The case $k = 1$ gives the result of this problem, while the case $k = 2$ gives

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{nm}\left(\frac{\pi^2}{6}-\sum_{p=1}^{n+m}\frac{1}{p^2}\right)=\frac{\pi^4}{45}.$$

To prove (∗), observe that $\int_0^1 t^{p-1}(-\ln t)\,dt = p^{-2}$ for each positive integer p. This is the base case of a proof by induction that $\int_0^1 t^{p-1}(-\ln t)^m\,dt = m!/p^{m+1}$ for all positive integers p and m. Hence

$$\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\cdots\sum_{n_k=1}^{\infty}\frac{1}{n_1n_2\ldots n_k}\sum_{p=1+n_1+n_2+\cdots+n_k}^{\infty}\frac{1}{p^2}$$
$$=\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\cdots\sum_{n_k=1}^{\infty}\frac{1}{n_1n_2\ldots n_k}\int_0^1\frac{t^{n_1+n_2+\cdots+n_k}}{1-t}(-\ln t)\,dt$$
$$=\int_0^1\frac{(-\ln(1-t))^k}{1-t}(-\ln t)\,dt=\int_0^1\frac{(-\ln s)^k}{s}(-\ln(1-s))\,ds$$
$$=\frac{1}{k+1}\Big[(-\ln s)^{k+1}\ln(1-s)\Big]_0^1+\frac{1}{k+1}\int_0^1\frac{(-\ln s)^{k+1}}{1-s}\,ds$$
$$=\frac{1}{k+1}\int_0^1\frac{(-\ln s)^{k+1}}{1-s}\,ds=\frac{1}{k+1}\sum_{n=1}^{\infty}\int_0^1 s^{n-1}(-\ln s)^{k+1}\,ds=k!\sum_{n=1}^{\infty}\frac{1}{n^{k+2}}.$$

Editorial comment. A closely related identity appeared as Problem 4431 [1951, 195; 1952, 471] in this MONTHLY. A generalization appeared as Problem 1302 in *Math. Mag.* **62** (1989) 275: For each integer $n \geq 3$, $\zeta(n) = \sum_{i=1}^{n-2}\sum_{p,q=1}^{\infty} p^{-i}(p+q)^{i-n}$.

Readers pointed out a large number of references to this problem and various generalizations, going back to work of Euler in 1743. Among these references were: W. E. Briggs, S. Chowla, A. J. Kempner, and W. E. Mientka, On some infinite series, *Scripta Math.* **21** (1955) 28; J. M. Borwein and R. Girgensohn, Evaluating triple Euler sums, *Electronic J. Comb.* **3** (1996) R23; and B. C. Berndt, *Ramanujan's Notebooks*, Part I (1985) Springer-Verlag, p. 252.

Solved also by P. J. Andersen (Canada), J. Anglesio (France), D. & J. Borwein (Canada), P. Bracken (Canada), D. M. Bradley (Canada), D. Callan, R. J. Chapman (U. K.), H. Chen, C. Georghiou (Greece), W. Janous (Austria), P. Khalili, V. Lucic (Canada), J. G. Merickel, S. Northshield, L. Quet, V. Schindler (Germany), H.-J. Seifert (Germany), M. Sharma (India), P. Simeonov, I. Sofair, A. Stenger, T. V. Trif (France), D. B. Tyler, J. J. van Lint (Netherlands), M. Vowe (Switzerland), GCHQ Problems Group (U. K.), and the proposer.

Essentially Discontinuous Functions

10668 [1998, 465]. *Proposed by Abram Kagan, University of Maryland, College Park, MD, and Larry Shepp, Rutgers University, New Brunswick, NJ.* Let H be an infinite-dimensional closed subspace of $L^2[0, 1]$. Prove that H contains a function f that is essentially discontinuous, meaning that there is no continuous function g on $[0, 1]$ equal to f almost everywhere. Does the conclusion remain true if g is required to be continuous only on $(0, 1]$?

Solution by Kenneth Schilling, University of Michigan, Flint, MI. Suppose, for purposes of contradiction, that H contains no essentially discontinuous function. Then, since every continuous function on $[0, 1]$ is bounded, every element of H is essentially bounded. Let $\|\cdot\|_2$ and $\|\cdot\|_\infty$ denote the L^2 and essential supremum norms, respectively. Since $\|f\|_2 \leq \|f\|_\infty$, the identity map from the Banach space $(H, \|\cdot\|_2)$ to the Banach space $(H, \|\cdot\|_\infty)$ is continuous. By the Open Mapping Theorem, the inverse function is also continuous, so there exists $K > 0$ such that $\|f\|_\infty \leq K\cdot\|f\|_2$ for all $f \in H$.

