10722. *Proposed by Richard F. McCoart, Loyola College, Baltimore, MD.*
(**a**) In how many ways can $2n$ indistinguishable balls be placed into n distinguishable urns, if the first r urns may contain at most $2r$ balls for each $r \in \{1, 2, \ldots, n\}$?
(**b**) Suppose that $0 \le m \le n$. In how many of the ways enumerated in part (**a**) are exactly m urns empty?

10723. *Proposed by Christopher J. Hillar, Yale University, New Haven, CT.* Let p be an odd prime. Prove that $\sum_{i=1}^{p-1} 2^i \cdot i^{p-2} \equiv \sum_{i=1}^{(p-1)/2} i^{p-2} \pmod{p}$.

10724. *Proposed by Serge Tabachnikov, University of Arkansas, Fayetteville, AR.*
(**a**) Let P be a convex plane polygon with vertices $A_1, \ldots, A_n$, and let l be a continuous transverse field of directions along the boundary ∂P. (This means that through every point $X \in \partial P$ there passes a line $l(X)$ that intersects the interior of P and depends continuously on X.) Let α_i and β_i be the angles between the line $l(A_i)$ and the adjacent sides A_iA_{i-1} and A_iA_{i+1}, respectively. Assume that $\prod_1^n \sin\alpha_i = \prod_1^n \sin\beta_i$. Prove that the lines $l(X)$ cover the interior of P twice, that is, every interior point of P belongs to at least two of these lines.
(**b**) Suppose $n \ge 3$, and let P be a convex polyhedron in n-dimensional space. As in (**a**), a continuous transverse line field l is given along the boundary ∂P. This field has the property that for every $(n-2)$-dimensional face E of P there exists a hyperplane $\pi(E)$ such that all the lines $l(X)$ with $X \in E$ belong to $\pi(E)$. Prove that the lines $l(X)$ cover the interior of P twice.

SOLUTIONS

Principal Ideals in Noetherian Rings

10534 [1996, 510]. *Proposed by Paul Arne Østvær, Oslo University, Oslo, Norway.* Suppose that R is a Noetherian ring in which all maximal ideals are principal. Show that all ideals in R are principal.

Solution by Robert Gilmer, Florida State University, Tallahassee, FL. If $M = (m)$ is a maximal ideal of R, then M/M^2 is a vector space over the field R/M of dimension at most 1. Hence there are no ideals of R properly between M and M^2. From this it follows (R. Gilmer, *Multiplicative Ideal Theory,* Queen's Papers Pure Appl. Math. **90** (1992), Theorem 39.2) that $R = D_1 \oplus \cdots \oplus D_n \oplus S_1 \oplus \cdots \oplus S_m$ is a finite direct sum of Dedekind domains D_i and special primary rings S_i. To show that each ideal of R is principal, it suffices to show that the D_i and S_i have this property. For S_i this is part of the definition of a special primary ring (Gilmer, p. 200). Moreover, D_i inherits from R the property that each of its maximal ideals is principal, and a Dedekind domain is a principal ideal domain whenever all of its maximal ideals are principal.

Editorial comment. D. D. Anderson mentions a stronger result that appears in R. Gilmer and W. Heinzer, Principal ideal rings and a condition of Kummer, *J. Algebra* **83** (1983) 285–292: If R has the ascending chain condition on *principal* ideals and each maximal ideal of R is principal, then every ideal of R is principal.

Solved also by Mahalal'el ben keinan (Israel), F. Calegari (Australia), J. E. Dawson (Australia), T. H. Foregger, O. Moubinool (France), S. Sertöz (Turkey), and M. Tabaâ (Morocco).

A Telescoping Constraint

10566 [1997, 68]. *Proposed by Gerry Myerson, Macquarie University, Australia.* Let S be a finite set of cardinality $n > 1$. Let f be a real-valued function on the power set of S, and suppose that $f(A \cap B) = \min\{f(A), f(B)\}$ for all subsets A and B of S. Prove that

$$\sum (-1)^{n-|A|} f(A) = f(S) - \max f(A),$$

where the sum is taken over all subsets A of S and the maximum is taken over all proper subsets A of S.

