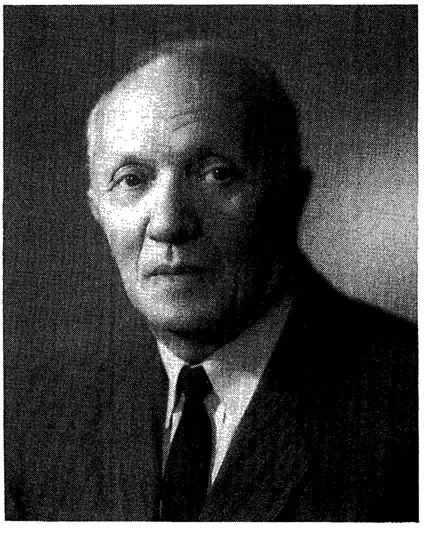
# The Bieberbach Conjecture and Milin's Functionals

**Arcadii Z. Grinshpan**

Dedicated to the memory of Isaak Moiseevich Milin (1919–1992)

The now-settled Bieberbach conjecture (1916) for the Taylor coefficients of univalent (1–1 analytic) functions is one of the most famous and inspirational problems of mathematics. The following events are milestones in the long history of this problem.



(i) Bieberbach's conjecture motivated the development of the Loewner parametric method (1923), which became a very useful tool in the theory of univalent functions.

(ii) In 1971, I. M. Milin constructed a sequence of functionals associated with his exponentiation approach and the Bieberbach conjecture. He conjectured that these functionals were nonpositive, and gave an elementary argument showing that his conjecture implies Bieberbach's.

(iii) In 1984, L. de Branges proved that Milin's functionals were nonpositive. He used Loewner's method together with results and ideas from several fields of mathematics. This achievement allowed de Branges to confirm the Bieberbach conjecture as a theorem.

de Branges' original argument was complicated. During 1984–97, analysts from several countries simplified every component of his approach. Using the ideas of various authors, we prove the Bieberbach and Milin conjectures in a way accessible to many readers. A helpful basic reference is [**4**].

## 1. THE BIEBERBACH CONJECTURE: SIMPLE TO STATE BUT HARD TO PROVE.

According to the Riemann mapping theorem (1851) [**4**, Section 1.5], every simply connected domain $D$ that is a proper subset of the complex plane can be mapped analytically and 1–1 onto the unit disk $E = \{z : |z| < 1\}$. Furthermore, there is a unique such mapping that takes a given point in $D$ into the origin and has a positive derivative there. This fundamental result allows complex analysts to formulate many extremal problems in the plane in terms of normalized univalent (*schlicht*) functions in $E$. Therefore the analytic and geometric properties of such functions are of interest. The behavior of their Taylor coefficients has been the subject of intensive research for most of the Twentieth Century.

Let $S$ (for *schlicht*) be the class of functions $f(z)$ that are analytic and 1–1 in $E$, and normalized by the conditions $f(0) = 0$, $f'(0) = 1$. An important example is the Koebe function $K(z) = z/(1-z)^2 = \sum_{n=1}^{\infty} nz^n$, which maps $E$ onto the plane slit along $(-\infty, -1/4]$. This function is extremal for many classical functionals on $S$. For example, for all $f \in S$ and all $z \in E$ we have

$$-K(-|z|) \leq |f(z)| \leq K(|z|), \tag{1}$$

which is known as the *growth theorem* [**4**, Section 2.3]. The Koebe function is of particular interest when dealing with coefficient estimates.

Let $\{f\}_n$ denote the coefficient of $z^n$ in the Taylor series expansion about $z = 0$ of a function $f(z)$. Note that $\{K\}_n = n$, $n = 1, 2, \ldots$ .

*The Bieberbach conjecture* (1916) [**2**] *asserts that*

$$|\{f\}_n| \leq n, \quad n = 2, 3, \ldots \tag{2}$$

*for each $f \in S$, and that equality holds for any given $n$ only for the Koebe function $K(z)$ and its rotations $\bar{\lambda}K(\lambda z)$, $|\lambda| = 1$.*

For nearly 70 years, many mathematicians, including the very best analysts, tried to prove or disprove this conjecture [**1**], [**4**], [**10**]. L. Bieberbach himself proved it for $n = 2$ [**2**]. Here is a short version of his proof based on the fact that area is always nonnegative.

