Solution by the proposer. If B_i is the midpoint of $A_{i+1}A_{i+2}$ for $i = 1, 2, 3$, then triangles $A_1A_2A_3$ and $B_1B_2B_3$ are similar, so $|B_1B_2| = (1/2)|A_1A_2|$, $|B_2B_3| = (1/2)|A_2A_3|$, and $|B_3B_1| = (1/2)|A_3A_1|$. Hence

$$|A_1B_2| + |B_1B_2| = \frac{1}{2}|A_1A_3| + \frac{1}{2}|A_1A_2| = |B_1B_3| + |A_1B_3|,$$

and similarly for the other conditions of both parts.

(a) We prove that for any triangle $B_1B_2B_3$ there exists exactly one triangle $A_1A_2A_3$ such that $|A_iB_{i+1}|+|B_iB_{i+1}| = |A_iB_{i+2}|+|B_iB_{i+2}|$ for $i = 1, 2, 3$. This implies our assertion. Fix a triangle $B_1B_2B_3$, and suppose that for a triangle $A_1A_2A_3$ the conditions are satisfied. Let (i, j, k) be a permutation of $(1, 2, 3)$. Consider the hyperbola with foci B_j and B_k passing through B_i. Since $|A_iB_j| + |B_iB_j| = |A_iB_k| + |B_iB_k|$, the hyperbola passes through A_i. Write h_i for the part of the branch of the hyperbola passing through A_i that is on the opposite side of the line B_jB_k from B_i. Since B_j and B_k are the foci of the hyperbola, h_i is entirely contained in the union of all lines joining A_i and some point on the segment B_jB_k.

Now suppose that A is any point on h_1 different from A_1. (This A is a candidate for the vertex A_1 in a new triangle satisfying the conditions.) If A is inside triangle $B_2A_1B_3$, then the line from A through B_2 intersects h_3 in a point P that is on the opposite side of the line A_2A_3 from A_1, and if A is outside of $B_2A_1B_3$ then P is on the same side of A_2A_3 as A_1. (Point P is the candidate for point A_3 of the new triangle.) The same holds for the intersection Q of the line AB_3 with h_2 (the candidate for A_2 of the new triangle). Therefore, the line segment PQ does not pass through B_1. We conclude that A cannot be a vertex of a triangle that satisfies our requirements. A similar argument shows that no point A outside triangle $B_2A_1B_3$ can be a vertex of a triangle that satisfies our requirements. Thus $A_1A_2A_3$ is the only triangle for which the conditions hold.

(b) Let $a_k = (1/2)|A_iA_j|$, $b_k = |B_iB_j|$, and $x_j = a_j - |A_iB_j|$, where (i, j, k) is an even permutation of $(1, 2, 3)$. By hypothesis, $a_i + x_i + a_k - x_k = b_i + b_k$. Adding two of these equations and subtracting the third yields $b_i = a_i - x_j + x_k$, so

$$b_i^2 = a_i^2 + x_j^2 + x_k^2 - 2a_ix_j - 2x_jx_k + 2a_ix_k \tag{1}$$

By the law of cosines we obtain $b_i^2 = (a_j + x_j)^2 + (a_k - x_k)^2 - 2(a_j + x_j)(a_k - x_k)\cos A_i$. Since $\cos A_i = a_j^2 + a_k^2 - a_i^2/2a_ja_k$ we get after simple transformations

$$b_i^2 = a_i^2 + x_j^2 + x_k^2 + \frac{x_j}{a_j}(a_j^2 + a_i^2 - a_k^2) - \frac{x_k}{a_k}(a_k^2 + a_i^2 - a_j^2) + \frac{x_j}{a_j}\frac{x_k}{a_k}(a_j^2 + a_k^2 - a_i^2) \tag{2}$$

Let $z_i = x_i/a_i$. Comparing expressions (1) and (2) for b_i^2, we get

$$z_j(a_j + a_i - a_k) - z_k(a_k + a_i - a_j) + z_jz_k(a_j + a_k - a_i) = 0\,.$$

If one of the z_i's is 0, then all of them vanish. If they are all nonzero, then dividing by z_jz_k and adding all three equalities we get $a_1 + a_2 + a_3 = 0$, which is evidently false. Therefore, all the x_i's vanish and the assertion is proved.

Solved also by M. Vowe (Switzerland) and GCHQ Problems Group (U. K.).

Binary Expansions and kth Powers

10596 [1997, 456]. *Proposed by Paul Bateman, University of Illinois, Urbana, IL, and David Bradley, Simon Fraser University, Burnaby, BC, Canada.*

(a) Prove the identity

$$\sum_{j=0}^{2^{k-1}-1} (-1)^{k-1-\eta(j)}(y + j)^k = k! \cdot 2^{(k-1)(k-2)/2}(y + (2^{k-1} - 1)/2),$$

where $\eta(j)$ is the number of ones in the binary expansion of the nonnegative integer j.
(b) Use part **(a)** to infer that there is a positive integer $s = s(k)$ such that every integer n is expressible in the form $n = \epsilon_1 x_1^k + \epsilon_2 x_2^k + \cdots + \epsilon_s x_s^k$ in infinitely many ways, where $\epsilon_i = \pm 1$ for $1 \le i \le s$ and where $x_1, x_2, \ldots, x_s$ are distinct positive integers.

