Catalan and Hankel: 10585

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Möbius and Riemann

10582 [1997, 270]. *Proposed by Peter Lindqvist and Kristian Seip, Norwegian University of Science and Technology, Trondheim, Norway.* Let $\mu(n)$ denote the Möbius function of number theory, and let $\zeta(s)$ denote the Riemann zeta function. Prove that

$$\zeta(s)\sum_{m=1}^{N}\sum_{n=1}^{N}\frac{(\gcd(m,n))^s}{(mn)^s}\mu(m)\mu(n) = 1+\sum_{j=2}^{\infty}\frac{1}{j^s}\left(\sum_{\substack{n|j\\ n>N}}\mu(n)\right)^2$$

when $s > 1$.

Solution by David M. Bradley, University of Maine, Orono, ME. We use the well-known fact that $\sum_{n|j}\mu(n) = 0$ for $j \geq 2$. We compute

$$1+\sum_{j=2}^{\infty}\frac{1}{j^s}\left(\sum_{\substack{n|j\\ n>N}}\mu(n)\right)^2 = 1+\sum_{j=2}^{\infty}\frac{1}{j^s}\left(\sum_{n|j}\mu(n)-\sum_{\substack{n|j\\ n\leq N}}\mu(n)\right)^2$$

$$= 1+\sum_{j=2}^{\infty}\frac{1}{j^s}\left(\sum_{\substack{n|j\\ n\leq N}}\mu(n)\right)^2 = 1+\sum_{j=2}^{\infty}\frac{1}{j^s}\left(\sum_{\substack{m|j\\ m\leq N}}\mu(m)\right)\left(\sum_{\substack{n|j\\ n\leq N}}\mu(n)\right).$$

In the inner sums, m and n both divide j if and only if $\operatorname{lcm}(m,n)|j$. Writing $j = k\cdot\operatorname{lcm}(m,n)$ and interchanging the order of summation yields

$$1+\sum_{j=2}^{\infty}\frac{1}{j^s}\left(\sum_{\substack{n|j\\ n>N}}\mu(n)\right)^2 = \sum_{m=1}^{N}\sum_{n=1}^{N}\frac{\mu(m)\mu(n)}{(\operatorname{lcm}(m,n))^s}\sum_{k=1}^{\infty}\frac{1}{k^s}.$$

Since $\operatorname{lcm}(m,n) = mn/\gcd(m,n)$, the result follows.

Editorial comment. The proposers' solution was quite different. They introduced the functions $f(x) = \sum_{k=1}^{\infty}\sin(kx)/k^s$ and $f_N(x) = \sum_{n=1}^{N}(\mu(n)/n^s)f(nx)$. For $N \in \mathbb{N}$, we have $f_N(x) \to \sin x$ as $s \to \infty$, so it is natural to compute the $L_2(0,\pi)$ norm of the "error" $\sin x - f_N(x)$. Doing this in two different ways yields the result.

Solved also by M. N. Balachandran (India), D. Callan, R. J. Chapman (U. K.), R. Holzsager, J. H. Lindsey II, R. Padma (India), P. Simeonov, NSA Problems Group, and the proposers.

Catalan and Hankel

10585 [1997, 361]. *Proposed by Alta Kellogg, Ormond Beach, FL.* A sequence $a_0, a_1, \ldots$ of real numbers is called *strictly totally positive* (STP) if every submatrix of the Hankel matrix $(a_{i+j})_{i,j\geq 0}$ has positive determinant.

(a) Show that the sequence $C_0, C_1, \ldots$ of Catalan numbers, defined by $C_n = \frac{1}{n+1}\binom{2n}{n}$, is STP.

(b) Show that the sequence of Catalan numbers is minimal in the following sense: If $a_0, a_1, a_2, \ldots$ is an STP sequence of positive integers with $a_n \leq C_n$ for every n, then $a_n = C_n$ for every n.

Solution to part **(a)** *by David Callan, Madison, WI.* Let C be the matrix (C_{i+j}). For sets of indices $\mathbf{u} = \{u_1 < \cdots < u_n\}$ and $\mathbf{v} = \{v_1 < \cdots < v_n\}$, let $C[\mathbf{u}|\mathbf{v}]$ denote the submatrix of C with rows indexed by $\mathbf{u}$ and columns indexed by $\mathbf{v}$. Recall that the Catalan number C_k is the number of Dyck paths ("mountain ranges") of length $2k$. (A Dyck path consists of northeast and southeast steps, starts on the x axis, ends on the x axis, and never falls below the x axis.) We claim that $\det C[\mathbf{u}|\mathbf{v}]$ is the number of n-tuples of pairwise nonintersecting Dyck paths in which the ith path extends from $(-2u_i, 0)$ to $(2v_i, 0)$.