**Theorem A.** (L. Bieberbach [**2**]). *If $f \in S$, then $|\{f\}_2| \leq 2$, with equality if and only if $f$ is a rotation of the Koebe function.*

*Proof:* The function $F(z) = [f(z^2)]^{-1/2} = z^{-1} - \frac{1}{2}\{f\}_2 z + \cdots$ is univalent in $E \setminus \{0\}$ [**4**, Section 2.1]. For $r \in (0,1)$, let $C_r$ be the image under $F$ of the circle $|z| = r$. Clearly, $C_r$ is a simple closed curve. Switching to polar coordinates, we write $F(re^{i\alpha}) = Re^{i\Psi}$, $0 \leq \alpha < 2\pi$. Since the area enclosed by $C_r$ is positive, we have

$$\frac{1}{2}\int_{C_r} R^2 \, d\Psi > 0,$$

where the integration is performed along $C_r$ in the counterclockwise sense. Using $R\Psi_\alpha = rR_r$, one of the Cauchy-Riemann equations in polar coordinates, we find

$d\Psi = (r/R)R_r\, d\alpha$. It follows that

$$-\frac{d}{dr}\int_0^{2\pi}|F(re^{i\alpha})|^2 d\alpha = 4\pi\left(r^{-3} - \left|\frac{1}{2}\{f\}_2\right|^2 r - \cdots\right) > 0.$$

As $r \to 1$ we deduce that $|\{f\}_2| \leq 2$. Equality is possible only if $F(z) = z^{-1} - \lambda z$, where $|\lambda| = |\frac{1}{2}\{f\}_2| = 1$, and thus $f(z) = \bar{\lambda}K(\lambda z)$. ■

The present proof of the Bieberbach conjecture for $n > 2$ is based on four theorems (A, B, C, D) and four lemmas (1, 2, 3, 4). The desired result is an immediate consequence of Milin's Theorem C and de Branges' Theorem D. Theorem C states that if certain functionals on $S$ are nonpositive (the Milin conjecture) then the Bieberbach conjecture is true. Theorem D asserts the truth of the Milin conjecture.

We follow Milin's argument to prove Theorem C. A particular case of his monotonicity lemma (Lemma 2) implies the so-called Lebedev-Milin exponential inequalities for formal power series. We obtain (2) by applying the Cauchy-Schwarz inequality and the Lebedev-Milin inequalities to the coefficients of a function in $S$. Theorem A is used to settle the case of equality.

The proof of Theorem D based on de Branges' idea requires some familiarity with Loewner's approach (Section 2). It is sufficient to prove the nonpositivity of Milin's functionals only for a dense subclass of $S$ (Lemma 1) associated with Loewner's differential equation (3) (Theorem B). We use a simplified coefficient form of de Branges' construction to produce for each $f$ in this subclass, and for each Milin functional $I$, a differentiable function $\varphi(t)$, $0 \leq t \leq T$, so that $\varphi(0) = I(f)$ and $\varphi(T) = 0$. We combine Loewner's equation, certain polynomial inequalities (Corollary of Lemma 3), and a simple lemma (Lemma 4) to show that $\varphi' \geq 0$; it follows that $I(f) \leq 0$.

All theorems and lemmas are arranged in historical order, which also turns out to be the logical one. Thus, a certain amount of energy and patience allows the reader to walk in the footprints of C. Loewner, I.M. Milin, L. de Branges, and many other mathematicians in proving the famous conjecture.

**2. THE LOEWNER METHOD: A POWERFUL BUT SELECTIVE TOOL.** The parametric method introduced by C. Loewner in 1923 [**8**] and later developed by other authors (see [**4**, Chapter 3]) permits one to solve many extremal problems on the class $S$ via reduction to a dense subclass associated with a partial differential equation.

Although Loewner's method has been very productive, it is far from being universal. Loewner used it to prove (2) for $n = 3$ (and $n = 2$) in his original paper [**8**]. However a proof for $n = 4$ based solely on Loewner's method was given (by Z. Nehari) only 50 years later, when the cases $n = 4, 5, 6$ had been settled by other means (see [**4**; Sections 3.5, 4.6 and Notes, pp. 69, 139]). Despite substantial efforts, no one was able to use Loewner's method in a direct proof of any case $n > 4$. In fact, prior to 1984 no method yielded the conjectured estimate for coefficients beyond the first six.

Fortunately, Loewner's representation theorem for single-slit mappings (functions that map $E$ onto the plane slit along a Jordan arc) can be applied to Milin's functionals. It is a consequence of the Carathéodory convergence theorem (1912) [**4**, Section 3.1] that the single-slit mappings are dense in $S$ with respect to uniform convergence on compact subsets of $E$. A proof of this important property can be found in [**4**, Section 3.2]. Changing it slightly we show that something stronger is true.

