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UNSOLVED PROBLEMS

Edited by **Richard Nowakowski**

In this department the MONTHLY presents easily stated unsolved problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial or related results. Typescripts should be sent to Richard Nowakowski, Department of Mathematics & Statistics & Computing Science, Dalhousie University, Halifax NS, Canada B3H 3J5, rjn@mscs.dal.ca

Periods in Taking and Splitting Games

Ian Caines, Carrie Gates, Richard K. Guy, and Richard J. Nowakowski

In 1997, J. H. Conway asked who wins *Couples-Are-Forever*? This game is played by two players playing alternately with heaps of counters. A move is to choose a heap and split it into two non-empty heaps, except that no heap of 2 may be split, hence the name. The player who moves last wins.

Without the exception, the game is easy to analyze. The next player wins if and only if there is an even number of heaps that contain an odd number of counters. This is the archetypal *she-loves-me-she-loves-me-not* game found in [**2**, pp. 113, 115].

Recall that the *nim-value* of a position G, denoted by $\mathcal{G}(G)$, is defined inductively. The end-positions, positions from which there are no moves, have value zero. Any other position has value the minimum excluded non-negative integer (*mex* for short) that does not occur among the values of its followers, i.e., those positions that can be reached in one move. If the nim-value of a position is zero then the next player cannot win; if it is non-zero then the next player does have a winning strategy.

Couples-are-Forever is a *disjunctive* sum of games, that is, a player must choose one heap and make a move in it. The nim-value of a disjunctive sum of games is the nim-sum of the nim-values. This is obtained by writing out the nim-values in binary and adding without carrying, denoted by $\oplus$. The single heap games of Couples-are-Forever have nim-values: $\mathcal{G}(1) = \mathcal{G}(2) = 0$ since these are end-positions. A heap of 3 can be split into heaps of size 1 and 2 so that $\mathcal{G}(3) = \text{mex}\{\mathcal{G}(1) \oplus \mathcal{G}(2)\} = \text{mex}\{0 \oplus 0\} = \text{mex}\{0\} = 1$. Similarly,

$$\begin{aligned}\mathcal{G}(4) &= \text{mex}\{\mathcal{G}(1) \oplus \mathcal{G}(3), \mathcal{G}(2) + \mathcal{G}(2)\}\\ &= \text{mex}\{0 \oplus 1, 0 + 0\} = \text{mex}\{1,0\} = 2.\end{aligned}$$

The first twenty-five nim-values are: $0, 0, 1, 2, 0, 1, 2, 3, 1, 2, 3, 4, 0, 3, 4, 2, 1, 3, 2, 1, 0, 2, 1, 4, 5$. By automating the work, we have found the nim-values for heaps of size 1 through 50 million and no pattern has emerged. The value 0 occurs fourteen times and 1 twenty-six times. The largest nim-value found is $\mathscr{G}(19,739,544) = 325$ and the most common values are:

nim-value:	256	257	129	128	134	135
no. of occurrences:	1,798,261	1,596,606	1,059,556	1,058,974	894,436	893,216

The behavior of the nim-values is quite striking when they are plotted against the heap size. The upper envelope of the graph rises in a linear fashion to approximately 300 at heap size 20,000. Then it is almost constant for the rest of the 50 million terms. A 'sparse-space-common-coset' phenomenon occurs; see [**5**]. A *rare* nim-value in this game is one which has an even number of 1-digits in its binary expansion if the digits in the 'ones' and 'sixteens' places are ignored. Values with an odd number of such 1-digits are called *common*. The nim-sum of two rare values, and likewise of two common values, is a rare value and so rare values tend to get excluded in the mex operation, keeping them rare. This allows for fast calculation of the nim-values: find the smallest common value that does not arise as the nim-sum from heaps, one with a rare value, the other with a common. Then the mex is either this, or is a smaller rare value, so compute sufficient 'common-common' (and 'rare-rare') values until all smaller rare values are excluded, or, exceptionally, you've discovered a new rare value.

If the rare values stop appearing and the sequence of nim-values remains bounded, then it eventually becomes periodic. For Couples-are-Forever, the last rare value found is $\mathscr{G}(20{,}628) = 277$. Unfortunately we cannot prove that there are no more.

Question: *Is Couples-are-Forever ultimately periodic*?

If it is ultimately periodic then the period length cannot divide the size of any heap that has nim-value zero. The nim-value zero occurs for heap sizes 1, 2, 5, 13, 21, 31, 47, 73, 99, 125, 151, 177, 315, and 409, so the period cannot be any of: 1, 2, 3, 5, 7, 9, 11, 15, 21, 25, 31, 33, 35, 45, 47, 59, 63, 73, 99, 105, 125, 151, 177, 315, or 409.