Solution I to part **(a)** *by David Callan, University of Wisconsin, Madison, WI.* With $n = k-1$, equating coefficients of y reduces the proposed identity to

$$\sum_{j=0}^{2^n-1} (-1)^{n-\eta(j)} j^r = \begin{cases} 0 & \text{if } r < n; \\ n!2^{n(n-1)/2} & \text{if } r = n; \\ (n+1)!2^{n(n-1)/2}(2^n-1)/2 & \text{if } r = n+1. \end{cases}$$

Let $S(j)$ denote the set of positions containing 1 in the binary representation of j, so that, for example, $S(13) = S((1101)_2) = \{1, 3, 4\}$). Write $j \preceq k$ when $S(j) \subseteq S(k)$. Consider a set of boxes labeled $1, \ldots, 2^n - 1$. For $0 \le i \le n-1$, let G_i be the set of boxes with labels $2^i, \ldots, 2^{i+1} - 1$. Note that $|G_i| = 2^i$ for $i \ge 0$.

Let $f(j)$ be the number of placements of r distinguishable balls into $\bigcup_{i \in S(j)} G_i$. Clearly $f(j) = j^r$. Let $g(j)$ be the number of such placements in which, for each $i \in S(j)$, at least one ball is in at least one box in G_i. By the Inclusion-Exclusion Principle, $g(k) = \sum_{j \preceq k} (-1)^{\eta(k)-\eta(j)} f(j)$. In particular, $g(2^n - 1) = \sum_{j=0}^{2^n-1} (-1)^{n-\eta(j)} j^r$.

Since $S(2^n - 1) = \{1, \ldots, n\}$, the distributions counted by $g(2^n - 1)$ are those with all n groups nonempty. When $r < n$, there are none. When $r = n$, one of the 2^i boxes in G_i is used, for each i. When $r = n + 1$, we distribute n balls and then one more, dividing by 2 to eliminate overcounting. Thus both sides of the identity equal $g(2^n - 1)$.

Solution II by Richard Stong, Rice University, Houston, TX.
(a) Letting $\Delta_r f(y) = f(y+r) - f(y)$, the left side of the identity is $\Delta_1 \Delta_2 \Delta_4 \cdots \Delta_{2^{k-2}} y^k$. If f is a polynomial of degree n with leading coefficient c, then $\Delta_r f$ is a polynomial of degree $n-1$ with leading coefficient crn. Since we have applied $k-1$ such operators, the left side of the identity is a polynomial of degree 1 with leading coefficient $k!2^{0+1+\cdots+(k-2)} = k!2^{(k-1)(k-2)/2}$.

Since $\eta(2^{k-1} - 1 - i) = k - 1 - \eta(i)$, the terms for $j = i$ and $j = 2^{k-1} - 1 - i$ in the sum exactly cancel if $y = -(2^{k-1} - 1)/2$. Thus the left side of the identity is the polynomial of degree 1 with leading coefficient $k!2^{(k-1)(k-2)/2}$ that vanishes at $y = -(2^{k-1} - 1)/2$. This agrees with the right side.
(b) If $k = 1$, then $s = 3$ suffices, because the identities $n = (n+2+m) - (1+m) - 1$ and $n = (m+2+n) - (m+1-n) - (1-n)$ give suitable representations for all $m \ge 1$ in the cases $n \ge 0$ and $n < 0$, respectively.

Now consider $k \ge 2$. Let $M = k!2^{(k-1)(k-2)/2}$, and let $s(k) = M + 2^{k-1}$. Given any integer n, choose integers q and r such that $n = (2^{k-1} - 1)M/2 + Mq + r$, where $0 \le r < M$. Let $x_1, \ldots, x_r$ be multiples of M, and let $x_{r+1}, \ldots, x_M$ be numbers congruent to 1 modulo M. Now $n + \sum_{i=1}^M x_i^k$ is congruent to $(2^{k-1} - 1)M/2$ modulo M. Thus for some y we have $n + \sum_{i=1}^M x_i^k = M(y + (2^{k-1} - 1)/2)$. The identity in **(a)** now yields

$$n = \sum_{j=0}^{2^{k-1}-1} (-1)^{k-1-\eta(j)} (y+j)^k - \sum_{i=1}^M x_i^k.$$

The numbers $x_1, \ldots, x_M$ were chosen arbitrarily subject to congruence conditions modulo M; there are infinitely many such choices. By fixing $x_1, \ldots, x_{M-1}$ and making x_M sufficiently large, we can ensure that y exceeds x_M, since $k \ge 2$. Thus we have infinitely many choices in which $x_1, \ldots, x_M, y, y+1, \ldots, y + 2^{k-1} - 1$ are distinct, as desired.

Solved also by K. McInturff, J. H. Lindsey II, GCHQ Problems Group (U. K.), R. J. Chapman (U. K.), and the proposers. Part **(a)** solved also by D. Beckwith.