**Lemma 1.** *To each $f \in S$ there corresponds a sequence of single-slit mappings $f_n \in S$, $n = 1, 2, \ldots$, such that $f_n \to f$ uniformly on compact subsets of $E$ as $n \to \infty$, and the boundary of each $f_n(E)$, $n \geq 1$, contains a subray of the negative real axis.*

*Proof:* Each function $f \in S$ can be approximated uniformly on closed subdisks of $E$ by the functions $r^{-1}f(rz) \in S$, $0 < r < 1$. Thus, it is sufficient to prove our statement for a function $f \in S$ that maps $E$ onto a domain $D$ bounded by an analytic Jordan curve $C$. In this case there exists a subray $L$ of the negative real axis that belongs to the complement of $\overline{D}$ except for its endpoint $w_L \in C$.

Let $J_n$ be a Jordan arc that runs from infinity along $L$ to the point $w_L$ and then along a portion of $C$ to a point $w_n$ (Figure 1). Let $G_n$ be the complement of $J_n$ and

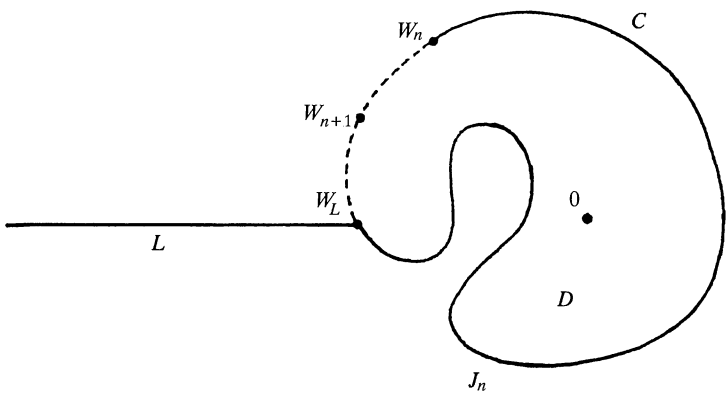



**Figure 1**

let $g_n$ denote the unique 1–1 analytic map of $E$ onto $G_n$ such that $g_n(0) = 0$ and $g_n'(0) > 0$. Choose the endpoints $w_n$, $n = 1, 2, \ldots$, so that $J_n \subset J_{n+1}$ and $w_n \to w_L$.

Then $D$ is the component of $\bigcap_{n=1}^{\infty} G_n$ containing the origin. According to the Carathéodory convergence theorem, $g_n \to f$ uniformly on compact subsets of $E$ as $n \to \infty$. Hence $g_n'(0) \to f'(0) = 1$ and we may take $f_n = g_n/g_n'(0)$, $n \geq 1$. ∎

Thus, the single-slit mappings that omit a subray of the negative real axis are dense in $S$. Similar slit mappings have been considered by G. M. Goluzin, P. P. Kufarev, W. K. Hayman, and other authors. Lemma 1 can be obtained without the Carathéodory theorem, or by just using some of its proof's components. For example, use the Schwarz lemma [**4**, Section 1.1] and the growth theorem (1) to show that the functions $g_n$, $n \geq 1$, are uniformly bounded on closed subdisks of $E$. Then it suffices to establish the existence of a subsequence of $\{g_n\}$ that converges uniformly to $f$ on compact sets [**4**; Sections 1.1 and 1.3].

We now state the representation theorem.

**Theorem B.** (C. Loewner [**8**]). *Let $f \in S$ map $E$ onto the complement of a given Jordan arc $J = \{V(t) : 0 \leq t \leq \infty\}$ ($V$ is 1–1 and continuous) extending from $V(0)$ to infinity. For each $t > 0$ let $f(z,t)$ denote the unique 1–1 analytic map of $E$ onto the* *plane less the portion of J from $V(t)$ to infinity such that $f(0,t) = 0$ and $f_z(0,t) > 0$, and let $f(z,0) = f(z)$.*

*The parametrization $V(t)$ can be chosen so that $f_z(0,t) = e^t$, $t > 0$. In this case $f(z,t)$ satisfies the partial differential equation*

$$f_t(z,t) = zf_z(z,t)\frac{\gamma(t)+z}{\gamma(t)-z}, \qquad z \in E, \quad t \geq 0, \tag{3}$$

*where $\gamma(t)$ is a continuous complex-valued function on $[0,\infty)$ with $|\gamma| = 1$.*

We refer the reader to [**4**; Section 3.3 and Exercise 8, p. 117] for a proof of this classic result. The family $\{f(z,t): t \geq 0\}$ is an example of a so-called *Loewner chain* starting at $f(z)$ and generated by a continuously increasing family of simply connected domains (Figure 2). The point $V(t)$ corresponds to $\gamma(t)$ under the map

