Decision Making: A Golden Rule

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1. INTRODUCTION. Suppose someone suggests to you the following game: You are able to take as many slips of paper as you please and on each slip write a different number without restrictions. Then, you turn the slips face down, shuffle them, and start turning them face up, one at a time. As the numbers present themselves one after the other, the proponent of the game is to interrupt their procession by speculating that some number just passed is the largest of the sequence. He is to make a single guess about a number being the largest right at the moment that it shows up. If all slips have been turned over and he has not yet pronounced a preference, he must "choose" the last number. The proponent of the game is courteous enough to play the game with any odds you consider to be fair. What would you suggest?

This problem first appeared in the February 1960 issue of *Scientific American* [**7**]. Since then, it has been extended [**1**–**2**] and generalized [**3**–**9**] in many directions by eminent probabilists and statisticians so one may justly claim that it now constitutes a distinct field of study within probability-optimization theories. It has come to be known as the beauty contest problem, the secretary problem, the marriage problem or fiancé problem, and the dowry problem.

The importance of the game is that it provides an artificial-idealized simulation of sequential decision processes. Indeed, everyday life reveals that almost all successful decisions are preceded by a learning period during which one observes, classifies, and ranks experiences. Given a finite life-span for some decision making, many alternate strategies can be pursued, every one of which is specified by the ratio between learning and acting-decision periods one agrees to employ. The solution offered by the idealized mathematical version of the problem is remarkable: The optimum strategy is attained when about e^{-1} of the available decision time is devoted to learning. The probability of success for this optimum strategy is also about e^{-1}, which is approximately 37%. This simple and elegant rule rightly deserves to be called *a golden rule for decision making*.

We explore the problem and construct its solution through an ongoing, developing learning process. At first, we elaborate on the alternative possible strategies by listing and enumerating the cases where particular strategies win. Thus, we form a preliminary conception of the optimization nature of the problem and we derive a general expression for a strategy's probability of success. This expression is then used to determine a probability spectrum for the winning strategies in cases where direct listing and counting are not possible. A pattern of optimal strategies emerges, ultimately expressed by general conditions. These in turn yield a further expansion of the horizons of the problem that culminates in the devising of (a) a practical guide for making optimal decisions and (b) a very efficient rule for estimating the decision span for which any particular strategy becomes optimal. Finally, exploration of the optimization conditions for the winning strategies leads to the golden rule for decisions.

2. PROBLEM DESCRIPTION. The original problem with the slips of paper may be restated more formally as follows: A known number N of items is to be presented to an observer one by one in random order, all possible orderings being equally likely. The observer is able at any time to rank without ties the items that have so far been presented in order of desirability. As each item is presented he must either accept it, in which case the process stops, or reject it, in which case the next item in the sequence is presented and the observer faces the same dilemma as before. If the last item is ever reached it must be accepted. The observer's aim is to find the best of the N items available by employing a strategy with as high a probability of success as possible.

3. ON THE POSSIBLE STRATEGIES. The observer must either accept or reject an item right at the moment that it is presented, i.e., he cannot go back and choose an already-presented item that, in retrospect, turns out to be best. He has to balance the risk of stopping too soon and accepting an apparently desirable item when an even better one might be still to come, against that of going on too long and discovering that the best item was rejected earlier.

All possible strategies range between two equally likely extremes that constitute the worst choices an observer can make. On the one hand, an observer might pick the first item. On the other hand, he might wait for the last item. The probability of success for both these trivial strategies is the same and equals $1/N$.

Consequently, getting some experience from the contest process—by observing, comparing, and ranking items—before reaching a decision, cannot make things worse. On the contrary, there is hope for improving one's chances. Let us define the following $N-2$ non trivial S_n strategies:

> The observer lets n items pass, $1 \leq n \leq N-2$, ranks them in order of desirability, and then among the next items selects the first one found with a higher rank.

Among all possible strategies S_n, the observer wants to select and employ the one with the maximum probability of success.

4. EXPLICIT SOLUTIONS. To gain some insight into the problem context, it is instructive to explore the simplest cases first, i.e., when the total number N of items is small.

