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An Elementary Proof of Binet's Formula for the Gamma Function

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The present note presents an elementary proof of the following important result of J. P. M. Binet [**3**, p. 249].

Theorem 1. *For* $x > 0$ *we have*

$$\Gamma(x+1) = \left(\frac{x}{e}\right)^x \sqrt{2\pi x} \cdot e^{\theta(x)} \tag{1}$$

where

$$\theta(x) = \int_0^\infty \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2}\right) e^{-xt} \frac{1}{t}\, dt.$$

Here Γ *denotes the gamma function defined by*

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt.$$

Since $\lim_{x\to\infty}\theta(x) = 0$, *from* (1) *we immediately obtain Stirling's formula*

$$n! = \Gamma(n+1) \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}\,.$$

Binet's formula can also be used to prove a more precise version of Stirling's asymptotic expansion

$$\log \frac{n!}{(n/e)^n \sqrt{2\pi n}} = \sum_{j=1}^{\infty} \frac{B_{2j}}{2j(2j-1)n^{2j-1}} = \frac{1}{12n} - \frac{1}{360n^3} + \frac{1}{1260n^5} - \cdots,$$

where the B_{2j}*'s denote the Bernoulli numbers defined by*

$$\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} = \sum_{j=1}^{\infty} \frac{B_{2j}}{(2j)!} t^{2j-1}.$$

For, by problem 154 *in Part I, Chapter* 4 *of* [**2**], *the inequalities*

$$\sum_{j=1}^{2N} \frac{B_{2j}}{(2j)!} t^{2j-1} < \frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} < \sum_{j=1}^{2N+1} \frac{B_{2j}}{(2j)!} t^{2j-1}$$

hold for each nonnegative integer N. From $j! = \int_0^\infty t^j e^{-t}\,dt$ and the definition of θ we immediately obtain

$$\sum_{j=1}^{2N} \frac{B_{2j}}{2j(2j-1)n^{2j-1}} < \log \frac{n!}{(n/e)^n \sqrt{2\pi n}} < \sum_{j=1}^{2N+1} \frac{B_{2j}}{2j(2j-1)n^{2j-1}}.$$

In this MONTHLY *several derivations of Stirling's formula and asymptotic expansion have been published. We mention here only the most recent* [**1**].

To prove Binet's formula, we define the function φ by the equation

$$e^{\varphi(x)} = \frac{1}{\sqrt{2\pi}} \int_0^\infty \sqrt{x}\,(te^{1-t})^x\,dt$$

so that

$$\Gamma(x+1) = \left(\frac{x}{e}\right)^x \sqrt{2\pi x} \cdot e^{\varphi(x)}. \tag{2}$$

Binet's formula is equivalent to $\theta(x) = \varphi(x)$. We prove this equality by showing that θ and φ both satisfy a certain difference equation and that $\theta(\frac{1}{2}) = \varphi(\frac{1}{2})$.

Our first lemma tells nothing new; we present a proof for the sake of completeness.

Lemma 1. *For all $x > 0$ and $a > -x$ we have*

$$\int_0^\infty \frac{e^{-xt} - e^{-(x+a)t}}{t}\,dt = \log\left(1 + \frac{a}{x}\right). \tag{3}$$

Proof: Denoting by $f(x)$ and $g(x)$ the left and right hand sides of (3), respectively, we have $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = 0$ and $f'(x) = g'(x)$. Consequently, $f(x) = g(x)$ for all $x > 0$. ■

Lemma 2. *For all $x > 0$ we have*

$$\varphi(x) - \varphi(x+1) = \theta(x) - \theta(x+1) = \left(x + \frac{1}{2}\right)\log\left(1 + \frac{1}{x}\right) - 1. \tag{4}$$

Proof: Denote by $g(x)$ the right-hand side of (4). That $\varphi(x) - \varphi(x+1) = g(x)$ follows immediately from (2) by using the equation $\Gamma(x+2) = (x+1)\Gamma(x+1)$. To prove the statement about θ, first note that $\lim_{x\to\infty}\theta(x) - \theta(x+1) = \lim_{x\to\infty} g(x) = 0$. Moreover,

$$\theta'(x) - \theta'(x+1) = \int_0^\infty \frac{e^{-xt} - e^{-(x+1)t}}{t} - \frac{e^{-xt} + e^{-(x+1)t}}{2}\,dt.$$

