REVIEWS

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An Accompaniment to Higher Mathematics. By George R. Exner. Springer-Verlag, 1996, xvii + 198 pp., $29.95.
Journey into Mathematics: An Introduction to Proofs. By Joseph Rotman. Prentice Hall, 1998, xiii + 237 pp., $66.67.
Mathematical Thinking: Problem Solving and Proofs. By John P. D'Angelo and Douglas B. West. Prentice Hall, 1997, xviii + 365 pp., $71.

Reviewed by **Joseph H. Silverman**

One of the highest hurdles facing erstwhile mathematics majors is the transitory leap from rote problem solving to proof creation. (Dissenting readers may substitute "discovery" for "creation" if they wish, but the fascinating question of whether proofs are discovered or created is largely irrelevant to our discussion.) By rote problem solving I mean, of course, the sort of process used in most calculus classes whereby students are shown standard problem templates and, after absorbing a sufficient number of examples, learn to solve similar problems in a color-by-number fashion. There is nothing inherently wrong with this activity, since for most students achieving competence requires hard work and a significant amount of mental effort, two essential ingredients of every good college education. Further, students who do master the material are left with a sense of accomplishment, and we hope that those students who actually need the calculus for their further studies in engineering, economics, or the hard sciences will solidify and deepen their understanding of the subject when they see it used in their other courses.

For mathematicians, problem solving of the sort just described is related to mathematics much as doing a crossword puzzle is related to writing a novel. Both activities require a good vocabulary and some mental agility, but only the latter requires creativity. However, just as there are writing guidelines and exercises designed to assist budding novelists, it is the thesis of these three books that there are teachable techniques through which aspiring mathematics majors can make the transition from mathematical watchers to mathematical doers.

What, roughly, are some of the meta-mathematical tools (as opposed to mathematical techniques such as induction) that every mathematician keeps close at hand when tackling a mathematical problem? In no particular order, the following (non-definitive and non-disjoint) list comes to mind:

- Do lots of examples, numerical or otherwise.
- Specialize the problem. Do special cases.
- Generalize the problem. Eliminate unnecessary hypotheses. This technique can be surprisingly effective, since with fewer hypotheses, there are fewer ways to proceed!

- Search for counterexamples to the original problem.
- Find counterexamples when each of the hypotheses is relaxed. Thus the origin of the phrase "the exception 'proves' the rule," using the original sense of the word 'prove' meaning 'test the limits of,' not 'verify the truth of.'
- Formulate and prove analogous results to provide "evidence" for the validity of the original conjecture.

In addition, every mathematician must acquire various meta-mathematical skills, such as:

- Take a poorly or incompletely posed problem and formulate precise statements to be studied.
- "Fiddle" with a problem, try this-and-that-and-the-other, until eventually some of the ideas that didn't work suddenly fit together to give the solution.

This last item is, in some sense, the most important lesson for a student to absorb. To return to our earlier analogy, if you want to be a writer, the first requirement is that you sit down and write. What you write doesn't have to be good, nor even grammatical, but you have to get some words down on paper. (The modern reader may substitute "computer screen" for piece of paper, although I think it is still true that for mathematical scribbling, nothing beats pencil and paper.) It's not enough to stare at a piece of paper, nor to read other people's novels. Similarly, the first requirement for doing mathematics is to work on it, even if the work doesn't seem to be getting anywhere. You can't *do* mathematics by staring at a blank piece of paper, nor by reading a textbook, nor by listening to a professor explain mathematics on a blackboard.

The books under review appear to have two principal goals. First, they attempt to provide a framework (or a collection of frameworks) on which students can hang the pieces of their proofs. Second, they try to force the students to do mathematical research, by which I mean to solve problems and to write proofs that are not exactly like problems and proofs they've seen before. In short, students who successfully read these books will have to think for themselves, a hard and painful process required for doing mathematics. Beyond that, the three books have different contents and strengths and weaknesses. Which book is to be preferred depends largely on the teacher's tastes and the expected mathematical ability of the students. Herewith are some remarks to help in making that choice.

Exner's book has the heaviest concentration of proof techniques and the lightest load of mathematical content. This is not meant as a criticism, because the techniques it stresses are fundamental, and it explains them in a way designed to grab and hold the students' attention. Virtually every page contains a signal for the student to stop reading and to do some personal work, starting with Exercise 1.1: "Ahem... why don't *you* pick a function and we'll see [if it's injective]" and ending with Exercise 3.116: "Take a shot at proving this [the Schröder-Bernstein theorem]." Used in conjunction with a standard text in a traditional first course involving proofs, whether it be algebra, real analysis, or topology, this book will certainly help most students to make the transition from spectator to participant. A lengthy appendix giving exercise hints (but not solutions) will further aid the reader. On the other hand, even with some Laboratories included as Chapter 4, Exner's book would need to be supplemented with additional mathematical content if it were to be used for the full semester transition or bridge course common in many mathematics departments.

Rotman's book contains an interesting blend of miscellaneous mathematical topics interspersed with numerous jokes, stories, and historical anecdotes. The emphasis is on teaching proofs by example using topics that are mathematically interesting but relatively elementary. Thus logic and set theory are largely ignored, since the author "finds this material rather dull and uninspiring, and imagines that this feeling is shared by most students." Some of the more interesting topics include:

- The Pythagorean theorem, Pythagorean triples, and trigonometry.
- Circles, areas, and π.
- Polynomials, including derivation of the formulas for the roots of quadratic, cubic, and quartic polynomials.
- Diophantine approximation and proofs of the irrationality of e and π.

As may be seen from this list, the topics covered are sufficiently elementary that proofs do not require much in the way of definitions or machinery, but they are sufficiently mathematically interesting to appeal to most instructors and most students. However, in contrast to Exner's book, the clearly written proofs do not include much in the way of motivation or discussion of meta-mathematical proof structure, so instructors may need to supply some of this material in class. On the plus side, the proofs are clear and provide excellent models for students who can learn by example, there are lots of exercises of varying degrees of difficulty, and the non-mathematical material is generally fun to read.

The book by D'Angelo and West is the longest and most ambitious of the three books under review. It contains all of the material needed for a transition course aimed at serious mathematics majors. In the first section (74 pages), the authors discuss set theory, logic, and other topics that are used in creating proofs. They then settle down to the main purpose of their book, namely to present several standard areas of mathematics with clear and complete definitions, proofs, and examples. Their topics include number theory, combinatorics, probability, graph theory, and real analysis (of one variable), so everyone will find something of interest. Each chapter ends with a substantial collection of exercises, and an appendix gives hints (but not solutions) for the more difficult exercises.

The three books under review are all worthwhile additions to the textbook market serving somewhat different audiences. Exner's book teaches proof techniques in a bright and lively manner, making it a good supplementary text for a first proof-based course. Rotman's offering provides mathematical content not normally covered in standard mathematics courses; it has carefully written proofs and much entertaining non-mathematical material. It would be especially good for students who hope to study higher mathematics, but who need some coddling before their leap into the cold waters of proof-laden upper-division courses. Finally, the book by D'Angelo and West contains everything needed for a "transition to proofs" course aimed at students with a serious intention of majoring in mathematics.

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