



## An Infinite Product: 10605

Jonathan M. Borwein; C. G. Pinner; David Bradley

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## An Infinite Product

**10605** [1997, 567]. *Proposed by Jonathan M. Borwein and C. G. Pinner, Simon Fraser University, Burnaby, BC, Canada.* Let  $r$  and  $m$  be positive integers and define

$$P_r(m) = \prod_{n \neq m} \frac{n^r - m^r}{n^r + m^r}.$$

(a) Show that  $P_1(m) = 0$  and that

$$P_3(m) = (-1)^{m+1} \frac{2}{3} (m!)^2 \prod_{n=1}^m \frac{n+m}{n^3+m^3}.$$

(b) Show that  $P_2(m) = (-1)^{m+1} \pi m / \sinh(\pi m)$  and that, more generally,  $P_{2s}(m)$  is given by

$$(-1)^{m+1} \frac{2^\epsilon m \pi}{s} (\sinh m \pi)^{(-1)^s} \prod_{j=1}^{s-1} \left( \cosh \left( 2\pi m \sin \left( \frac{j\pi}{2s} \right) \right) - \cos \left( 2\pi m \cos \left( \frac{j\pi}{2s} \right) \right) \right)^{(-1)^j}$$

where  $\epsilon = (1 + (-1)^s)/2$ .

*Solution by David Bradley, University of Maine, Orono, ME.*

(a) First, for the case  $r = 1$ , the infinite product “diverges” to 0 because of the divergence of the harmonic series. Next consider the case  $r = 3$ . Let  $f(n) = n(n-m)/(n^2 - mn + m^2)$ . The product becomes  $\prod_{n \neq m} f(n)/f(n+m)$ . The product now telescopes, and since  $f(n) \rightarrow 1$  as  $n \rightarrow \infty$ , it reduces to  $f(2m) \prod_{n=1}^{m-1} f(n)$  and then to the given expression.

(b) For each positive integer  $r$ , define  $f_r(x) = \prod_{n \geq 1} (n^r - x^r)/(n^r + x^r)$  when  $x$  is not an integer. Then for positive integers  $s$  and  $m$ , we have

$$P_{2s}(m) = \lim_{x \rightarrow m} \frac{m^{2s} + x^{2s}}{m^{2s} - x^{2s}} f_{2s}(x).$$

Let  $\omega = \exp(i\pi/s)$  and  $y = x \exp(-i\pi/2s)$ . Then

$$f_{2s}(x) = \prod_{n \geq 1} \frac{n^{2s} - x^{2s}}{n^{2s} + x^{2s}} = \prod_{n \geq 1} \prod_{k=1}^{2s} \frac{n - x\omega^k}{n - y\omega^k}.$$

Using Gauss’s infinite product expansion  $\Gamma(1+z) = \prod_{n \geq 1} \left(1 + \frac{1}{n}\right)^z \left(1 + \frac{z}{n}\right)^{-1}$ , we obtain

$$f_{2s}(x) = \prod_{k=1}^{2s} \frac{\Gamma(1 - y\omega^k)}{\Gamma(1 - x\omega^k)} = \prod_{k=1}^s \frac{\Gamma(1 - y\omega^k)\Gamma(1 + y\omega^k)}{\Gamma(1 - x\omega^k)\Gamma(1 + x\omega^k)}.$$

The reflection formula  $\Gamma(1-z)\Gamma(1+z) = \pi z / \sin(\pi z)$  — a consequence of Euler’s formula  $\prod_{n \geq 1} (1 - z^2/n^2) = \sin(\pi z)/(\pi z)$  — now gives

$$\begin{aligned} f_{2s}(x) &= \prod_{k=1}^s \frac{\sin(\pi x \omega^k) \pi y \omega^k}{\pi x \omega^k \sin(\pi y \omega^k)} = e^{-i\pi/2} \prod_{j=1}^{2s} \left( \sin(\pi x e^{i\pi j/2s}) \right)^{(-1)^j} \\ &= -i (\sin(i\pi x))^{(-1)^s} \sin(-\pi x) \prod_{j=1}^{s-1} \left( \sin(\pi x e^{i\pi j/2s}) \sin(\pi x e^{i\pi(2s-j)/2s}) \right)^{(-1)^j} \\ &= i 2^\epsilon (i \sinh(\pi x))^{(-1)^s} \sin(\pi x) \prod_{j=1}^{s-1} \left( 2 \sin(\pi x e^{i\pi j/2s}) \sin(-\pi x e^{-i\pi j/2s}) \right)^{(-1)^j} \end{aligned}$$

where  $\epsilon = (1 + (-1)^s)/2$ . We now use the addition formulæ for the cosine to express a product of two sines as a difference of two cosines and simplify to obtain

$$f_{2s}(x) = 2^\epsilon (\sinh(\pi x))^{(-1)^s} \sin(\pi x) \prod_{j=1}^{s-1} \left( \cosh\left(2\pi x \sin\left(\frac{\pi j}{2s}\right)\right) - \cos\left(2\pi x \cos\left(\frac{\pi j}{2s}\right)\right) \right)^{(-1)^j}$$

