Solid Angles of a Tetrahedron: 10598

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Finally, in the last expression set $l = m - n$.

Editorial comment. William Seaman and the proposer proved that both sides equal the value at $x = -1$ of $\sum_{m=0}^{2n} \left(\frac{d}{dx}\right)^m \left(1 - x^2\right)^n$.

Solved also by J. C. Binz (Switzerland), R. J. Chapman (U. K.), Q. H. Darwish (Oman), J. E. Dawson (Australia), M. Ismail & P. Simeonov (U. K.), M. Omarjee (France), L. Pebody (U. K.), C. R. Pranesachar (India), R. Richberg (Germany), W. J. Seaman, H.-J. Seiffert (Germany), A. Tissier (France), and the proposer.

A Large Bipartite Subgraph

10580 [1997, 270]. *Proposed by Stephen C. Locke, Florida Atlantic University, Boca Raton, FL.* Let G be a simple graph with v vertices and e edges and with maximum degree at most 3. Suppose that no component of G is a complete graph on 4 vertices. Prove that G contains a bipartite subgraph with at least $e - v/3$ edges.

Solution by James M. Benedict and Gerald Thompson, Augusta State University, Augusta, GA. When G is bipartite, the claim holds trivially, so we may assume that the chromatic number of G is at least 3. Since G does not have a complete graph of order 4 as a component, Brooks's Theorem implies that G is 3-colorable. Consider a proper 3-coloring using colors red, white, and blue; we may assume that blue appears least often.

Each blue vertex has at most 3 neighbors, all red or white. In either red or white it has at most one neighbor. After removing that edge, we can change the blue vertex to that color and still have a proper coloring. Doing this for each blue vertex deletes at most $v/3$ edges and produces a 2-colored (that is, bipartite) subgraph.

Editorial comment. Brooks's Theorem states that a graph with maximum degree k has a proper k-coloring if $k \geq 3$ and no component is a complete graph of order $k + 1$ (see for example J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications*, North-Holland, 1976, p. 122). An inductive solution that avoids Brooks's Theorem is also possible.

Solved also by R. J. Chapman (U. K.), C. P. Rupert, P. Tracy, and the proposer.

Solid Angles of a Tetrahedron

10598* [1997, 457]. *Proposed by Jeffrey C. Lagarias, AT&T Research, and Thomas J. Richardson, Bell Laboratories, Murray Hill, NJ.* Let F_1, F_2, F_3, F_4 denote the faces of a tetrahedron. For $i = 1, 2, 3, 4$, let α_i denote the solid angle of the vertex opposite face F_i, where the measure of a solid angle is normalized so that a full solid angle is 1, and let β_i denote the area of F_i, where the unit of area is normalized so that the tetrahedron has surface area 1.

(a) Prove that $\beta_i \geq \alpha_i$.

(b) Generalize to m dimensions.

Solution by John H. Lindsey II, Ft. Myers, FL.

(a) We prove the sharper claim that $\beta_i > f(\pi\alpha_i)$, where $f(\theta) = \sec\theta\tan\theta - \tan^2\theta = 1/(\csc\theta + 1)$. To see that this bound is sharper, note that $\alpha_i < 1/2$, since $1/2$ is the normalized solid angle of a plane and each angle of the tetrahedron lies on one side of a plane. Since $f(0) = 0$, $f(\pi/2) = 1/2$, and $f''(\theta) = \sec^4\theta(\sin\theta - 1)^2(\sin\theta - 2) < 0$, we have $f(\pi\alpha) \geq \alpha$ for $0 < \alpha < 1/2$.

Suppose that a counterexample exists. We relabel and translate to arrange that the counterexample occurs for $i = 1$, the vertex opposite F_1 is the origin O, and the other vertices are xU, yV, zW, where U, V, W are unit vectors and x, y, z are positive. Then

$$\frac{1}{1/\beta_1 - 1} = \frac{\beta_1}{\beta_2 + \beta_3 + \beta_4} = \frac{|(xU - zW) \times (yV - zW)|}{xy|U \times V| + xz|U \times W| + yz|W \times V|}. \quad (1)$$

Varying x, y, and z does not change α_1; therefore we may choose a sequence of counterexamples $(O, x_n U, y_n V, z_n W)$ with $0 < x_n, y_n, z_n$ for which β_1 converges to its infimum. Some ordering of (x_n, y_n, z_n) must occur infinitely often, so after reordering the vertices, passing to a subsequence, and rescaling, we may assume $1 = x_n \geq y_n \geq z_n$.