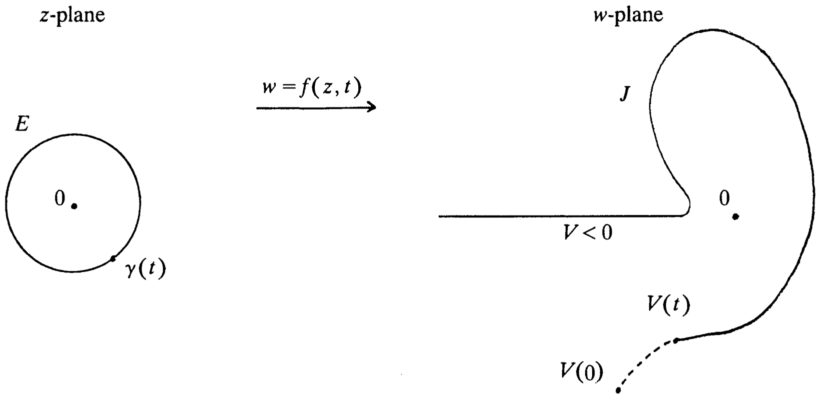



**Figure 2**

$f(z,t)$. One can use the Schwarz lemma to show that if $V(t) < 0$ for $t \geq T \geq 0$ then $f(z,t) = e^t K(z)$ for these values of $t$. We also note that the Taylor coefficients of $f(z,t)$ are differentiable in the parameter $t$. This may be shown by using Cauchy's integral formula and differentiation with respect to $t$ under the integral sign.

**3. THE MILIN THEOREM AND CONJECTURE: SOMETIMES IT'S BETTER TO ATTACK A HARDER PROBLEM.** Since $f(z)/z$ is analytic and zero-free in $E$ for any $f \in S$, we may take $\log[f(z)/z]$ to be the (analytic) branch that vanishes at $z = 0$. With this understanding, the logarithmic coefficients of any $f \in S$ are $\{\log[f(z)/z]\}_n$, $n = 1, 2, \ldots$. For example, $\{\log[K(z)/z]\}_n = 2/n$.

In 1971, I. M. Milin established the following far-reaching connection between the Bieberbach conjecture and the logarithmic coefficients of univalent functions.

**Theorem C.** (I.M. Milin [**10**, discussion before Theorem 3.2]). *For $f \in S$ and $n \geq 1$, define*

$$I_n(f) \equiv \sum_{m=1}^{n} (n+1-m)\Big(m\big|\{\log[f(z)/z]\}_m\big|^2 - 4/m\Big). \tag{4}$$

*If*

$$I_n(f) \leq 0 \tag{5}$$

*for each $f \in S$ and each $n \geq 1$, then the Bieberbach conjecture is true.*

The functionals $I_n$ in (4) are called *Milin's functionals* and (5) is known as *Milin's conjecture*. Since $I_n(K) = 0$ for all $n \geq 1$, Milin's conjecture involves an extremal property of the Koebe function deeper than that of Bieberbach's conjecture. Although the Milin conjecture was verified in certain cases in 1972 [**6**], few seriously believed at the time that one could prove the Bieberbach conjecture through (5). However it was Theorem C that ultimately led researchers out of the dead end they had faced for many years.

Milin's theorem and conjecture go hand in hand with the exponentiation approach developed by Milin in his book [**10**, Chapters 2 and 3]. Properties of formal power series generated by the exponential function play a crucial role in Milin's theory, and the monotonicity lemma [**10**, Lemma 2.2] is the deepest known result of this kind. It has many applications to univalent functions, including a proof of Theorem C. Here we need only a particular case of this lemma, and our proof is similar to the one in [**10**].

**Lemma 2.** *Let $\{A_m\}_1^\infty$ be an arbitrary sequence of complex numbers, and let the sequence $\{D_m\}_0^\infty$ be defined by the formal expansion*

$$\sum_{m=0}^{\infty} D_m z^m = \exp\left( \sum_{m=1}^{\infty} A_m z^m \right). \tag{6}$$

*Let*

$$\Theta_n = n^{-1} \sum_{\nu=0}^{n-1} |D_\nu|^2 \exp\left[ n^{-1} \sum_{m=1}^{n} (n-m)\left(m^{-1} - m|A_m|^2\right) \right], \quad n \geq 1.$$

*Then* $\cdots \leq \Theta_n \leq \cdots \leq \Theta_2 \leq \Theta_1 = 1$, *and* $\Theta_n = 1$ *for some* $n > 1$ *if and only if there is some* $\lambda$ *with* $|\lambda| = 1$ *such that* $A_m = \lambda^m/m$, $m = 1, \ldots, n-1$.

