where the sum is taken over all subsets $A$ of $S$ and the maximum is taken over all proper subsets $A$ of $S$.

*Composite solution by Reiner Martin, Deutsche Bank, London, U. K., and Walter Stromquist, Berwyn, PA.* Since $f(A)$ cannot increase when $A$ decreases, the maximum over proper subsets occurs at some set $U$ of size $n-1$; let $x$ be the missing element. For $B \subseteq U$, let $B' = B \cup \{x\}$. Since $B = B' \cap U$, we have $f(B) = \min\{f(B'), f(U)\} = f(B')$ unless $B = U$. Since $B$ and $B'$ differ in size by 1, the terms of the summation other than for $S$ and $U$ cancel in pairs, and we have $\sum(-1)^{n-|A|} f(A) = f(S) - f(U) = f(S) - \max f(A)$.

Solved also by R. Barbara (France), D. Beckwith, M. Benedicty, J. C. Binz (Switzerland), M. Bowron, M. A. Brodie, D. Callan, R. J. Chapman (U. K.), J. E. Dawson (Australia), D. Donini (Italy), D. Dwyer, R. Ehrenborg, G. Gordon, R. Holzsager, T. Jager, W. Janous, K. S. Kedlaya, N. Komanda, O. Kouba (Syria), J. Kuplinsky, C. Lanski, J. H. Lindsey II, O. P. Lossers (The Netherlands), A. Nijenhuis, K. Schilling, Z. Shan & E. Wang (Canada), M. Shemesh (Israel), J. Simpson (Australia), J. H. Steelman, H.-T. Wee (Singapore), GCHQ Problems Group (U. K.), WMC Problems Group, and the proposer.

## A Summation Identity

**10575** [1997, 169]. *Proposed by Xiaokang Yu, Penn State Altoona Campus, Altoona, PA.* Prove that

$$\sum_{l=0}^{n}(-1)^l\binom{n}{l}(2l)!\sum_{m=0}^{2l}\frac{(-1)^m}{m!} = \sum_{l=0}^{n}(-1)^l\binom{n}{l}2^{n-l}(n+l)!$$

for every nonnegative integer $n$.

*Solution I by Richard Holzsager, American University, Washington, DC.* The expression $(2l)!\sum_{m=0}^{2l}(-1)^m/m!$ on the left side is the well-known *derangement number* $D_{2l}$, the number of ways to return hats to $2l$ people so that no hat returns to its owner. When hats are returned to $n$ couples, let $d(n,l)$ be the number of permutations in which the hats for the people in $l$ couples are deranged, while the remaining hats all go to their owners. The left side is $\sum_{l=0}^{n}(-1)^l d(n,l)$, counting such permutations positively or negatively depending on the parity of the number of deranged couples.

The expression $\binom{n}{l}2^{n-l}(n+l)!$ on the right side is the number of ways to permute the $2n$ hats so that one specified person in each of $n-l$ couples receives the right hat (others may also receive the right hat). In this sum, these terms are weighted by the parity of the number of couples with no member specified. In order to prove the desired identity, it suffices to prove that, in the sum on the right, each permutation of the $2n$ hats is counted with total weight 1, $-1$, or 0, depending on whether the set of those who get the wrong hat forms an even number of couples, an odd number of couples, or includes some "isolated" individuals whose mates get the right hat.

Consider a permutation of hats in which the correct hats are received by exactly $i$ couples and $j$ isolated individuals. This permutation is counted for each of the $2^r\binom{i}{r}\binom{j}{s}$ choices of one partner from each of $r$ of the couples and $s$ of the individuals; such a choice contributes weight $(-1)^{n-(r+s)}$. The total weight for this permutation is therefore

$$\sum_{r\le i}\sum_{s\le j}(-1)^{n-(r+s)}2^r\binom{i}{r}\binom{j}{s} = (-1)^n\sum_{r\le i}(-2)^r\binom{i}{r}\sum_{s\le j}(-1)^s\binom{j}{s}$$
$$= (-1)^n(1-2)^i(1-1)^j = (-1)^{n-i}0^j,$$

with $0^0 = 1$. This gives $\pm1$ or 0 in precisely the cases we need.

