An Infinite Product

10605 [1997, 567]. *Proposed by Jonathan M. Borwein and C. G. Pinner, Simon Fraser University, Burnaby, BC, Canada.* Let r and m be positive integers and define

$$P_r(m) = \prod_{n \neq m} \frac{n^r - m^r}{n^r + m^r}.$$

(**a**) Show that $P_1(m) = 0$ and that

$$P_3(m) = (-1)^{m+1}\frac{2}{3}(m!)^2 \prod_{n=1}^{m} \frac{n+m}{n^3+m^3}.$$

(**b**) Show that $P_2(m) = (-1)^{m+1}\pi m/\sinh(\pi m)$ and that, more generally, $P_{2s}(m)$ is given by

$$(-1)^{m+1}\frac{2^\epsilon m\pi}{s}(\sinh m\pi)^{(-1)^s}\prod_{j=1}^{s-1}\left(\cosh\left(2\pi m\sin\left(\frac{j\pi}{2s}\right)\right) - \cos\left(2\pi m\cos\left(\frac{j\pi}{2s}\right)\right)\right)^{(-1)^j}$$

where $\epsilon = (1+(-1)^s)/2$.

Solution by David Bradley, University of Maine, Orono, ME.

(**a**) First, for the case $r = 1$, the infinite product "diverges" to 0 because of the divergence of the harmonic series. Next consider the case $r = 3$. Let $f(n) = n(n-m)/(n^2 - mn + m^2)$. The product becomes $\prod_{n\neq m} f(n)/f(n+m)$. The product now telescopes, and since $f(n) \to 1$ as $n \to \infty$, it reduces to $f(2m)\prod_{n=1}^{m-1} f(n)$ and then to the given expression.

(**b**) For each positive integer r, define $f_r(x) = \prod_{n\geq 1}(n^r - x^r)/(n^r + x^r)$ when x is not an integer. Then for positive integers s and m, we have

$$P_{2s}(m) = \lim_{x\to m} \frac{m^{2s} + x^{2s}}{m^{2s} - x^{2s}} f_{2s}(x).$$

Let $\omega = \exp(i\pi/s)$ and $y = x\exp(-i\pi/2s)$. Then

$$f_{2s}(x) = \prod_{n\geq 1} \frac{n^{2s} - x^{2s}}{n^{2s} + x^{2s}} = \prod_{n\geq 1}\prod_{k=1}^{2s} \frac{n - x\omega^k}{n - y\omega^k}.$$

Using Gauss's infinite product expansion $\Gamma(1+z) = \prod_{n\geq 1}\left(1+\frac{1}{n}\right)^z\left(1+\frac{z}{n}\right)^{-1}$, we obtain

$$f_{2s}(x) = \prod_{k=1}^{2s} \frac{\Gamma(1-y\omega^k)}{\Gamma(1-x\omega^k)} = \prod_{k=1}^{s} \frac{\Gamma(1-y\omega^k)\Gamma(1+y\omega^k)}{\Gamma(1-x\omega^k)\Gamma(1+x\omega^k)}.$$

The reflection formula $\Gamma(1-z)\Gamma(1+z) = \pi z/\sin(\pi z)$ — a consequence of Euler's formula $\prod_{n\geq 1}(1 - z^2/n^2) = \sin(\pi z)/(\pi z)$ — now gives

$$\begin{aligned} f_{2s}(x) &= \prod_{k=1}^{s} \frac{\sin(\pi x\omega^k)\pi y\omega^k}{\pi x\omega^k \sin(\pi y\omega^k)} = e^{-i\pi/2}\prod_{j=1}^{2s}\left(\sin\left(\pi x e^{i\pi j/2s}\right)\right)^{(-1)^j} \\ &= -i(\sin(i\pi x))^{(-1)^s}\sin(-\pi x)\prod_{j=1}^{s-1}\left(\sin\left(\pi x e^{i\pi j/2s}\right)\sin\left(\pi x e^{i\pi(2s-j)/2s}\right)\right)^{(-1)^j} \\ &= i2^\epsilon(i\sinh(\pi x))^{(-1)^s}\sin(\pi x)\prod_{j=1}^{s-1}\left(2\sin\left(\pi x e^{i\pi j/2s}\right)\sin\left(-\pi x e^{-i\pi j/2s}\right)\right)^{(-1)^j} \end{aligned}$$

where $\epsilon = (1 + (-1)^s)/2$. We now use the addition formulæ for the cosine to express a product of two sines as a difference of two cosines and simplify to obtain

$$f_{2s}(x) = 2^\epsilon (\sinh(\pi x))^{(-1)^s} \sin(\pi x) \prod_{j=1}^{s-1} \left(\cosh\big(2\pi x \sin(\frac{\pi j}{2s})\big) - \cos\big(2\pi x \cos(\frac{\pi j}{2s})\big)\right)^{(-1)^j}$$

