+\cos 2\alpha)}{\sin^2 t-\cos^2 2\alpha}\right)dt$$

$$-\int_{\alpha_1}^{\alpha}\arccos\left(\frac{\cos 2t}{1-2\cos 2t}\right)dt+4\pi\left(\alpha-\frac{\pi}{4}\right)_+$$

valid for $\alpha_1 \le \alpha \le \pi/2$, where $\alpha_0 = \arccos(\cot\alpha\sqrt{1-2\cos 2\alpha})$, $\alpha_1 = \operatorname{arccot}\sqrt{2}$, and $\left(\alpha-\frac{\pi}{4}\right)_+$ is Heaviside's function, namely $\alpha-\frac{\pi}{4}$ for $\alpha \ge \frac{\pi}{4}$ and 0 for $\alpha \le \frac{\pi}{4}$.

Shattuck and Cooper (tbp) have found divergent RATS sequences [1989, 425] in bases 50, 99, 148, 962, $18n+1$, $18n+10$, and $(2^t-1)^2+1$, where t is a prime or pseudoprime, base 2. Conway's conjecture, that in base 10, all RATS (Reverse, Add, Then Sort) sequences either cycle or are tributary to the sequence

$$1\,2\,3^m\,4^4\,5^2\,6^m\,7^4,\ 1\,2\,3^{m+1}\,4^4\,5^2\,6^{m+1}\,7^4,\ldots,$$

remains an open question.

Scott Hochwald corrected [1993, 947] a result of Tony Gardiner [1988, 927] and now has further results. Let $S(n,p)=\sum_{k=1}^{p-1}(k^n)$ and $H(n,p)=\sum_{k=1}^{p}(1/k^n)$. Let $A(n)$ be the set of primes, p, such that the numerator of $H(p^n,p-1)$ is divisible by p^{n+3} and let $B(n)$ be the set of primes, p, such that $S(p^{2n+1}-p^{2n}-p^n,p)$ is divisible by p^{n+3}. Gardiner showed that

$$\{\text{primes } p: \text{the numerator of } H(1,p-1) \text{ is divisible by } p^3\}$$
$$=\{\text{primes } p: \text{ the numerator of } H(2,p-1) \text{ is divisible by } p^2\}.$$

Hochwald has shown that these two sets are equal to $A(n)$ and $B(n)$ for $n=1,2,3,4,\ldots$.

Also, he has shown that if p is a prime larger than 3, and if m and n are positive integers chosen so that m is not divisible by $p-1$, then the numerator of $H(mp^n,p-1)$ is divisible by p^{n+1}; and the numerator of $H(mp^n,p-1)$ is divisible by p^{n+2} whenever m is odd and $m+1$ is not divisible by $p-1$.

Further coin-weighing [1995, 164] results where given by Wan and Du (1997). Suppose there are n coins of which d are light. Let $M_A(n;d)$ ($M_A(n,d)$) denote the maximum number of tests needed by an algorithm A to sort n coins where the number, d, of light coins is unknown (known). The algorithm A has a competitive ratio of c if there is a constant b such that for all $0<d<n$, $M_A(n;d)\le c\cdot M_A(n,d)+b$. Wan et al. found an algorithm with competitive ratio $1/2+\ln 3$. This improves on the earlier values, $(3/2)\ln 3$, $2\ln 3$, and $3\ln 3$, of c found by Wan, Yang, and Kelley (1997), Hu and Hwang (1994), and Hu, Chen, and Hwang (1995).

Andrew Bremner (tbp) has continued the search for a 3×3 magic square whose entries are distinct squares [1995, 925]. He approaches the problem from two different directions: to find a magic square with a maximum number of square entries; to find a square with square entries and a maximum number of magic sums. He gives a parametric solution with 7 of the 8 magic sums equal, but can achieve a truly magic square only in fields of degrees $4, 8, 16, 20, 24, 27, 28, 32, 34, \ldots$. Lee Sallows (1997) has a relevant article.

In writing about the problem [1997, 359] of finding solutions to the equation $\phi(n) + \sigma(n) = kn$ proposed by Zhang, Lin, and Wang, where $\sigma(n)$ is the sum of divisors function and $\phi(n)$ is Euler's totient function, we omitted C. A. Nicol's (1966) paper in which he shows that if $k \geq 3$, then n is not squarefree, and if k is odd, then n is even or the square of an odd composite integer. He also shows that, for $k = 3$, if $q = 7 \cdot 2^{r-2} - 1$ is prime, then $n = 2^r \cdot 3q$ is a solution; this is so for $r = 3, 7, 11, 19, 23, 31, 47, 179, 18383, 22531, 24559, 26111, 34859, 41959, 67423$, and 70211, but no one is likely to show that this gives an infinity of solutions. In a 98-04-26 email, James Ordway sent the solution $n = 2^6 \cdot 3 \cdot 113 \cdot 6343$ for the case $k = 3$.

Irving Kaplansky notes that not every integer is the sum of three cubes, as was asked in [1998, 953], but those that are not $\pm 4 \bmod 9$ may be.