Let us evaluate the probability of success for each S_n strategy explicitly when $N = 3, 4, 5$. Our goal will be achieved by brute force, i.e., by listing in every such case all possible orderings and their associated winning strategies (the items are represented by their ranks). The optimal strategy in all cases will be the one that wins most often.

TABLE 1. Orderings and Winning Strategies ($N = 3$)

1	1	2	3	
2	1	3	2	S_1
3	2	1	3	S_1
4	2	3	1	S_1
5	3	1	2	
6	3	2	1	

Case $N = 3$ There are 6 possible orderings $(\mathbf{1}, \mathbf{2}, \ldots, \mathbf{6})$ and one non-trivial strategy, S_1 (see Table 1). Thus S_1 wins in the orderings **2**, **3**, **4** and loses in all others. Therefore, the probability of success for S_1 is $3/6 = 50\%$.

Case $N = 4$ There are 24 orderings in this case and there are two non-trivial strategies: S_1, S_2 (see Table 2). Strategies S_1 and S_2, are successful with probabilities $P(S_1) = 11/24$ and $P(S_2) = 10/24$. Therefore, S_1 is the optimal strategy.

TABLE 2. Orderings and Winning Strategies ($N = 4$)

1	1	2	3	4	
2	1	2	4	3	S2
3	1	3	2	4	S2
4	1	3	4	2	S2
5	1	4	2	3	S1
6	1	4	3	2	S1
7	2	1	3	4	
8	2	1	4	3	S1 S2
9	2	3	1	4	S2
10	2	3	4	1	S2
11	2	4	1	3	S1
12	2	4	3	1	S1
13	3	1	2	4	S1 S2
14	3	1	4	2	S1 S2
15	3	2	1	4	S1 S2
16	3	2	4	1	S1 S2
17	3	4	1	2	S1
18	3	4	2	1	S1
19	4	1	2	3	
20	4	1	3	2	
21	4	2	1	3	
22	4	2	3	1	
23	4	3	1	2	
24	4	3	2	1	

Case $N = 5$ There are 120 orderings here and there are three non-trivial strategies: S_1, S_2, and S_3 with probabilities of success $P(S_1) = 50/120$, $P(S_2) = 52/120$, and $P(S_3) = 42/120$ (see Table 3). Therefore, the optimal strategy is S_2.

After this direct listing and enumeration of cases, we see that

(i) the non-trivial strategies do indeed improve odds compared to a chance selection, and
(ii) among these strategies, some are better than others.

5. A STRATEGY'S PROBABILITY OF SUCCESS. The search for good strategies by listing of all orderings becomes a forbidding task for larger N. Even for $N = 10$ there are about 3.5 million, while for $N = 15$ there are one trillion! Consequently, we must figure out some abstract way to estimate the probability of success for strategies.

Let us start with some general observations. A strategy S_n may fail in two ways:

- The best item may be included in the n items defining S_n. As an example, for $N = 5$ we have three strategies: S_1, S_2, and S_3. Strategy S_3 loses whenever number 5 appears first or second or third, S_2 loses whenever number 5 appears first or second and, finally, S_1 loses whenever number 5 appears first.
- The best item may be preceded by at least one item whose rank exceeds those of the first n items. For example, when $N = 4$ strategy S_2 loses in orderings **1** and **7** where one chooses as best number 3 instead of 4. Similarly, for $N = 5$ strategy S_2 loses in 68 orderings, (e.g., **26** and **52**), while S_3 loses in 78 orderings (e.g., **31** and **55**).

TABLE 3. Orderings and Winning Strategies ($N = 5$)