and hence

$$P_{2s}(m) = \lim_{x\to m} \frac{m^{2s} + x^{2s}}{m^{2s} - x^{2s}} f_{2s}(x) = 2^\epsilon \big(\sinh(\pi m)\big)^{(-1)^s} \frac{2\pi m^{2s}\cos(\pi m)}{-2sm^{2s-1}} \times$$

$$\prod_{j=1}^{s-1}\left(\cosh\big(2\pi m \sin(\frac{\pi j}{2s})\big) - \cos\big(2\pi m \cos(\frac{\pi j}{2s})\big)\right)^{(-1)^j}$$

$$=(-1)^{m+1}\frac{2^\epsilon \pi m}{s}\big(\sinh(\pi m)\big)^{(-1)^s} \prod_{j=1}^{s-1} \left(\cosh\big(2\pi m \sin(\frac{\pi j}{2s})\big) - \cos\big(2\pi m \cos(\frac{\pi j}{2s})\big)\right)^{(-1)^j}$$

as required. Note that this formula gives $P_2(m) = (-1)^{m+1}\pi m/\sinh(\pi m)$ when $s = 1$.

Solved also by R. J. Chapman (U. K.), K.-K. Choi, R. Mortini (France), H.-J. Seiffert (Germany), and the proposers.

Monomial Bounds for Polynomials

10613 [1997, 767]. *Proposed by F. J. Flanigan, San Jose State University, San Jose, CA.* Fix a positive real number ν. Find all polynomials $P(x)$ with nonnegative real coefficients such that

(**a**) $P(0) = 0$, $P(1) = 1$, and $P(x) \le x^\nu$ for all $x \ge 0$.
(**b**) $P(0) = 0$, $P(1) = 1$, and $P(x) \ge x^\nu$ for all $x \ge 0$.

Solution by Roberto Tauraso, Firenze, Italy. Let $P(x) = \sum_{i=m}^n a_i x^i$ with nonnegative real coefficients, $a_m > 0$, and $a_n > 0$. The conditions $P(0) = 0$, $P(1) = 1$ imply immediately that $m \ge 1$ and $\sum_{i=m}^n a_i = 1$.
(**a**) If $P(x) \le x^\nu$ for all $x \ge 0$, then necessarily

$$\lim_{x\to+\infty} \frac{P(x)}{x^\nu} = \lim_{x\to+\infty} \frac{a_n x^n}{x^\nu} \le 1 \qquad \text{and} \qquad \lim_{x\to 0^+} \frac{P(x)}{x^\nu} = \lim_{x\to 0^+} \frac{a_n x^m}{x^\nu} \le 1,$$

which imply $n \le \nu$ and $\nu \le m$, respectively. Hence $m = \nu = n$, and condition (**a**) is satisfied if and only if ν is a positive integer and $P(x) = x^\nu$.
(**b**) If $P(x) \ge x^\nu$ for all $x \ge 0$, then the function $\varphi(x) = P(x) - x^\nu$ is nonnegative, differentiable, and satisfies $\varphi(1) = 0$. Hence φ has a minimum at $x = 1$, so $\varphi'(1) = \left(\sum_{i=m}^n i a_i\right) - \nu = 0$. Thus ν is a convex combination of the integers $m, \ldots, n$.

On the other hand, suppose that a polynomial $P(x) = \sum_{i=m}^n a_i x^i$ has nonnegative real coefficients such that $\sum_{i=m}^n a_i = 1$ and $\sum_{i=m}^n i a_i = \nu$. Then $P(0) = 0$, $P(1) = 1$, and, by the weighted arithmetic-geometric mean inequality, $P(x) = \sum_{i=m}^n a_i x^i \ge x^\nu$ for all $x \ge 0$. Thus condition (**b**) is satisfied if and only if $\nu \ge 1$ and $P(x) = \sum_{i=m}^n a_i x^i$, with $\sum_{i=m}^n a_i = 1$ and $\sum_{i=m}^n i a_i = \nu$.

Editorial comment. Erik I. Verriest provided a generalization to the case in which $P(x)$ is a power series. The results are the same as in the selected solution, except that in part (**b**) the upper limit of summation n may be infinite.

Solved also by P. Alsholm (Denmark), K. F. Andersen (Canada), T. Armstrong, M. Babilonová & J. Kupka (Czech Republik), R. J. Chapman (U. K.), J. H. Lindsey II, A. Nijenhuis, C. Popescu (Belgium), H.-J. Seiffert (Germany), E. I. Verriest, GCHQ Problems Group (U. K.), NSA Problems Group, WMC Problems Group, and the proposer.