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The American Mathematical Monthly, Vol. 106, No. 2. (Feb., 1999), pp. 99-111.

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Sat Dec 22 05:16:17 2007

Does Mathematics Need New Axioms?

Solomon Feferman

The question, "Does mathematics need new axioms?," is ambiguous in practically every respect.

- What do we mean by "mathematics"?
- What do we mean by "need"?
- What do we mean by "axioms"?

You might even ask, What do we mean by "does"?

Part of the ambiguity lies in the various points of view from which this question might be considered. The crudest difference lies between the point of view of the working mathematician and that of the logician concerned with the foundations of mathematics. Some logicians might protest this distinction since they identify themselves as (working) mathematicians who happen to specialize in mathematical logic. Certainly, modern logic has established itself as a very respectable branch of mathematics, and there are quite a few highly technical journals in logic, such as *The Journal of Symbolic Logic* and the *Annals of Pure and Applied Logic*, whose contents, from a cursory inspection, look just like those of other mathematical journals, setting subjects aside. Looking even closer, you can pick up a paper on, say, the semi-lattice of degrees of unsolvability or the model theory of fields and not see it as any different in general character from a paper on combinatorial graph theory or cohomology of groups; they belong to the same big frame of mind, so to speak. But if you pick up Gödel's paper on the incompleteness of axiom systems for mathematics, or his work and that of Cohen on the consistency and independence of the Axiom of Choice relative to the axioms of set theory, we're in a different frame of mind, because we are doing what Hilbert called *metamathematics*: the study of mathematics itself by the means of mathematical logic through its formalization in axiomatic systems. And it's that stance I want to distinguish from that of the mathematician working on analysis or algebra or topology or degrees of unsolvability, and so on. It's awkward to keep talking about the logician as metamathematician, and I won't keep qualifying it that way, but that's what I intend.

Though I won't at all neglect the viewpoint of the working mathematician, for most of this article I will be looking at the leading question from the point of view of the logician, and for a substantial part of that, from the perspective of one supremely important logician: Kurt Gödel. From the time of his stunning incompleteness results in 1931 to the end of his life, Gödel called for the pursuit of new axioms to settle undecided arithmetical problems. And from 1947 on, with the publication of his unusual article, "What is Cantor's continuum problem? **[12]**, he called in addition for the pursuit of new axioms to settle Cantor's famous conjecture about the cardinal number of the continuum. In both cases, he pointed primarily to schemes of higher infinity in set theory as the direction in which to seek these new principles. In recent years logicians have learned a great deal that is relevant to Gödel's program, but there is considerable disagreement about what conclusions to draw from their results. I'm far from unbiased in this respect, and you'll see how I come out on these matters by the end of this essay, but I will try to give you a fair presentation of other positions along the way so you can decide for yourself which you favor.

The *Oxford English Dictionary* defines 'axiom' as used in Logic and Mathematics, by: "A self-evident proposition requiring no formal demonstration to prove its truth, but received and assented to as soon as mentioned." I think it's fair to say that something like this definition is the first thing we have in mind when we speak of axioms for mathematics: I'll call this the *ideal* sense of the word. It's surprising how far the meaning of axiom has been stretched from the ideal sense in practice, both by mathematicians and logicians. Some even take it to mean *an arbitrary assumption*, and so refuse to take seriously what purpose axioms are supposed to serve.

When the working mathematician speaks of axioms, he or she usually means those for some particular part of mathematics such as groups, rings, vector spaces, topological spaces, Hilbert spaces, etc. These axioms have nothing to do with self-evident propositions, nor are they arbitrary starting points. They are simply *definitions of kinds of structures* that have been recognized to recur in various mathematical situations. But they *act* as axioms in the sense that they provide a framework in which certain kinds of operations and lines of reasoning are appropriate whereas others are not. And once we run into a structure meeting one of these axiom systems—for example, a group associated with some equation or with a topological space—we can call on a vast body of previously established consequences for our further work. Without trying to argue this further, I take it that the value of these kinds of *structural axioms* for the organization of mathematical work is now indisputable. Moreover, we seem to keep coming up with new axioms of this sort, and I think the case can be made that they come up due to a continuing need to package and communicate our knowledge in digestible ways.

