



Binary Expansions and k th Powers: 10596

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Solution by the proposer. If B_i is the midpoint of $A_{i+1}A_{i+2}$ for $i = 1, 2, 3$, then triangles $A_1A_2A_3$ and $B_1B_2B_3$ are similar, so $|B_1B_2| = (1/2)|A_1A_2|$, $|B_2B_3| = (1/2)|A_2A_3|$, and $|B_3B_1| = (1/2)|A_3A_1|$. Hence

$$|A_1B_2| + |B_1B_2| = \frac{1}{2}|A_1A_3| + \frac{1}{2}|A_1A_2| = |B_1B_3| + |A_1B_3|,$$

and similarly for the other conditions of both parts.

(a) We prove that for any triangle $B_1B_2B_3$ there exists exactly one triangle $A_1A_2A_3$ such that $|A_iB_{i+1}| + |B_iB_{i+1}| = |A_iB_{i+2}| + |B_iB_{i+2}|$ for $i = 1, 2, 3$. This implies our assertion. Fix a triangle $B_1B_2B_3$, and suppose that for a triangle $A_1A_2A_3$ the conditions are satisfied. Let (i, j, k) be a permutation of $(1, 2, 3)$. Consider the hyperbola with foci B_j and B_k passing through B_i . Since $|A_iB_j| + |B_iB_j| = |A_iB_k| + |B_iB_k|$, the hyperbola passes through A_i . Write h_i for the part of the branch of the hyperbola passing through A_i that is on the opposite side of the line B_jB_k from B_i . Since B_j and B_k are the foci of the hyperbola, h_i is entirely contained in the union of all lines joining A_i and some point on the segment B_jB_k .

Now suppose that A is any point on h_1 different from A_1 . (This A is a candidate for the vertex A_1 in a new triangle satisfying the conditions.) If A is inside triangle $B_2A_1B_3$, then the line from A through B_2 intersects h_3 in a point P that is on the opposite side of the line A_2A_3 from A_1 , and if A is outside of $B_2A_1B_3$ then P is on the same side of A_2A_3 as A_1 . (Point P is the candidate for point A_3 of the new triangle.) The same holds for the intersection Q of the line AB_3 with h_2 (the candidate for A_2 of the new triangle). Therefore, the line segment PQ does not pass through B_1 . We conclude that A cannot be a vertex of a triangle that satisfies our requirements. A similar argument shows that no point A outside triangle $B_2A_1B_3$ can be a vertex of a triangle that satisfies our requirements. Thus $A_1A_2A_3$ is the only triangle for which the conditions hold.

(b) Let $a_k = (1/2)|A_iA_j|$, $b_k = |B_iB_j|$, and $x_j = a_j - |A_iB_j|$, where (i, j, k) is an even permutation of $(1, 2, 3)$. By hypothesis, $a_i + x_i + a_k - x_k = b_i + b_k$. Adding two of these equations and subtracting the third yields $b_i = a_i - x_j + x_k$, so

$$b_i^2 = a_i^2 + x_j^2 + x_k^2 - 2a_ix_j - 2x_jx_k + 2a_ix_k \quad (1)$$

By the law of cosines we obtain $b_i^2 = (a_j + x_j)^2 + (a_k - x_k)^2 - 2(a_j + x_j)(a_k - x_k) \cos A_i$. Since $\cos A_i = \frac{a_j^2 + a_k^2 - a_i^2}{2a_ja_k}$ we get after simple transformations

$$b_i^2 = a_i^2 + x_j^2 + x_k^2 + \frac{x_j}{a_j}(a_j^2 + a_i^2 - a_k^2) - \frac{x_k}{a_k}(a_k^2 + a_i^2 - a_j^2) + \frac{x_j}{a_j} \frac{x_k}{a_k}(a_j^2 + a_k^2 - a_i^2) \quad (2)$$

Let $z_i = x_i/a_i$. Comparing expressions (1) and (2) for b_i^2 , we get

$$z_j(a_j + a_i - a_k) - z_k(a_k + a_i - a_j) + z_jz_k(a_j + a_k - a_i) = 0.$$

If one of the z_i 's is 0, then all of them vanish. If they are all nonzero, then dividing by z_jz_k and adding all three equalities we get $a_1 + a_2 + a_3 = 0$, which is evidently false. Therefore, all the x_i 's vanish and the assertion is proved.

Solved also by M. Vowe (Switzerland) and GCHQ Problems Group (U. K.).

Binary Expansions and k th Powers

10596 [1997, 456]. *Proposed by Paul Bateman, University of Illinois, Urbana, IL, and David Bradley, Simon Fraser University, Burnaby, BC, Canada.*

(a) Prove the identity

$$\sum_{j=0}^{2^{k-1}-1} (-1)^{k-1-\eta(j)} (y+j)^k = k! \cdot 2^{(k-1)(k-2)/2} (y + (2^{k-1} - 1)/2),$$

where $\eta(j)$ is the number of ones in the binary expansion of the nonnegative integer j .

