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UNSOLVED PROBLEMS

Edited by **Richard Nowakowski**

In this department the MONTHLY presents easily stated unsolved problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial or related results. Typescripts should be sent to Richard Nowakowski, Department of Mathematics and Statistics, Dalhousie University, Halifax NS, Canada B3H 3J5, rjn@cs.dal.ca

Beggar My Neighbour

Marc M. Paulhus

> "What do you play, boy?" asked Estella of myself, with the greatest disdain.
> "Nothing but beggar my neighbour, miss."
> "Beggar him," said Miss Havisham to Estella. So we sat down to cards.
> ...
> I played the game to an end with Estella, and she beggared me. She threw the cards down on the table when she had won them all, as if she despised them for having been won of me.
>
> *Charles Dickens, Great Expectations (1860)*

Many readers can probably recall childhood hours that were whiled away playing a simple card game as Pip and Estella did, although we hope your luck was better than poor Pip's.

Dickens knew the game as "Beggar my Neighbour" but "Strip Jack Naked" is also common. It has been suggested that it is also called "Beat your Neighbour out of Doors" and it may even be the same game as "Knave out of Doors" as mentioned in John Haywood's "A Woman Killed with Kindness" (1607) [**1**]. Similar games include an Italian version called "Camicia" which is played with a different deck and "Egyptian Ratscrew" (or "Egyptian War" or "Bloodystump" or "Egyptian Rhapsody"), which has added elements of speed and violence [**2**].

You may recall that the game often went on for a very long time, with first one person accumulating a lot of cards, then another, so that bedtime or boredom arrived before a winner could be decided. A question that has been asked, perhaps as long as the game has been played, but certainly by John Conway, is: can the game go on forever? On this topic, Conway wrote:

> This was one of my "anti-Hilbert problems". With the standard pack of 52 cards, I just don't know, and it's not for want of trying. What I do know is that there are cycles in some smaller packs ...
>
> My guess is that there ARE cycles (after all, their existence with small packs shows that there's no magic reason why there shouldn't be), and that a clever enough computer search would probably find one. I've played some games that went on very long and seemed to be cyclic, but they've always ended. It's not only true that "of course, any mistake produces gravitational waves"—it also beggars either me or my neighbor.

How is it played? An ordinary deck of 52 cards is divided as equally as possible among the players, who hold their respective shares face down. Players in rotation take one card from the top of their stack and place it face up on a stack in the center of the table. Play continues until a court card (J, Q, K, or A) is played, whereupon the next player is required to contribute respectively 1, 2, 3, or 4 cards to the central stack. If one of these 1, 2, 3, or 4 cards is a court card, then the player stops contributing, and the onus to supply the appropriate number of cards to the central stack passes to the next player. If none of the 1, 2, 3, or 4 cards is a court card, then the last player to play a court card collects the whole of the central stack, turns it over, and adds it to the underside of his own stack. This player then starts play again by turning the top card of his stack and placing it face up in the center of the table. Any player who runs out of cards, drops out of the game. If this happens during the contribution process, then the obligation to complete the contribution passes to the next player. The winner is the player who accumulates the whole deck.

As Conway says, there are cycles with small decks. Here is one:

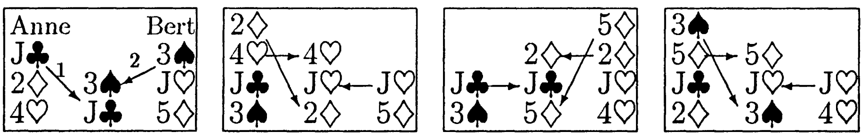

Figure 1. A never-ending mini-game of Beggar-My-Neighbor

and this extends to more players, if Carla, Dan, ... each start with a hand similar to Bert's. A slightly less trivial example, in which a court card does change hands, is the following in which the tops of the hands are on the left. Anne starts with the 9♣.

Anne 9 ♣ 3 ♦ J ♥ 8 ♠

Bert K ♥ 7 ♣ J ♠ 4 ♦

where the King could as well have been a Queen or an Ace.

A much less trivial unending game is

Anne 9♣ 4♣ J♣ 6♣ 3♣ 10♣ Q♣ 2♣ A♣ 5♣ 7♣ K♣ A♦ 8♣

Bert K♦ 3♦ 9♦ J♦ 8♦ 4♦ Q♦ 7♦ 2♦ 10♦ 6♦ 5♦ 2♠ 2♥

You may wonder what those extra 2♠ and 2♥ are doing in there. An exhaustive search on two-player games with half a deck (just two 13-card suits) revealed no cycles at all. But if we add in or remove just two common cards, then there are cycles!

A *deal* is a legal starting position, i.e., the cards are distributed as equally as possible between the players, who are ordered, and one player is designated to start. A *move* in a game is when the cards in the middle are collected by one of the players. Periods and preperiods are reported in terms of moves. A *position* is an ordered set of hands and a bullet to indicate who is to play. To save space, positions are reported without suits and with 0 for common cards, 1 for the Jack, 2 for the Queen, etc...