Now, in contrast to the working mathematician's structural axioms, when the logician speaks of axioms, he or she means, first of all, laws of valid reasoning that are supposed to apply to *all* parts of mathematics, and, secondly, axioms for such fundamental concepts as number, set, and function that underlie *all* mathematical concepts; I call the latter *foundational axioms*. I won't get into the question here of whether mathematics needs such axioms at all, and let the historical development of mathematics speak for that. Certainly, these correspond to such basic parts of our subject that they hardly need any mention at all in daily practice, and many mathematicians can carry on their entire work without calling on them even once. But that doesn't mean that they are not needed in the end to justify that practice, nor that they can safely be ignored in all situations. At any rate, I will take the necessity of foundational axioms for mathematics for granted in the following.

In particular, I will be concentrating on two axiom systems at conceptual extremes, the Dedekind-Peano Axioms for number theory and the Zermelo-Fraenkel axioms for set theory. I assume general familiarity with these, and so will skimp over the specifics of their formulations, which are in any case not important for the following; but I do have to say something about their development and the reasons for their acceptance. In both cases, one started with an informal "naive" system, which was later transformed into a formal system in the precise sense of metamathematics.

Dedekind's axioms [**3**] for the natural numbers $\mathbf{N} = \{0, 1, 2, \dots\}$ simply took the initial element 0 [Dedekind started with 1] and the successor operation $x \mapsto x'(= x + 1)$ as basic, with the evident axioms that 0 is not a successor and that successor is one-one. Induction was formulated set-theoretically, in the form: $\mathbf{N}$ is the smallest set that contains 0 and is closed under the successor operation. This takes the informal notion of *arbitrary set* of natural numbers for granted, and in those terms the axioms are categorical and hence complete. Dedekind used the induction principle to show that one can define functions by simple recursion on **N**, i.e., where we prescribe how the function is defined at 0 and how it is defined at any successor element x' in terms of how it is defined at x; in particular this is used in the proof of categoricity. The functions of one or more arguments from **N** to **N** generated by explicit and simple recursive definition are nowadays called the *primitive recursive functions*.

Peano [**24**] made a first stab at adding axioms (to those of Dedekind) about which sets exist. He stated that every property determines a set, and then gave some closure conditions on properties. In the Peano axioms, induction is equivalent to the statement that any property of natural numbers that holds of 0 and is closed under successor holds of all natural numbers.

If one takes the notions of set or property as not needing any further analysis, then it seems to me that the Dedekind-Peano axioms come as close as anything we have to meeting the ideal dictionary sense of the word. But the metamathematical view is that *all* notions used in an axiomatic system and *all* assumptions concerning these must be fully spelled out. That is done by fixing a formal language for number theory, and taking for the properties in the induction principle just those expressed by a well-formed formula of that language. The resulting formal system nowadays is called Peano Arithmetic and denoted PA. In the formal language of PA, whose basic relation is that of equality, $=$, we need to add symbols for the operations $+$ and $\cdot$ to those for 0 and $'$, with their recursive defining equations as axioms; though $+$ and $\cdot$ are set-theoretically definable in terms of the latter by Dedekind's result, they are not definable from them in the first-order language used for formal arithmetic. However, Gödel showed in [**10**] that once we have 0, $'$, $+$, and $\cdot$, all primitive recursive functions are definable in PA.

Unlike the origin of the Dedekind-Peano axioms in a clear intuitive concept, Zermelo's axioms arose out of a need to give some sort of foundation to Cantor's revolutionary work in set theory, which many people regarded as problematic. In particular, Cantor made essential use of the Well-Ordering Principle (WO), according to which any set can be well-ordered, in order to establish various facts about cardinal arithmetic, in particular that for infinite cardinals κ, μ,

$$\kappa + \mu = \kappa \cdot \mu = \max(\kappa, \mu), \text{ while } \kappa < 2^{\kappa}.$$

Moreover, he used WO to show that the infinite cardinals can be laid out in a scale indexed by ordinals α,

$$\aleph_0 < \aleph_1 < \aleph_2 < \cdots < \aleph_\alpha < \aleph_{\alpha+1} < \cdots < \aleph_\lambda < \cdots,$$

where each $\aleph_{\alpha+1}$ is the least cardinal greater than $\aleph_\alpha$, and for limit λ, $\aleph_\lambda$ is the limit of all $\aleph_\alpha$ for $\alpha < \lambda$. This scale and the fact that $\aleph_0 < 2^{\aleph_0}$ immediately led to the conjecture known as the Continuum Hypothesis,

$$\text{(CH)} \qquad 2^{\aleph_0} = \aleph_1,$$

since $2^{\aleph_0}$ is the cardinal number of the continuum **R**. The extension of this conjecture to all α is called the Generalized Continuum Hypothesis,