*Solution II by M. A. Prasad, Mumbai, India.* We first note that

$$\frac{(2l)!}{m!} = \binom{2l}{m}(2l-m)! = \binom{2l}{m}\int_0^\infty e^{-t}t^{2l-m}\,dt,$$

using the integral representation of the gamma function. Thus, we can rewrite the left side $L$ of the desired equation as

$$L=\sum_{l=0}^{n}(-1)^{l}\binom{n}{l}\sum_{m=0}^{2l}(-1)^{m}\binom{2l}{m}\int_{0}^{\infty}e^{-t}t^{2l-m}\,dt.$$

Interchanging the order of integration and summation and summing over $m$ by the Binomial Theorem yields

$$L=\int_{0}^{\infty}e^{-t}\sum_{l=0}^{n}(-1)^{l}\binom{n}{l}\sum_{m=0}^{2l}(-1)^{m}\binom{2l}{m}t^{2l-m}\,dt=\int_{0}^{\infty}e^{-t}\sum_{l=0}^{n}(-1)^{l}\binom{n}{l}(1-t)^{2l}\,dt.$$

By the Binomial Theorem, the inner sum is $\left(1-(1-t)^{2}\right)^{n}$, which simplifies to $t^{n}(2-t)^{n}$. Expanding $(2-t)^{n}$ and then interchanging the order of summation and integration yields

$$L=\int_{0}^{\infty}e^{-t}t^{n}\sum_{l=0}^{n}\binom{n}{l}(-1)^{l}2^{n-l}t^{l}\,dt=\sum_{l=0}^{n}(-1)^{l}\binom{n}{l}2^{n-l}\int_{0}^{\infty}e^{-t}t^{n+l}\,dt.$$

Since the remaining integral is the integral formula for $(n+l)!$, we have the desired result.

*Solution III by Doron Zeilberger, Temple University, Philadelphia, PA.* We use the umbral calculus (the calculus of finite differences). Define the shift operator $E$ by $Ef(x)=f(x+1)$. Since $E$ commutes with multiplication by constants, we can apply the binomial theorem to expressions involving $E$ and operators that multiply by constants.

Let $\Delta=1-E$. Working solely with operators, we have

$$\begin{aligned}R&:=\sum(-1)^{l}\binom{n}{l}2^{n-l}E^{n+l}=2^{n}E^{n}(1-\frac{1}{2}E)^{n}=E^{n}(2-E)^{n}\\&=(1-\Delta)^{n}(1+\Delta)^{n}=(1-\Delta^{2})^{n}=\sum(-1)^{l}\binom{n}{l}\Delta^{2l}.\end{aligned}$$

Let $F$ denote the factorial function. The right side of the desired equation is $RF$ evaluated at 0. To complete the proof, it suffices to show that $\Delta^{2l}F$ evaluated at 0 yields $(2l)!\sum_{m=0}^{2l}(-1)^{m}/m!$. Since $\Delta^{2l}=(E-1)^{2l}=\sum_{m=0}^{2l}(-1)^{m}\binom{2l}{m}E^{2l-m}$, we have

$$\Delta^{2l}F(0)=(2l)!\sum_{m=0}^{2l}\frac{(-1)^{m}}{m!}\frac{1}{(2l-m)!}E^{2l-m}F(0)=(2l)!\sum_{m=0}^{2l}\frac{(-1)^{m}}{m!}.$$

*Solution IV (composite) by Yong Kong, Washington University, St. Louis, MO, and John Henry Steelman, Indiana University of Pennsylvania, Indiana, PA.* We use an auxiliary identity

$$\sum_{l=0}^{n}\binom{n}{l}\binom{2l}{m}(-1)^{l}=(-1)^{n}2^{2n-m}\binom{n}{m-n}.$$

The two sides of this identity agree for all $n$ when $m\in\{0,1\}$, and both satisfy the recurrence $f(n,m)=-2f(n-1,m-1)-f(n-1,m-2)$. Thus they are always equal.

To prove the desired identity, we replace $m$ by $2l-m$ on the left side, then interchange the order of summation, and apply the auxiliary identity as follows:

$$\begin{aligned}\sum_{l=0}^{n}(-1)^{l}\binom{n}{l}(2l)!\sum_{m=0}^{2l}\frac{(-1)^{m}}{m!}&=\sum_{l=0}^{n}(-1)^{l}\binom{n}{l}\sum_{m=0}^{2l}\binom{2l}{m}m!(-1)^{2l-m}\\&=\sum_{m}(-1)^{m}m!\sum_{l}\binom{n}{l}\binom{2l}{m}(-1)^{l}=\sum_{m}(-1)^{m}m!(-1)^{n}2^{2n-m}\binom{n}{m-n}\end{aligned}$$

Finally, in the last expression set $l = m - n$.