Applying (3), we obtain

$$\theta'(x) - \theta'(x+1) = \log\left(1 + \frac{1}{x}\right) - \frac{1}{2}\left(\frac{1}{x} + \frac{1}{x+1}\right) = g'(x).$$

Since the limits at ∞ and the derivatives are equal, $g(x) = \theta(x) - \theta(x+1)$. ■

Remark. Differentiating under the integral sign in the previous proofs can be avoided by replacing $\frac{1}{t}$ by $\int_0^\infty e^{-st}\,ds$ and then using Fubini's theorem.

Lemma 3. $\varphi(\frac{1}{2}) = \theta(\frac{1}{2}) = \frac{1}{2} - \frac{1}{2}\log 2$.

Proof: Since $\Gamma(\frac{3}{2}) = \frac{1}{2}\sqrt{\pi}$, (2) yields $\varphi(\frac{1}{2}) = \frac{1}{2} - \frac{1}{2}\log 2$.

As to $\theta(\frac{1}{2})$, we follow an idea of A. Pringsheim [3, p. 249]. By an obvious substitution,

$$\theta(1) = \int_0^\infty \left(\frac{1}{e^{\frac{1}{2}t} - 1} - \frac{2}{t} + \frac{1}{2} \right) e^{-\frac{1}{2}t} \frac{1}{t}\, dt.$$

Using this, we obtain

$$\theta(1/2) = (\theta(1/2) - \theta(1)) + \theta(1)$$

$$= \int_0^\infty \left(\frac{e^{-\frac{1}{2}t}}{t} - \frac{1}{e^t - 1} \right) \frac{1}{t}\, dt + \int_0^\infty \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) e^{-t} \frac{1}{t}\, dt$$

$$= \int_0^\infty \left(\frac{e^{-\frac{1}{2}t} - e^{-t}}{t} - \frac{1}{2}e^{-t} \right) \frac{1}{t}\, dt = \int_0^\infty -\frac{d}{dt}\left(\frac{e^{-\frac{1}{2}t} - e^{-t}}{t} \right) - \frac{e^{-\frac{1}{2}t} - e^{-t}}{2t}\, dt.$$

Applying (3) we obtain the desired result. ■

Proof of Theorem 1: We have to show that $\varphi(x) = \theta(x)$. By (4), $\theta(x) - \theta(x+1) = \varphi(x) - \varphi(x+1)$. Applying this to $x, x+1, \ldots, x+n-1$ and summing these equations, we see that $\theta(x) - \theta(x+n) = \varphi(x) - \varphi(x+n)$. Since $\lim_{n\to\infty}\theta(x+n) = 0$, we immediately obtain

$$\theta(x) = \varphi(x) - \lim_{n\to\infty}\varphi(x+n) =: \varphi(x) - h(x). \tag{5}$$

Next we show that the function h is decreasing. If $0 \le y \le x$ and $0 \le p \le 1$ then

$$\sqrt{x+n}\, p^{x+n} - \sqrt{y+n}\, p^{y+n}$$
$$\le \sqrt{x+n}\, p^{y+n} - \sqrt{y+n}\, p^{y+n} \le \left(\sqrt{x+n} - \sqrt{y+n}\right)p$$

for all $n \ge 1$. Noting that $0 \le te^{1-t} \le 1 (t \ge 0)$ and using the definition of φ, we conclude that

$$e^{\varphi(x+n)} - e^{\varphi(y+n)} \le \left(\sqrt{x+n} - \sqrt{y+n}\right)e^{\varphi(1)}.$$

Taking the limit as $n \to \infty$, we obtain $e^{h(x)} - e^{h(y)} \le 0$, i.e., $h(x) \le h(y)$. Since the function h is also periodic with period 1, it must be constant. Applying (5) and Lemma 3, we obtain that $h(x) = h(\frac{1}{2}) = 0$ for all $x > 0$. ■

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