Now let $f_1, \dots, f_n$ be continuous functions that are orthonormal in H. For all real numbers $a_1, \dots, a_n$ and all $x \in [0, 1]$, we have

$$\sum_{i=1}^{n} a_i f_i(x) \le K \cdot \left\| \sum_{i=1}^{n} \alpha_i f_i \right\|_2 = K \sqrt{\sum_{i=1}^{n} \alpha_i^2}.$$

Fix $x \in [0, 1]$, and let $\alpha_i = f_i(x)$. Then $\sum_{i=1}^{n}(f_i(x))^2 \le K \cdot \sqrt{\sum_{i=1}^{n}(f_i(x))^2}$, so $\sum_{i=1}^{n}(f_i(x))^2 \le K^2$. Integrating both sides from 0 to 1 gives $n \le K^2$. Thus every orthonormal set of continuous functions in H has at most K^2 elements. This contradicts the assumption that H is infinite-dimensional.

The conclusion does not follow with $(0, 1]$ in place of $[0, 1]$. For $n = 1, 2, \dots$, let $f_n\colon [0, 1] \to \mathbf{R}$ be a continuous function with $\|f_n\|_2 = 1$ and support in $(1/(n+1), 1/n)$. Then $\{f_n\}$ is an orthonormal set, so the map Φ: $l^2 \to L^2[0, 1]$ given by $\Phi(\alpha) = \sum_{n=1}^{\infty} \alpha_n f_n$ is a linear isometry. In addition, each $\Phi(\alpha)$ is continuous on $(0, 1]$, since for all $x \in (0, 1]$ there exists an open interval I about x such that $f_n \neq 0$ on I for at most one n. Thus the range of Φ is a closed, infinite-dimensional subspace of $L^2(0, 1]$ whose elements are continuous functions.

The first part of this problem is contained in problems 28 and 55 in Chapter 10 of H. L. Royden, *Real Analysis*, Third Edition, Macmillan, 1988. The solution here follows Royden's generous hints.

Solved also by P. J. Fitzsimmons, P. M. Jarvis, J. H. Lindsey II, A. Sasane (The Netherlands), and the proposers.

Two Recurrence Relations, One Easy, One Hard

10670 [1998, 559]. *Proposed by Salomon Benchimol and Elliott Cohen, Paris, France.*
(a) For which values of $u_0 > 0$ and $u_1 > 0$ does the sequence defined by $u_{n+2} = 1+u_{n+1}/u_n$ for $n \ge 0$ converge?
(b) For which values of $u_0 > 0$ and $u_1 > 0$ does the sequence defined by $u_{n+2} = 1+u_n/u_{n+1}$ for $n \ge 0$ converge?

Solution of part **(a)** *by Con Amore Problems Group, Copenhagen, Denmark.* This sequence converges to 2 for every choice of $u_0, u_1 > 0$. Clearly $u_n > 0$ for all n, so $u_n = 1 + u_{n-1}/u_{n-2} > 1$ for $n \ge 2$. If $n \ge 5$, then $u_n = 1+u_{n-1}/u_{n-2} = 1+1/u_{n-3}+1/u_{n-2} < 3$. This proves the $k = 0$ case of the following claim: For any $k \ge 0$,

$$u_n > \frac{2^{2k+2}-1}{2^{2k+1}+1} \text{ for } n \ge 6k+2, \qquad \text{and} \qquad u_n < \frac{2^{2k+3}+1}{2^{2k+2}-1} \text{ for } n \ge 6k+5.$$

This proves convergence, since both of these bounds converge to 2 as $k \to \infty$. We prove the claim by induction. Choose $k \ge 1$ and assume that the claim holds for smaller values of k. For $n \ge 6(k-1)+5 = 6k-1$, we have

$$u_n < \frac{2^{2(k-1)+3}+1}{2^{2(k-1)+2}-1} = \frac{2^{2k+1}+1}{2^{2k}-1}.$$

Therefore, for $n \ge 6k+2$, we have

$$u_n = 1 + \frac{1}{u_{n-2}} + \frac{1}{u_{n-3}} > 1 + 2\,\frac{2^{2k}-1}{2^{2k+1}+1} = \frac{2^{2k+2}-1}{2^{2k+1}+1},$$

as required. For $n \ge 6k+5$, we then have

$$u_n = 1 + \frac{1}{u_{n-2}} + \frac{1}{u_{n-3}} < 1 + 2\,\frac{2^{2k+1}+1}{2^{2k+2}-1} = \frac{2^{2k+3}+1}{2^{2k+2}-1},$$

as required.