$$\text{(GCH)} \qquad 2^{\aleph_\alpha} = \aleph_{\alpha+1}.$$

The question of justifying the Well-Ordering Principle was worrisome to Cantor. At first he argued that it is a "Law of Thought"; then he sought a proof of it on the basis of a more evident principle, but failed to come up with anything satisfactory. Such a principle was first offered in 1904 by Zermelo [**31**] in the form of the Axiom of Choice (AC). Zermelo proved that AC implies WO; in fact, they are equivalent, but Zermelo argued that AC is evident in a way that WO is not. Following publication of this work, there were objections not only to the acceptance of AC but also to the correctness of his proof of the implication. In order to meet the latter objections, Zermelo introduced axioms in [**32**] that spelled out just which principles on sets were employed in his argument. These are the axioms of: Extensionality, Empty set, Unordered pair, Power set, Union, Infinity and Separation. The latter axiom says that for any *definite property* $P(x)$ of objects and any given set a, the set $b = \{x : x \in a \ \& \ P(x)\}$ also exists. This principle was objected to as being vague on what counts as a definite property, so, not long after, precise proposals were made independently by Weyl, Skolem, and Fraenkel to tie these down. Their proposals all essentially amount to taking for the definite properties just those expressed in the formal language for set theory, with basic symbols $=$ and $\in$. An additional modification was that Zermelo's axioms did not allow one to establish the existence of $\aleph_\alpha$ for infinite α; Fraenkel added his Replacement Axiom to accomplish this. As a formal system, the Zermelo-Fraenkel axioms are denoted ZF, and the same axioms to which AC is adjoined are denoted ZFC. A small point to note is that Zermelo allowed the existence of *urelements*, i. e., objects (other than the empty set) without elements. These have been dispensed with in ZF since they are not necessary for the foundations of set theory.

What was left unsettled by this development is an explanation of what, exactly, the Zermelo-Fraenkel axioms are axioms *for*. If they are to be considered to be axioms in the ideal, dictionary sense, they should be evident for some pre-axiomatic concept that we have in mind. The concept of *arbitrary set*, so to speak at large, which might first be offered as a candidate for this is unsatisfactory, because it seems to be an evident characteristic of this concept that for any property $P(x)$ the set of *all* x satisfying P exists. But as we know, this results in contradictions, the simplest being that due to Russell, using the property: $\neg(x \in x)$, where $\neg$ is the *negation* symbol. And it is just this sort of contradictory construction that Zermelo's Axiom of Separation avoids, by applying P only to elements of a "pre-existing" set a. What justifies that, but not the more general, contradictory concept of set? An answer was first offered by Zermelo [**33**] in 1930, in terms of what has since come to be called *the cumulative hierarchy of sets*. In this picture, sets are conceived of as being *built up* from below in stages, starting with the urelements at the lowest stage. Since we have dispensed with those, nowadays we simply start with the empty set, usually denoted 0. At each stage, we gather together all the sets obtained at preceding stages into a single set a. Then at the next stage we adjoin all members of the *power set* of a, $\wp(a)$, i.e., the *set of all subsets* of a. Finally, this process is iterated transfinitely. But to spell out what this model is, we need set theoretical notions themselves, as follows: the stages are indexed by ordinals. The set (or partial universe) of objects obtained at stage α is denoted V_α.

$$V_0 = 0,$$

$$V_{\alpha+1} = V_\alpha \cup \wp(V_\alpha), \quad \text{and}$$

$$V_\lambda = \text{the union of } \{V_\alpha : \alpha < \lambda\} \quad \text{for limit ordinals } \lambda.$$

It is argued by set-theorists nowadays that the axioms of ZFC are evident for the universe V of sets consisting of all objects in some V_α. But the intuition for that is a far cry from what leads one to accept the Dedekind-Peano axioms. Among other things, what this takes for granted is that there is an objective notion of arbitrary subset of a given set. This is the Platonistic conception of mathematics applied to set theory, a conception which is philosophically controversial; we shall have more to say about that later on.

We return now to the origins of Gödel's program for new axioms in his 1931 paper [**10**] on the incompleteness for formal systems extending arithmetic. I want to remind you briefly of these results, for which we need some slightly technical notions. The simplest statements of number-theoretical interest are those of *purely universal form* $(\forall x) f(x) = 0$ and *purely existential form* $(\exists x) f(x) = 0$, where f is a primitive recursive function; these are dual forms under negation in the sense that $\neg(\forall x)f(x) = 0$ is equivalent to $(\exists x)g(x) = 0$ where $g(x)$ is 0 if $f(x) \neq 0$ and is 1 otherwise. A formal system S whose language contains that of PA is said to be *sound* for a class K of statements if whenever $S \vdash \phi$ (S proves ϕ) and $\phi \in K$, then ϕ is true in the natural numbers. It is easily shown that if S is *consistent* and contains PA (or even a weak fragment thereof) then it is sound for (purely) universal statements but (as Gödel showed) it need not be sound for existential statements. S is called 1-*consistent* just in case it is sound for existential statements. Note that 1-consistency implies consistency. (Gödel himself used a slightly stronger notion called *ω-consistency*.)