1	1	2	3	4	5			
2	1	2	3	5	4			S_3
3	1	2	4	3	5			S_3
4	1	2	4	5	3			S_3
5	1	2	5	3	4		S_2	
6	1	2	5	4	3		S_2	
7	1	3	2	4	5			
8	1	3	2	5	4		S_2	S_3
9	1	3	4	2	5			S_3
10	1	3	4	5	2			S_3
11	1	3	5	2	4		S_2	
12	1	3	5	4	2		S_2	
13	1	4	2	3	5		S_2	S_3
14	1	4	2	5	3		S_2	S_3
15	1	4	3	2	5		S_2	S_3
16	1	4	3	5	2		S_2	S_3
17	1	4	5	2	3		S_2	
18	1	4	5	3	2		S_2	
19	1	5	2	3	4	S_1		
20	1	5	2	4	3	S_1		
21	1	5	3	2	4	S_1		
22	1	5	3	4	2	S_1		
23	1	5	4	2	3	S_1		
24	1	5	4	3	2	S_1		
25	2	1	3	4	5			
26	2	1	3	5	4			S_3
27	2	1	4	3	5			S_3
28	2	1	4	5	3			S_3
29	2	1	5	3	4	S_1	S_2	
30	2	1	5	4	3	S_1	S_2	
31	2	3	1	4	5			
32	2	3	1	5	4		S_2	S_3
33	2	3	4	1	5			S_3
34	2	3	4	5	1			S_3
35	2	3	5	1	4		S_2	
36	2	3	5	4	1		S_2	
37	2	4	1	3	5		S_2	S_3
38	2	4	1	5	3		S_2	S_3
39	2	4	3	1	5		S_2	S_3
40	2	4	3	5	1		S_2	S_3
41	2	4	5	1	3		S_2	
42	2	4	5	3	1		S_2	
43	2	5	1	3	4	S_1		
44	2	5	1	4	3	S_1		
45	2	5	3	1	4	S_1		
46	2	5	3	4	1	S_1		
47	2	5	4	1	3	S_1		
48	2	5	4	3	1	S_1		
49	3	1	2	4	5			
50	3	1	2	5	4	S_1	S_2	S_3
51	3	1	4	2	5			S_3
52	3	1	4	5	2			S_3
53	3	1	5	2	4	S_1	S_2	
54	3	1	5	4	2	S_1	S_2	
55	3	2	1	4	5			
56	3	2	1	5	4	S_1	S_2	S_3
57	3	2	4	1	5			S_3
58	3	2	4	5	1			S_3
59	3	2	5	1	4	S_1	S_2	
60	3	2	5	4	1	S_1	S_2	
61	3	4	1	2	5		S_2	S_3
62	3	4	1	5	2		S_2	S_3
63	3	4	2	1	5		S_2	S_3
64	3	4	2	5	1		S_2	S_3
65	3	4	5	1	2		S_2	
66	3	4	5	2	1		S_2	
67	3	5	1	2	4	S_1		
68	3	5	1	4	2	S_1		
69	3	5	2	1	4	S_1		
70	3	5	2	4	1	S_1		
71	3	5	4	1	2	S_1		
72	3	5	4	2	1	S_1		
73	4	1	2	3	5	S_1	S_2	S_3
74	4	1	2	5	3	S_1	S_2	S_3
75	4	1	3	2	5	S_1	S_2	S_3
76	4	1	3	5	2	S_1	S_2	S_3
77	4	1	5	2	3	S_1	S_2	
78	4	1	5	3	2	S_1	S_2	
79	4	2	1	3	5	S_1	S_2	S_3
80	4	2	1	5	3	S_1	S_2	S_3
81	4	2	3	1	5	S_1	S_2	S_3
82	4	2	3	5	1	S_1	S_2	S_3
83	4	2	5	1	3	S_1	S_2	
84	4	2	5	3	1	S_1	S_2	
85	4	3	1	2	5	S_1	S_2	S_3
86	4	3	1	5	2	S_1	S_2	S_3
87	4	3	2	1	5	S_1	S_2	S_3
88	4	3	2	5	1	S_1	S_2	S_3
89	4	3	5	1	2	S_1	S_2	
90	4	3	5	2	1	S_1	S_2	
91	4	5	1	2	3	S_1		
92	4	5	1	3	2	S_1		
93	4	5	2	1	3	S_1		
94	4	5	2	3	1	S_1		
95	4	5	3	1	2	S_1		
96	4	5	3	2	1	S_1		
97	5	1	2	3	4			
98	5	1	2	4	3			
99	5	1	3	2	4			
100	5	1	3	4	2			
101	5	1	4	2	3			
102	5	1	4	3	2			
103	5	2	1	3	4			
104	5	2	1	4	3			
105	5	2	3	1	4			
106	5	2	3	4	1			
107	5	2	4	1	3			
108	5	2	4	3	1			
109	5	3	1	2	4			
110	5	3	1	4	2			
111	5	3	2	1	4			
112	5	3	2	4	1			
113	5	3	4	1	2			
114	5	3	4	2	1			
115	5	4	1	2	3			
116	5	4	1	3	2			
117	5	4	2	1	3			
118	5	4	2	3	1			
119	5	4	3	1	2			
120	5	4	3	2	1			

Consequently, S_n is a winning strategy if

(a) *the best item is a candidate for selection, and*
(b) *the ranks of items, if any, preceding the best, do not exceed those of the first n items.*

These two conditions lead to a general expression for the probability of success of strategy S_n, henceforth denoted as $P_N(S_n)$.