Given a deck of cards C, which is not necessarily the standard deck, we can construct a directed graph $D_n(C)$ where n is the number of players. Each node in the graph is a position and has one outarrow which points to the position which would result after one move in the game. Some nodes have invalence zero. Naturally, we are most interested in the portion of $D_n(C)$ that is reachable from a deal, which we will call $D'_n(C)$. If the deck C has a unending game then it appears as a cycle in $D'_n(C)$. Note that, with the exception of positions with zero invalence, $D_n(C)$ appears as a subgraph of $D_{n+1}(C)$.

Table 1 shows the results on some exhaustive searches on two-player deals. The last column reports that there is at least one cycle with 2J, 2Q, 2K, 2A, and 20 common cards. This cycle was found by random sampling.

TABLE 1 Results of an exhaustive search of two-player games on decks of cards with 2J, 2Q, 2K, 2A, and n common cards. The second row reports the number of deals with cycles and the final row reports the probability the second player wins.

n	0	2	4	6	8	10	12	14	16	18	20
	0	0	0	1260	11928	4308	0	0	36	0	≥ 1
	1	.60	.533	.514	.508	.506	.505	.504	.504	.503	?

Table 2 shows the 36 deals that never end when two players play with a deck of 2J, 2Q, 2K, 2A, and 16 common cards. Remarkably, there is is essentially only one cycle, which all these games eventually enter. That cycle has a period of 11 moves.

If C is a full deck of cards, does $D'_2(C)$ have a cycle? We leave this question unanswered except to say that we have been unable to find one in 3.2 billion randomly chosen deals. Of course we have searched only a very small portion of the $52!/(36!(4!)^4) = 653{,}534{,}134{,}886{,}878{,}245{,}000$ starting positions. Note that there are an equal number of terminal positions. The longest game we have found is

●0 0 0 0 0 0 0 0 0 0 0 0 3 4 2 0 0 0 0 1 0 0 0 0 0 0
0 1 2 2 3 0 0 0 3 0 0 0 0 1 3 0 0 2 4 0 4 0 1 4 0 0

which requires that 4791 cards be played before terminating. The average game plays about 254 cards before terminating and there appears to be a nearly nonexistent advantage to going second.

We also played 1 billion random deals with 4 players and were unable to find a cycle. The average game plays about 364 cards before terminating and you should prefer to go 4th rather than 3rd, and 3rd rather than 2nd, and 2nd rather than 1st, although position makes only a small difference to your chances of winning.

Let's look more closely at the graph $D_2(C)$. It consists of a large number of connected components. Every terminal position is a member of a tree component and every tree component contains exactly one terminal position. If there are cycles in $D_2(C)$ then the average size of a tree is less than 52. Hence, by randomly sampling terminal nodes and determining the size of their corresponding trees, you can try to establish statistical evidence to support the existence of cycles. As you might expect, when C is the standard deck, the results are inconclusive.

For the record, the Italian game of "Camicia" is played with a 40 card deck with 12 court cards (4 each of values 1, 2, and 3). Playing a billion random deals of Camicia also failed to produce any cycles.

TABLE 2 The 36 deals that cycle when C is a deck with 2J, 2Q, 2K, 2A, and 16 common cards. As an exercise, the reader may want to draw the subgraph of $D_2(C)$ that is reachable from these deals to see that they all reach essentially the same cycle.

Deal (preperiod)	Deal (preperiod)	Deal (preperiod)
●100020000320 (2) 001000004403	●100040342030 (2) 002100000000	●100000000204 (2) 042030100300
●010000000002 (2) 400420301003	●010000044030 (2) 002000032010	●001000040342 (2) 300021000000
●201000004034 (2) 030000210000	●021000000000 (1) 004034203010	●0301000000200 (2) 020310000044
●030140000200 (2) 020031000004	●320100000040 (2) 403000002100	●0203100000444 (2) 301000002000
●003010400002 (2) 002040310000	●2030100400000 (3) 00200403100	●000021000000 (2) 100004034203
●300021000000 (2) 010000403420	●020031000004 (2) 301400002000	●420301003000 (0) 000000020410
●020301000400 (3) 000020004031	●003201000000 (2) 440300000021	●044030100000 (2) 003201200000
●000320120000 (2) 044030100000	●002030130004 (3) 000002010040	●00204310000 (2) 030104000020
●000002010040 (3) 020301300040	●000032010200 (2) 004403001000	●400420301003 (2) 100000000020
●004403001000 (2) 000320102000	●000003201002 (2) 000440300010	●403000002100 (2) 201000000403
●002000032010 (2) 100000440300	●004034203010 (1) 210000000000	●0000440300010 (1) 000032010020
●040300000021 (2) 320100000004	●000020004031 (3) 203010004000	●440300000021 (2) 032010000000

ACKNOWLEDGMENT. We thank Richard K. Guy for his help with this work.

Added in Proof. Since this paper was written we have learned that Michael Kleber has independently established the results in Table 1. He also discovered a longer full-deck game, namely

- 00012000304000040103000230
- 01000000000004124000030002

which requires 5790 cards (805 moves) to terminate.

REFERENCES

1. David Parlett, *Oxford History of Card Games*, Oxford Univ Press, 1990.
2. John McLeod's webpage: www.pagat.com.

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