A Random Distance: 10592

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the x axis.) We claim that $\det C[\mathbf{u}|\mathbf{v}]$ is the number of n-tuples of pairwise nonintersecting Dyck paths in which the ith path extends from $(-2u_i, 0)$ to $(2v_i, 0)$.

To see this, let $S = S(\mathbf{u}, \mathbf{v})$ denote the set of all n-tuples of Dyck paths such that the ith path extends from $(-2u_i, 0)$ to $(2v_{\phi(i)}, 0)$ and ϕ is a permutation of $\{1, \ldots, n\}$. To each such n-tuple, assign the weight $\operatorname{sgn}\phi$. From the definition of $\det C[\mathbf{u}|\mathbf{v}]$ as a sum of signed products, it is immediate that $\det C[\mathbf{u}|\mathbf{v}]$ is the sum of the weights of the n-tuples in S. The n-tuples that have intersections cancel via a sign-reversing involution: on the lexicographically lowest-indexed pair of intersecting paths, locate the first point of intersection and switch the tails of these two paths after this intersection point. This changes the sign of the associated permutation ϕ and is an involution.

Thus only the nonintersecting n-tuples contribute to the sum, and their weight is 1, since avoiding intersections forces ϕ to be the identity. This establishes the claim.

It follows that $\det C[\mathbf{u}|\mathbf{v}]$ is positive, since there is always the nonintersecting n-tuple consisting of paths in which all northeast steps precede all southeast steps.

Solution to part **(b)** *by Robin J. Chapman, University of Exeter, Exeter, U. K.* In the cases in which $\mathbf{u} = \mathbf{v} = \{0, 1, \ldots, n-1\}$ or $\mathbf{u} = \{0, 1, \ldots, n-1\}$ and $\mathbf{v} = \{1, 2, \ldots, n\}$, there is only one nonintersecting n-tuple of paths. Therefore, the matrices $H_{n-1} = (C_{i+j})_{i,j=0}^{n-1}$ and $H'_{n-1} = (C_{i+j+1})_{i,j=0}^{n-1}$ both have determinant 1.

Now suppose that (a_{i+j}) is an STP sequence of positive integers with $a_n \le C_n$ for every n. Since $C_0 = C_1 = 1$, we have $a_0 = a_1 = 1$. For an inductive proof, we suppose that $a_i = C_i$ for $i < n$.

When $n = 2m$ is even, let $A = (a_{i+j})_{i,j=0}^{m}$. Expanding along the bottom row yields

$$1 - \det(A) = \det(H_m) - \det(A) = (C_n - a_n)\det(H_{m-1}) = C_n - a_n \ge 0.$$

Thus $\det(A) \le 1$, but by hypothesis $\det(A) \ge 1$. Hence $\det(A) = 1$ and $a_n = C_n$.

When $n = 2m + 1$ is odd, the argument is similar, setting $A = (a_{i+j+1})_{i,j=0}^{m}$ and considering H'_m rather than H_m.

Editorial comment. The interpretation of a minor of a Hankel determinant in terms of nonintersecting Dyck paths is due to X. G. Viennot, *Une théorie combinatoire des polynômes orthogonaux généraux* (lecture notes), Université du Québec à Montréal, 1983. The Catalan numbers are essentially moments of the Chebyshev polynomials of the second kind.

Another evaluation of the determinant $\det(C_{i+j})_{i,j=0}^{n-1}$ appears in C. Radoux, Nombres de Catalan généralisés, *Bull. Belg. Math. Soc.* 4 (1997) 289–292.

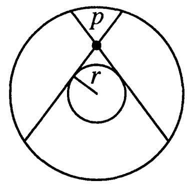
Solved also by the proposer.

A Random Distance

10592 [1997, 456]. *Proposed by Roger Pinkham, Hoboken, NJ.* Three points are selected independently and at random in a disk of radius one. What is the average distance of the third from the line determined by the first two?

Solution by Richard Holzsager, American University, Washington, DC. Let f be the probability density function of the distance r from the chord through the first two points to the origin and let $\mu = \mu(r)$ be the mean distance from a random point in the circle to a chord at distance r from the origin. Then $\int_0^1 f(r)\mu(r)\,dr$ gives the expected value we want.

We can find f by calculating the probability of a chord being at distance greater than r and differentiating. Let p be the first point, q the second point, and R the distance of the chord through them from the origin. Then $f(r) = -(d/dr)\,P(R > r)$. To find $P(R > r)$, first consider a point p at distance s from the origin. The point q determines a chord at distance greater than r if q is between the two tangents from p to the circle $x^2 + y^2 = r^2$. Thus, given p, the probability that the chord pq is at distance greater than r is $1/\pi$ times the area of the butterfly-shaped region between the tangents, as shown in the figure at right. To compute this area, consider a variable chord in this region through p.