*Proof:* Differentiation of (6) and coefficient comparison yield

$$nD_n = \sum_{m=1}^{n} mA_m D_{n-m}, \quad n \geq 1.$$

By the Cauchy–Schwarz inequality,

$$|D_n|^2 \leq \sigma_n \sum_{m=0}^{n-1} |D_m|^2/n, \quad \text{where} \quad \sigma_n = \sum_{m=1}^{n} |mA_m|^2/n. \tag{7}$$

As

$$\Theta_{n+1}/\Theta_n = \frac{n}{n+1}\left( \sum_{\nu=0}^{n} |D_\nu|^2 \Big/ \sum_{\nu=0}^{n-1} |D_\nu|^2 \right) \exp\left[ \sum_{m=1}^{n} \left( \frac{1}{n} - \frac{1}{n+1} \right)\left(1 - |mA_m|^2\right) \right],$$

(7) and the inequality $(1-x)e^x \leq 1$ with $x = (1-\sigma_n)/(n+1)$ give

$$\Theta_{n+1}/\Theta_n \leq \frac{n}{n+1}(1 + \sigma_n/n)\exp[(1-\sigma_n)/(n+1)] \leq 1, \quad n \leq 1.$$

If $\Theta_{n+1} = 1$, then $|A_1|^2 = \sigma_1 = 1$ and there are constants $\lambda_\nu$ $(\nu = 1, \ldots, n)$ such that $mA_m = \lambda_\nu \overline{D}_{\nu-m}$, $m = 1, \ldots, \nu$. Since $D_0 = 1$ and $D_1 = A_1$, we get $\nu A_\nu = \lambda_\nu$, $|\lambda_1| = 1$, and $\lambda_\nu = \lambda_{\nu-1}\lambda_1$. It follows that $A_m = \lambda_1^m/m$, $m = 1, \ldots, n$. ■

*Proof of Theorem C:* Let $f \in S$ and set

$$D_0 = 1, \quad D_m = \left\{[f(z)/z]^{1/2}\right\}_m, \qquad m \geq 1.$$

Then

$$\{f\}_n = \sum_{m=0}^{n-1} D_m D_{n-1-m}, \quad n > 1.$$

The Cauchy-Schwarz inequality gives

$$|\{f\}_n| \leq \sum_{m=0}^{n-1} |D_m|^2. \tag{8}$$

To prove (2) for any given $n > 1$ use (8), the inequality $\Theta_n \leq 1$ from Lemma 2 with $A_m = \{\log[f(z)/z]\}_m/2$ ($m \geq 1$), and (5) for the preceding value of $n$.

If $|\{f\}_n| = n$, then Lemma 2 ensures that $\{f\}_2 = \{\log[f(z)/z]\}_1 = 2A_1 = 2\lambda$, where $|\lambda| = 1$. Since the Bieberbach conjecture is already known to be true for $n = 2$ (Theorem A), it follows that $f(z) = \bar{\lambda}K(\lambda z)$. ■

The inequalities $\Theta_n \leq 1$ in Lemma 2 are known as the *Lebedev-Milin exponential inequalities*. In 1967, Milin presented them in a more general form but without proof as a joint work with N. A. Lebedev **[9]**. Milin used his monotonicity lemma to prove them in **[10]**. All published proofs of the Lebedev-Milin inequalities to date are similar to the one in **[10]**.

**4. DE BRANGES' DISCOVERY: THE EAGLE HAS LANDED.** In 1984, L. de Branges proved a subtle multiparameter inequality for bounded univalent functions that implies the nonpositivity of Milin's functionals (the final version of his proof was published in **[3]**). This result and Milin's Theorem C allowed de Branges to confirm the truth of the Bieberbach conjecture. It came as a

**Milin's place, St. Petersburg, Russia (1984). de Branges' proof of Milin's conjecture has been verified. L. de Branges is about to become a mathematical hero. Left to right: seated are Isaak Milin and Louis de Branges; standing are Mrs. Asya Grinshpan, Mrs. Evdokiya Milin, Arcadii Grinshpan, and Evgenii Emelyanov.**

sensation to the mathematical world. So far no one has proved the Bieberbach conjecture in any other way, nor has anyone given an essentially different proof of Milin's conjecture.