Let E_k denote the event that the best item is at some position k, i.e., it is the kth term in the N-item sequence. Since all orderings are assumed equally likely, then by condition (a), the respective probability is $P(E_k) = 1/N$ with $k > n$.

Let F_k denote the event described by condition (b). Then the probability that F_k occurs, i.e., that the highest rank of the first $k - 1$ terms appears in the first n terms, is $P(F_k) = n/(k-1)$.

S_n is a winning strategy if both conditions (a) and (b) are satisfied. Since events E_k and F_k are independent, and all E_k events are exclusive and exhaustive alternatives, we have

$$P_N(S_n) = \sum_{k=n+1}^{N} P(E_k \cap F_k)$$
$$= \sum_{k=n+1}^{N} P(E_k) \cdot P(F_k) = \frac{n}{N} \cdot \sum_{k=n+1}^{N} \left(\frac{1}{k-1}\right) \tag{1}$$

This is the desired expression for the probability of success for any strategy S_n and for any number N of items one cares to choose from. The $P_N(S_n)$ expression is also valid for $n = N - 1$, in which case it equals

$$P_N(S_{N-1}) = [(N-1)/N] \cdot [1/(N-1)] = 1/N,$$

as it should be.

Since we have made no assumptions about the distribution of items, we safely say that the S_n decision rule is general.

6. THE PROBABILITY SPECTRUM FOR THE WINNING STRATEGIES. The $P_N(S_n)$ general expression in (1) provides a powerful tool for evaluating the probabilities of winning for all S_n strategies without actually having to list the possible orderings and count the corresponding winning frequencies. Table 4 exemplifies this point for $N = 1, 2, \ldots, 20$ and all possible strategies S_n, $0 \leq n \leq N - 1$. Every entry is the probability of success of a strategy S_n when the number of items is N, expresed with four significant digits.

7. OPTIMAL STRATEGIES. Each column in Table 4 possesses a maximum probability for some value of N. For example, in columns S_4 and S_7, the probabilities of success are 0.3984 for $N = 11$ and 0.3850 for $N = 19$. Furthermore, we see that every row corresponding to a particular N-value possesses a maximum probability of success for some strategy. For example, the maximum probabilities of success for $N = 10$ and $N = 16$ are 0.3987 and 0.3881, respectively, and they correspond to strategies S_3 and S_6, respectively.

The emerging pattern of optimal strategies may be expressed by the following statements:

(a) *To every number N of items there corresponds one strategy S_n with a maximum probability of success, and conversely,*
(b) *every possible strategy S_n is optimal for a particular number N of items.*

TABLE 4. Probabilities of Success for Strategies (N = 1 to 20)