Composite solution by Reiner Martin, Deutsche Bank, London, U. K., and Walter Stromquist, Berwyn, PA. Since $f(A)$ cannot increase when A decreases, the maximum over proper subsets occurs at some set U of size $n-1$; let x be the missing element. For $B \subseteq U$, let $B' = B \cup \{x\}$. Since $B = B' \cap U$, we have $f(B) = \min\{f(B'), f(U)\} = f(B')$ unless $B = U$. Since B and B' differ in size by 1, the terms of the summation other than for S and U cancel in pairs, and we have $\sum(-1)^{n-|A|} f(A) = f(S) - f(U) = f(S) - \max f(A)$.

Solved also by R. Barbara (France), D. Beckwith, M. Benedicty, J. C. Binz (Switzerland), M. Bowron, M. A. Brodie, D. Callan, R. J. Chapman (U. K.), J. E. Dawson (Australia), D. Donini (Italy), D. Dwyer, R. Ehrenborg, G. Gordon, R. Holzsager, T. Jager, W. Janous, K. S. Kedlaya, N. Komanda, O. Kouba (Syria), J. Kuplinsky, C. Lanski, J. H. Lindsey II, O. P. Lossers (The Netherlands), A. Nijenhuis, K. Schilling, Z. Shan & E. Wang (Canada), M. Shemesh (Israel), J. Simpson (Australia), J. H. Steelman, H.-T. Wee (Singapore), GCHQ Problems Group (U. K.), WMC Problems Group, and the proposer.

A Summation Identity

10575 [1997, 169]. *Proposed by Xiaokang Yu, Penn State Altoona Campus, Altoona, PA.* Prove that

$$\sum_{l=0}^{n}(-1)^l\binom{n}{l}(2l)!\sum_{m=0}^{2l}\frac{(-1)^m}{m!} = \sum_{l=0}^{n}(-1)^l\binom{n}{l}2^{n-l}(n+l)!$$

for every nonnegative integer n.

Solution I by Richard Holzsager, American University, Washington, DC. The expression $(2l)!\sum_{m=0}^{2l}(-1)^m/m!$ on the left side is the well-known *derangement number* D_{2l}, the number of ways to return hats to $2l$ people so that no hat returns to its owner. When hats are returned to n couples, let $d(n, l)$ be the number of permutations in which the hats for the people in l couples are deranged, while the remaining hats all go to their owners. The left side is $\sum_{l=0}^{n}(-1)^l d(n, l)$, counting such permutations positively or negatively depending on the parity of the number of deranged couples.

The expression $\binom{n}{l}2^{n-l}(n+l)!$ on the right side is the number of ways to permute the $2n$ hats so that one specified person in each of $n-l$ couples receives the right hat (others may also receive the right hat). In this sum, these terms are weighted by the parity of the number of couples with no member specified. In order to prove the desired identity, it suffices to prove that, in the sum on the right, each permutation of the $2n$ hats is counted with total weight 1, -1, or 0, depending on whether the set of those who get the wrong hat forms an even number of couples, an odd number of couples, or includes some "isolated" individuals whose mates get the right hat.

Consider a permutation of hats in which the correct hats are received by exactly i couples and j isolated individuals. This permutation is counted for each of the $2^r\binom{i}{r}\binom{j}{s}$ choices of one partner from each of r of the couples and s of the individuals; such a choice contributes weight $(-1)^{n-(r+s)}$. The total weight for this permutation is therefore

$$\sum_{r\le i}\sum_{s\le j}(-1)^{n-(r+s)}2^r\binom{i}{r}\binom{j}{s} = (-1)^n\sum_{r\le i}(-2)^r\binom{i}{r}\sum_{s\le j}(-1)^s\binom{j}{s}$$
$$= (-1)^n(1-2)^i(1-1)^j = (-1)^{n-i}0^j,$$

with $0^0 = 1$. This gives ±1 or 0 in precisely the cases we need.

Solution II by M. A. Prasad, Mumbai, India. We first note that

$$\frac{(2l)!}{m!} = \binom{2l}{m}(2l-m)! = \binom{2l}{m}\int_0^\infty e^{-t}t^{2l-m}\,dt,$$