This chord is divided by p into two segments, one in each wing of the butterfly. Using the polar integral, we can get the area of one of the wings by integrating half the square of the length of one of these segments. It turns out, however, to be convenient to handle both at the same time, rather than finding one area and doubling.

Let θ be the angle between the chord and the radius through p, which is between $\theta_0 = \sin^{-1}(r/s)$ and $\theta_1 = \pi - \sin^{-1}(r/s)$. The lengths of the two parts of the chord satisfy $l_1+l_2 = 2\sqrt{1-s^2\sin^2\theta}$ and $l_1-l_2 = 2s\cos\theta$. Squaring and adding these equations yields $l_1^2+l_2^2 = 2(1+s^2\cos^2(2\theta))$. The area of the region is then $A = \int_{\theta_2}^{\theta_1} 1+s^2\cos(2\theta)\,d\theta = \pi - 2\sin^{-1}(r/s) - 2r\sqrt{s^2-r^2}$. Given the point p, the probability that the chord is at distance greater than r from the origin is, therefore, A/π. The probability distribution function for s is $g(s) = 2s$, so the overall probability that the chord is at distance greater than r is $P(R>r) = 2\int_r^1 s - (2s/\pi)(\sin^{-1}(r/s) + r\sqrt{s^2-r^2})\,ds$.

Fortunately we do not have to calculate this integral to get its derivative. The Fundamental Theorem and differentiation under the integral sign yield

$$f(r) = 2\left(r - \frac{2r}{\pi}\left(\frac{\pi}{2}\right)\right) - 2\int_r^1 -\frac{2s}{\pi}\left(\frac{1}{\sqrt{s^2-r^2}} + \sqrt{s^2-r^2} - \frac{r^2}{\sqrt{s^2-r^2}}\right) ds$$

$$= \frac{4}{\pi}\left(\sqrt{s^2-r^2} + \frac{1}{3}(s^2-r^2)^{3/2} - r^2\sqrt{s^2-r^2}\right)\Bigg|_r^1 = \frac{16}{3\pi}(1-r^2)^{3/2}.$$

Next, consider a chord C at distance r from the origin. A symmetric pair of points at distance less than r from the diameter parallel to C have average distance r from C, while a symmetric pair at distance $x > r$ from the diameter have average distance x from C. Denote by $A(r)$ the area of the smaller region cut off by C within the circle. The overall mean distance from points in the circle to C is $\mu(r) = (1/\pi)\left((\pi - 2A(r))r + 2\int_r^1 x \cdot 2\sqrt{1-x^2}\,dx\right) = r - (2/\pi)rA(r) + (4/3\pi)(1-r^2)^{3/2}$.

Combining the two results gives

$$\int_0^1 f(r)\mu(r)\,dr = \frac{16}{3\pi}\int_0^1 (1-r^2)^{3/2}\left(r - \frac{2}{\pi}rA(r) + \frac{4}{3\pi}(1-r^2)^{3/2}\right) dr$$

$$= \frac{16}{3\pi}\int_0^1 r(1-r^2)^{3/2}dr - \frac{32}{3\pi^2}\int_0^1 r(1-r^2)^{3/2}A(r)dr + \frac{64}{9\pi^2}\int_0^1 (1-r^2)^3 dr.$$

Using integration by parts on the middle integral, this works out to

$$-\frac{16}{15\pi}(1-r^2)^{5/2}\Bigg|_0^1 + \frac{32}{3\pi^2}\left(\frac{1}{5}A(r)(1-r^2)^{5/2}\Bigg|_0^1 + \frac{2}{5}\int_0^1 (1-r^2)^3 dr\right)$$

$$+ \frac{64}{9\pi^2}\int_0^1 (1-r^2)^3 dr$$

$$= \frac{16}{15\pi} - \frac{32}{3\pi^2}\left(\frac{\pi}{10}\right) + \left(\frac{64}{15\pi^2} + \frac{64}{9\pi^2}\right)\left(1 - 1 + \frac{3}{5} - \frac{1}{7}\right) = \frac{8192}{1575\pi^2} \approx 0.527.$$

Solved also by D. Callan, K. McInturff, A. Pechtl (Germany), H. A. Steinberg, R. Stong, GCHQ Problems Group (U. K.), and the proposer.