10730. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.* Fix an integer $n \geq 2$. Determine the largest constant $C(n)$ such that

$$\sum_{1 \leq i < j \leq n} (x_j - x_i)^2 \geq C(n) \cdot \min_{1 \leq i < n} (x_{i+1} - x_i)^2$$

for all real numbers $x_1 < x_2 < \cdots < x_n$.

10731. *Proposed by M. J. Pelling, London, England.* Let A be an n-by-n real symmetric matrix, and consider the quadratic form $Q(x) = x^T Ax$ for $x \in \mathbb{R}^n$. Let C be the cube $[-1, 1]^n$. Prove that $\max_{x \in C} Q(x)$ is at least as large as the sum of the positive real eigenvalues of A.

SOLUTIONS

Connected Sets of Periodic Functions

10434 [1995, 170]. *Proposed by Daniel Goffinet, Saint Étienne, France.* Let P be the set of nonconstant periodic mappings from $\mathbb{R}$ to $\mathbb{R}$, endowed with the topology derived from the supremum norm. Find the components of P.

Composite solution I by Kiran S. Kedlaya, Massachusetts Institute of Technology, Cambridge, MA, Kenneth Schilling, University of Michigan, Flint, MI, and Arlo W. Schurle, University of Guam, Mangilao, Guam. For any function $f: \mathbb{R} \to \mathbb{R}$, define $\|f\|$ to be $\sup\{\,|f(x)| : x \in \mathbb{R}\}$, which is taken to be ∞ when the set of values of f is unbounded.

We first show that f and g are in different components of P if $\|f - g\| = \infty$. Let $B_g = \{k \in P\colon \|k - g\| < \infty\}$. By the triangle inequality B_g is an open set, and if $h \notin B_g$, then the triangle inequality again shows that $\{z\colon \|z - h\| < 1\} \cap B_g = \emptyset$. Consequently B_g is both open and closed, and so the component of P containing any given $g \in P$ must lie in B_g.

Conversely, if $f - g$ is bounded for $f, g \in P$, then there is an arc in P joining f to g. First, suppose that f and g have a common period p. The standard path $k_t(x) = (1 - t)f(x) + tg(x)$ for $0 \leq t \leq 1$ consists of functions having p as a period, and since $\|f - g\|$ is finite, k_t depends continuously on t. There is a danger that some $k_t(x)$ is a constant function, but this can happen only if f is an affine function of g, that is, there are constants A and B with $f = Ag + B$. In this case, the function $h(x)$ that is equal to $f(x)$ except at integer multiples of p, where it is $f(x) + 1$, is at bounded distance from both f and g and is not an affine function of either. A path from f to g can be obtained by taking the standard path from f to h followed by the standard path from h to g.

Suppose now that f and g have no common period. Let r be a period of f and let s be a period of g. We wish to construct h that has both r and s as periods such that $\|f - h\|$ (and hence also $\|g - h\|$) is finite. To do this, pick an arbitrary set of coset representatives for $\mathbb{R}/(r\mathbb{Z} + s\mathbb{Z})$, define h to agree with f at these values, and extend by periodicity. Then for any x, let $x = y + rm + sn$, where y represents the coset containing x. Then

$$\begin{aligned}|h(x) - f(x)| &= |f(y) - f(y + sn)| \\ &= |f(y) - g(y) + g(y + sn) - f(y + sn)| \leq 2\,\|f - g\|\end{aligned}$$

Since f and h have common period r and $\|f - h\|$ is finite, there is a path from f to h, and since h and g have common period s and $\|h - g\|$ is finite, there is a path from h to g.

Composite solution II by Fredric D. Ancel, University of Wisconsin, Milwaukee, WI, Phil Bowers and John Bryant, The Florida State University, Tallahassee, FL, and the proposer. We assume that "mapping" means "continuous function". Then two functions in P belong to the same component if and only if they have commensurate periods. As in solution I, the components are path-components.