To see this, let $S = S(\mathbf{u}, \mathbf{v})$ denote the set of all n-tuples of Dyck paths such that the ith path extends from $(-2u_i, 0)$ to $(2v_{\phi(i)}, 0)$ and ϕ is a permutation of $\{1, \ldots, n\}$. To each such n-tuple, assign the weight $\operatorname{sgn} \phi$. From the definition of $\det C[\mathbf{u}|\mathbf{v}]$ as a sum of signed products, it is immediate that $\det C[\mathbf{u}|\mathbf{v}]$ is the sum of the weights of the n-tuples in S. The n-tuples that have intersections cancel via a sign-reversing involution: on the lexicographically lowest-indexed pair of intersecting paths, locate the first point of intersection and switch the tails of these two paths after this intersection point. This changes the sign of the associated permutation ϕ and is an involution.

Thus only the nonintersecting n-tuples contribute to the sum, and their weight is 1, since avoiding intersections forces ϕ to be the identity. This establishes the claim.

It follows that $\det C[\mathbf{u}|\mathbf{v}]$ is positive, since there is always the nonintersecting n-tuple consisting of paths in which all northeast steps precede all southeast steps.

Solution to part **(b)** *by Robin J. Chapman, University of Exeter, Exeter, U. K.* In the cases in which $\mathbf{u} = \mathbf{v} = \{0, 1, \ldots, n-1\}$ or $\mathbf{u} = \{0, 1, \ldots, n-1\}$ and $\mathbf{v} = \{1, 2, \ldots, n\}$, there is only one nonintersecting n-tuple of paths. Therefore, the matrices $H_{n-1} = (C_{i+j})_{i,j=0}^{n-1}$ and $H'_{n-1} = (C_{i+j+1})_{i,j=0}^{n-1}$ both have determinant 1.

Now suppose that (a_{i+j}) is an STP sequence of positive integers with $a_n \le C_n$ for every n. Since $C_0 = C_1 = 1$, we have $a_0 = a_1 = 1$. For an inductive proof, we suppose that $a_i = C_i$ for $i < n$.

When $n = 2m$ is even, let $A = (a_{i+j})_{i,j=0}^{m}$. Expanding along the bottom row yields

$$1 - \det(A) = \det(H_m) - \det(A) = (C_n - a_n)\det(H_{m-1}) = C_n - a_n \ge 0.$$

Thus $\det(A) \le 1$, but by hypothesis $\det(A) \ge 1$. Hence $\det(A) = 1$ and $a_n = C_n$.

When $n = 2m+1$ is odd, the argument is similar, setting $A = (a_{i+j+1})_{i,j=0}^{m}$ and considering H'_m rather than H_m.

Editorial comment. The interpretation of a minor of a Hankel determinant in terms of nonintersecting Dyck paths is due to X. G. Viennot, *Une théorie combinatoire des polynômes orthogonaux généraux* (lecture notes), Université du Québec à Montréal, 1983. The Catalan numbers are essentially moments of the Chebyshev polynomials of the second kind.

Another evaluation of the determinant $\det(C_{i+j})_{i,j=0}^{n-1}$ appears in C. Radoux, Nombres de Catalan généralisés, *Bull. Belg. Math. Soc.* 4 (1997) 289–292.

Solved also by the proposer.

A Random Distance

10592 [1997, 456]. *Proposed by Roger Pinkham, Hoboken, NJ.* Three points are selected independently and at random in a disk of radius one. What is the average distance of the third from the line determined by the first two?

Solution by Richard Holzsager, American University, Washington, DC. Let f be the probability density function of the distance r from the chord through the first two points to the origin and let $\mu = \mu(r)$ be the mean distance from a random point in the circle to a chord at distance r from the origin. Then $\int_0^1 f(r)\mu(r)\,dr$ gives the expected value we want.

We can find f by calculating the probability of a chord being at distance greater than r and differentiating. Let p be the first point, q the second point, and R the distance of the chord through them from the origin. Then $f(r) = -(d/dr)\,P(R > r)$. To find $P(R > r)$, first consider a point p at distance s from the origin. The point q determines a chord at distance greater than r if q is between the two tangents from p to the circle $x^2 + y^2 = r^2$. Thus, given p, the probability that the chord pq is at distance greater than r is $1/\pi$ times