A system S is called *formally complete* if for each closed formula ϕ, either $S \vdash \phi$ or $S \vdash \neg \phi$. Hilbert had two fundamental conjectures about PA: that its consistency can be proved finitarily and that it is formally complete. Both conjectures were dashed by Kurt Gödel's incompleteness theorems of 1931 [**10**]. Moreover, they apply to (effectively presented) formal systems S extending PA much more generally. Gödel associated with each such system a purely universal statement θ_S, which expresses of itself, via its Gödel number, that it is not provable in S. Gödel's *first incompleteness theorem* has two parts. The first part tells us that if S is consistent then θ_S is indeed not provable in S, so by its very construction, it is true. The second part tells us that if S is 1-consistent then $\neg \theta_S$ is also not provable in S. For otherwise, being equivalent to an existential statement, if $\neg \theta_S$ were provable in S it would be true, contrary to the first part. Gödel's *second incompleteness theorem* tells us that the number-theoretic statement $Con(S)$ expressing the consistency of S is not provable in S if S is consistent. This comes about by formalizing the proof of the first part of the first theorem. It follows that if S is a system in which all finitary reasoning can be formalized, then the general Hilbert finitary consistency program cannot be carried out for S. It is now generally accepted that all finitary reasoning can already be formalized in PA, if not in much weaker systems, and that's where Hilbert's finitary consistency program has its limits.

Not only were Gödel's results stunning, but also his own explanation of why they hold was surprising. This was given in a footnote that was apparently included in the paper [**10**] only as an afterthought, since it is numbered 48[a]. But it expressed a fundamental conviction of Gödel's which he reiterated throughout the rest of his life, and this conviction brings us close to the heart of our leading question. There is evidence that he thought such a view would be unacceptable to the Hilbert school, and that he must have hesitated to say anything of this sort at all. The footnote reads:

> ...the true reason for the incompleteness inherent in all formal systems of mathematics is that the formation of ever higher types can be continued into the transfinite...[since] the undecidable propositions constructed here become decidable whenever appropriate higher types are added. [**10**, p. 191]

However, the connection of incompleteness with set theory is not explained here;

the unstated reason is that the consistency of a system S can be proved in a system that uses variables for sets ranging over arbitrary subsets of the universe of discourse of S, by means of which a truth definition for the language of S can be introduced. Nothing more like footnote 48[a] was said by Gödel until the mid 1940s, by which time he was safely ensconced at the Institute for Advanced Study in Princeton, and Hilbert was dead and gone.

In the meantime, Gödel had established in **[11]** his second ground-breaking result, that if ZF is consistent then it remains consistent when we add AC and GCH. Gödel's method of proof for this was to produce a *new* cumulative hierarchy as a model of set theory, formally defined within set theory, by restricting the sets introduced at each stage to be all and only those subsets of the preceding stage which are definable in the language of set theory over that stage. The sets constructed in this way at stage α are denoted L_α, and their definition looks exactly like the sets V_α, except at the successor stages, where we take

$$L_{\alpha+1} = L_\alpha \cup Def(L_\alpha),$$

where $Def(a)$ for any set a is the *set of all definable subsets* b of a. A set is called *constructible* by Gödel if it belongs to some L_α; then L is used for the collection of all constructible sets. The so-called *Axiom of Constructibility* asserts that all sets are constructible, and is symbolized by $V = L$. This "axiom" served as a convenient intermediary in Gödel's relative consistency proof, as follows:

1. If ZF is consistent then ZF + $V = L$ is consistent.
2. ZF + $V = L \vdash$ AC & GCH.

Aside from the formal positioning of $V = L$ in 1 and 2, in what sense is its statement an acceptable axiom for set theory? At the time of his proof (circa 1938) Gödel stated that it provides a kind of natural completion of the axioms of set theory, since it ties down—in a way that ZF does not—exactly which sets we are talking about. But within a decade he was clearly rejecting it as an axiom, on the basis of a strongly Platonistic point of view of what set theory is supposed to be about. This position first emerged in an article on Russell's mathematical logic in 1944, but it was only stated forthrightly and with specific reference to open set-theoretical problems by Gödel in his 1947 article, "What is Cantor's continuum problem?" **[12]**, along the following lines:

1° Set theory is about a universe V of objects existing independently of human thoughts and constructions. It consists of the result of iterating into the transfinite the full power set operation, i.e., the operation of forming the set of arbitrary subsets of any given set. (So, on the basis of this, there is no reason to accept $V = L$, which says that all sets are introduced by successive definitions.)
2° Statements of set theory have a determinate truth value (in V). In particular, all axioms of ZFC are true in V.
3° So, also, CH has a determinate truth value. According to Gödel in **[12]** it is probably false.
4° Thus CH should be independent of ZFC. (Indeed, this was eventually demonstrated in 1963 by Paul Cohen **[2]**.)
5° And thus, in order to fix the position $\aleph_\alpha$ of $2^{\aleph_0}$ in the scale of alephs, we will [no doubt] need to add new axioms to ZFC.
6° These new axioms may be formulated and accepted by a direct extension of the informal reasoning that led us to accept ZFC in the first place. More

precisely:

> The simplest of these [new axioms]... assert the existence of [strongly] inaccessible numbers... > $\aleph_0$. [This] axiom, roughly speaking, means nothing else but that the totality of sets obtainable by exclusive use of the processes of formation of sets expressed in the other axioms forms again a set (and, therefore, a new basis for a further application of these processes). Other axioms of infinity have been formulated by P. Mahlo. ...these axioms show clearly, not only that the axiomatic system of set theory as known today is incomplete, but also that it can be supplemented without arbitrariness by new axioms which are only the natural continuation of those set up so far. [**12**, p. 520]

An uncountable cardinal is called (*strongly*) *inaccessible* if it is closed under exponentiation and limits of smaller cardinals. It follows that if κ is inaccessible then V_κ is a model of the ZFC axioms. Thus, if it is assumed that there exists an inaccessible cardinal then the consistency of ZFC, Con(ZFC), is a consequence and so, by Gödel's second incompleteness theorem, it is not provable in ZFC (if ZFC is consistent). Similarly, if one assumes there are a certain number of inaccessible cardinals, then one will not be able to prove the existence of larger inaccessibles. The Mahlo axioms assert the existence, to begin with, of arbitrarily large inaccessibles, and then of arbitrarily large inaccessible fixed points of the enumeration of the inaccessibles, and so on, iterated into the transfinite. An informal way of justifying their existence, and, indeed, of infinite cardinals at all, is by reference to "Cantor's Absolute": the universe of all sets is beyond being captured by any closure condition on sets; instead, any such condition always closes off at a set. A bit more explicitly, whatever closure property P one recognizes to be satisfied by the universe V of all sets, there will exist arbitrarily large κ for which V_κ satisfies P. Formal versions of this, introduced by Azriel Levy [**20**] and Paul Bernays [**1**], are called *Reflection Principles* in set theory. They are behind Gödel's reason for saying that we are led to new axioms, such as those of Mahlo type, "without arbitrariness" and as a "natural continuation" of those axioms previously accepted. But, he continued [from the preceding quote],

> [a]s for the continuum problem, there is little hope of solving it by means of those axioms of infinity which can be set up on the basis of principles known today... [ibid.]

The reason is that the Mahlo axioms are consistent with $V = L$, and since GCH is true in L, and Gödel believed CH to be false, its falsity could not be proved in this way. "In the face of this," he continued on,

> ... probably there exist other [axioms] based on hitherto unknown principles...which a more profound understanding of the concepts underlying logic and mathematics would enable us to recognize as implied by these concepts. [ibid.]

A candidate for a larger kind of cardinal had in fact been suggested some time before, by Stanislaw Ulam, in 1930. Ulam called an uncountable cardinal κ *measurable* if there exists a two-valued κ-additive measure on $\wp(\kappa)$. Not much was known about the strength of this until 1961, when Dana Scott proved in [**25**] that the existence (MC) of measurable cardinals implies $V \neq L$, so MC then became a viable possibility to settle CH. A few years later, Alfred Tarski with his students William Hanf and H. Jerome Keisler proved ([**16**], [**19**]) that if κ is a measurable cardinal then it is very large, since V_κ satisfies the axioms of Mahlo type and other powerful axioms of infinity. Their work led further to a notion of *strongly compact cardinal*, whose existence was shown to imply the existence of measurable cardinals. But then both Tarski and Gödel had qualms about the assumption of the existence of such enormous cardinals. To quote Tarski:

> the belief in the existence of inaccessible cardinals...(and even of arbitrarily large cardinals of this kind) seems to be a natural consequence of basic intuitions underlying the "naive" set theory and referring to what can be called "Cantor's absolute." On the contrary, we see at this moment no cogent intuitive reasons which could induce us to believe in the existence of [strongly compact] cardinals, or which at least would make it very plausible that the hypothesis stating the existence of such cardinals is consistent with familiar axiom systems of set theory. [**27**, p. 134]

In his 1964 revision [**13**] of his 1947 article, Gödel seconded this view of Tarski's in full, but then added:

> However, [the new axioms] are supported by rather strong argument from *analogy*... ([**13**, p. 264, ftn. 20], italics mine)

Moreover, Gödel had already proposed in 1947 still another kind of argument that might lead one to accept certain statements as new axioms, even though they do not rest on the same kind of evidence that led one to accept ZFC in the first place, to wit:

> [Finally, we may look for axioms which are] so abundant in their verifiable consequences...that *quite irrespective of their intrinsic necessity* they would have to be assumed in the same sense as any well-established physical theory. ([**12**, p. 521], italics mine)

Higher axioms of infinity, or so-called "large cardinals" in set theory have been the subject of intensive investigation since the 1960s and many new kinds of cardinals with special set-theoretical properties have emerged in these studies.[1] A complicated web of relationships has been established, as witnessed by charts to be found in the recent book by Aki Kanamori, *The Higher Infinite* [**17**, p. 471], and the earlier expository article by Kanamori and Menachem Magidor [**18**]. A rough distinction is made between "small" large cardinals, and "large" large cardinals, according to whether they are weaker or stronger, in some logical measure or other, than measurable cardinals. Attempts to justify acceptance of both kinds of cardinals have been made by set theorists involved in this development. The philosopher, Penelope Maddy, in two interesting articles called "Believing the axioms," analyzed the various kinds of arguments for these and other kinds of strong axioms and summarized the evidence for them [**21**]. Broadly speaking, the

[1]The elaboration of this subject has almost outrun the names that have been introduced for various large cardinal notions, witness (in roughly increasing order of strength): "inaccessible," "Mahlo," "weakly compact," "indescribable," "subtle," "ineffable," "Ramsey," "measurable," "strong," "Woodin," "superstrong," "strongly compact," "supercompact," "almost huge," "huge," and "super-huge."

arguments are classified as being based on intrinsic or extrinsic reasons. The above-mentioned reflection principles are examples of intrinsic reasons, but these do not take us beyond the "small" large cardinals. Among the extrinsic reasons for going higher are that the assumption of "large" large cardinals has been fruitful—through the dazzling work of Solovay, Martin, Foreman, Magidor, Shelah, Steel, Woodin and others—in extending "standard" properties of Borel and analytic subsets of the continuum, such as Lebesgue measurability, the Baire property, the perfect subset property, determinateness of associated infinite games, etc. to substantially larger classes.

But the striking thing, despite all this progress, is that contrary to Gödel's hopes, the Continuum Hypothesis is *still* undecided by these further axioms, since it has been shown to be independent of all remotely plausible axioms of infinity, including MC, that have been considered so far (assuming their consistency)[2]. That may lead one to raise doubts not only about Gödel's program but also about its very presumptions. Is CH a definite problem as Gödel and many current set-theorists believe? Is the continuum itself a definite mathematical entity? If it has only *Platonic* existence, how can we access its properties? Alternatively, one might argue that the continuum has *physical* existence in space and/or time. But then one must ask whether the *mathematical structure* of the real number system can be identified with the *physical structure*, or whether it is instead simply an *idealized mathematical model* of the latter, much as the laws of physics formulated in mathematical terms are highly idealized models of aspects of physical reality. (Hermann Weyl raised just such questions in his 1918 monograph *Das Kontinuum*, **[29]**.) But even if we grant some kind of independent existence, abstract or physical, to the continuum, in order to formulate CH we need to refer to arbitrary *subsets of the continuum* and possible mappings between them, and then we are dealing with objects of a higher level of abstraction, the nature of whose existence is even more problematic than that of the continuum. Here we are skirting deep philosophical waters; let us retreat from them for the moment.