N	S_0	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}	S_{11}	S_{12}	S_{13}	S_{14}	S_{15}	S_{16}	S_{17}	S_{18}	S_{19}
1	1.0000																			
2	0.5000	0.5000																		
3	0.3333	**0.5000**	0.3333																	
4	0.2500	0.4583	0.4167	0.2500																
5	0.2000	0.4167	**0.4333**	0.3500	0.2000															
6	0.1667	0.3806	0.4278	0.3917	0.3000	0.1667														
7	0.1429	0.3500	0.4143	0.4071	0.3524	0.2619	0.1429													
8	0.1250	0.3241	0.3982	**0.4098**	0.3798	0.3185	0.2321	0.1250												
9	0.1111	0.3020	0.3817	0.4060	0.3931	0.3525	0.2897	0.2083	0.1111											
10	0.1000	0.2829	0.3658	0.3987	0.3983	0.3728	0.3274	0.2653	0.1889	0.1000										
11	0.0909	0.2663	0.3507	0.3897	**0.3984**	0.3844	0.3522	0.3048	0.2444	0.1727	0.0909									
12	0.0833	0.2517	0.3366	0.3800	0.3955	0.3902	0.3683	0.3324	0.2847	0.2265	0.1591	0.0833								
13	0.0769	0.2387	0.3236	0.3700	0.3907	**0.3923**	0.3784	0.3517	0.3141	0.2668	0.2110	0.1474	0.0769							
14	0.0714	0.2272	0.3114	0.3600	0.3848	0.3917	0.3843	0.3651	0.3356	0.2972	0.2508	0.1973	0.1374	0.0714						
15	0.0667	0.2168	0.3002	0.3503	0.3782	0.3894	0.3873	0.3741	0.3513	0.3202	0.2817	0.2366	0.1853	0.1286	0.0667					
16	0.0625	0.2074	0.2898	0.3409	0.3712	0.3859	**0.3881**	0.3799	0.3627	0.3377	0.3058	0.2676	0.2238	0.1747	0.1208	0.0625				
17	0.0588	0.1989	0.2801	0.3319	0.3641	0.3816	0.3873	0.3832	0.3708	0.3509	0.3246	0.2923	0.2547	0.2122	0.1652	0.1140	0.0588			
18	0.0556	0.1911	0.2711	0.3233	0.3569	0.3767	0.3854	0.3848	0.3763	0.3608	0.3392	0.3120	0.2798	0.2429	0.2018	0.1567	0.1078	0.0556		
19	0.0526	0.1840	0.2626	0.3150	0.3498	0.3715	0.3827	**0.3850**	0.3799	0.3682	0.3506	0.3278	0.3001	0.2681	0.2321	0.1923	0.1490	0.1023	0.0526	
20	0.0500	0.1774	0.2548	0.3072	0.3429	0.3661	0.3793	0.3842	0.3820	0.3734	0.3594	0.3403	0.3167	0.2889	0.2573	0.2221	0.1836	0.1420	0.0974	0.0500

For fixed N, the winning probabilities for any two successive strategies, S_n and S_{n+1}, differ by

$$\Delta P_{(n)} = P_N(S_{n+1}) - P_N(S_n) = \frac{1}{N}\left[\sum_{k=n+2}^{N}\left(\frac{1}{k-1}\right) - 1\right]. \tag{2}$$

The optimal strategy corresponds to the smallest n that makes $\Delta P_{(n)}$ negative. Therefore, the best n for a given fixed N is the least n such that

$$\sum_{k=n+2}^{N}\left(\frac{1}{k-1}\right) = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{N-1} < 1. \tag{3}$$

For fixed n, the winning probabilities of S_n for any two consecutive N-values, N and $N+1$, differ by

$$\Delta P_{(N)} = P_{N+1}(S_n) - P_N(S_n) = \frac{n}{N(N+1)}\left[1 - \sum_{k=n+1}^{N}\left(\frac{1}{k-1}\right)\right]. \tag{4}$$

It follows that S_n is best for the smallest N-value that makes $\Delta P_{(N)}$ negative. Therefore, the best N for a given fixed strategy S_n is the least N such that

$$\sum_{k=n+1}^{N}\left(\frac{1}{k-1}\right) = \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{N-1} > 1. \tag{5}$$

The latter condition expands our computational and conceptual horizons of the problem considerably. Table 5 presents the N-values where specific S_n strategies (fixed n) become most appropriate, i.e., they attain the maximum possible probability of success. This probability is also maximum when all possible S_n strategies are compared for the same N.

8. GEOMETRIC EVALUATION OF PROBABILITY FOR OPTIMAL STRATEGIES. The treatment of the problem may be extended still more with quite interesting results. In what follows we shall consider condition (5) for optimal strategies within the realm of the real number continuum. Consequently, we shall deduce an efficient rule for identifying the best N-value for any S_n strategy.