(b) Use part (a) to infer that there is a positive integer $s = s(k)$ such that every integer n is expressible in the form $n = \epsilon_1 x_1^k + \epsilon_2 x_2^k + \cdots + \epsilon_s x_s^k$ in infinitely many ways, where $\epsilon_i = \pm 1$ for $1 \leq i \leq s$ and where x_1, x_2, \dots, x_s are distinct positive integers.

Solution I to part (a) by David Callan, University of Wisconsin, Madison, WI. With $n = k - 1$, equating coefficients of y reduces the proposed identity to

$$\sum_{j=0}^{2^n-1} (-1)^{n-\eta(j)} j^r = \begin{cases} 0 & \text{if } r < n; \\ n!2^{n(n-1)/2} & \text{if } r = n; \\ (n+1)!2^{n(n-1)/2}(2^n-1)/2 & \text{if } r = n+1. \end{cases}$$

Let $S(j)$ denote the set of positions containing 1 in the binary representation of j , so that, for example, $S(13) = S((1101)_2) = \{1, 3, 4\}$. Write $j \leq k$ when $S(j) \subseteq S(k)$. Consider a set of boxes labeled $1, \dots, 2^n - 1$. For $0 \leq i \leq n - 1$, let G_i be the set of boxes with labels $2^i, \dots, 2^{i+1} - 1$. Note that $|G_i| = 2^i$ for $i \geq 0$.

Let $f(j)$ be the number of placements of r distinguishable balls into $\bigcup_{i \in S(j)} G_i$. Clearly $f(j) = j^r$. Let $g(j)$ be the number of such placements in which, for each $i \in S(j)$, at least one ball is in at least one box in G_i . By the Inclusion-Exclusion Principle, $g(k) = \sum_{j \leq k} (-1)^{\eta(k)-\eta(j)} f(j)$. In particular, $g(2^n - 1) = \sum_{j=0}^{2^n-1} (-1)^{n-\eta(j)} j^r$.

Since $S(2^n - 1) = \{1, \dots, n\}$, the distributions counted by $g(2^n - 1)$ are those with all n groups nonempty. When $r < n$, there are none. When $r = n$, one of the 2^i boxes in G_i is used, for each i . When $r = n + 1$, we distribute n balls and then one more, dividing by 2 to eliminate overcounting. Thus both sides of the identity equal $g(2^n - 1)$.

Solution II by Richard Stong, Rice University, Houston, TX.

(a) Letting $\Delta_r f(y) = f(y+r) - f(y)$, the left side of the identity is $\Delta_1 \Delta_2 \Delta_4 \cdots \Delta_{2^{k-2}} y^k$. If f is a polynomial of degree n with leading coefficient c , then $\Delta_r f$ is a polynomial of degree $n - 1$ with leading coefficient crn . Since we have applied $k - 1$ such operators, the left side of the identity is a polynomial of degree 1 with leading coefficient $k!2^{0+1+\cdots+(k-2)} = k!2^{(k-1)(k-2)/2}$.

Since $\eta(2^{k-1} - 1 - i) = k - 1 - \eta(i)$, the terms for $j = i$ and $j = 2^{k-1} - 1 - i$ in the sum exactly cancel if $y = -(2^{k-1} - 1)/2$. Thus the left side of the identity is the polynomial of degree 1 with leading coefficient $k!2^{(k-1)(k-2)/2}$ that vanishes at $y = -(2^{k-1} - 1)/2$. This agrees with the right side.

(b) If $k = 1$, then $s = 3$ suffices, because the identities $n = (n + 2 + m) - (1 + m) - 1$ and $n = (m + 2 + n) - (m + 1 - n) - (1 - n)$ give suitable representations for all $m \geq 1$ in the cases $n \geq 0$ and $n < 0$, respectively.

Now consider $k \geq 2$. Let $M = k!2^{(k-1)(k-2)/2}$, and let $s(k) = M + 2^{k-1}$. Given any integer n , choose integers q and r such that $n = (2^{k-1} - 1)M/2 + Mq + r$, where $0 \leq r < M$. Let x_1, \dots, x_r be multiples of M , and let x_{r+1}, \dots, x_M be numbers congruent to 1 modulo M . Now $n + \sum_{i=1}^M x_i^k$ is congruent to $(2^{k-1} - 1)M/2$ modulo M . Thus for some y we have $n + \sum_{i=1}^M x_i^k = M(y + (2^{k-1} - 1)/2)$. The identity in (a) now yields

$$n = \sum_{j=0}^{2^{k-1}-1} (-1)^{k-1-\eta(j)} (y+j)^k - \sum_{i=1}^M x_i^k.$$

The numbers x_1, \dots, x_M were chosen arbitrarily subject to congruence conditions modulo M ; there are infinitely many such choices. By fixing x_1, \dots, x_{M-1} and making x_M sufficiently large, we can ensure that y exceeds x_M , since $k \geq 2$. Thus we have infinitely many choices in which $x_1, \dots, x_M, y, y + 1, \dots, y + 2^{k-1} - 1$ are distinct, as desired.

Solved also by K. McInturff, J. H. Lindsey II, GCHQ Problems Group (U. K.), R. J. Chapman (U. K.), and the proposers. Part (a) solved also by D. Beckwith.