As a result of discussions that took place during a series of talks given by de Branges to the Goluzin seminar in geometric function theory, de Branges' original proof was verified and reformulated in a simpler form in collaboration with I. M. Milin, E. G. Emelyanov, the author, and others in St. Petersburg, Russia in May, 1984 (see Milin's comments [**11**]). C. FitzGerald and Ch. Pommerenke [**5**], and later L. Weinstein [**12**], further simplified the proof of Milin's conjecture. However their key steps remain the same: apply Loewner's method and use the fact that certain functions introduced by de Branges (de Branges' functions) are nondecreasing in Loewner's parameter $t$. In general, every de Branges function comes from a Loewner chain starting at some mapping $f$. In Milin's case it serves as a delicate link between the value of a Milin functional at $f$ and 0 (discussion after Theorem A, Section 2, and (17)).

**5. THE AUXILIARY POLYNOMIAL INEQUALITIES: COULD THEY HAVE BEEN PROVED TWO HUNDRED YEARS AGO?** The proof that de Branges' functions are nondecreasing is based on the nonnegativity of certain polynomials on the interval $[0, 1]$. de Branges recognized this nonnegativity property as a particular case of the Askey-Gasper inequalities for hypergeometric series (1976) [**3**]. Later, Weinstein's simplification [**12**] led to the same property via the addition theorem for Legendre polynomials (1785). See also [**13**], where a mathematician (D. Zeilberger) collaborated with a computer (Shalosh B. Ekhad). More recently, the author and M. E. H. Ismail [**7**] found an elementary and self-contained proof of the polynomial inequalities in question. Here we give a simplified version of this proof.

**Lemma 3.** *The polynomials* $B_{m,n}(x)$ *defined by the formal expansion*

$$\left[1 - \left(2x^2 + (1 - x^2)(\zeta + \zeta^{-1})\right)z + z^2\right]^{-1/2} = \sum_{n=0}^{\infty} \left[B_{0,n}(x) + \sum_{m=1}^{n} B_{m,n}(x)(\zeta^m + \zeta^{-m})\right] z^n \qquad (9)$$

*can be expressed as*

$$B_{m,n}(x) = 4^{-n} \frac{(n-m)!}{(n+m)!}(1 - x^2)^m \left[\frac{1}{n!}\left(\frac{d}{dx}\right)^{n+m} (x^2 - 1)^n\right]^2. \qquad (10)$$

*Proof:* Taking advantage of an elementary observation used by Th. Clausen (1828), we show that one function is the square of another by means of two linear differential equations.

Given a function $u = u(x)$ and two integers $m$ and $n$ $(0 \le m \le n)$, define

$$H[u] \equiv (x^2 - 1)u'' + 2(1 + m)xu' - (n - m)(n + m + 1)u.$$

Coefficient comparison reveals that $u_0(x) = (d/dx)^{n+m}(x^2 - 1)^n$ satisfies the second-order linear differential equation $H[u] = 0$, and hence the equation

$$(x^2 - 1)u(H[u])' + \left[(2 + 4m)xu + 3(x^2 - 1)u'\right]H[u] = 0. \qquad (11)$$

But we can rewrite (11) as a third-order equation for $h = u^2$ satisfied by $h_0 = u_0^2$:

$$(x^2-1)^2 h''' + 6(1+m)x(x^2-1)h'' + 2\big[(3(2m+1)(m+1) - 2n^2 - 2n)x^2$$
$$-(2m+1)(m+1) + 2n^2 + 2n\big]h' - 4(n-m)(n+m+1)(2m+1)xh = 0. \quad (12)$$

Using Leibniz's rule for the $(n+m)$-th derivative of the product $(x-1)^n(x+1)^n$ we find

$$h_0(1) = u_0^2(1) = 4^{n-m}\left[\frac{n!\,(n+m)!}{m!\,(n-m)!}\right]^2.$$

The next step is to show that each polynomial $B_{m,n}$ differs from a solution of (12) by the $m$-th power of $(1-x^2)$. Indeed, the left-hand side of (9) can be written as

$$\left[1-(1-x^2)(\zeta^{1/2}-\zeta^{-1/2})^2 z(1-z)^{-2}\right]^{-1/2}\Big/(1-z),$$

so we can use the binomial expansion and the identities

$$(\zeta^{1/2}-\zeta^{-1/2})^{2\nu} = \sum_{m=1}^{\nu}(-1)^{\nu-m}\binom{2\nu}{\nu-m}(\zeta^m+\zeta^{-m}) + (-1)^{\nu}\binom{2\nu}{\nu}, \quad \nu \ge 1,$$

to obtain $B_{m,n}(x) = (1-x^2)^m g(x^2-1)$, where

$$g(x) = \sum_{\nu=0}^{n-m}\frac{x^\nu}{\nu!}\,\frac{(n+m+\nu)!}{(n-m-\nu)!\,(\nu+2m)!}\,\frac{(1/2)_{\nu+m}}{(\nu+m)!}.$$