Every term of the sum in (5), say the term $1/m$, may be represented as the area of the rectangle $m_-M_-M_+m_+$ (see Figure 1) with base $(m_-m_+) = 1$ and height $(Mm) = 1/m$. Evidently, M is a point on the hyperbola $y = 1/x$. The area (LM_-M) below the hyperbola is

$$E \equiv (LM_-M) = \int_{m-1/2}^{m}\left(\frac{1}{x} - \frac{1}{m}\right)dx = \ln\left(\frac{2m}{2m-1}\right) - \frac{1}{2m}. \tag{6}$$

Similarly, the area (MM_+N) above the hyperbola is

$$\varepsilon \equiv (MM_+N) = \int_{m}^{m+1/2}\left(\frac{1}{m} - \frac{1}{x}\right)dx = \frac{1}{2m} - \ln\left(\frac{2m+1}{2m}\right). \tag{7}$$

TABLE 5. A Practical Guide on Making the Optimal Decision when Choosing the Best out of a Number N of Contestants Presenting Themselves in Sequence

n	N	Pr (%)	n	N	Pr (%)	n	N	Pr (%)	n	N	Pr (%)
1	**3**	50.00	26	**70**	37.24	51	**138**	37.02	76	**206**	36.94
2	**5**	43.33	27	**73**	37.22	52	**141**	37.01	77	**209**	36.94
3	**8**	40.98	28	**76**	37.21	53	**144**	37.01	78	**212**	36.94
4	**11**	39.84	29	**78**	37.19	54	**146**	37.00	79	**214**	36.94
5	**13**	39.23	30	**81**	37.18	55	**149**	37.00	80	**217**	36.93
6	**16**	38.81	31	**84**	37.17	56	**152**	37.00	81	**220**	36.93
7	**19**	38.50	32	**87**	37.15	57	**155**	36.99	82	**223**	36.93
8	**21**	38.28	33	**89**	37.14	58	**157**	36.99	83	**225**	36.93
9	**24**	38.12	34	**92**	37.13	59	**160**	36.99	84	**228**	36.93
10	**27**	37.98	35	**95**	37.12	60	**163**	36.98	85	**231**	36.93
11	**30**	37.87	36	**98**	37.11	61	**165**	36.98	86	**233**	36.92
12	**32**	37.78	37	**100**	37.10	62	**168**	36.98	87	**236**	36.92
13	**35**	37.70	38	**103**	37.10	63	**171**	36.97	88	**239**	36.92
14	**38**	37.63	39	**106**	37.09	64	**174**	36.97	89	**242**	36.92
15	**40**	37.57	40	**108**	37.08	65	**176**	36.97	90	**244**	36.92
16	**43**	37.53	41	**111**	37.07	66	**179**	36.96	91	**247**	36.92
17	**46**	37.48	42	**114**	37.07	67	**182**	36.96	92	**250**	36.91
18	**49**	37.44	43	**117**	37.06	68	**184**	36.96	93	**252**	36.91
19	**51**	37.41	44	**119**	37.05	69	**187**	36.96	94	**255**	36.91
20	**54**	37.38	45	**122**	37.05	70	**190**	36.95	95	**258**	36.91
21	**57**	37.35	46	**125**	37.04	71	**193**	36.95	96	**261**	36.91
22	**59**	37.32	47	**127**	37.04	72	**195**	36.95	97	**263**	36.91
23	**62**	37.30	48	**130**	37.03	73	**198**	36.95	98	**266**	36.91
24	**65**	37.28	49	**133**	37.03	74	**201**	36.95	99	**269**	36.91
25	**68**	37.26	50	**136**	37.02	75	**204**	36.94	100	**271**	36.90

Decision Rule: *Let n of out N contestants pass, rank them in order of desirability, and then choose the first contestants who rank higher.*

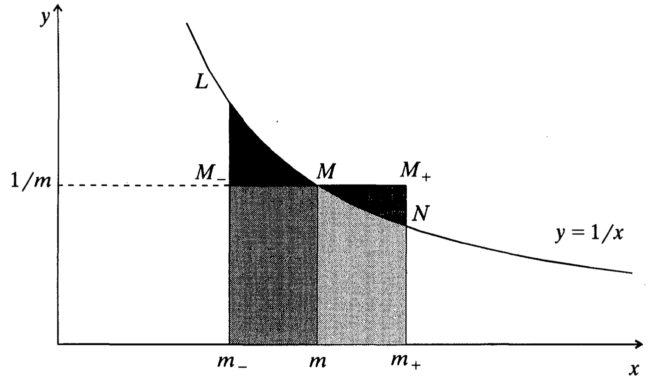

Figure 1

Consequently, the area of the region $LM_{-}M$ is larger than the area of the region $MM_{+}N$, i.e., $E > \varepsilon$, since

$$E - \varepsilon = \ln\left(\frac{2m+1}{2m-1}\right) - \frac{1}{m} = 2\left[\frac{1}{3(2m)^3} + \frac{1}{5(2m)^5} + \frac{1}{7(2m)^7} + \cdots\right] > 0. \tag{8}$$