Suppose $z_n \to 0$. Then terms that are small relative to $x_n y_n |U \times V|$ do not affect the limit of (1). Ignoring them, we are left with

$$\frac{|x_n U \times (y_n V - z_n W)|}{x_n y_n |U \times V| + x_n z_n |U \times W|}.$$

This is a 2-dimensional analogue $((1/\beta'_{1,n}) - 1)^{-1}$ for the triangle with edges $x_n y_n U \times V$ and $x_n z_n U \times W$. Assuming the 2-dimensional version, we have

$$\lim \frac{1}{\frac{1}{\beta_{1,n}} - 1} = \lim \frac{1}{\frac{1}{\beta'_{1,n}} - 1} \geq \lim \frac{1}{\frac{1}{f(\pi\alpha'_{1,n})} - 1} = \frac{1}{\frac{1}{f(\pi\alpha'_1)} - 1} > \frac{1}{\frac{1}{f(\pi\alpha_1)} - 1},$$

since α'_1, the angle between $U \times V$ and $U \times W$, is one of the dihedral angles of the tetrahedron that meet at O. Thus we do not have a counterexample. Therefore we may assume that z_n does not converge to zero, and passing to a subsequence we get a limiting nondegenerate tetrahedron that is a counterexample and that is a minimum for β_1.

Let P_1, P_2, P_3, P_4 be the vertices of this tetrahedron, with P_1 the origin. We may assume that F_1 is parallel to the x,y-plane and at distance a from it. Let P_1^* be the projection of P_1 onto the plane containing F_1. Imagine moving P_i toward P_1 at a constant rate. By minimality, the derivative of the ratio of areas defining $(1/\beta_1 - 1)^{-1}$ with respect to time is 0 at the start of this motion. The component of the movement orthogonal to the plane containing F_1 has no first order effect on the area of F_1 at the start, so if we replace the area of F_1 by its projection on the original plane of F_1, then the derivative of our new ratio is again 0 at the start. However, the new ratio is a quotient of linear functions of time, so, since it has derivative 0 at the start, it must be constant. If P_1^* lies outside F_1, say outside the edge P_3P_4, then while P_2 is en route to P_1, P_2^* (the projection of P_2 onto the plane containing F_1) crosses the extended edge, at which point our new ratio is 0. This is impossible since the ratio is constant. When P_i reaches P_1, our new ratio is the ratio of the area of the projection of F_i onto the plane of F_1 to the area of F_i. This ratio is $b_i(a^2 + b_i^2)^{-1/2}$, where b_i is the distance from P_1^* to the edge of F_1 opposite P_i. It follows that $(1/\beta_1 - 1)^{-1} = b_i(a^2 + b_i^2)^{-1/2}$ for every $i \in \{2, 3, 4\}$. Thus P_1^* and $b = b_i$ are the incenter and inradius of F_1.

Let $g(\gamma)$ be the solid angle, normalized so that the full solid angle is 4π, from P_1 spanned by a right triangle $P_1^* RS$ in the plane of F_1, with $|P_1^* R| = b$, $\angle P_1^* RS = \pi/2$, and $\angle RP_1^* S = \gamma$. A calculation shows that

$$g(\gamma) = \gamma - \int_0^\gamma \frac{a\, d\theta}{\sqrt{a^2 + b^2 \sec^2\theta}} = \gamma - \arcsin\left(\frac{a \sin\gamma}{\sqrt{a^2 + b^2}}\right).$$

Since $g(\gamma)$ is concave upward, $g(0) = 0$, and $g(\pi/2) = \arctan(b/a)$, it follows that $g(\gamma) < (2\gamma/\pi)\arctan(b/a)$ for $\gamma \in (0, \pi/2)$. Since F_1 is a union of six such triangles $P_1^* RS$, with angles γ summing to 2π, we see that $4\pi\alpha_1 = \sum g(\gamma) < (2\sum \gamma/\pi)\arctan(b/a) = 4\arctan(b/a)$, where the summations are taken over the 6 values of γ. Hence $\tan\pi\alpha_1 < b/a$, and

$$\beta_1 = \frac{b}{b + \sqrt{a^2 + b^2}} > \frac{1}{\csc\pi\alpha_1 + 1} = f(\pi\alpha_1).$$

Thus a counterexample cannot exist.

(b) The argument is similar. In dimension $m > 3$ we again get strict inequality. To see this, consider a counterexample in dimension m. Arrange that $\beta_1 \leq f(\pi\alpha_1)$ and that the vertices are $P_1 = O$ and $P_i = z_i U_i$ for $2 \leq i \leq m+1$, where U_i are unit vectors and $z_i > 0$. Again varying the z_i does not change α_1, so we may choose a sequence for which β_1 approaches its infimum. A subsequence either degenerates to a lower dimensional simplex or leads to a counterexample with β_1 minimal. If the limit is degenerate, then a computation shows that there is a counterexample for lower m, contradicting the minimality of m.

