10773. *Proposed by Jean Anglesio, Garches, France.* Let $a_0, a_1, \ldots, a_k$ be positive integers. For $0 \le i \le k$, let p_i/q_i be the fraction in lowest terms with continued fraction expansion $[a_0, a_1, \ldots, a_i]$. Find the continued fraction expansions of

$$\sqrt{\frac{p_k\,p_{k-1}}{q_k\,q_{k-1}}},\ \sqrt{\frac{p_k\,q_k}{p_{k-1}\,q_{k-1}}},\ \sqrt{\frac{p_k^2+p_{k-1}^2}{q_k^2+q_{k-1}^2}},\ \text{and}\ \sqrt{\frac{p_k^2+q_k^2}{p_{k-1}^2+q_{k-1}^2}}$$

in terms of $a_0, a_1, \ldots, a_k$.

SOLUTIONS

Tracking the Incenters

10631 [1997, 975]. *Proposed by Greg Huber, University of Chicago, Chicago, IL.* Given a triangle T, let the *intriangle* of T be the triangle whose vertices are the points where the circle inscribed in T touches T. Given a triangle T_0, form a sequence of triangles $T_0, T_1, T_2, \ldots$ in which each T_{n+1} is the intriangle of T_n. Let d_n be the distance between the incenters of T_n and T_{n+1}. Find $\lim_{n\to\infty} d_{n+1}/d_n$ when T_0 is not equilateral.

Solution by the GCHQ Problems Group, Cheltenham, U. K. We show that $d_{n+1}/d_n \to 1/4$. Let A, B, C be the angles of a triangle, r its inradius, R its circumradius, and d the distance from its incenter to its circumcenter. Then

$$d^2 = R^2 - 2Rr \tag{1}$$

and

$$r = 4R\sin(A/2)\sin(B/2)\sin(C/2). \tag{2}$$

(H. S. M. Coxeter and S. L. Greitzer, *Geometry Revisited,* MAA, 1967). Now let A', B', C' be the angles of the intriangle of ABC (with A' on side BC, etc.). Then $A' = \pi/2 - A/2$, so

$$A' - \pi/3 = (-1/2)(A - \pi/3), \tag{3}$$

and similarly for B' and C'. From (3) we infer that triangle T_n approaches equilateral as $n \to \infty$. For the triangle T_n, with angles A_n, B_n, C_n, define $a_n = A_n - \pi/3$, $b_n = B_n - \pi/3$, $c_n = C_n - \pi/3$, and $S_n = a_n^2 + b_n^2 + c_n^2$. Then (3) implies that $S_{n+1}/S_n = 1/4$. Also, $a_n + b_n + c_n = 0$, so $(a_n + b_n + c_n)^2 = 0$, and therefore

$$S_n = -2(a_nb_n + b_nc_n + c_na_n). \tag{4}$$

Now define $U_n = 1 - 8\sin(A_n/2)\sin(B_n/2)\sin(C_n/2)$. Using (1) and (2) and observing that $R_{n+1} = r_n$, we obtain

$$\left(\frac{d_{n+1}}{d_n}\right)^2 = \frac{R_{n+1}^2}{R_n^2}\,\frac{U_{n+1}}{U_n} = 16\sin^2(A_n/2)\sin^2(B_n/2)\sin^2(C_n/2)\,\frac{U_{n+1}}{U_n}. \tag{5}$$

Note that

$$\begin{aligned} 2\sin(A_n/2) &= 2\sin(a_n/2 + \pi/6) = \sqrt{3}\,\sin(a_n/2) + \cos(a_n/2) \\ &= 1 + \frac{\sqrt{3}}{2}\,a_n - \frac{1}{8}\,a_n^2 + O(a_n^3). \end{aligned}$$

Therefore

$$\begin{aligned} U_n &= 1 - \left(1 + \frac{\sqrt{3}}{2}\,a_n - \frac{1}{8}\,a_n^2 + \cdots\right)\left(1 + \frac{\sqrt{3}}{2}\,b_n - \frac{1}{8}\,b_n^2 + \cdots\right)\left(1 + \frac{\sqrt{3}}{2}\,c_n - \frac{1}{8}\,c_n^2 + \cdots\right) \\ &= \frac{1}{8}\,S_n - \frac{3}{4}\big(a_nb_n + b_nc_n + c_na_n\big) + \text{ terms of degree 3 or higher} \\ &= \frac{1}{2}\,S_n + \text{ terms of degree 3 or higher,} \end{aligned}$$

by (4). Therefore $\lim_{n\to\infty} U_{n+1}/U_n = \lim_{n\to\infty} S_{n+1}/S_n = 1/4$. Putting $\lim_{n\to\infty} A_n = \lim_{n\to\infty} B_n = \lim_{n\to\infty} C_n = \pi/3$ into (5) yields $d_{n+1}^2/d_n^2 \to 1/16$, or $d_{n+1}/d_n \to 1/4$.

