Apery's Constant: 10635

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by (4). Therefore $\lim_{n\to\infty} U_{n+1}/U_n = \lim_{n\to\infty} S_{n+1}/S_n = 1/4$. Putting $\lim_{n\to\infty} A_n = \lim_{n\to\infty} B_n = \lim_{n\to\infty} C_n = \pi/3$ into (5) yields $d_{n+1}^2/d_n^2 \to 1/16$, or $d_{n+1}/d_n \to 1/4$.

Solved also by J. Anglesio (France), G. L. Body (U. K.), R. J. Chapman (U. K.), J. E. Dawson (Australia), N. Lakshmanan, J. H. Lindsey II, P. G. Poonacha (India), V. Schindler (Germany), A. Tissier (France), and the proposer.

An Appearance of the Beta Function

10632 [1997, 975]. *Proposed by William F. Trench, Trinity University, San Antonio, TX.* For given nonnegative integers m and n, evaluate

$$\sum_{k=0}^{m} \frac{(-1)^k}{n+k+1}\binom{m}{k}(1-y)^{n+k+1} + \sum_{k=0}^{n} \frac{(-1)^k}{m+k+1}\binom{n}{k} y^{m+k+1}.$$

Solution by Ronald A. Kopas, Clarion University of Pennsylvania, Clarion, PA. The sum is $m!\,n!/(m+n+1)!$. To see this, note that

$$\begin{aligned}\int_0^y t^m(1-t)^n\,dt &= \int_0^y t^m \sum_{k=0}^{n}\binom{n}{k}(-1)^k t^k\,dt \\ &= \sum_{k=0}^{n}\binom{n}{k}(-1)^k \int_0^y t^{m+k}\,dt = \sum_{k=0}^{n}\binom{n}{k}(-1)^k \frac{y^{m+k+1}}{m+k+1}.\end{aligned}$$

Substituting $1-t$ for t and then computing in the same way yields

$$\int_y^1 t^m(1-t)^n\,dt = \int_0^{1-y} t^n(1-t)^m\,dt = \sum_{k=0}^{m}\binom{m}{k}(-1)^k \frac{(1-y)^{n+k+1}}{n+k+1}.$$

Hence the desired sum equals $\int_0^1 t^m(1-t)^n\,dt$, which repeated integration by parts reduces to $m!\,n!/(m+n+1)!$.

Editorial comment. Most solvers first differentiated the given expression to show that it was independent of y. They then evaluated the expression at $y=0$ or $y=1$ and got the final result either by induction or by reducing it to the beta integral that appears in the published solution.

Solved also by U. Abel (Germany), K. F. Andersen (Canada), P. J. Anderson (Canada), J. Anglesio (France), G. W. Arnold, G. Bach (Germany), D. Beckwith, J. C. Binz (Switzerland), G. L. Body (U. K.), P. Bracken (Canada), D. Callan, R. J. Chapman (U. K.), Q. H. Darwish (Oman), M. N. Deshpande (India), P. Devaraj & R. S. Deodhar (India), S. B. Ekhad, Z. Franco & M. Wood, R. García-Pelayo (Spain), C. Georghiou (Greece), T. Hermann, V. Hernández & J. Martín (Spain), D. Huang, G. Kesselman, M. S. Klamkin (Canada), R. A. Leslie, N. F. Lindquist, J. H. Lindsey II, S. McDonald & K. Adzievski, J. G. Merickel, C. A. Minh, D. A. Morales (Venezuela), R. G. Mosier, A. Nijenhuis, M. Omarjee (France), G. Peng, H. Qin, V. Schindler (Germany), H.-J. Seiffert (Germany), P. Simeonov, N. C. Singer, J. H. Steelman, R. F. Swarttouw (The Netherlands), A. Tissier (France), E. I. Verriest, M. Vowe (Switzerland), H. Widmer (Switzerland), M. Woltermann, Y. Yang, Q. Yao, Anchorage Math Solutions Group, BARC Problems Group (India), GCHQ Problems Group (U. K.), NSA Problems Group, WMC Problems Group, and the proposer.

Apéry's Constant

10635 [1998, 68]. *Proposed by Nicholas R. Farnum, California State University, Fullerton, CA.* Show that the value of $\zeta(s) = \sum_{k=1}^{\infty} k^{-s}$ at $s=3$, also called Apéry's constant, can be expressed as $\zeta(3) = \sum_{n=1}^{\infty} r_n/n$, where $r_n = (\pi^2/6) - \sum_{k=1}^{n} k^{-2}$ is the nth remainder of the series expansion of $\zeta(2)$.