In describing the 'greedy odd algorithm' for Egyption fractions [1998, 953], it should have been made clear that the $1/n$ that was to be subtracted from a given rational number should have the smallest **odd** n that left a non-negative remainder. The question is, does repetition of the process always lead to a zero remainder? A more spectacular example, $2/24631$, was found by Broadhurst; the numerators are

$$2, 3, 4, 5, 6, \ldots, 25, 2, 3, 4, 5, 2, 1.$$

Even more spectacular examples were found by Broadhurst:

fraction	with	numerator
$2/588391$	28	$2 \ldots 28\ 1$
$4/538199$	28	$4 \ldots 25\ 2 \ldots 5\ 2\ 1$
$3/46547$	29	$3 \ldots 25\ 2 \ldots 5\ 2\ 1$
$2/24631$	30	$2 \ldots 25\ 2 \ldots 5\ 2\ 1$
$6/104651$	30	$6 \ldots 32\ 3\ 2\ 1$
$5/1444613$	37	$5\ 2 \ldots 34\ 43\ 64\ 1$

where the last denominator is a number of 384122451172 digits. David Eppstein (wrc) mentions that $7/1113923414579765333660423$ also has a long expansion.

Kevin Brown has a method for constructing fractions with arbitrarily long odd greedy expansions at
`http://www.seanet.com/ ksbrown/kmath478.htm.`

Gary Mulkey (wrc) and Tom Hagedorn [tbp] have each proved the Hardin-Sloane conjecture [1998, 953] that if $n > 3$ is odd and not a multiple of 3, then $3/n$ can be expressed as the sum of the reciprocals of three distinct odd positive integers.

Marc Paulhus [1999, 162] should have referred to Beasley (1989), who devoted four pages to Beggar-My-Neighbour, giving a computer simulation and a probabilistic heuristic that there is at least a 90% of there being a loop in the game, but noting the common feature of many combinatorial problems, that, as the numbers increase, the size of the haystack increases exponentially relative to the size of the needle we're looking for. Our interest in the problem was restimulated by independent enquiries from Reg Allenby and John Mackay; further interest may be generated by the recent television production of *Great Expectations*.

We are indebted to numerous correspondents for help with this compilation.

John D. Beasley, *The Mathematics of Games*, Oxford University Press, 1989, esp. pp. 149–153. [1999, 162]

Andrew Bremner, On squares of squares, *Acta Arith.* (to appear). [1995, 925]

Ralph H. Buchholz and James A. MacDougall, Rational-derived polynomials and their extensions to quadratic fields, preprint. [1989, 129]

H. S. M. Coxeter, Trisecting an orthoscheme; symmetry 2: unifying human understanding, Part 1, *Comput. Math. Appl.* **17** (1989) 59–71; *MR* **90d**:51034. [1988, 330]

H. T. Croft, K. J. Falconer, and R. K. Guy, *Unsolved Problems in Geometry*, Springer-Verlag, New York, 1991, §**G5**. [1976, 188]

J. H. Davenport, A "piano movers" problem, *SIGSAM* **76** (1986) 15–17. [1976, 188]

Gerd Eriksson, Henrik Eriksson and Kimmo Eriksson, Moving a food trolley around a corner, *Theoret. Comput. Sci.* **191** (1998) 193–203. [1976, 188]

Thomas R. Hagedorn, A proof of the Hardin-Sloane conjecture, preprint. [1998, 953]

X.-D. Hu, P.-D. Chen, and F. K. Hwang, A new competitive algorithm for the counterfeit coin problem, *Inform. Process Lett.* **51** (1994) 213–218. [1995, 164]

X.-D. Hu and F. K. Hwang, A competitive algorithm for the counterfeit coin problem, in *D.-Z. Du and P. M. Pardalos, eds.*, Minimax and Applications, Kluwer Academic Publishers, 1995, pp. 241–250. [1995, 164]

C. A. Nicol, Some diophantine equations involving arithmetic functions, *J. Math. Anal. Appl.* **15** (1966) 154–161; *MR* **33** #4007. [1997, 358]

Lee Sallows, The lost theorem, *Math. Intelligencer* **19** (1997) 51–54. [1995, 925]

Tomás Oliveira e Silva, Maximum excursion and stopping time record-holders for the $3x + 1$ problem: computational results, *Math. Comput.* **68** (1999) 371–384. [1983, 36]

Steven Shattuck and Curtis Cooper, Divergent RATS sequences, *Proc. Ninth Internat. Conf. Fibonacci Numbers Appl.*, Luxembourg, 2000 (to appear). [1989, 425]

James A. Sommers, On some convex solutions to the sofa problem (98-05-25 preprint). [1976, 188]

Peter Wagner, Solution to a problem posed by H. S. M. Coxeter, *C. R. Math. Rep. Acad. Sci. Canada* **18** (1996) 273–277. [1988, 330]

Wan Peng-Jun, Yang Qi-Fan, and Dean Kelley, A $(3/2)\log 3$-competitive algorithm for the counterfeit coin problem, *Theor. Comput. Sci.* **181** (1997) 347–356. [1995, 164]

Wan Peng-Jun and Du Ding-Zhu, A $(\log_2 3 + 1/2)$-competitive algorithm for the counterfeit coin problem, *Discrete Math.* **163** (1997) 173–200. [1995, 164]