10621 [1997, 870]. *Proposed by Harold G. Diamond and Bruce Reznick, University of Illinois, Urbana-Champaign, IL.* Let $F(x)$ denote the Cantor singular function, that is, the unique non-decreasing function on [0, 1] such that, if $x = \sum_{j=1}^{\infty} 2\epsilon_j/3^j$ with $\epsilon_j \in \{0, 1\}$, then $F(x) = \sum_{j=1}^{\infty} \epsilon_j/2^j$. It is clear by symmetry that $\int_0^1 F(x)\,dx = 1/2$. Prove that

$$\int_0^1 (F(x))^2\,dx = \frac{3}{10} \quad \text{and} \quad \int_0^1 (F(x))^3\,dx = \frac{1}{5}.$$

More generally, evaluate $\int_0^1 (F(x))^n\,dx$ for every positive integer n.

Solution I by Kenneth F. Andersen, University of Alberta, Edmonton, Alberta. We prove that

$$\int_0^1 (F(x))^n\,dx = \frac{2}{3(n+1)} \sum_{j=0}^{n} \binom{n+1}{j} \frac{B_j}{3 \cdot 2^{j-1} - 1} \tag{1}$$

for all positive integers n, where B_j denotes the j^{th} Bernoulli number given by $B_0 = 1$ and $(j+1)B_j = -\sum_{m=0}^{j-1} \binom{j+1}{m} B_m$ for $j \geq 1$.

The Cantor set C is given by $[0, 1] \setminus \bigcup_{k=1}^{\infty} \bigcup_{j=1}^{2^{k-1}} I_{k,j}$, where $I_{1,1}$ is the open interval $(1/3, 2/3)$ and the open intervals $I_{k,1}, I_{k,2}, \ldots, I_{k,2^{k-1}}$ are the middle thirds of the 2^{k-1} component intervals of $[0, 1] \setminus \bigcup_{m=1}^{k-1} \bigcup_{j=1}^{2^{m-1}} I_{m,j}$. The $I_{k,j}$ are pairwise disjoint, F is constant on each $I_{k,j}$, and the range of F on $\bigcup_{j=1}^{2^{k-1}} I_{k,j}$ is given by $\{(2j-1)/2^k : 1 \leq j \leq 2^{k-1}\}$. Thus the function F takes the value $1/2$ for $x \in [1/3, 2/3]$, an interval of length $1/3$, the value $1/4$ for $x \in [1/9, 2/9]$ and $3/4$ for $x \in [7/9, 8/9]$, intervals of length $1/9$, and so forth.

To prove (1), let $\sigma_n(m) = \sum_{j=1}^{m} j^n$. Note that

$$\sum_{j=1}^{2^{k-1}} (2j-1)^n = \sigma_n(2^k) - 2^n \sigma_n(2^{k-1}). \tag{2}$$

Since

$$\sigma_n(m) = \frac{1}{n+1} \sum_{j=0}^{n} \binom{n+1}{j} B_j (m+1)^{n+1-j}$$

(L. Comtet, *Advanced Combinatorics,* Riedel, 1974, p. 155), we have

$$\sigma_n(2^k) = 2^{kn} + \sigma_n(2^k - 1) = 2^{kn} + \frac{1}{n+1} \sum_{j=0}^{n} \binom{n+1}{j} B_j 2^{k(n+1-j)}. \tag{3}$$

The Cantor set C has measure zero, so we have

$$\int_0^1 (F(x))^n\,dx = \sum_{k=1}^{\infty} \sum_{j=1}^{2^{k-1}} \int_{I_{k,j}} (F(x))^n\,dx = \sum_{k=1}^{\infty} \frac{1}{3^k} \sum_{j=1}^{2^{k-1}} \left(\frac{2j-1}{2^k}\right)^n$$

$$= \sum_{k=1}^{\infty} \frac{\sigma_n(2^k) - 2^n \sigma_n(2^{k-1})}{3^k 2^{nk}} = \sum_{k=1}^{\infty} \frac{1}{3^k} \left(\frac{\sigma_n(2^k)}{2^{nk}} - \frac{\sigma_n(2^{k-1})}{2^{n(k-1)}} \right)$$

$$= \sum_{k=1}^{\infty} \frac{1}{3^k} \frac{\sigma_n(2^k)}{2^{nk}} - \frac{\sigma_n(1)}{3} - \sum_{k=1}^{\infty} \frac{1}{3^{k+1}} \frac{\sigma_n(2^k)}{2^{nk}} = -\frac{1}{3} + \frac{2}{3} \sum_{k=1}^{\infty} \frac{\sigma_n(2^k)}{3^k \cdot 2^{nk}}. \tag{4}$$