The symbol $(1/2)_k$ denotes the *shifted factorial* of $1/2$:

$$(1/2)_0 = 1 \quad\text{and}\quad (1/2)_k = (1/2)(1/2+1)\cdots(1/2+k-1), \quad k = 1,2,\ldots.$$

To see that $G(x) = g(x^2-1)$ satisfies (12), substitute $G(x)$ for $h(x)$ in the left-hand side of (12), divide through by $x$, and consider the result as a polynomial in $(x^2-1)$. It is identically zero if and only if all of its coefficients are zero, i.e., if and only if

$$2\nu(\nu-1)(\nu-2) + 3\nu(\nu-1) + 6(1+m)\nu(\nu-1) + 3(1+m)\nu$$
$$+\left[3(2m+1)(m+1) - 2n^2 - 2n\right]\nu - (2m+1)(n-m)(n+m+1)$$
$$+\left[\nu(\nu-1) + 3(1+m)\nu + (2m+1)(m+1)\right](n-m-\nu) \quad (13)$$
$$\cdot(2\nu+2m+1)(n+\nu+m+1)/\left[(\nu+2m+1)(\nu+m+1)\right] = 0$$

for fixed $n$ and $m$ and corresponding values of the index $\nu$. Since

$$\nu(\nu-1) + 3(1+m)\nu + (2m+1)(m+1) = (\nu+2m+1)(\nu+m+1),$$

(13) is easy to verify.

Finally, we observe that a polynomial solution of (12) is uniquely determined by its value at $x = 1$. Indeed, we can write this solution as a polynomial in $(x-1)$. Then (12) allows us to find all of its coefficients step-by-step. Because $G$ and $h_0$ are polynomials and

$$G(1) = \frac{(n+m)!}{(n-m)!}\,\frac{(1/2)_m}{(2m)!\,m!} = \frac{4^{-n}}{(n!)^2}\,\frac{(n-m)!}{(n+m)!}h_0(1),$$

(10) follows. ■

**Corollary 1.** *The polynomials* $P_{m,n}(x)$ *defined by the formal expansion*

$$\left[1-\left(2(1-x)+x(\zeta+\zeta^{-1})\right)z+z^2\right]^{-1}$$
$$=\sum_{n=0}^{\infty}\left[\sum_{m=0}^{n}P_{m,n}(x)(\zeta^m+\zeta^{-m})\right]z^n \tag{14}$$

*are nonnegative for* $x\in[0,1]$.

**6. DE BRANGES' THEOREM (THE PROOF OF MILIN'S CONJECTURE): A FUNCTION WITH NONNEGATIVE DERIVATIVE IS NONDECREASING.** In this section we combine the main idea of de Branges with a coefficient representation of de Branges' functions to present a short proof of Milin's conjecture. We take into account some observations from [**5**] and [**12**] on treating Loewner's equation and the auxiliary polynomials.

**Theorem D.** (L. de Branges [**3**]). *For each* $f\in S$ *and each* $n\geq 1$, *Milin's functionals* (4) *satisfy*

$$I_n(f)\leq 0. \tag{15}$$

To prove Theorem D we need one more lemma.

**Lemma 4.** *Let* $a_m$, $m=1,2,\dots$, *be given and define* $b_m=2(1+\sum_{\nu=1}^{m}a_\nu)-a_m$, $m=1,2,\dots$ . *Then*

$$4\mathrm{Re}\left(1+\sum_{\nu=1}^{m}\bar{a}_\nu b_\nu\right)=|a_m+b_m|^2,\quad m\geq 1. \tag{16}$$

*Proof:* We use induction on $m$. Since $a_k+b_k=b_{k+1}-a_{k+1}$, (16) holds for $m=(k+1)$ if it is valid for $m=k$, and (16) holds for $m=1$ because $b_1-a_1=2$. ∎

*Proof of Theorem D:* Fix a natural number $n$.

Since $I_n$ is a continuous functional on $S$ [**4**, Section 1.4], it is sufficient to prove (15) for the dense subclass of $S$ consisting of all single-slit mappings that omit a subray of the negative real axis (Lemma 1). Fix $f(z)$ in this subclass and construct the Loewner chain $\{f(z,t):t\geq 0\}$ as in Theorem B. Then there exists some $T\geq 0$ such that $f(z,t)=e^tK(z)$ for $t\geq T$ (discussion after Theorem B). Define the differentiable function

$$\varphi_n(t)=\left\{K(z)\sum_{m=1}^{n}\left(m|c_m(t)|^2-4/m\right)w^m(z,t)\right\}_{n+1},\quad t\in[0,T], \tag{17}$$

where $c_m(t)=\{\log[f(z,t)/z]\}_m$ and $w(z,t)$ is the Pick function defined implicitly by the equation

$$e^tK(w(z,t))=K(z),\qquad z\in E,\quad t\geq 0.$$

Clearly, $w(z,0)=z$ and hence $\varphi_n(0)=I_n(f)$. Also $\varphi_n(T)=0$ since $c_m(T)=\{\log[e^TK(z)/z]\}_m=2/m$, $m\geq 1$. Thus it is enough to show that $\varphi_n'\geq 0$.