Solved also by J. Anglesio (France), G. L. Body (U. K.), R. J. Chapman (U. K.), J. E. Dawson (Australia), N. Lakshmanan, J. H. Lindsey II, P. G. Poonacha (India), V. Schindler (Germany), A. Tissier (France), and the proposer.

An Appearance of the Beta Function

10632 [1997, 975]. *Proposed by William F. Trench, Trinity University, San Antonio, TX.* For given nonnegative integers m and n, evaluate

$$\sum_{k=0}^{m} \frac{(-1)^k}{n+k+1}\binom{m}{k}(1-y)^{n+k+1} + \sum_{k=0}^{n} \frac{(-1)^k}{m+k+1}\binom{n}{k} y^{m+k+1}.$$

Solution by Ronald A. Kopas, Clarion University of Pennsylvania, Clarion, PA. The sum is $m!\,n!/(m+n+1)!$. To see this, note that

$$\int_0^y t^m(1-t)^n\,dt = \int_0^y t^m \sum_{k=0}^{n}\binom{n}{k}(-1)^k t^k\,dt$$
$$= \sum_{k=0}^{n}\binom{n}{k}(-1)^k \int_0^y t^{m+k}\,dt = \sum_{k=0}^{n}\binom{n}{k}(-1)^k \frac{y^{m+k+1}}{m+k+1}.$$

Substituting $1-t$ for t and then computing in the same way yields

$$\int_y^1 t^m(1-t)^n\,dt = \int_0^{1-y} t^n(1-t)^m\,dt = \sum_{k=0}^{m}\binom{m}{k}(-1)^k\frac{(1-y)^{n+k+1}}{n+k+1}.$$

Hence the desired sum equals $\int_0^1 t^m(1-t)^n\,dt$, which repeated integration by parts reduces to $m!\,n!/(m+n+1)!$.

Editorial comment. Most solvers first differentiated the given expression to show that it was independent of y. They then evaluated the expression at $y=0$ or $y=1$ and got the final result either by induction or by reducing it to the beta integral that appears in the published solution.

Solved also by U. Abel (Germany), K. F. Andersen (Canada), P. J. Anderson (Canada), J. Anglesio (France), G. W. Arnold, G. Bach (Germany), D. Beckwith, J. C. Binz (Switzerland), G. L. Body (U. K.), P. Bracken (Canada), D. Callan, R. J. Chapman (U. K.), Q. H. Darwish (Oman), M. N. Deshpande (India), P. Devaraj & R. S. Deodhar (India), S. B. Ekhad, Z. Franco & M. Wood, R. García-Pelayo (Spain), C. Georghiou (Greece), T. Hermann, V. Hernández & J. Martín (Spain), D. Huang, G. Kesselman, M. S. Klamkin (Canada), R. A. Leslie, N. F. Lindquist, J. H. Lindsey II, S. McDonald & K. Adzievski, J. G. Merickel, C. A. Minh, D. A. Morales (Venezuela), R. G. Mosier, A. Nijenhuis, M. Omarjee (France), G. Peng, H. Qin, V. Schindler (Germany), H.-J. Seiffert (Germany), P. Simeonov, N. C. Singer, J. H. Steelman, R. F. Swarttouw (The Netherlands), A. Tissier (France), E. I. Verriest, M. Vowe (Switzerland), H. Widmer (Switzerland), M. Woltermann, Y. Yang, Q. Yao, Anchorage Math Solutions Group, BARC Problems Group (India), GCHQ Problems Group (U. K.), NSA Problems Group, WMC Problems Group, and the proposer.

Apéry's Constant

10635 [1998, 68]. *Proposed by Nicholas R. Farnum, California State University, Fullerton, CA.* Show that the value of $\zeta(s) = \sum_{k=1}^{\infty} k^{-s}$ at $s=3$, also called Apéry's constant, can be expressed as $\zeta(3) = \sum_{n=1}^{\infty} r_n/n$, where $r_n = (\pi^2/6) - \sum_{k=1}^{n} k^{-2}$ is the nth remainder of the series expansion of $\zeta(2)$.

Solution by Alain Tissier, Montfermeil, France. We prove a generalization: For each positive integer k,

$$k!\,\zeta(k+2) = \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\cdots\sum_{n_k=1}^{\infty}\frac{1}{n_1 n_2 \ldots n_k}\sum_{p=1+n_1+n_2+\cdots+n_k}^{\infty}\frac{1}{p^2}. \qquad (*)$$