Therefore, we have

$$\ln\left(\frac{2m+1}{2m-1}\right) = \int_{m-1/2}^{m+1/2} \left(\frac{1}{x}\right) dx > \frac{1}{m}. \tag{9}$$

Summing up for $m = n, n+1, \ldots, N-1$ gives

$$\int_{n-1/2}^{N-1/2} \left(\frac{1}{x}\right) dx > \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{N-1}. \tag{10}$$

For optimal strategies, (5) ensures that the right-hand side of (10) is larger than 1. Thus, the optimal N-value for a fixed S_n strategy is the smallest N that satisfies

$$\ln\left(\frac{2N-1}{2n-1}\right) > 1.$$

Thus, the best N for a given fixed strategy S_n is the least N such that

$$N > e\left(n - \frac{1}{2}\right) + \frac{1}{2} \tag{11}$$

The corresponding probability of winning $P_N(S_n)$ has an upper bound

$$P_N(S_n) \equiv Pr(\Sigma) < Pr(\textstyle\int) \equiv \frac{n}{N}\ln\left(\frac{2N-1}{2n-1}\right). \tag{12}$$

9. THE ASYMPTOTIC BEHAVIOR OF STRATEGIES. Having established two alternative ways of evaluating the probabilities of success for strategies—sums (1) and integrals (12)—we are now in a position to compute these probabilities for large values of N and compare the results. Table 6 displays:

(i) the characteristic optimal (n, N) pairs for large N,
(ii) the respective ration (n/N),
(iii) the corresponding probability of success derived by sums, denoted as $Pr(\Sigma)$, and
(iv) the probability of success derived by the integral approximation, denoted as $Pr(\int)$ in (12).

From Table 6 we see that the probabilities calculated by (1) and (12) both decrease as n increases and they start to (a) coincide (within six significant figures) from the entry ($n = 300$, $N = 815$) onwards, and (b) converge to 0.367879 from the entry ($n = 2{,}000{,}000$, $N = 5{,}436{,}563$). We also observe that the ratio (n/N) converges to the very same number 0.367879. This latter number is e^{-1}.

Thus, we conclude that the two distinct characteristic quantities of optimal strategies, i.e., the ratio (n/N) and the probability of success, both converge to the same limit as $N \to \infty$.

This conclusion may also be derived formally. Since the best N for any S_n strategy is defined as the least N-value satisfying (11), we have $2N > 2en + (1-e)$ for N, and $2en + (1-e) \geq 2(N-1)$ for the immediately lower value, $N-1$.

TABLE 6. Probabilities of Success for Strategies when N is Large

n	N	n/N	Pr(Σ)	Pr(∫)
100	271	0.369004	0.369045	0.369046
200	543	0.368324	0.368461	0.368462
300	815	0.368098	0.368267	0.368267
400	1087	0.367985	0.368170	0.368170
500	1359	0.367918	0.368112	0.368112
600	1631	0.367872	0.368073	0.368073
700	1902	0.368034	0.368046	0.368046
800	2174	0.367985	0.368025	0.368025
900	2446	0.367948	0.368009	0.368009
1000	2718	0.367918	0.367996	0.367996
2000	5436	0.367918	0.367938	0.367938
3000	8154	0.367918	0.367918	0.367918
4000	10873	0.367884	0.367909	0.367909
5000	13591	0.367891	0.367903	0.367903
6000	16309	0.367895	0.367899	0.367899
7000	19028	0.367879	0.367896	0.367896
8000	21746	0.367884	0.367894	0.367894
9000	24464	0.367888	0.367892	0.367892
10000	27182	0.367891	0.367891	0.367891
20000	54365	0.367884	0.367885	0.367885
30000	81548	0.367881	0.367883	0.367883
40000	108731	0.367880	0.367882	0.367882
50000	135914	0.367880	0.367882	0.367882
60000	163097	0.367879	0.367881	0.367881
70000	190279	0.367881	0.367881	0.367881
80000	217462	0.367880	0.367881	0.367881
90000	244645	0.367880	0.367881	0.367881
100000	271828	0.367880	0.367881	0.367881
200000	543656	0.367880	0.367880	0.367880
300000	815484	0.367880	0.367880	0.367880
400000	1087312	0.367880	0.367880	0.367880
500000	1359141	0.367879	0.367880	0.367880
600000	1630969	0.367879	0.367880	0.367880
700000	1902797	0.367879	0.367880	0.367880
800000	2174625	0.367880	0.367880	0.367880
900000	2446453	0.367880	0.367880	0.367880
1000000	2718281	0.367880	0.367880	0.367880
2000000	5436563	0.367879	0.367879	0.367879