While Gödel's program to find new axioms to settle CH has not been realized, what about the origins of his program in the incompleteness results for number theory? As we saw, throughout his life Gödel said we would need new, ever-stronger set-theoretical axioms to settle open arithmetical problems of even the simplest, purely universal, form—problems he called of *Goldbach type*. Indeed, the Goldbach conjecture can be written in that form. But the incompleteness theorem by itself gives no evidence that any open arithmetical problems—or, equivalently, finite combinatorial problems—of *mathematical interest* will require new such axioms. I emphasize the 'mathematical interest', because Gödel's own examples of undecidable statements for each consistent S extending PA were of two kinds: the first, θ_S, cooked up by a diagonal construction in order to establish incompleteness and evidently true by the very theorem that it is used to prove, and the second, Con(S), of definite *metamathematical interest*, but not of mathematical interest in the ordinary sense of the word. Beginning in the mid-1970s, logicians began trying to rectify this situation by producing finite combinatorial statements of prima-facie mathematical interest that are independent of such S. The first example was provided by Jeff Paris and Leo Harrington who showed in **[23]** that a modified form (PH) of the finite Ramsey theorem concerning existence of homogeneous sets for certain kinds of partitions is not provable in PA. PH is recognized to be true as

[2]Cf. Martin **[22]**; the situation reported there in 1976 remains unchanged to date.

a simple consequence of the infinite Ramsey theorem; its independence rests on showing that PH implies Con(PA); in fact PH is equivalent to 1-Con(PA). Moving up to a stronger system, a few years later, Harvey Friedman, Ken McAloon, and Stephen Simpson produced a finite combinatorial version FGP of the Galvin-Prikry theorem GP that is independent of the Feferman-Schütte system of predicative analysis (call it FS for present purposes)[3]. It happens that GP is itself a considerable strengthening of the infinite Ramsey Theorem, and FGP has certain analogies to PH. Again, this finitary version FGP is proved to be true as a simple consequence of GP, while its independence rests on showing that it implies Con(FS); in fact, FGP is equivalent to the 1-consistency of predicative analysis. Further results of this type have been obtained by these researchers and others for still stronger systems of analysis[4]. While in each case, the statement ϕ shown independent of S is equivalent to its 1-consistency, the argument for the truth of ϕ is by ordinary mathematical reasoning.

For some years, Friedman has been trying to go much farther, by producing mathematically perspicuous finite combinatorial statements ϕ whose proof requires the existence of many Mahlo axioms or even stronger axioms of infinity and has come up with various candidates for that ([7] contains the latest work in this direction). From the point of view of metamathematics, this kind of result is of the same character as the earlier work just mentioned; that is, for certain very strong systems S of set theory, the ϕ produced is equivalent to the 1-consistency of S. But the conclusion to be drawn is not nearly as clear as for the earlier work, since the truth of ϕ is now *not* a result of ordinary mathematical reasoning, but depends essentially on acceptance of 1-Con(S). It is begging the question to claim this shows we need axioms of large cardinals in order to settle the truth of such ϕ, since our *only* reason for accepting that truth lies in our belief in the 1-consistency of those axioms. However plausible we might find that, perhaps by some sort of picture we can form of the models of such axioms, it doesn't follow that we should accept *those axioms themselves* as first-class mathematical principles. Finally, we must take note of the fact that up to now, *no previously formulated open problem* from number theory or finite combinatorics, such as the Goldbach conjecture or the Riemann Hypothesis or the twin prime conjecture or the P = NP problem, is known to be independent of the kinds of formal systems we have been talking about, not even of PA. If such were established in the same way as the examples (PH, FGP, etc.) mentioned above, then their truth would at the same time be verified. I think it is more likely, as has been demonstrated in the case of the Fermat "last theorem," that the truth of these will eventually be settled—if at all—by ordinary mathematical reasoning without any passage through metamathematics, and that only afterward might we see just which basic axiomatic principles are required for their proofs.

[3] Friedman, McAloon, and Simpson work with a system ATR_0 which is shown to be proof-theoretically equivalent to the FS system of ramified analysis up to the ordinal Γ_0. Friedman later found a finite version of Kruskal's theorem KT which is independent of ATR_0. The infinitary theorem KT, a staple of graph-theoretic combinatorics, asserts the well-quasi-ordering of the embeddability relation between finite trees. Friedman's work in this respect is reported in [**26**].

[4] The systems involved and associated independent statements are more complicated to explain and would go beyond the scope of this article to do so, but at least one result is worth indicating in connection with footnote 3. Friedman found an extended version EKT of KT which is independent of the impredicative Π^1_1 comprehension principle in analysis (cf. [**26**]). EKT later turned out to have close mathematical and metamathematical relationships with the graph minor theorem of Robertson and Seymour, as shown in [9].

