10773

Jean Anglesio

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10773. *Proposed by Jean Anglesio, Garches, France.* Let $a_0, a_1, \dots, a_k$ be positive integers. For $0 \le i \le k$, let p_i/q_i be the fraction in lowest terms with continued fraction expansion $[a_0, a_1, \dots, a_i]$. Find the continued fraction expansions of

$$\sqrt{\frac{p_k\, p_{k-1}}{q_k\, q_{k-1}}},\ \sqrt{\frac{p_k\, q_k}{p_{k-1}\, q_{k-1}}},\ \sqrt{\frac{p_k^2 + p_{k-1}^2}{q_k^2 + q_{k-1}^2}},\ \text{and}\ \sqrt{\frac{p_k^2 + q_k^2}{p_{k-1}^2 + q_{k-1}^2}}$$

in terms of $a_0, a_1, \dots, a_k$.

SOLUTIONS

Tracking the Incenters

10631 [1997, 975]. *Proposed by Greg Huber, University of Chicago, Chicago, IL.* Given a triangle T, let the *intriangle* of T be the triangle whose vertices are the points where the circle inscribed in T touches T. Given a triangle T_0, form a sequence of triangles $T_0, T_1, T_2, \dots$ in which each T_{n+1} is the intriangle of T_n. Let d_n be the distance between the incenters of T_n and T_{n+1}. Find $\lim_{n\to\infty} d_{n+1}/d_n$ when T_0 is not equilateral.

Solution by the GCHQ Problems Group, Cheltenham, U. K. We show that $d_{n+1}/d_n \to 1/4$. Let A, B, C be the angles of a triangle, r its inradius, R its circumradius, and d the distance from its incenter to its circumcenter. Then

$$d^2 = R^2 - 2Rr \tag{1}$$

and

$$r = 4R\sin(A/2)\sin(B/2)\sin(C/2). \tag{2}$$

(H. S. M. Coxeter and S. L. Greitzer, *Geometry Revisited,* MAA, 1967). Now let A', B', C' be the angles of the intriangle of ABC (with A' on side BC, etc.). Then $A' = \pi/2 - A/2$, so

$$A' - \pi/3 = (-1/2)(A - \pi/3), \tag{3}$$

and similarly for B' and C'. From (3) we infer that triangle T_n approaches equilateral as $n \to \infty$. For the triangle T_n, with angles A_n, B_n, C_n, define $a_n = A_n - \pi/3$, $b_n = B_n - \pi/3$, $c_n = C_n - \pi/3$, and $S_n = a_n^2 + b_n^2 + c_n^2$. Then (3) implies that $S_{n+1}/S_n = 1/4$. Also, $a_n + b_n + c_n = 0$, so $(a_n + b_n + c_n)^2 = 0$, and therefore

$$S_n = -2(a_n b_n + b_n c_n + c_n a_n). \tag{4}$$

Now define $U_n = 1 - 8\sin(A_n/2)\sin(B_n/2)\sin(C_n/2)$. Using (1) and (2) and observing that $R_{n+1} = r_n$, we obtain

$$\left(\frac{d_{n+1}}{d_n}\right)^2 = \frac{R_{n+1}^2}{R_n^2}\,\frac{U_{n+1}}{U_n} = 16\sin^2(A_n/2)\sin^2(B_n/2)\sin^2(C_n/2)\,\frac{U_{n+1}}{U_n}. \tag{5}$$

Note that

$$\begin{aligned} 2\sin(A_n/2) &= 2\sin(a_n/2 + \pi/6) = \sqrt{3}\,\sin(a_n/2) + \cos(a_n/2) \\ &= 1 + \frac{\sqrt{3}}{2}\,a_n - \frac{1}{8}\,a_n^2 + O(a_n^3). \end{aligned}$$

Therefore

$$\begin{aligned} U_n &= 1 - \left(1 + \frac{\sqrt{3}}{2}\,a_n - \frac{1}{8}\,a_n^2 + \cdots\right)\left(1 + \frac{\sqrt{3}}{2}\,b_n - \frac{1}{8}\,b_n^2 + \cdots\right)\left(1 + \frac{\sqrt{3}}{2}\,c_n - \frac{1}{8}\,c_n^2 + \cdots\right) \\ &= \frac{1}{8}\,S_n - \frac{3}{4}\big(a_n b_n + b_n c_n + c_n a_n\big) + \text{ terms of degree 3 or higher} \\ &= \frac{1}{2}\,S_n + \text{ terms of degree 3 or higher,} \end{aligned}$$