and hence

$$\begin{aligned} P_{2s}(m) &= \lim_{x \rightarrow m} \frac{m^{2s} + x^{2s}}{m^{2s} - x^{2s}} f_{2s}(x) = 2^\epsilon (\sinh(\pi m))^{(-1)^s} \frac{2\pi m^{2s} \cos(\pi m)}{-2sm^{2s-1}} \times \\ &\quad \prod_{j=1}^{s-1} \left( \cosh\left(2\pi m \sin\left(\frac{\pi j}{2s}\right)\right) - \cos\left(2\pi m \cos\left(\frac{\pi j}{2s}\right)\right) \right)^{(-1)^j} \\ &= (-1)^{m+1} \frac{2^\epsilon \pi m}{s} (\sinh(\pi m))^{(-1)^s} \prod_{j=1}^{s-1} \left( \cosh\left(2\pi m \sin\left(\frac{\pi j}{2s}\right)\right) - \cos\left(2\pi m \cos\left(\frac{\pi j}{2s}\right)\right) \right)^{(-1)^j} \end{aligned}$$

as required. Note that this formula gives  $P_2(m) = (-1)^{m+1} \pi m / \sinh(\pi m)$  when  $s = 1$ .

Solved also by R. J. Chapman (U. K.), K.-K. Choi, R. Mortini (France), H.-J. Seiffert (Germany), and the proposers.

### Monomial Bounds for Polynomials

**10613** [1997, 767]. *Proposed by F. J. Flanigan, San Jose State University, San Jose, CA.* Fix a positive real number  $\nu$ . Find all polynomials  $P(x)$  with nonnegative real coefficients such that

- (a)  $P(0) = 0$ ,  $P(1) = 1$ , and  $P(x) \leq x^\nu$  for all  $x \geq 0$ .  
 (b)  $P(0) = 0$ ,  $P(1) = 1$ , and  $P(x) \geq x^\nu$  for all  $x \geq 0$ .

*Solution by Roberto Tauraso, Firenze, Italy.* Let  $P(x) = \sum_{i=m}^n a_i x^i$  with nonnegative real coefficients,  $a_m > 0$ , and  $a_n > 0$ . The conditions  $P(0) = 0$ ,  $P(1) = 1$  imply immediately that  $m \geq 1$  and  $\sum_{i=m}^n a_i = 1$ .

(a) If  $P(x) \leq x^\nu$  for all  $x \geq 0$ , then necessarily

$$\lim_{x \rightarrow +\infty} \frac{P(x)}{x^\nu} = \lim_{x \rightarrow +\infty} \frac{a_n x^n}{x^\nu} \leq 1 \quad \text{and} \quad \lim_{x \rightarrow 0^+} \frac{P(x)}{x^\nu} = \lim_{x \rightarrow 0^+} \frac{a_m x^m}{x^\nu} \leq 1,$$

which imply  $n \leq \nu$  and  $\nu \leq m$ , respectively. Hence  $m = \nu = n$ , and condition (a) is satisfied if and only if  $\nu$  is a positive integer and  $P(x) = x^\nu$ .

(b) If  $P(x) \geq x^\nu$  for all  $x \geq 0$ , then the function  $\varphi(x) = P(x) - x^\nu$  is nonnegative, differentiable, and satisfies  $\varphi(1) = 0$ . Hence  $\varphi$  has a minimum at  $x = 1$ , so  $\varphi'(1) = (\sum_{i=m}^n i a_i) - \nu = 0$ . Thus  $\nu$  is a convex combination of the integers  $m, \dots, n$ .

On the other hand, suppose that a polynomial  $P(x) = \sum_{i=m}^n a_i x^i$  has nonnegative real coefficients such that  $\sum_{i=m}^n a_i = 1$  and  $\sum_{i=m}^n i a_i = \nu$ . Then  $P(0) = 0$ ,  $P(1) = 1$ , and, by the weighted arithmetic-geometric mean inequality,  $P(x) = \sum_{i=m}^n a_i x^i \geq x^\nu$  for all  $x \geq 0$ . Thus condition (b) is satisfied if and only if  $\nu \geq 1$  and  $P(x) = \sum_{i=m}^n a_i x^i$ , with  $\sum_{i=m}^n a_i = 1$  and  $\sum_{i=m}^n i a_i = \nu$ .

*Editorial comment.* Erik I. Verriest provided a generalization to the case in which  $P(x)$  is a power series. The results are the same as in the selected solution, except that in part (b) the upper limit of summation  $n$  may be infinite.

Solved also by P. Alsholm (Denmark), K. F. Andersen (Canada), T. Armstrong, M. Babilonová & J. Kupka (Czech Republik), R. J. Chapman (U. K.), J. H. Lindsey II, A. Nijenhuis, C. Popescu (Belgium), H.-J. Seiffert (Germany), E. I. Verriest, GCHQ Problems Group (U. K.), NSA Problems Group, WMC Problems Group, and the proposer.