PROBLEMS AND SOLUTIONS

Proposed problems and solutions should be sent in duplicate to the MONTHLY *problems address on the inside front cover. Submitted problems should include solutions and relevant references. Submitted solutions should arrive at that address before September 30, 1999; Additional information, such as generalizations and references, is welcome. The problem number and the solver's name and address should appear on each solution. An acknowledgement will be sent only if a mailing label is provided. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.*

PROBLEMS

10725. *Proposed by Vasile Mihai, Toronto, ON, Canada.* Fix a positive integer n. Given a permutation α of $\{1, 2, \ldots, n\}$, let $f(\alpha) = \sum_{i=1}^{n}\big(\alpha(i)-\alpha(i+1)\big)^2$, where $\alpha(n+1) = \alpha(1)$. Find the extreme values of $f(\alpha)$ as α ranges over all permutations of $\{1, 2, \ldots, n\}$.

10726. *Proposed by Donald E. Knuth, Stanford University, Stanford, CA.* Start in state 0. For every nonnegative integer k, stay in state k for X_k units of time, then go to state $k+1$. What is the probability of being in state s after t units of time, assuming that X_k is distributed exponentially **(a)** with mean $1/(k+1)$? **(b)** with mean $1/2^k$?

10727. *Proposed by Jean Anglesio, Garches, France.* Let m be a fixed positive integer. For a positive integer n, let $s_m(n)$ be the sum of the mth powers of the decimal digits of n. For example, $s_3(172) = 1^3 + 7^3 + 2^3 = 352$. Starting with any positive integer n_0, construct a sequence of positive integers by setting $n_k = s_m(n_{k-1})$ for every $k \geq 1$.
(a) Show that $n_0, n_1, n_2, \ldots$ is eventually periodic.
(b) Show that only finitely many periods are possible as n_0 varies.

10728. *Proposed by Titu Andreescu, American Mathematics Competitions, Lincoln, NE.* Determine all functions $f\colon \mathbb{Z} \to \mathbb{Z}$ satisfying

$$f(x^3 + y^3 + z^3) = \big(f(x)\big)^3 + \big(f(y)\big)^3 + \big(f(z)\big)^3$$

for all integers x, y, and z.

10729. *Proposed by David P. Bellamy and Felix Lazebnik, University of Delaware, Newark, DE.* Let $I \subset \mathbb{R}$ be an open interval, and let n be a positive integer. Characterize the functions $f\colon I \to \mathbb{R}$ that have a continuous nth derivative and satisfy

$$f^{(n)} + p_1 f^{(n-1)} + \cdots + p_{n-1} f' + p_n f = 0$$

for some continuous functions $p_1, p_2, \ldots, p_n$ on I.