*Editorial comment.* William Seaman and the proposer proved that both sides equal the value at $x = -1$ of $\sum_{m=0}^{2n} \left(\frac{d}{dx}\right)^m \left(1 - x^2\right)^n$.

Solved also by J. C. Binz (Switzerland), R. J. Chapman (U. K.), Q. H. Darwish (Oman), J. E. Dawson (Australia), M. Ismail & P. Simeonov (U. K.), M. Omarjee (France), L. Pebody (U. K.), C. R. Pranesachar (India), R. Richberg (Germany), W. J. Seaman, H.-J. Seiffert (Germany), A. Tissier (France), and the proposer.

### A Large Bipartite Subgraph

**10580** [1997, 270]. *Proposed by Stephen C. Locke, Florida Atlantic University, Boca Raton, FL.* Let $G$ be a simple graph with $v$ vertices and $e$ edges and with maximum degree at most 3. Suppose that no component of $G$ is a complete graph on 4 vertices. Prove that $G$ contains a bipartite subgraph with at least $e - v/3$ edges.

*Solution by James M. Benedict and Gerald Thompson, Augusta State University, Augusta, GA.* When $G$ is bipartite, the claim holds trivially, so we may assume that the chromatic number of $G$ is at least 3. Since $G$ does not have a complete graph of order 4 as a component, Brooks's Theorem implies that $G$ is 3-colorable. Consider a proper 3-coloring using colors red, white, and blue; we may assume that blue appears least often.

Each blue vertex has at most 3 neighbors, all red or white. In either red or white it has at most one neighbor. After removing that edge, we can change the blue vertex to that color and still have a proper coloring. Doing this for each blue vertex deletes at most $v/3$ edges and produces a 2-colored (that is, bipartite) subgraph.

*Editorial comment.* Brooks's Theorem states that a graph with maximum degree $k$ has a proper $k$-coloring if $k \geq 3$ and no component is a complete graph of order $k + 1$ (see for example J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications*, North-Holland, 1976, p. 122). An inductive solution that avoids Brooks's Theorem is also possible.

Solved also by R. J. Chapman (U. K.), C. P. Rupert, P. Tracy, and the proposer.

### Solid Angles of a Tetrahedron

**10598*** [1997, 457]. *Proposed by Jeffrey C. Lagarias, AT&T Research, and Thomas J. Richardson, Bell Laboratories, Murray Hill, NJ.* Let $F_1, F_2, F_3, F_4$ denote the faces of a tetrahedron. For $i = 1, 2, 3, 4$, let $\alpha_i$ denote the solid angle of the vertex opposite face $F_i$, where the measure of a solid angle is normalized so that a full solid angle is 1, and let $\beta_i$ denote the area of $F_i$, where the unit of area is normalized so that the tetrahedron has surface area 1.

**(a)** Prove that $\beta_i \geq \alpha_i$.

**(b)** Generalize to $m$ dimensions.

*Solution by John H. Lindsey II, Ft. Myers, FL.*

**(a)** We prove the sharper claim that $\beta_i > f(\pi\alpha_i)$, where $f(\theta) = \sec\theta\tan\theta - \tan^2\theta = 1/(\csc\theta + 1)$. To see that this bound is sharper, note that $\alpha_i < 1/2$, since $1/2$ is the normalized solid angle of a plane and each angle of the tetrahedron lies on one side of a plane. Since $f(0) = 0$, $f(\pi/2) = 1/2$, and $f''(\theta) = \sec^4\theta(\sin\theta - 1)^2(\sin\theta - 2) < 0$, we have $f(\pi\alpha) \geq \alpha$ for $0 < \alpha < 1/2$.

Suppose that a counterexample exists. We relabel and translate to arrange that the counterexample occurs for $i = 1$, the vertex opposite $F_1$ is the origin $O$, and the other vertices are $xU, yV, zW$, where $U, V, W$ are unit vectors and $x, y, z$ are positive. Then

$$\frac{1}{1/\beta_1 - 1} = \frac{\beta_1}{\beta_2 + \beta_3 + \beta_4} = \frac{|(xU - zW) \times (yV - zW)|}{xy|U \times V| + xz|U \times W| + yz|W \times V|}. \quad (1)$$