Therefore we may consider a counterexample that has β_1 minimal under varying the z_i. Assume that F_1 lies in the affine subspace $S = \{(a, x_2, \ldots, x_m)\}$, and let P_1^* be the projection of P_1 into this subspace. Arguing as for the 3-dimensional case, we see that P_1^* is in the interior of F_1 and is equidistant from all the faces of F_1. Let this common distance be b. Let F be a face of F_1, let T be the $(m-2)$-dimensional affine subspace containing F, and let Q be the orthogonal projection of P_1 into T. Let $f(r)dr$ be the solid angle generated from P_1^* by the points of T whose distance from Q is between r and $r+dr$. Let $S(r)$ be the sphere of radius r about Q in T, and define $g_F(r) = \text{area}(F \cap S(r))/\text{area}(S(r))$. Note that $g_F(r)$ is nonincreasing, by convexity of F. If a solid angle Φ in S with vertex P_1^* meets T at a distance from Q of between r and $r+dr$, then let $h_b(r)$ be the measure of the solid angle from P_1 generated by the portion of Φ bounded by T. With these definitions, F generates a solid angle of $\int_0^\infty g_F(r)f(r)dr$ from P_1^* in S, and the portion of F_1 between F and P_1^* generates a solid angle of $\int_0^\infty g_F(r)h_b(r)f(r)dr$ from P_1.

Since $g_F(r)$ is nonincreasing and nonconstant, and since $h_b(r)$ is increasing, we have

$$\int_0^\infty f(r)dr \int_0^\infty g_F(r)h_b(r)f(r)dr < \int_0^\infty h_b(r)f(r)dr \int_0^\infty g_F(r)f(r)dr.$$

Let A_t be the $(t-1)$-dimensional area of the t-dimensional sphere of radius 1. Summing the last inequality over all faces of F_1 gives

$$A_m\alpha_1 \int_0^\infty f(r)dr < A_{m-1} \int_0^\infty h_b(r)f(r)dr. \tag{2}$$

The same calculation applies if F_1 is replaced by the slab $G = \{(a, x_2, \ldots, x_m) \colon |x_2| \le b\}$, except that (1) we now get equality, since for both faces H of G, the function g_H is identically 1, and (2) α_1 is replaced by ϕ, the probability that the ray from O through a random point $v = (y_1, \ldots, y_m)$ on the unit sphere hits G. Hence

$$A_m\phi \int_0^\infty f(r)dr = A_{m-1} \int_0^\infty h_b(r)f(r)dr. \tag{3}$$

From (2) and (3), we infer that $\alpha_1 < \phi$. The random ray from O hits G if and only if $y_1 > 0$ and $|y_2|/y_1 \le b/a$. This depends only on the direction of (y_1, y_2), which is uniformly distributed. Thus $\alpha_1 < \phi = \alpha_1'$, where α_1' is the value of α_1 for the (2-dimensional) isosceles triangle J with altitude a and base $2b$. Since J has the same value of β_1 as G has, we are reduced to the 2-dimensional case. Since we can approximate G by F_1, $f(\pi\alpha_1)$ is the best possible lower bound for β_1.

A Tricky Convergence

10614 [1997, 767]. *Proposed by Grigore-Raul Tataru, University of Bucharest, Bucharest, Romania.* Fix $p > 1$. Suppose that $a_1, a_2, \ldots$ is a sequence of positive real numbers such that $a_n a_{n+1} a_{n+2}^p + a_{n+2} - a_n = 0$ for all $n \ge 1$. Show that $\{a_n\}$ is convergent.

Solution by the GCHQ Problems Group, Cheltenham, U. K. Since $a_n - a_{n+2} = a_n a_{n+1} a_{n+2}^p$ is positive, the even and odd subsequences are decreasing and therefore convergent, say to x and y respectively. Taking limits gives $yx^{p+1} = 0 = xy^{p+1}$, so at least one of x and y must be 0. Without loss of generality, we may assume $x = 0$. If $y > 0$, then $a_{2n-1} - a_{2n+1} > y^{p+1}a_{2n}$, so the series $\sum a_{2n}$ converges. Let m be large enough that $a_{2m+1} < 2y$ and $a_{2m} < 1$. Let $\epsilon = a_{2m}$. For $n \ge m$, we have $a_{2n} - a_{2n+2} < 2y\epsilon^{p+1}$, so the number of integers n with $\epsilon/2 \le a_{2n} < \epsilon$ is at least $(\epsilon/2)/(2y\epsilon^{p+1}) = 1/(4y\epsilon^p) > 1/(4y\epsilon)$, and