Are there any other reasons to believe that Couples-are-Forever is periodic? Some variants are easily shown to be periodic. If heaps of size 1 through n are not allowed to be split then: the period is $0, 1, 2 \ldots, n$ with a pre-period of $n - 1$ zeros if n is odd; the period is $n + 1, 1, 2, \ldots, n$ with a pre-period of n zeros followed by $1, 2, \ldots, n, 1, 2, \ldots, n + 1, 1, 2, \ldots, n + 1, 1, 2, \ldots, n - 1, 0$ if $n = 2^k - 4$ for $k \geq 3$. However, the other values of n give games as intransigent as the original; see [**3**] for other results.

Couples-are-Forever is very similar to *Grundy's Game* (split a heap into two *unequal* heaps) in that the game ends with heaps of just one or two. This also defies analysis, in spite of several energetic attempts. It was during his computation of the first quarter million nim-values that Elwyn Berlekamp discovered the sparse-space-common-coset phenomenon (see [**2**, p. 111]); here the sparse space comprises vectors of even weight after deleting the units digit. The 1,272nd rare value is $\mathscr{G}(36{,}184) = 157$ and then, apart from $\mathscr{G}(82{,}860) = 108$, no more. Nor did Mike Guy find any more among the first ten million. But Anil Gangolli found $\mathscr{G}(47{,}468{,}481) = 261$, $\mathscr{G}(48{,}142{,}376) = 265$ and 11 other rare values between. Dan Hoey has reached 11 billion, but found neither a pattern nor any more rare values.

Wild conjecture: *All finite octal games are ultimately periodic.*

An *octal game* is a 'take-and-break' game with Guy-Smith code [6] $.\mathbf{d}_1\mathbf{d}_2\mathbf{d}_3\ldots$, where the base 8 digits $\mathbf{d}_i = 2^2h_{i2} + 2^1h_{i1} + 2^0h_{i0}$ with $h_{ij} = 1$ or 0, signify that you may(1) or may not(0) take i counters from a heap, provided that you leave exactly j nonempty heaps in its place. For example, in the game **.172**, you may take one counter if it's on its own, two counters in any case, leaving the remainder, if any, in one or two heaps, or three counters from a heap provided that it is not the whole heap. The analysis of this game has not been completed, nor has it for Officers, **.6** (take one counter from a larger heap, leaving the rest as one or two heaps), nor for **.007** (Treblecross or one-dimensional tic-tac-toe; remove three contiguous skittles from a row), nor for any of the following games ([7, pp. 475–476], where **.644** should not appear, since its period length, 442, was discovered by Richard Austin [1]): **.06, .14, .36, .64, .74, .76, .004, .005, .006, .016, .104, .106, .114, .135, .136, .142, .143, .146, .162, .163, .324, .336, .342, .362, .371, .374, .404, .414, .416, .444, .454, .564, .604, .606, .744, .764, .774, .776**, and plenty of games where you're allowed to take more than three counters.

We return to the splitting games. A *pure* period is a sequence of numbers $(a_1, a_2, \ldots, a_k)$ where

$$a_j = \text{mex}\left\{a_i \oplus a_{j-i} | i = 1, 2, \ldots, k-1, \quad \text{where the subscripts are taken cyclically}\right\}.$$

When a pure period exists, the corresponding splitting game has $\mathcal{G}(n)$ prescribed for $n = 1, 2, \ldots, k$ and thereafter, $\mathcal{G}(n+k) = \mathcal{G}(n)$. For example, 0 1 is a pure period. There are pure periods $2^n\ 3^2\ 2^n\ 1$ of length $2n+3$, and $1\ (3\ 0)^n\ 3\ 1\ 1$ of length $2n+4$, for every $n \geq 1$, where $(*)^n$ means $*$ repeated n times.

A pure period of length k can have no value greater than $\lfloor k/2 \rfloor$, since there are only $\lceil k/2 \rceil$ terms when taking the *mex* and one of the terms must be zero for the maximum nim-value to occur. We have calculated all pure periods up to length 11, of which there are many.

For period lengths one through eleven, there are 1, 2, 1, 2, 3, 6, 7, 18, 13, 58, 50 pure periods respectively. All have the form $SS^{-1}1$ if of odd length and $SaS^{-1}1$ if even, where S^{-1} is the sequence S in reverse and a is a single number.

What finite sequences of nim-values are pure periods? Must they have the form $SaS^{-1}1$ or $SS^{-1}1$?

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Dalhousie University, Halifax, Nova Scotia, Canada, B3H 3J5.
ian.caines@dal.ca; carrie.gates@dal.ca; rjn@mscs.dal.ca

University of Calgary, Calgary, Alberta, Canada T2N 1N4
rkg@cpsc.ucalgary.ca