Solution by Alain Tissier, Montfermeil, France. We prove a generalization: For each positive integer k,

$$k!\,\zeta(k+2) = \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\cdots\sum_{n_k=1}^{\infty} \frac{1}{n_1 n_2 \ldots n_k} \sum_{p=1+n_1+n_2+\cdots+n_k}^{\infty} \frac{1}{p^2}. \qquad (*)$$

The case $k = 1$ gives the result of this problem, while the case $k = 2$ gives

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{nm}\left(\frac{\pi^2}{6}-\sum_{p=1}^{n+m}\frac{1}{p^2}\right)=\frac{\pi^4}{45}.$$

To prove (∗), observe that $\int_0^1 t^{p-1}(-\ln t)\,dt = p^{-2}$ for each positive integer p. This is the base case of a proof by induction that $\int_0^1 t^{p-1}(-\ln t)^m\,dt = m!/p^{m+1}$ for all positive integers p and m. Hence

$$\begin{aligned}
&\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\cdots\sum_{n_k=1}^{\infty}\frac{1}{n_1n_2\ldots n_k}\sum_{p=1+n_1+n_2+\cdots+n_k}^{\infty}\frac{1}{p^2}\\
&=\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\cdots\sum_{n_k=1}^{\infty}\frac{1}{n_1n_2\ldots n_k}\int_0^1\frac{t^{n_1+n_2+\cdots+n_k}}{1-t}(-\ln t)\,dt\\
&=\int_0^1\frac{(-\ln(1-t))^k}{1-t}(-\ln t)\,dt=\int_0^1\frac{(-\ln s)^k}{s}(-\ln(1-s))\,ds\\
&=\frac{1}{k+1}\Big[(-\ln s)^{k+1}\ln(1-s)\Big]_0^1+\frac{1}{k+1}\int_0^1\frac{(-\ln s)^{k+1}}{1-s}\,ds\\
&=\frac{1}{k+1}\int_0^1\frac{(-\ln s)^{k+1}}{1-s}\,ds=\frac{1}{k+1}\sum_{n=1}^{\infty}\int_0^1 s^{n-1}(-\ln s)^{k+1}\,ds=k!\sum_{n=1}^{\infty}\frac{1}{n^{k+2}}.
\end{aligned}$$

Editorial comment. A closely related identity appeared as Problem 4431 [1951, 195; 1952, 471] in this MONTHLY. A generalization appeared as Problem 1302 in *Math. Mag.* **62** (1989) 275: For each integer $n \geq 3$, $\zeta(n) = \sum_{i=1}^{n-2}\sum_{p,q=1}^{\infty} p^{-i}(p+q)^{i-n}$.

Readers pointed out a large number of references to this problem and various generalizations, going back to work of Euler in 1743. Among these references were: W. E. Briggs, S. Chowla, A. J. Kempner, and W. E. Mientka, On some infinite series, *Scripta Math.* **21** (1955) 28; J. M. Borwein and R. Girgensohn, Evaluating triple Euler sums, *Electronic J. Comb.* **3** (1996) R23; and B. C. Berndt, *Ramanujan's Notebooks*, Part I (1985) Springer-Verlag, p. 252.

Solved also by P. J. Andersen (Canada), J. Anglesio (France), D. & J. Borwein (Canada), P. Bracken (Canada), D. M. Bradley (Canada), D. Callan, R. J. Chapman (U. K.), H. Chen, C. Georghiou (Greece), W. Janous (Austria), P. Khalili, V. Lucic (Canada), J. G. Merickel, S. Northshield, L. Quet, V. Schindler (Germany), H.-J. Seifert (Germany), M. Sharma (India), P. Simeonov, I. Sofair, A. Stenger, T. V. Trif (France), D. B. Tyler, J. J. van Lint (Netherlands), M. Vowe (Switzerland), GCHQ Problems Group (U. K.), and the proposer.

Essentially Discontinuous Functions

10668 [1998, 465]. *Proposed by Abram Kagan, University of Maryland, College Park, MD, and Larry Shepp, Rutgers University, New Brunswick, NJ.* Let H be an infinite-dimensional closed subspace of $L^2[0, 1]$. Prove that H contains a function f that is essentially discontinuous, meaning that there is no continuous function g on $[0, 1]$ equal to f almost everywhere. Does the conclusion remain true if g is required to be continuous only on $(0, 1]$?

Solution by Kenneth Schilling, University of Michigan, Flint, MI. Suppose, for purposes of contradiction, that H contains no essentially discontinuous function. Then, since every continuous function on $[0, 1]$ is bounded, every element of H is essentially bounded. Let $\|\cdot\|_2$ and $\|\cdot\|_\infty$ denote the L^2 and essential supremum norms, respectively. Since $\|f\|_2 \leq \|f\|_\infty$, the identity map from the Banach space $(H, \|\cdot\|_2)$ to the Banach space $(H, \|\cdot\|_\infty)$ is continuous. By the Open Mapping Theorem, the inverse function is also continuous, so there exists $K > 0$ such that $\|f\|_\infty \leq K \cdot \|f\|_2$ for all $f \in H$.