Hence we have

$$\frac{1}{e}\left(1-\frac{1}{N}\right) \le \frac{n}{N} - \frac{1}{2eN}\left(\frac{1}{e}-1\right) < \frac{1}{e}. \tag{13}$$

It follows that $\lim_{N\to\infty} (n/N) = 1/e$ and, consequently, (12) gives $\lim_{N\to\infty} P_N(S_n) = 1/e$.

Concluding, we may state the following *Golden Rule* for decisions:

The optimum strategy is to wait until e^{-1} of the items pass and then select the next relatively best one. The probability of success for the optimum strategy is e^{-1}.

TABLE 7. The Pyramid e-Expansion of N for Optimal Strategies

K	$ne - (e-1)/2$ $\{n = 10^K\}$	n(rounded up)
1	26.32	27
2	270.97	271
3	2717.42	2718
4	27181.96	27182
5	271827.32	271828
6	2718280.97	2718281
7	27182817.43	27182818
8	271828181.99	271828182
9	2718281827.60	2718281828
10	27182818283.73	27182818284
11	271828182845.05	271828182846
12	2718281828458.19	2718281828459
13	27182818284589.59	27182818284590
14	271828182845903.66	271828182845904
15	2718281828459044.38	2718281828459045
16	27182818284590451.49	27182818284590452
17	271828182845904522.68	271828182845904523
18	2718281828459045234.50	2718281828459045235
19	27182818284590452352.74	27182818284590452353
20	271828182845904523535.17	271828182845904523536
21	2718281828459045235359.43	2718281828459045235360
22	27182818284590452353602.02	27182818284590452353603
23	271828182845904523536027.89	271828182845904523536028
24	2718281828459045235360286.61	2718281828459045235360287
25	27182818284590452353602873.85	27182818284590452353602874
26	271828182845904523536028746.28	271828182845904523536028747
27	2718281828459045235360287470.49	2718281828459045235360287471
28	27182818284590452353602874712.67	27182818284590452353602874713
29	271828182845904523536028747134.41	271828182845904523536028747135
30	2718281828459045235360287471351.80	2718281828459045235360287471352
31	27182818284590452353602874713525.77	27182818284590452353602874713526
32	271828182845904523536028747135265.39	271828182845904523536028747135266
33	2718281828459045235360287471352661.64	2718281828459045235360287471352662
34	27182818284590452353602874713526624.12	27182818284590452353602874713526625
35	271828182845904523536028747135266248.92	271828182845904523536028747135266249
36	2718281828459045235360287471352662496.90	2718281828459045235360287471352662497
37	27182818284590452353602874713526624976.71	27182818284590452353602874713526624977
38	271828182845904523536028747135266249774.87	271828182845904523536028747135266249775
39	2718281828459045235360287471352662497756.39	2718281828459045235360287471352662497757
40	27182818284590452353602874713526624977571.61	27182818284590452353602874713526624977572

Having established that the golden rule for decisions is associated with e, it is interesting to note that when n takes as values the successive integer powers of 10, the corresponding values of N generate the decimal expansion of e. This is illustrated in Table 7.

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There is a sibling rivalry
between this conjecture and its negation
and I, poor mother
throw up my hands.
"Anything, anything.
"Whatever you decide.
"Just please
"hurry up
"and make up your mind."

Contributed by Marion Cohen, Drexel University, Philadelphia, PA