Editorial comment. No correct solutions of **(b)** were received. It appears that the set of pairs (x, y) such that the sequence defined by $u_0 = x, u_1 = y, u_{n+2} = 1 + u_n/u_{n+1}$ converges is a curve through (2, 2) of the form

$$y = 2 + \frac{1}{2}(x-2) - \frac{1}{20}(x-2)^2 + \frac{7}{600}(x-2)^3 - \frac{71}{20400}(x-2)^4 + \cdots.$$

Part **(a)** solved also by S. S. Kim and the proposer.

The Number of Zeros of a Maclaurin Polynomial

10671 [1998, 559]. *Proposed by F. Rothe, University of North Carolina, Charlotte, NC.* Let

$$P_n(x) = \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

be the Maclaurin polynomial of order $2n+1$ of the sine function. Let c_n be the number of real zeros of P_n. Determine $\lim_{n\to\infty} c_n/(2n+1)$.

Composite solution by Sung Soo Kim, Hanyang University, Ansan, Kyunggi, Korea, and the editors. The integral form of Taylor's theorem tells us that

$$P_n(x) = \sin x + \frac{(-1)^n}{(2n+1)!} e_{2n+1}(x), \qquad \text{where} \qquad e_k(x) = \int_0^x (x-t)^k \sin t \, dt.$$

Now $e_1(x) = x - \sin x$ and is positive for all $x > 0$, and $e_k'(x) = k e_{k-1}(x)$ for $k > 1$. Thus for $k \geq 3$, $e_k(x)$ is positive, increasing, and convex (concave up) on $(0, \infty)$.

Let $f_n(x) = e_{2n+1}(x)/(2n+1)!$. We now consider the intervals $[a, b]$ on which $\sin x$ is monotone. Suppose first that n is even and $f_n(b) < 1$. If $a = (2m - 1/2)\pi$ and $b = (2m + 1/2)\pi$, then $P_n(x)$ is negative at a and positive at b and strictly increasing on $[a, b]$ so there is exactly one zero of P_n in $[a, b]$. If instead $a = (2m + 1/2)\pi$ and $b = (2m + 3/2)\pi$, then $P_n(x)$ is positive at a and negative at b. If $c = (2m+1)\pi$, then P_n is positive on $[a, c]$. Thus P_n has at least one real zero in $[c, b]$. If there were more than one zero in $[c, b]$, there would have to be some $z \in [c, b]$ with $P_n''(z) < 0$: a convex function cannot be zero at more than one point on an interval if it is positive at one end and negative at the other. But $\sin(x)'' > 0$ on $[c, b]$, so also $P_n'' > 0$ on $[c, b]$, which is a contradiction. The case where n is odd is similar. The final case to be considered is when $f_n(a) < 1 < f_n(b)$. Here there can be two zeros in the interval, but again considerations of convexity forbid more.

This shows that the number of real zeros of P_n differs by at most a constant from the number of intervals $(k - \pi/2, k + \pi/2)$ in which $f_n < 1$. That number is given to within a bounded error by $2B(n)/\pi$, where $B(n)$ is the unique positive solution to $f_n(x) = 1$. But

$$e_k(x) = \int_0^x (x-t)^k \sin t \, dt < \int_0^\pi (x-t)^k \sin t \, dt < \pi x^k,$$

while

$$e_k(x) > \int_0^{2\pi} (x-t)^k \sin t \, dt = \sum_{j=0}^{k} \binom{k}{j} (x-\pi)^{k-j} \int_{-\pi}^{\pi} u^j \sin u \, du > 2\pi k (x-\pi)^{k-1}.$$

Thus $B(n)$ lies between the solutions to $x^{2n+1} = (2n+1)!/\pi$ and $(x-\pi)^{2n} = (2n)!/(2\pi)$. Both are asymptotically $2n/e$ by Stirling's formula, so $B(n) \approx 2n/e$. Thus, the number c_n of real zeros of $P_n(x)$ is asymptotic to $4n/e\pi$, so that $c_n/(2n+1) \approx 2/e\pi$.

Editorial comment. David Bradley pointed out that the result is known and may be found (with details for the cosine function) in G. Szegő, Über eine Eigenschaft der Exponentialreihe, in *Gábor Szegő: Collected Papers 1915–1927*, Birkhauser, 1982, p. 659).

Solved also by J. H. Lindsey II, GCHQ Problems Group, and the proposer.