**10730.** *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.* Fix an integer $n \geq 2$. Determine the largest constant $C(n)$ such that

$$\sum_{1 \leq i < j \leq n} \left(x_j - x_i\right)^2 \geq C(n) \cdot \min_{1 \leq i < n} \left(x_{i+1} - x_i\right)^2$$

for all real numbers $x_1 < x_2 < \cdots < x_n$.

**10731.** *Proposed by M. J. Pelling, London, England.* Let $A$ be an $n$-by-$n$ real symmetric matrix, and consider the quadratic form $Q(x) = x^T Ax$ for $x \in \mathbb{R}^n$. Let $C$ be the cube $[-1, 1]^n$. Prove that $\max_{x \in C} Q(x)$ is at least as large as the sum of the positive real eigenvalues of $A$.

# SOLUTIONS

### Connected Sets of Periodic Functions

**10434** [1995, 170]. *Proposed by Daniel Goffinet, Saint Étienne, France.* Let $P$ be the set of nonconstant periodic mappings from $\mathbb{R}$ to $\mathbb{R}$, endowed with the topology derived from the supremum norm. Find the components of $P$.

*Composite solution I by Kiran S. Kedlaya, Massachusetts Institute of Technology, Cambridge, MA, Kenneth Schilling, University of Michigan, Flint, MI, and Arlo W. Schurle, University of Guam, Mangilao, Guam.* For any function $f\colon \mathbb{R} \to \mathbb{R}$, define $\|f\|$ to be $\sup\{\,|f(x)|: x \in \mathbb{R}\}$, which is taken to be $\infty$ when the set of values of $f$ is unbounded.

We first show that $f$ and $g$ are in different components of $P$ if $\|f - g\| = \infty$. Let $B_g = \{\, k \in P\colon \|k - g\| < \infty\}$. By the triangle inequality $B_g$ is an open set, and if $h \notin B_g$, then the triangle inequality again shows that $\{\, z\colon \|z - h\| < 1\} \cap B_g = \emptyset$. Consequently $B_g$ is both open and closed, and so the component of $P$ containing any given $g \in P$ must lie in $B_g$.

Conversely, if $f - g$ is bounded for $f, g \in P$, then there is an arc in $P$ joining $f$ to $g$. First, suppose that $f$ and $g$ have a common period $p$. The standard path $k_t(x) = (1 - t)f(x) + tg(x)$ for $0 \leq t \leq 1$ consists of functions having $p$ as a period, and since $\|f - g\|$ is finite, $k_t$ depends continuously on $t$. There is a danger that some $k_t(x)$ is a constant function, but this can happen only if $f$ is an affine function of $g$, that is, there are constants $A$ and $B$ with $f = Ag + B$. In this case, the function $h(x)$ that is equal to $f(x)$ except at integer multiples of $p$, where it is $f(x) + 1$, is at bounded distance from both $f$ and $g$ and is not an affine function of either. A path from $f$ to $g$ can be obtained by taking the standard path from $f$ to $h$ followed by the standard path from $h$ to $g$.

Suppose now that $f$ and $g$ have no common period. Let $r$ be a period of $f$ and let $s$ be a period of $g$. We wish to construct $h$ that has both $r$ and $s$ as periods such that $\|f - h\|$ (and hence also $\|g - h\|$) is finite. To do this, pick an arbitrary set of coset representatives for $\mathbb{R}/(r\mathbb{Z} + s\mathbb{Z})$, define $h$ to agree with $f$ at these values, and extend by periodicity. Then for any $x$, let $x = y + rm + sn$, where $y$ represents the coset containing $x$. Then

$$\begin{aligned} |h(x) - f(x)| &= |f(y) - f(y + sn)| \\ &= |f(y) - g(y) + g(y + sn) - f(y + sn)| \leq 2\,\|f - g\| \end{aligned}$$

Since $f$ and $h$ have common period $r$ and $\|f - h\|$ is finite, there is a path from $f$ to $h$, and since $h$ and $g$ have common period $s$ and $\|h - g\|$ is finite, there is a path from $h$ to $g$.

*Composite solution II by Fredric D. Ancel, University of Wisconsin, Milwaukee, WI, Phil Bowers and John Bryant, The Florida State University, Tallahassee, FL, and the proposer.* We assume that "mapping" means "continuous function". Then two functions in $P$ belong to the same component if and only if they have commensurate periods. As in solution I, the components are path-components.