First we use the definitions of $w = w(z,t)$ and $K(z)$ to get

$$K(z)\frac{1-w}{1+w}\left[1+\sum_{m=1}^{\infty} w^m(\zeta^m+\zeta^{-m})\right]$$

$$= z\left[1-\left(2(1-e^{-t})+e^{-t}(\zeta+\zeta^{-1})\right)z+z^2\right]^{-1},$$

provided $\zeta \neq 0$ and $|w| < |\zeta|, |\zeta|^{-1}$. This equation and (14) give

$$P_{m,n}(e^{-t}) = \left\{K(z)\frac{1-w(z,t)}{1+w(z,t)}w^m(z,t)\right\}_{n+1}, \qquad 1 \le m \le n. \tag{18}$$

Now we observe that $w(z,t)$ satisfies the equation $w_t/w = (w-1)/(w+1)$, and use the expansion $(1+w)/(1-w) = 1 + 2w + 2w^2 + \cdots$ and (18) to compute $\varphi_n'$:

$$\begin{aligned}\varphi_n'(t) &= \frac{d}{dt}\left\{K(z)\left[\sum_{m=1}^{n}\left(m|c_m(t)|^2 - 4/m\right)w^m(z,t)\right]\right\}_{n+1}\\ &= \left\{K(z)\sum_{m=1}^{n}\left[2\operatorname{Re}\left(mc_m'(t)\overline{c_m(t)}\right)\right.\right.\\ &\qquad\left.\left.+\left(|mc_m(t)|^2-4\right)\frac{w(z,t)-1}{w(z,t)+1}\right]w^m(z,t)\right\}_{n+1} \qquad (19)\\ &= \sum_{m=1}^{n}P_{m,n}(e^{-t})\operatorname{Re}\left[4\left(1+\sum_{\nu=1}^{m}\nu c_\nu'(t)\overline{c_\nu(t)}\right)-2mc_m'(t)\overline{c_m(t)}-|mc_m(t)|^2\right].\end{aligned}$$

Dividing (3) by $f(z,t)$ and comparing the coefficients of $z^m$ on both sides, we find

$$c_m'\gamma^m = 2\left(1+\sum_{\nu=1}^{m}\nu c_\nu\gamma^\nu\right) - mc_m\gamma^m, \quad m \ge 1.$$

We are now in a position to apply Lemma 4 with $a_m = mc_m\gamma^m$ and $b_m = c_m'\gamma^m$, $m = 1, \ldots, n$, to the last line in (19). We have

$$\varphi_n'(t) = \sum_{m=1}^{n}P_{m,n}(e^{-t})|c_m'(t)|^2. \tag{20}$$

Corollary 1 and (20) show that $\varphi_n'(t) \ge 0$, $t \in [0,T]$. It follows that $I_n(f) \le 0$. ■

Although our proof did not require a geometric description of the Pick function, we mention that for each $t > 0$, $w(z,t) = e^{-t}z + \cdots$ maps $E$ onto $E$ cut along the negative real axis from $-1$ to $1 + 2[(e^{2t} - e^t)^{1/2} - e^t]$.

**7. PROOF OF THE BIEBERBACH CONJECTURE.** Theorems C and D imply that the Bieberbach conjecture is true:

**Main Theorem.** (L. de Branges [3]). *Let $f \in S$. Then*

$$|\{f\}_n| \le n, \quad n = 2, 3, \ldots.$$

*Equality holds for any given n only for the Koebe function $K(z) = z/(1-z)^2$ and its rotations $\overline{\lambda}K(\lambda z)$, $|\lambda| = 1$.*

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**ARCADII Z. GRINSHPAN** was born (1945) and educated (Ph.D., 1973) in St. Petersburg, Russia. Since his student years at St. Petersburg University, he has been engaged in both pure and applied mathematical research: geometric function theory, inequalities, mathematical modeling in diesel engineering, and numerical analysis. Now his experience is used to teach future mathematicians and engineers at the University of South Florida.
*University of South Florida, Tampa, FL 33620*
*azg@math.usf.edu*


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