Moving beyond the domains of arithmetic and finite combinatorics, what is the evidence that we might need new axioms for everyday mathematics? Here it is certainly the case that various parts of descriptive set theory have been shown to require higher axioms of infinity, in some cases well beyond the range of "small" large cardinals. But again we are in a question-begging situation, since our belief in the truth of these new results depends essentially on our belief in the consistency or correctness to some extent or other of these "higher" statements. Also, I think it is fair to say that these kinds of results are at the margin of ordinary mathematics, that is of what mathematicians deal with in daily practice.[5] What is *not* at the margin can be readily formalized within ZFC, and in fact in much weaker systems, as has been demonstrated by many case studies in recent years.

Let's look more specifically at the part of mathematics that is indispensable to scientific applications, which clearly includes vast tracts of analysis, among other subjects. One of the arguments for accepting any set theory at all, if one is not a Platonist, has been advanced by the philosophers Willard van Orman Quine and Hilary Putnam, along the lines that some set theory is necessary for the foundations of analysis, and that the resulting mathematics is justified by its essential and successful use in established physical theory. But this argument is undermined by a series of case studies, beginning with that of Hermann Weyl in 1918, in his famous monograph *Das Kontinuum* **[29]**, in which he showed in principle how all of nineteenth-century analysis of piecewise continuous functions could be formalized in a system S reducible to PA; this has been continued since the mid-70s with work by Gaisi Takeuti, Harvey Friedman, Stephen Simpson, and myself among others, to extend this to substantial portions of twentieth-century analysis including much of measure theory and functional analysis. As a result of these studies, I have come to conjecture that practically all scientifically applicable mathematics can be formalized in systems reducible to PA, or, as I have sloganized it in **[4]**: *a little bit goes a long way*. Against this, I have learned of a couple of cases in some approaches to the foundations of quantum field theory where it appears one must go beyond the resources of PA; but the physical theories that require such additional strength are rather speculative. In any case, the mathematics needed for these cases can be carried out in relatively weak subsystems of impredicative analysis, even if PA does not suffice. I am not by any means arguing that everyday mathematical practice should be restricted to working in such subsystems. The instrumental value of "higher" and less restricted set-theoretical concepts and principles is undeniable. The main concern here is, rather, to see: *what*, *fundamentally, is needed for what*?

To conclude, I hope I have given you some food for thought that will help you come to your own conclusions about whether questions like the Continuum Hypothesis are determinate, and, if so, what is going to settle them, given that present axioms are insufficient. At the beginning of this piece I promised to tell you my own views of these matters. By now, you have probably guessed what these are, but let me say them out loud: I am convinced that the Continuum Hypothesis is an inherently vague problem that *no* new axiom will settle in a convincingly definite way[6]. Moreover, I think the Platonistic philosophy of mathematics that is

[5]For an opposite point of view and beautiful exposition of the need for new axioms in that respect, cf. Woodin [30].

[6]CH is just the most prominent example of many set-theoretical statements that I consider to be inherently vague. Of course, one may reason confidently *within* set theory (e.g., in ZFC) about such statements *as if* they had a definite meaning.

currently claimed to justify set theory and mathematics more generally is thoroughly unsatisfactory and that some other philosophy grounded in inter-subjective *human* conceptions will have to be sought to explain the apparent objectivity of mathematics. Finally, I see no evidence for the practical need for new axioms to settle open arithmetical and finite combinatorial problems. The example of the solution of the Fermat problem shows that we just have to work harder with the basic axioms at hand. However, there is considerable theoretical interest for logicians to try to explain what new axioms *ought to be accepted* if one has already accepted a given body of principles, much as Gödel thought the axioms of inaccessibles and Mahlo cardinals *ought to be accepted* once one has accepted the Zermelo-Fraenkel axioms. In fact this is something I've worked on in different ways for over thirty years; during the last year I have arrived at what I think is the most satisfactory general formulation of that idea, in what I call the *unfolding of a schematic formal system* **[5]**. And this returns in an essential respect to the original "naive" schematic formulation of principles such as induction in number theory and separation in set theory, in their use of the pre-theoretic notion of arbitrary "definite" property. That is in closer accord with everyday practice, where such principles are taken in an *open-ended* way, without advance restriction on what specific language they are formulated in. But we can systematically enlarge what we regard as meaningful in a given subject, by using those very principles in a kind of feed-back way, for example in the use of induction to prove that a function or predicate of natural numbers defined implicitly by recursion equations is total and thus can be added to our language. There are already some definitive results for specific systems on what can be obtained by the unfolding process, in joint work with Thomas Strahm **[6]**, with a host of new and interesting problems waiting to be tackled. But that's another story for another occasion.

ACKNOWLEDGMENTS. Text of an invited AMS-MAA lecture, Joint Annual Meeting, San Diego, Jan. 10, 1997.

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