A Supremum of Sine Differences

10604 [1997, 567]. *Proposed by Joseph Rosenblatt, University of Illinois, Urbana, IL.*
(**a**) Determine positive constants c and C such that if $0 < a < b$ then

$$c\left(1-\frac{a}{b}\right) \le \sup_{x>0}\left|\frac{\sin(ax)}{ax}-\frac{\sin(bx)}{bx}\right| \le C\left(1-\frac{a}{b}\right). \qquad (*)$$

(**b**)* What are the largest constant c and smallest constant C such that $(*)$ holds whenever $0 < a < b$?

Solution of part (**a**) *by Thomas Hermann, SDRC, Milford, OH.* We may take $c = 2/\pi$ and $C = 4$. To see this, let $\varphi(x) = \sin x/x$, $\lambda = a/b$, and $\mu(\lambda, x) = \big(\varphi(x) - \varphi(\lambda x)\big)/(1-\lambda)$. The problem is to find positive lower and upper bounds for $M(\lambda) = \sup_{x>0}|\mu(\lambda, x)|$. Since $|\varphi(x)| \le 1$ for all x,

$$|\mu(\lambda, x)| \le \frac{2}{1-\lambda} \le 4 \qquad (1)$$

when $0 < \lambda \le 1/2$. By the Mean Value Theorem, there is a $y \in [\lambda x, x]$, such that $\mu(\lambda, x)/x = \varphi'(y)$. Now $x\varphi'(y) = (x/y)(\cos y - \varphi(y))$, so

$$|\mu(\lambda, x)| \le 2\,\frac{x}{y} \le \frac{2}{\lambda} \le 4 \qquad (2)$$

when $\frac{1}{2} \le \lambda < 1$. Combining (1) and (2), we obtain $M(\lambda) \le 4$ for all λ in $(0, 1)$.

To get a lower bound, observe that $\mu(\lambda, \pi) = \sin(\lambda\pi)/\,(\lambda(1-\lambda)\pi) = \mu(1-\lambda, \pi)$, so it is enough to consider the case when $\lambda \in (0, 1/2]$. Since $\varphi(x)$ is decreasing on $(0, \pi/2]$, $\mu(\lambda, \pi) \ge \big(1/(1-\lambda)\big)\,\big(\sin(\pi/2)/(\pi/2)\big) \ge 2/\pi$. Therefore $2/\pi \le M(\lambda) \le 4$.

Editorial comment. For part (**b**), John H. Lindsey II and the GCHQ Problems Group independently computed that the largest value for c is approximately 1.0631036 and the smallest value for C is approximately 1.3805662. Lindsey used Maple to search for these values and used estimates on derivatives to prove that the optimal value of C satisfies $1.380566167 \le C \le 1.380577012$.

Part (a) also solved by R. J. Chapman (U.K.), T. Hermann, J. H. Lindsey II, GCHQ Problems Group (U.K.), and the proposer.

A Convergent Series

10657 [1998, 366]. *Proposed by Jovan Vukmirović, University of Belgrade, Belgrade, Yugoslavia.* Let ϕ be a strictly increasing function from $(0, \infty)$ onto $(0, \infty)$. Prove that if $\sum_{n=1}^{\infty} 1/(n\phi^{-1}(n))$ converges, then $\sum_{n=1}^{\infty}\phi(n)x^n$ converges for $|x| < 1$.

Solution by Kenneth Schilling, University of Michigan, Flint, MI. For $x > 1$,

$$\sum_{n=\lfloor\sqrt{x}\rfloor}^{\lfloor x\rfloor}\frac{1}{n} > \int_{\sqrt{x}}^{x}\frac{1}{t}dt = \frac{1}{2}\ln x.$$

Thus, since ϕ is increasing, we have

$$\frac{\ln x}{2\phi^{-1}(x)} < \sum_{n=\lfloor\sqrt{x}\rfloor}^{\lfloor x\rfloor}\frac{1}{n\phi^{-1}(x)} < \sum_{n=\lfloor\sqrt{x}\rfloor}^{\lfloor x\rfloor}\frac{1}{n\phi^{-1}(n)}.$$

By hypothesis, this expression converges to 0 as $x \to \infty$. Hence $\lim_{x\to\infty}\ln x/\phi^{-1}(x) = 0$. Since $\phi(x) \to \infty$ as $x \to \infty$, we get $\lim_{x\to\infty}\ln\phi(x)/x = 0$. Exponentiation yields $\lim_{x\to\infty}(\phi(x))^{1/x} = 1$. Thus the power series converges for $|x| < 1$ by the root test.

Solved also by S. Amghibech (France), J. Arregui, G. L. Body (U. K.), D. Borwein (Canada), D. Bradley (Canada), K. Dale & I. Skau (Norway), J. Fitch, K. Ford, G. L. Isaacs, P. Lang, J. H. Lindsey II, A. Stenger, T. V. Trif (Romania), T. Trimble, R. Vermes (Canada), C. Y. Yildirim (Turkey), GCHQ Problems Group (U. K.), and the proposer.