We have used (2) in going from the first line to the second. Substituting (3) into (4) yields

$$\int_0^1 (F(x))^n\,dx = -\frac{1}{3} + \frac{2}{3}\sum_{k=1}^{\infty}\frac{1}{3^k}\left(1 + \frac{1}{n+1}\sum_{j=0}^{n}\binom{n+1}{j}B_j 2^{(1-j)k}\right)$$
$$= \frac{2}{3(n+1)}\sum_{k=1}^{\infty}\frac{1}{3^k}\sum_{j=0}^{n}\binom{n+1}{j}B_j 2^{(1-j)k},$$

since $\sum_{k=1}^{\infty} 3^{-k} = 1/2$. An interchange in the order of summation now yields (1), since $\sum_{k=1}^{\infty}(3\cdot 2^{j-1})^{-k} = 1/(3\cdot 2^{j-1}-1)$. Putting $B_1 = -1/2$, $B_2 = 1/6$, $B_3 = 0$, and $B_4 = -1/30$ into (1) yields

$$\int_0^1 (F(x))^2\,dx = \frac{3}{10}, \quad \int_0^1 (F(x))^3\,dx = \frac{1}{5}, \quad \text{and} \quad \int_0^1 (F(x))^4\,dx = \frac{33}{230}.$$

Solution II by Omran Kouba, Higher Institute of Applied Sciences and Technology, Damascus, Syria. The function $F(x)$ satisfies the following self-similarity property: For every $x \in [0,1]$, we have

$$F(x) = 2F\left(\frac{x}{3}\right) = 2F\left(\frac{2}{3}+\frac{x}{3}\right) - 1.$$

Let $A(t) = \int_0^1 \exp(tF(x))\,dx$ for $t \in \mathbb{R}$. Using the self-similarity property and $F(1/3) = F(2/3) = 1/2$ yields

$$A(2t) = \int_0^{1/3}\exp(2tF(x))\,dx + \int_{1/3}^{2/3}\exp(2tF(x))\,dx + \int_{2/3}^{1}\exp(2tF(x))\,dx$$
$$= \frac{1}{3}\int_0^1 \exp\left(2tF\left(\frac{x}{3}\right)\right)dx + \frac{1}{3}e^t + \frac{1}{3}\int_0^1 \exp\left(2tF\left(\frac{2}{3}+\frac{x}{3}\right)\right)dx$$
$$= \frac{1}{3}(A(t) + e^t + e^t A(t)).$$

Thus

$$1 + 3A(2t) - (1+e^t)(1+A(t)) = 0. \tag{5}$$

On the other hand, letting $J_n = \int_0^1 (F(x))^n\,dx$, we have $A(z) = \sum_{n=0}^{\infty} z^n J_n/n!$. Substituting this in (5) gives

$$\sum_{n=0}^{\infty}\left((3\cdot 2^n - 1)J_n - 1 - \sum_{k=0}^{n}\binom{n}{k}J_k\right)\frac{t^n}{n!} = 0.$$

It follows that we may evaluate the sequence $(J_n)_{n\geq 0}$ by the recursion

$$J_0 = 1, \quad J_1 = \frac{1}{2}, \quad \text{and} \quad J_n = \frac{1}{3\cdot 2^n - 2}\left(2 + \sum_{k=1}^{n-1}\binom{n}{k}J_k\right) \quad \text{for} \quad n \geq 2.$$

Editorial comment. The recurrence is a special case of equation (5) of J. R. M. Hosking, Moments of order statistics of the Cantor distribution, *Stat. and Prob. Letters* **19** (1994) 161–165. Javier Duoandikoetxea notes that the integral $J_t = \int_0^1 (F(x))^t\,dx$ converges for all $t > -\log 3/\log 2$, and that $J_{-1} = \sum_{k=0}^{\infty} J_k$. Can the precise value of J_{-1} be computed?

Solved also by B. Burdick, R. J. Chapman (U. K.), J. E. Dawson (Australia), M. Desjarlais, J. Duoandikoetxea (Spain) T. Hermann, J. R. M. Hosking, J. H. Lindsey II, O. P. Lossers (Netherlands), V. Lucic (Canada), S. Mahajan, K. Schilling, N. C. Singer, A. Stenger, F. W. Steutel (Netherlands), D. C. Terr, A. Tissier (France), D. B. Tyler, Anchorage Math Solutions Group, GCHQ Problems Group (U. K.), WMC Problems Group, and the proposers.